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STOCHASTIC COMPARISONS OF INTERFAILURE TIMES UNDER A RELEVATION REPLACEMENT POLICY

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Abstract

We provide some results for the comparison of the failure times and interfailure times of two systems based on a replacement policy proposed by Kapodistria and Psarrakos (2012). In particular, we show that when the first failure times are ordered in terms of the dispersive order (or, the excess wealth order), then the successive interfailure times are ordered in terms of the usual stochastic order (respectively, the increasing convex order). As a consequence, we provide comparison results for the cumulative residual entropies of the systems and their dynamic versions.

Keywords: Stochastic order; dispersive order; excess wealth order; increasing convex order; cumulative residual entropy

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1. Introduction and motivation

Stochastic processes are important tools used in many areas of science and engineering to describe the evolution of systems over time. In this framework, the failure time and the mean time between failures are often used as a basis to make decisions regarding the reliability of a system. Therefore, results which give stochastic comparisons of the failure times or of the interfailure times can be useful in reliability theory.

In this paper we focus on a stochastic process introduced by Kapodistria and Psarrakos (2012) which has recently received some interest among researchers (see, e.g. Psarrakos and Navarro (2013), Burkschat and Navarro (2014), Di Crescenzo and Longobardi (2015), and Di Crescenzo and Toomaj (2015)). For a system governed by this process, the mean time between failures coincides with the cumulative residual entropy, a measure of uncertainty introduced in Rao *et al.* (2004) as an alternative to the Shannon entropy. The purpose of this paper is to provide conditions on the first failure times of two such processes under which the successive failure times, the interfailure times, and, consequently, the cumulative residual entropies, are stochastically ordered in various senses. We also provide conditions to compare the dynamic versions of the cumulative residual entropies of the systems.

Specifically, let X be a nonnegative continuous random variable with distribution function F. Based on X, Kapodistria and Psarrakos (2012) constructed a sequence of stochastically increasing random variables $\{X_n,\ n \geq 1\}$, where n is a positive integer, thus,

$$X_1 \stackrel{\mathrm{D}}{=} X, \qquad [X_{n+1} - X_n \mid X_n = t] \stackrel{\mathrm{D}}{=} [X_n - t \mid X_n > t], \qquad n \geq 1,\ t > 0, \qquad (1.1)$$

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where $[X \mid A]$ denotes a random variable having the same distribution of X conditioned on A and '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution. Note that the construction of this sequence is based on the relevation transform introduced by Krakowski (1973); see also Baxter (1982). The tail distributions associated to this process satisfy the recursive scheme

$$\bar{F}_1(t) = \bar{F}(t), \qquad \bar{F}_{n+1}(t) = [1 + \Lambda_{X_n}(t)]\bar{F}_n(t), \qquad n \geq 1,$$

where $\Lambda_{X_n}(t) = -\log \bar{F}_n(t)$ is the cumulative hazard function of X_n. In reliability, this sequence of random variables represents a process that describes the successive failures of a component, which, on failure, is replaced by a component of equal age, but the life distribution of the nth component is assumed to be identical to the distribution of the time until the nth failure. In this sense, at every failure instant, the component is instantly restored to its condition immediately prior to failure and its lifetime distribution function is prolonged.

It has been proved (see Corollary 4.1 in Kapodistria and Psarrakos (2012)) that the means of the sequence of random variables $\{X_n,\ n \geq 1\}$ satisfy the equality

$$\mathbb{E}[X_{n+1}] = \mathbb{E}[X_n] + \varepsilon(X_n), \tag{1.2}$$

where $\varepsilon(X) = -\int_0^\infty \bar{F}(t) \log \bar{F}(t)\,\mathrm{d}t$ is the cumulative residual entropy (CRE) of X, a measure of uncertainty introduced in Rao *et al.* (2004) and studied, among others, by Rao (2005), Asadi and Zohrevand (2007), and Navarro *et al.* (2010). The CRE is a measure that extends the notion of Shannon entropy from the discrete to the continuous setting and has some important properties. For example, it is always nonnegative, it vanishes if and only if X is degenerated at some value, and it is shift-independent (that is, if $Y = aX + b$, with $a > 0$ and $b \geq 0$, then $\varepsilon(Y) = a\varepsilon(X)$); see Di Crescenzo and Longobardi (2009). These properties suggest that $\varepsilon(X)$ could also be considered as a measure of the dispersion of the random variable X. At this point, following the requirements suggested by Bickel and Lehmann (1976) for a functional to be a dispersion measure, it is of interest to study the consistency of CRE with the dispersive order, a partial order that formalizes the intuition that a random variable X is less variable than Y.

Definition 1.1. Let X and Y be two random variables with respective distribution functions F and G. Then X is said to be smaller than Y in the dispersive order (denoted by $X \leq_{\mathrm{disp}} Y$) if

$$F^{-1}(p) - F^{-1}(q) \leq G^{-1}(p) - G^{-1}(q) \quad \text{for all } 0 < q < p < 1.$$

This order compares X and Y by variability, because it requires the difference between any two quantiles of X to be smaller than the corresponding quantiles of Y.

