J. Appl. Probab. **59,** 983–991 (2022)
*doi:*10.1017/jpr.2021.100

EXPONENTIAL CONVERGENCE TO A QUASI-STATIONARY DISTRIBUTION FOR BIRTH–DEATH PROCESSES WITH AN ENTRANCE BOUNDARY AT INFINITY

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Abstract

We study the quasi-stationary behavior of the birth–death process with an entrance boundary at infinity. We give by the h-transform an alternative and simpler proof for the exponential convergence of conditioned distributions to a unique quasi-stationary distribution in the total variation norm. In addition, we also show that starting from any initial distribution the conditional probability converges to the unique quasi-stationary distribution exponentially fast in the ψ-norm.

Keywords: Birth–death processes; quasi-stationary distribution; h-transform; rate of convergence

2020 Mathematics Subject Classification: Primary 60J80
Secondary 60B10; 37A25

1. Introduction and main results

Let $X = (X_t, t \geq 0)$ be a continuous-time birth–death process taking values in $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$, where 0 is an absorbing state and $\mathbb{N} = \{1, 2, \ldots\}$ is an irreducible transient class. Its jump rate matrix $(q_{ij}, i, j \in \mathbb{Z}_+)$ satisfies

$$q_{ij} = \begin{cases} b_i & \text{if } j = i+1, i \geq 0, \\ d_i & \text{if } j = i-1, i \geq 1, \\ -(b_i + d_i) & \text{if } j = i, i \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where the birth rates $(b_i, i \in \mathbb{N})$ and death rates $(d_i, i \in \mathbb{N})$ are strictly positive, and $d_0 = b_0 = 0$.

Consider $\pi = (\pi_i, i \in \mathbb{N})$ with the coefficients

$$\pi_1 = 1, \qquad \pi_i = \frac{b_1 b_2 \cdots b_{i-1}}{d_2 d_3 \cdots d_i}, \qquad i \geq 2. \tag{1.1}$$

Received 16 February 2021; revision received 26 November 2021.
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Then, we have $b_i\pi_i = d_{i+1}\pi_{i+1}$ for $i \in \mathbb{N}$, which implies that the process X is reversible with respect to π, that is, for all $i, j \in \mathbb{N}$, $\pi_i q_{ij} = \pi_j q_{ji}$. Put

$$A = \sum_{i=1}^{\infty} \frac{1}{b_i\pi_i}, \quad B = \sum_{i=1}^{\infty} \pi_i, \quad R = \sum_{i=1}^{\infty} \frac{1}{b_i\pi_i} \sum_{j=1}^{i} \pi_j, \quad S = \sum_{i=1}^{\infty} \frac{1}{b_i\pi_i} \sum_{j=i+1}^{\infty} \pi_j, \tag{1.2}$$

so we have

$$R + S = AB, \qquad A = \infty \Rightarrow R = \infty, \qquad S < \infty \Rightarrow B < \infty.$$

In this paper, we assume that the birth–death process is surely killed at 0, that is, for all $i \in \mathbb{N}$, $\mathbb{P}_i(T_0 < \infty) = 1$, where $T_0 = \inf\{t \geq 0 : X_t = 0\}$ is the absorption time of the process X, and $\mathbb{P}_i$ denotes the probability measure of the process when the initial state is i. It is well known (see, e.g., [10]) that the assumption on sure killing, $\mathbb{P}_i(T_0 < \infty) = 1$ for all $i \in \mathbb{N}$, is equivalent to $A = \infty$. Note that $A = \infty$ implies the process X is non-explosive. Let $(P_t)_{t\geq 0}$ be the semigroup of the process X before killing at 0. Then, for all $i \in \mathbb{N}$, $P_t f(i) = \mathbb{E}_i[f(X_t), T_0 > t]$, and it acts on the set of bounded measurable functions defined on $\mathbb{N}$. Here, $\mathbb{E}_i$ denotes the expectation with respect to $\mathbb{P}_i$.

For such a process, we are interested in the asymptotic behavior of the process conditioned on long-term survival. A well-studied object (see, e.g., [6, 12]) is the *quasi-stationary distribution*, that is, a probability measure α on $\mathbb{N}$ such that, for any $t \geq 0$,

$$\mathbb{P}_\alpha(X_t \in \cdot | T_0 > t) = \alpha. \tag{1.3}$$

Here, as usual, $\mathbb{P}_\alpha := \sum_{i\in\mathbb{N}} \alpha_i \mathbb{P}_i$. If there exists a probability measure μ on $\mathbb{N}$ such that

$$\lim_{t\to\infty} \mathbb{P}_\mu(X_t \in \cdot | T_0 > t) = \alpha, \tag{1.4}$$

then we say that μ is attracted to α, or is in the domain of attraction of α, for the conditional evolution. For any bounded and measurable function f on $\mathbb{N}$, (1.3) can also be written as

$$\frac{\alpha(P_t f)}{\alpha(P_t \mathbf{1})} = \alpha(f),$$

where $\mathbf{1} = \mathbb{1}_{\mathbb{N}}$ and $\alpha(f) = \sum_{i\in\mathbb{N}} \alpha_i f(i)$.