The starting point of this paper is the observation that the CRE is, in fact, consistent with the dispersive order, that is, if $X \leq_{\mathrm{disp}} Y$ then $\varepsilon(X) \leq \varepsilon(Y)$. Moreover, if $\{X_n,\ n \geq 1\}$ is the sequence (1.1) based on X, and $\{Y_n,\ n \geq 1\}$ is a similar sequence based on Y defined by

$$Y_1 \stackrel{\mathrm{D}}{=} Y, \qquad [Y_{n+1} - Y_n \mid Y_n = t] \stackrel{\mathrm{D}}{=} [Y_n - t \mid Y_n > t], \qquad n \geq 1,\ t > 0, \tag{1.3}$$

with tail distributions

$$\bar{G}_1(t) = \bar{G}(t), \qquad \bar{G}_{n+1}(t) = [1 + \Lambda_{Y_n}(t)]\bar{G}_n(t), \qquad n \geq 1,$$

it will be shown in Section 2 that

$$X \leq_{\mathrm{disp}} Y \quad \Longrightarrow \quad \varepsilon(X_n) \leq \varepsilon(Y_n) \quad \text{for } n \geq 2. \tag{1.4}$$

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This suggests, taking into account (1.2), that the size of the random variable $X_{n+1} - X_n$ is a measure of the variability of X (the more variable X, the bigger differences $X_{n+1} - X_n$, for $n \geq 1$) which leads us to observe that (1.4) has a nice extension of the form

$$X \leq_{(1)} Y \quad \Longrightarrow \quad X_{n+1} - X_n \leq_{(2)} Y_{n+1} - Y_n \quad \text{for } n \geq 1, \tag{1.5}$$

where '$\leq_{(1)}$' denotes some variability order and '$\leq_{(2)}$' denotes some location order. It will be shown that conjecture (1.5) holds when $\leq_{(1)}$ is the dispersive order (or, the excess wealth order) and $\leq_{(2)}$ is the stochastic order (respectively, the increasing convex order). We define these orders below.

In reliability systems, a matter of interest is how the age t of an item affects the information about its residual life. To study this topic, Asadi and Zohrevand (2007) generalized the CRE by considering a dynamic cumulative residual entropy (DCRE) defined by

$$\varepsilon(X, t) = -\int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} \log \frac{\bar{F}(x)}{\bar{F}(t)} \,\mathrm{d}x, \qquad t \geq 0.$$

It is easy to see that $\varepsilon(X, t) = \varepsilon(X_t)$, where $X_t = \{X - t \mid X > t\}$ is the residual lifetime of X at t (in particular, when $F(0) = 0$, $\varepsilon(X, 0) = \varepsilon(X)$). Applications and properties of these measures can be found in Navarro *et al.* (2010), Baratpour (2010), Kapodistria and Psarrakos (2012), Baratpour and Habibi Rad (2016), and Chamany and Baratpour (2014). We provide conditions in Section 3 under which $\varepsilon(X, t)$ is a measure consistent with the dispersive order. In addition, given the sequences $\{X_n,\ n \geq 1\}$ and $\{Y_n,\ n \geq 1\}$ defined, respectively, by (1.1) and (1.3), we study sufficient conditions to order, for all t, the DCRE of X_n and Y_n.

Next, we define the partial stochastic orders considered in this paper (see the books by Shaked and Shanthikumar (2007), Müller and Stoyan (2002), and Belzunce *et al.* (2016a) for general properties and applications).

Definition 1.2. Let X and Y be two random variables with respective distribution functions F and G and survival functions $\bar{F}$ and $\bar{G}$.

(i) X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{\mathrm{st}} Y$) if $\bar{F}(t) \leq \bar{G}(t)$ for all t or, equivalently, if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ for all increasing functions ϕ such that the expectations exist.

(ii) X is said to be smaller than Y in the hazard rate order (denoted by $X \leq_{\mathrm{hr}} Y$) if $\bar{G}(t)/\bar{F}(t)$ increases in t. When X and Y are two absolutely continuous random variables with densities $f(t)$ and $g(t)$ and hazard rate functions $r_X(t) = f(t)/\bar{F}(t)$ and $r_Y(t) = g(t)/\bar{G}(t)$, this is equivalent to saying that $r_X(t) \geq r_Y(t)$ for all t.

(iii) X is said to be smaller than Y in the increasing convex order (denoted by $X \leq_{\mathrm{icx}} Y$) if

$$\int_t^\infty \bar{F}(x)\,\mathrm{d}x \leq \int_t^\infty \bar{G}(x)\,\mathrm{d}x \quad \text{for all } t,$$

or, equivalently, if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ for all increasing convex functions ϕ such that the expectations exist.

(iv) X is said to be smaller than Y in the excess wealth order (denoted by $X \leq_{\mathrm{ew}} Y$) if

$$\int_{F^{-1}(p)}^\infty \bar{F}(x)\,\mathrm{d}x \leq \int_{G^{-1}(p)}^\infty \bar{G}(x)\,\mathrm{d}x \quad \text{for } p \in (0, 1). \tag{1.6}$$

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It is well known that

$$X \le_{\rm hr} Y \quad\Longrightarrow\quad X \le_{\rm st} Y \quad\Longrightarrow\quad X \le_{\rm icx} Y \quad\text{and}\quad X \le_{\rm disp} Y \quad\Longrightarrow\quad X \le_{\rm ew} Y.$$

The rest of the paper is organized as follows. In Section 2 we consider the processes (1.1) and (1.3) and obtain conditions, in terms of the dispersive order and the excess wealth order of their first failure times, to compare, in various stochastic senses, the successive failure times, the interfailure times, and the cumulative residual entropies of the systems. In Section 3 we compare their corresponding DCREs and provide examples that shed further light on the theoretical results. Section 4 contains the conclusions. Throughout this paper, the random variables X and Y are assumed to be nonnegative.

2. Comparing the time between successive failures

Consider the sequences $\{X_n,\ n \ge 1\}$ and $\{Y_n,\ n \ge 1\}$ given, respectively, by (1.1) and (1.3). Kapodistria and Psarrakos (2012) showed that

$$\varepsilon(X_n) = \mathbb{E}[X_{n+1} - X_n] = \int_0^\infty |\bar{F}_n(t) - \bar{F}_{n+1}(t)|\,\mathrm{d}t, \qquad n \ge 1. \tag{2.1}$$

Observe that (2.1) can be written as

$$\varepsilon(X_n) = \int_0^\infty |\bar{F}_n(t) - h(\bar{F}_n(t))|\,\mathrm{d}t, \qquad n \ge 1,$$

where $h(t) = t - t\log t$ is a distortion function, that is, a nondecreasing function from [0, 1] to [0, 1] such that $h(0) = 0$ and $h(1) = 1$. It has been shown by López-Díaz *et al.* (2012) that the Wasserstein distance between a random variable X and its distortion is a measure of the variability of X that satisfies the conditions of Bickel and Lehmann (1976). In particular, from Proposition 2.7 in López-Díaz *et al.* (2012), it follows that if $X_n \le_{\rm disp} Y_n$ then $\varepsilon(X_n) \le \varepsilon(Y_n)$ for $n \ge 1$. Moreover, taking into account that $h(t)$ is a concave distortion function, it follows from Yang *et al.* (2014), the next stronger result:

$$X_n \le_{\rm ew} Y_n \quad\Longrightarrow\quad \varepsilon(X_n) \le \varepsilon(Y_n) \quad \text{for } n \ge 1. \tag{2.2}$$

The implication (2.2) tells us that $\varepsilon(X_n)$ is a measure of the variability of X_n. In order to assert that $\varepsilon(X_n)$ measures the variability of the initial X (rather than X_n), we require the following result.

Theorem 2.1. *Let X and Y be two continuous random variables with strictly increasing distribution functions F and G, respectively. Then, the following hold.*

(i) *If $X \le_{\rm disp} Y$ then $X_n \le_{\rm disp} Y_n$ for $n \ge 2$.*

(ii) *If $X \le_{\rm ew} Y$ then $X_n \le_{\rm ew} Y_n$ for $n \ge 2$.*

Proof. In order to prove (i), note that

$$G_n^{-1}F_n(x) = G^{-1}F(x) \quad \text{for all } x \text{ and for } n \ge 1. \tag{2.3}$$

The result follows from (2.3) by using the fact that $X \le_{\rm disp} Y$ is equivalent to saying that $G^{-1}F(x) - x$ increases in x (see Section 3.B. in Shaked and Shanthikumar (2007)). To prove (ii), assume that (1.6) holds. A change of variable shows that this is the same as

$$\int_t^\infty \bar{F}(x)\,\mathrm{d}[G^{-1}F(x) - x] \ge 0.$$

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From Proposition 2.1 in Kapodistria and Psarrakos (2012) we know that the sequence $\{X_n, n \geq 1\}$ is stochastically ordered in the likelihood ratio order, that is, it verifies $X_1 \leq_{\text{lr}} X_2 \leq \cdots$. This implies (by using Theorem 1.C.1 in Shaked and Shanthikumar (2007)) that $\bar{F}_2(x)/\bar{F}(x)$ increases in x. By Lemma 7.1(a) of Barlow and Proschan (1975), we have

$$\int_t^\infty \bar{F}_2(x)\,\mathrm{d}[G^{-1}F(x) - x] \geq 0,$$

or, equivalently,

$$\int_t^\infty \bar{F}_2(x)\,\mathrm{d}[G_2^{-1}F_2(x) - x] \geq 0,$$

which means that $X_2 \leq_{\text{ew}} Y_2$. The result follows by induction. □

From Theorem 2.1, together with (2.2), we have

$$X \leq_{\text{ew}} Y \quad \Longrightarrow \quad \varepsilon(X_n) \leq \varepsilon(Y_n) \quad \text{for } n \geq 1. \tag{2.4}$$

Since the dispersive order implies the excess wealth order, it follows from (2.4) that $\varepsilon(X_n)$ is a measure of the variability of X in the sense of Bickel and Lehmann (1976) for all $n \geq 1$. In order to obtain a stronger result, we first provide the following characterization of the dispersive order. The proof follows along the same lines as the proof of Lemma 3.1 in Belzunce *et al.* (2016b), and is, therefore, omitted.

Lemma 2.1. *Let X and Y be two continuous random variables with strictly increasing distribution functions F and G, respectively, and let $h = F^{-1}G$. Then $X \leq_{\text{disp}} Y$ if and only if*

$$\mathbb{E}[\phi(h(Y) - h(x)) \mid Y > x] \leq \mathbb{E}[\phi(Y - x) \mid Y > x]$$

for all x and for any increasing function ϕ.

We also need the following lemma.

Lemma 2.2. *Let X and Y be two continuous random variables. Then, given the sequences of random variables (1.1) and (1.3), the random vectors (X_{n-1}, X_n) and (Y_{n-1}, Y_n), with $n \geq 2$, have the same copula.*

Proof. First observe that, from the continuity of X and Y, it follows that the marginal distributions of the vectors (X_{n-1}, X_n) and (Y_{n-1}, Y_n) are also continuous, and the copula is unique. Now, for $0 \leq u,\ v \leq 1$, the copula of (Y_{n-1}, Y_n) is given by

$$\begin{aligned} C_{(Y_{n-1},Y_n)}(u, v) &= \mathbb{P}[Y_{n-1} \leq G_{n-1}^{-1}(u), Y_n \leq G_n^{-1}(v)] \\ &= \int_0^{G_{n-1}^{-1}(u)} \mathbb{P}[Y_n \leq G_n^{-1}(v) \mid Y_{n-1} = s]\,\mathrm{d}G_{n-1}(s). \end{aligned} \tag{2.5}$$

Using the change of variable $G_{n-1}(s) = t$ and taking into account that

$$[Y_n \mid Y_{n-1} = s] \stackrel{\text{D}}{=} [Y_{n-1} \mid Y_{n-1} > s], \qquad n \geq 1,\ s > 0,$$

we see that the integral (2.5) is the same as

$$\int_0^u \mathbb{P}[Y_{n-1} \leq G_n^{-1}(v) \mid Y_{n-1} > G_{n-1}^{-1}(t)]\,\mathrm{d}t.$$

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Similarly, the copula $C_{(X_{n-1},X_n)}(u,v)$ of (X_{n-1},X_n) is given by

$$\int_0^u \mathbb{P}[X_{n-1} \le F_n^{-1}(v) \mid X_{n-1} > F_{n-1}^{-1}(t)]\,\mathrm{d}t.$$

Using (2.3), we see that $G_{n-1}G_n^{-1}(v) = F_{n-1}F_n^{-1}(v)$ for $0 \le v \le 1$, and, therefore,

$$\begin{aligned}\mathbb{P}[Y_{n-1} \le G_n^{-1}(v) \mid Y_{n-1} > G_{n-1}^{-1}(t)] &= \begin{cases} \dfrac{G_{n-1}(G_n^{-1}(v)) - t}{1-t}, & t < G_{n-1}(G_n^{-1}(v)), \\ 0, & t > G_{n-1}(G_n^{-1}(v)), \end{cases} \\ &= \mathbb{P}[X_{n-1} \le F_n^{-1}(v) \mid X_{n-1} > F_{n-1}^{-1}(t)],\end{aligned}$$

which means that $C_{(X_{n-1},X_n)}(u,v) = C_{(Y_{n-1},Y_n)}(u,v)$ for $0 \le u, v \le 1$. □

From (2.1) and the chain of implications

$$X \le_{\mathrm{st}} Y \quad\Longrightarrow\quad X \le_{\mathrm{icx}} Y \quad\Longrightarrow\quad \mathbb{E}[X] \le \mathbb{E}[Y],$$

it is clear that the following result is stronger than (2.4).

Theorem 2.2. *Let X and Y be two continuous random variables with strictly increasing distribution functions F and G, respectively. Then, the following hold.*

(i) *If $X \le_{\mathrm{disp}} Y$ then $X_n - X_{n-1} \le_{\mathrm{st}} Y_n - Y_{n-1}$ for $n \ge 2$.*

(ii) *If $X \le_{\mathrm{ew}} Y$ then $X_n - X_{n-1} \le_{\mathrm{icx}} Y_n - Y_{n-1}$ for $n \ge 2$.*

Proof. (i) First we prove that for an increasing function ϕ, it holds that

$$\mathbb{E}[\phi(X_n - X_{n-1})] \le \mathbb{E}[\phi(Y_n - Y_{n-1})] \quad \text{for } n \ge 2.$$

In order to establish this, let us consider the strictly increasing function $h = F^{-1}G = F_n^{-1}G_n$ for $n \ge 2$. Observe that, on the one hand,

$$X_n \overset{\mathrm{D}}{=} h(Y_n), \qquad n \ge 1,$$

and, on the other hand, the vectors

$$(X_{n-1}, X_n) \quad \text{and} \quad (h(Y_{n-1}), h(Y_n)), \qquad n \ge 2,$$

have the same copula (this follows from Lemma 2.2 and Theorem 2.4.3 in Nelsen (1999)). Consequently,

$$\phi(X_n - X_{n-1}) \overset{\mathrm{D}}{=} \phi(h(Y_n) - h(Y_{n-1})), \qquad n \ge 2.$$

Therefore, for $n \ge 2$, we have

$$\begin{aligned}\mathbb{E}[\phi(X_n - X_{n-1})] &= \mathbb{E}[\phi(h(Y_n) - h(Y_{n-1}))] \\ &= \int \mathbb{E}[\phi(h(Y_n) - h(Y_{n-1})) \mid Y_{n-1} = t]\,\mathrm{d}G_{n-1}(t).\end{aligned}$$

By using (1.3), this integral is the same as

$$\int \mathbb{E}[\phi(h(Y_{n-1}) - h(t)) \mid Y_{n-1} > t]\,\mathrm{d}G_{n-1}(t). \tag{2.6}$$

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From $X \leq_{\text{disp}} Y$, Theorem 2.1(i), and Lemma 2.1, we have

$$\mathbb{E}[\phi(h(Y_{n-1}) - h(t)) \mid Y_{n-1} > t] \leq \mathbb{E}[\phi(Y_{n-1} - t) \mid Y_{n-1} > t]. \tag{2.7}$$

Now, it follows from (2.7) that (2.6) is less than or equal to

$$\int \mathbb{E}[\phi(Y_{n-1} - t) \mid Y_{n-1} > t] \, \mathrm{d}G_{n-1}(t). \tag{2.8}$$

By using (1.3) again, we see that (2.8) is equal to

$$\int \mathbb{E}[\phi(Y_n - Y_{n-1}) \mid Y_{n-1} = t] \, \mathrm{d}G_{n-1}(t) = \mathbb{E}[\phi(Y_n - Y_{n-1})],$$

which completes the proof of (i).

(ii) The proof follows the same lines as the proof of (i) by considering an increasing and convex function ϕ (rather than simply increasing) and using Theorem 2.1(ii) and Lemma 3.1 of Belzunce *et al.* (2016b) rather than Theorem 2.1(i) and Lemma 2.1. □

Remark 2.1. As an anonymous referee pointed out, Theorem 2.2(i) admits an alternative proof which is independent from Lemma 2.1, based on the following considerations. It follows from the first lines of the proof of Theorem 2.2(i) that

$$X_n - X_{n-1} \stackrel{\mathrm{D}}{=} h(Y_n) - h(Y_{n-1}), \qquad n \geq 2,$$

with $h = F^{-1}G$. Since $Y_n \geq Y_{n-1}$ almost sure, the desired result is then obtained using Equation (3.B.15) in Shaked and Shanthikumar (2007) with $h = \varphi$.

3. Comparing dynamic cumulative residual entropies

When the hazard rate of X or Y is increasing and X and Y are ordered in the dispersive order, then the DCRE of X_n and Y_n can be ordered for all n and for all t. We can even prove a stronger result which is useful when the initial X or Y are not increasing failure rate (IFR) but, after some iterations, we observe that X_n or Y_n is IFR (we provide examples of this case below).

Theorem 3.1. *Let X and Y be two absolutely continuous random variables with the same left-end points of the supports. If $X \leq_{\text{disp}} Y$ and there exists a positive integer n_0 such that X_{n_0} or Y_{n_0} is IFR, then $\varepsilon(X_n, t) \leq \varepsilon(Y_n, t)$ for all t and $n \geq n_0$.*

Proof. If we denote by $r_{X_n}(t)$ the hazard rate function of X_n, it is easy to prove that

$$r_{X_{n+1}}(t) = \frac{\Lambda_{X_n}(t)}{1 + \Lambda_{X_n}(t)} r_{X_n}(t), \qquad n \geq 1, \tag{3.1}$$

where $\Lambda_{X_n}(t) = -\log \bar{F}_n(t)$. Recalling that $\Lambda_{X_n}(t)$ is increasing and rewriting (3.1) as

$$r_{X_{n+1}}(t) = \frac{1}{1/\Lambda_{X_n(t)} + 1} r_{X_n}(t), \qquad n \geq 1,$$

it follows that if $r_{X_n}(t)$ is increasing then $r_{X_{n+1}}(t)$ is increasing and, by induction, $r_{X_m}(t)$ is increasing for some positive integer $m > n$. Consequently, if X_{n_0} is IFR (the case when Y_{n_0} is IFR is analogous) then

$$X_n \text{ is IFR} \quad \text{for } n \geq n_0. \tag{3.2}$$

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On the other hand, from Theorem 2.1, it follows that if $X \leq_{\text{disp}} Y$ then

$$X_n \leq_{\text{disp}} Y_n \quad \text{for all } n. \tag{3.3}$$

Under the assumptions on the supports, it follows from (3.2), (3.3), and Theorem 3.4 of Belzunce *et al.* (1997) that $X_{n,t} \leq_{\text{disp}} Y_{n,t}$ for $n \geq n_0$, where $X_{n,t} = [X_n - t \mid X_n > t]$ and $Y_{n,t} = [Y_n - t \mid Y_n > t]$ are, respectively, the residual lifetimes of X_n and Y_n at t. Now, it follows from (2.4) that $\varepsilon(X_{n,t}) \leq \varepsilon(Y_{n,t})$ for all t, or, equivalently, $\varepsilon(X_n, t) \leq \varepsilon(Y_n, t)$ for all t. $\square$

In what follows, we provide two examples where the random variable $X \overset{\text{D}}{=} X_1$ is not IFR, while the distribution X_n is IFR for some $n \geq n_0 > 1$.

Example 3.1. Let $X \overset{\text{D}}{=} X_1$ be a mixture of an Exp(2) and a gamma(2, 1) random variable with density function

$$f_1(t) = \left(\tfrac{2}{3}\right)2\mathrm{e}^{-2t} + \tfrac{1}{3}t\mathrm{e}^{-t}, \qquad t \geq 0,$$

tail distribution

$$\bar{F}_1(t) = \tfrac{2}{3}\mathrm{e}^{-2t} + \tfrac{1}{3}(1+t)\mathrm{e}^{-t},$$

and hazard rate function $r_{X_1}(t) = f_1(t)/\bar{F}_1(t)$. In Figure 1 we illustrate the hazard rate functions $r_{X_1}(t)$, $r_{X_2}(t)$, and $r_{X_3}(t)$ for $0 \leq t \leq 20$. We observe that $r_{X_1}(t)$ is not monotonic; in particular, it is decreasing for $t < t_0$ and then increasing for $t > t_0$, where $t_0 \cong 2.37$. Moreover, $r_{X_2}(t)$ is increasing, and, hence, $r_{X_3}(t)$ is also increasing. Furthermore,

$$r_{X_1}(t) \geq r_{X_2}(t) \geq r_{X_3}(t), \qquad r_{X_1}(0) = \tfrac{4}{3}, \qquad r_{X_2}(0) = r_{X_3}(0) = 0,$$

and

$$\lim_{t\to\infty} r_{X_1}(t) = \lim_{t\to\infty} r_{X_2}(t) = \lim_{t\to\infty} r_{X_3}(t) = 1.$$

Example 3.2. Let $X \overset{\text{D}}{=} X_1$ be a mixture of a Rayleigh(1) and a gamma(2, 1) random variable with density function

$$f_1(t) = \left(\tfrac{1}{2}\right)2t\mathrm{e}^{-t^2} + \tfrac{1}{2}t\mathrm{e}^{-t}, \qquad t \geq 0,$$

tail distribution

$$\bar{F}_1(t) = \tfrac{1}{2}\mathrm{e}^{-t^2} + \tfrac{1}{2}(1+t)\mathrm{e}^{-t},$$

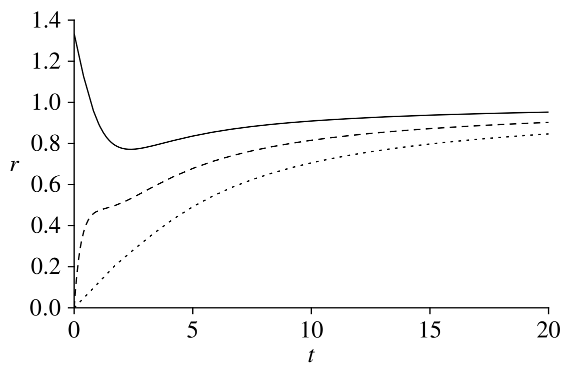

FIGURE 1: The hazard rate functions $r_{X_1}(t)$ (*solid*), $r_{X_2}(t)$ (*dashed*), and $r_{X_3}(t)$ (*dotted*), where $r_{X_1}(t) \geq r_{X_2}(t) \geq r_{X_3}(t)$, in Example 3.1, for $0 \leq t \leq 20$.

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and hazard rate function $r_{X_1}(t) = f_1(t)/\bar{F}_1(t)$. In Figure 2, we illustrate, for $0 \le t \le 20$, the hazard rate functions $r_{X_1}(t)$, $r_{X_2}(t)$, $r_{X_3}(t)$, and $r_{X_4}(t)$. We see that $r_{X_1}(t) \ge r_{X_2}(t) \ge r_{X_3}(t) \ge r_{X_4}(t)$. Moreover, the hazard rates $r_{X_1}(t)$ and $r_{X_2}(t)$ are not monotonic, while $r_{X_3}(t)$ and $r_{X_4}(t)$ are increasing functions. Furthermore,

$$r_{X_1}(0) = r_{X_2}(0) = r_{X_3}(0) = r_{X_4}(0) = 0,$$
$$\lim_{t\to\infty} r_{X_1}(t) = \lim_{t\to\infty} r_{X_2}(t) = \lim_{t\to\infty} r_{X_3}(t) = \lim_{t\to\infty} r_{X_4}(t) = 1.$$

When $n_0 = 1$, from Theorem 3.1 we have the following corollary.

Corollary 3.1. *Let X and Y be two continuous random variables with the same left-end points of the supports. If X or Y is IFR and $X \le_{\mathrm{disp}} Y$ then $\varepsilon(X_n, t) \le \varepsilon(Y_n, t)$ for all t.*

The proof of the following result is similar to the proof of Theorem 3.1 using Theorem 3.2 (rather than Theorem 3.4) in Belzunce *et al.* (1997), and is, therefore, omitted.

Theorem 3.2. *Let X and Y be two continuous random variables with the same left-end points of the supports. If X or Y is decreasing failure rate (DFR) and $X \le_{\mathrm{hr}} Y$ then $\varepsilon(X_n, t) \le \varepsilon(Y_n, t)$ for all t.*

In the following example we show that the sufficient conditions in Corollary 3.1 and Theorem 3.2 are not necessary. In particular, we provide an example where X is IFR, Y is DFR, and $X \not\le_{\mathrm{hr}} Y$, but $\varepsilon(X, t) \le \varepsilon(Y, t)$ for all t.

Example 3.3. We consider the random variables X and Y following a Rayleigh(1) and a Pareto(1, 3) supported on $[0, \infty)$, with density, tail, and hazard rate functions given by

$$f(t) = 2t\mathrm{e}^{-t^2}, \qquad \bar{F}(t) = \mathrm{e}^{-t^2}, \qquad r_X(t) = 2t,$$

and

$$g(t) = \frac{3}{(t+1)^4}, \qquad \bar{G}(t) = \frac{1}{(t+1)^3}, \qquad r_Y(t) = \frac{1}{t+1},$$

respectively. The dynamic cumulative entropies of X and Y (see Navarro and Psarrakos (2017)), are given by

$$\varepsilon(X, t) = -\int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} \log \frac{\bar{F}(x)}{\bar{F}(t)} \,\mathrm{d}x = \frac{1}{2}\int_0^\infty x(x+t^2)^{-1/2}\mathrm{e}^{-x}\,\mathrm{d}x$$

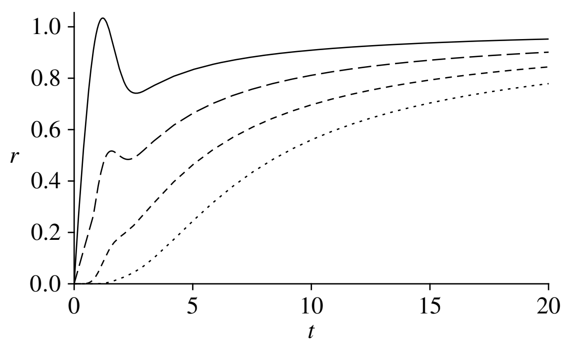

FIGURE 2: The hazard rate functions $r_{X_1}(t)$ (*solid*), $r_{X_2}(t)$ (*long dashed*), $r_{X_3}(t)$ (*short dashed*), and $r_{X_4}(t)$ (*dotted*), where $r_{X_1}(t) \ge r_{X_2}(t) \ge r_{X_3}(t) \ge r_{X_4}(t)$, in Example 3.2, for $0 \le t \le 20$.

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and

$$\varepsilon(Y,t) = -\int_t^\infty \frac{\bar{G}(x)}{\bar{G}(t)} \log \frac{\bar{G}(x)}{\bar{G}(t)}\, \mathrm{d}x = \frac{3}{4}(t+1).$$

It is clear that the function $\varepsilon(X,t)$ is decreasing, while the function $\varepsilon(Y,t)$ is (linearly) increasing with respect to t. Moreover, since $\varepsilon(X,0) = \sqrt{\pi}/4 < \frac{3}{4} = \varepsilon(Y,0)$, we have $\varepsilon(X,t) < \varepsilon(Y,t)$ for all t. In the Figure 3(a) we illustrate, for $0 \le t \le 5$, the hazard rate functions $r_X(t)$ and $r_Y(t)$, while in Figure 3(b), we illustrate the respective dynamic cumulative residual entropies. We observe that the hazard rate functions have an intersection point, while $\varepsilon(Y,t)$ is always greater than $\varepsilon(X,t)$. It follows from Theorem 3.B.20 of Shaked and Shanthikumar (2007) that $X \not\le_{\text{disp}} Y$.

In what follows, to avoid confusion, we denote by $\bar{F}_{X_n}$ and $\bar{G}_{Y_n}$ (rather than $\bar{F}_n$ and $\bar{G}_n$) the respective survival functions of X_n and Y_n. Our purpose is to provide a sufficient condition for the stochastic comparisons of the differences between two successive mean residual life functions of the sequences X_n and Y_n, denoted by

$$m_{X_n}(t) = \mathbb{E}(X_n - t \mid X_n > t) = \frac{\int_t^\infty \bar{F}_{X_n}(x)\,\mathrm{d}x}{\bar{F}_{X_n}(t)}, \qquad t \ge 0, \tag{3.4}$$

and

$$m_{Y_n}(t) = \mathbb{E}(Y_n - t \mid Y_n > t) = \frac{\int_t^\infty \bar{G}_{Y_n}(x)\,\mathrm{d}x}{\bar{G}_{Y_n}(t)}, \qquad t \ge 0,$$

respectively. Recall, from Proposition 4.3 in Kapodistria and Psarrakos (2012), that the equilibrium tail of X_{n+1} is written as

$$\bar{F}^{\mathrm{e}}_{X_{n+1}}(t) = \frac{\int_t^\infty \bar{F}_{X_{n+1}}(x)\,\mathrm{d}x}{\mathbb{E}(X_{n+1})} = \frac{\mathbb{E}(X_n)[1+\Lambda_{X_n}(t)]\bar{F}^{\mathrm{e}}_{X_n}(t) + \varepsilon(X_n,t)\bar{F}_{X_n}(t)}{\mathbb{E}(X_n)+\varepsilon(X_n,t)}. \tag{3.5}$$

Analogously, the equilibrium tail of Y_{n+1} is written as

$$\bar{G}^{\mathrm{e}}_{Y_{n+1}}(t) = \frac{\int_t^\infty \bar{G}_{Y_{n+1}}(x)\,\mathrm{d}x}{\mathbb{E}(Y_{n+1})} = \frac{\mathbb{E}(Y_n)[1+\Lambda_{Y_n}(t)]\bar{G}^{\mathrm{e}}_{Y_n}(t) + \varepsilon(Y_n,t)\bar{G}_{Y_n}(t)}{\mathbb{E}(Y_n)+\varepsilon(Y_n,t)}.$$

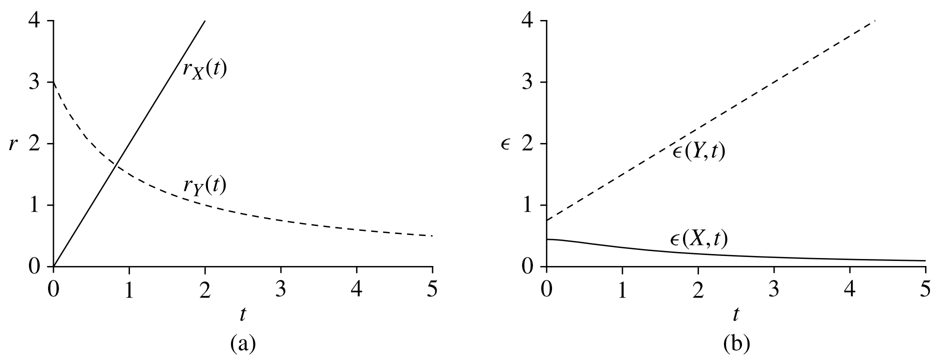

Figure 3: (a) The hazard rate functions $r_X(t)$ (*solid*) and $r_Y(t)$ (*dashed*); (b) and the dynamic residual cumulative entropies $\varepsilon(X,t)$ and $\varepsilon(Y,t)$, in Example 3.3, for $0 \le t \le 5$.

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Theorem 3.3. *For $t \geq 0$ and a fixed $n \geq 1$, if $X_n \leq_{\mathrm{st}} Y_n$ and $\varepsilon(X_n, t) \leq \varepsilon(Y_n, t)$, then $m_{X_{n+1}}(t) - m_{X_n}(t) \leq m_{Y_{n+1}}(t) - m_{Y_n}(t)$.*

Proof. By (1.2) and (3.5), we obtain

$$\begin{aligned} m_{X_{n+1}}(t) &= \mathbb{E}(X_{n+1} - t \mid X_{n+1} > t) \\ &= \mathbb{E}(X_{n+1}) \frac{\bar{F}^{\mathrm{e}}_{X_{n+1}}(t)}{\bar{F}_{X_{n+1}}(t)} \\ &= \frac{\mathbb{E}(X_n)[1 + \Lambda_{X_n}(t)]\bar{F}^{\mathrm{e}}_{X_n}(t) + \varepsilon(X_n, t)\bar{F}_{X_n}(t)}{[1 + \Lambda_{X_n}(t)]\bar{F}_{X_n}(t)} \\ &= m_{X_n}(t) + \frac{\varepsilon(X_n, t)}{1 + \Lambda_{X_n}(t)}. \end{aligned} \tag{3.6}$$

Similarly, we have

$$m_{Y_{n+1}}(t) = m_{Y_n}(t) + \frac{\varepsilon(Y_n, t)}{1 + \Lambda_{Y_n}(t)}. \tag{3.7}$$

Since, by hypothesis, $X_n \leq_{\mathrm{st}} Y_n$ and $\varepsilon(X_n, t) \leq \varepsilon(Y_n, t)$, it follows that $\Lambda_{X_n}(t) \geq \Lambda_{Y_n}(t)$ and $[1 + \Lambda_{X_n}(t)]^{-1}\varepsilon(X_n, t) \leq [1 + \Lambda_{Y_n}(t)]^{-1}\varepsilon(Y_n, t)$. By (3.6) and (3.7) the result follows. $\square$

4. Conclusions

For a stochastic process satisfying (1.1), it is well known (see Kapodistria and Psarrakos (2012)) that the CRE coincides with the mean time between failures. In this paper we have considered two processes satisfying such conditions, and we have shown that when the first failure times are ordered in terms of the dispersive order, or the excess wealth order, respectively, then the corresponding mean time between failures are ordered in the usual stochastic order and the increasing convex order, respectively. Consequently, the respective cumulative residual entropies are also ordered. We have also provided conditions to order their respective dynamic cumulative residual entropies for all t when one of the first failure rates are IFR or DFR.

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