The existence, uniqueness, and other properties of quasi-stationary distributions for birth–death processes has been extensively studied in past decades. On quasi-stationary distributions of birth–death processes, van Doorn [16] gave the following picture of the situation: there is no quasi-stationary distribution, a unique quasi-stationary distribution, or an infinite continuum of quasi-stationary distributions. Zhang and Zhu [18] proved that the unique quasi-stationary distribution attracts all initial distributions supported in $\mathbb{N}$. Villemonais [17] provided some new results on the domain of attraction of the minimal quasi-stationary distribution. Until now, no one has completely solved the problem of the domains of attraction of the infinite continuum of quasi-stationary distributions for birth–death processes. However, existence and uniqueness of a quasi-stationary distribution or attraction of all initial distributions do not imply uniform convergence. This paper is devoted to studying the speed of convergence of the conditional probability measure $\mathbb{P}_\mu(X_t \in \cdot | T_0 > t)$, for some initial measures μ on $\mathbb{N}$, towards the quasi-stationary distribution α when t goes to infinity. The *total variation distance* is usually used to quantify the weak convergence (1.4) – see, e.g., [2, 3, 11] – defined as $\|\mu - \nu\|_{TV} := \sup_{f\in\mathcal{B}_1(\mathbb{N})} |\mu(f) - \nu(f)|$, where μ, ν are any two probability measures on $\mathbb{N}$,

$\mathcal{B}_1(\mathbb{N})$ denotes the set of bounded measurable functions defined on $\mathbb{N}$ such that $\|f\|_\infty \leq 1$, and $\|f\|_\infty = \sup_{i\in\mathbb{N}} |f(i)|$. Other distances, for example the 1-Wasserstein distance [13], can also be used to quantify the weak convergence (1.4).

The boundary point ∞ is called the entrance boundary if $R = \infty$, $S < \infty$. When ∞ is an entrance boundary, the following result was obtained by Martnez et al. [11, Theorem 2]; this work proves that implication (ii) implies (iii) by using the h-transform (or Doob's h-transform). Here, h is the eigenfunction of the first nontrivial eigenvalue of the infinitesimal operator of the original absorbed process.

Theorem 1.1. *For the birth–death process X satisfying $A = \infty$, the following statements are equivalent:*

(i) $S < \infty$.

(ii) There exists a unique quasi-stationary distribution.

(iii) There exist a probability measure α and two constants C', $\gamma > 0$ such that, for all $t \geq 0$ and all probability measures μ on $\mathbb{N}$, $\left\|\mathbb{P}_\mu(X_t \in \cdot | T_0 > t) - \alpha\right\|_{\mathrm{TV}} \leq C' \mathrm{e}^{-\gamma t}$.

Moreover, in (iii) *the distribution α is the unique quasi-stationary distribution of X.*

It is already known (see [16]) that (i) and (ii) are equivalent. This equivalence is only mentioned here for completeness. For any measurable function $\psi : \mathbb{N} \to [1, +\infty)$, the ψ-norm of a signed measure μ is defined as $\|\mu\|_\psi := \sup_{|f|\leq\psi} |\mu(f)|$. If $\psi = 1$, then the ψ-norm is the total variation norm. Let $\|\cdot\|_2$ be the $\mathbb{L}^2(m)$-norm, defined by $\|f\|_2 = \left(\sum_{i\in\mathbb{N}} m_i f^2(i)\right)^{1/2}$. Here, m is the unique stationary distribution of the Q-process Y defined in Section 2. For convenience, we denote

$$\eta(i) := Q_i(\lambda_c), \qquad \eta \circ \mu_i := \frac{\mu_i \eta(i)}{\mu(\eta)}, \tag{1.5}$$

where $Q_i(\lambda_c)$ is defined in Section 2 and denotes the eigenfunction of the first nontrivial eigenvalue of the infinitesimal operator of the process X. We write $\mathbb{E}_\mu$ for the expectation with respect to $\mathbb{P}_\mu$. Further, we have the following result.

Theorem 1.2. *Let X be a birth and death process satisfying $A = \infty$ and $S < \infty$. Assume that there exists a function $\psi : \mathbb{N} \to [1, +\infty)$ such that $\alpha(\psi^2) < +\infty$, where α is the unique quasi-stationary distribution of X. Then, for any probability measure μ on $\mathbb{N}$, there exist t_μ and $\varepsilon > 0$ such that, for any $t \geq t_\mu$,*

$$\sup_{|f|\leq\psi} |\mathbb{E}_\mu[f(X_t) \mid T_0 > t] - \alpha(f)| \leq \max\{C_1, C_2\}\left[\alpha\left(\frac{\psi^2}{\eta}\right)\right]^{\frac{1}{2}} \left\|\frac{\mathrm{d}(\eta\circ\mu)}{\mathrm{d}(\eta\circ\alpha)} - 1\right\|_2 \mathrm{e}^{-\varepsilon t},$$

where

$$C_1 = \left(1 + \frac{1+\alpha(\psi)}{1-b}\right), \qquad C_2 = 2 + \alpha(\psi),$$

and b is a constant on $(0, 1)$.

The rest of this paper is organized as follows. In Section 2 we present some preliminaries that will be needed later. In Section 3 we give the proof of Theorem 1.1. The proof of Theorem 1.2 is given in Section 4.

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2. Preliminaries

In this section we introduce the Q-process as an h-transform for the sub-Markovian semigroup $(P_t)_{t\geq 0}$, and preliminary facts which will be used later.

Let $(Q_i(x), i \geq 0)$ be a sequence of birth–death polynomials satisfying the recurrence relation

$$\begin{aligned} &Q_0(x) = 0, \qquad Q_1(x) = 1, \\ &b_i Q_{i+1}(x) - (b_i + d_i) Q_i(x) + d_i Q_{i-1}(x) = -x Q_i(x), \qquad i \in \mathbb{N}. \end{aligned} \tag{2.1}$$

We write $P_{ij}(t) = \mathbb{P}_i(X_t = j)$. Under our assumptions, from [1, Theorem 5.1.9] we know that there exists a parameter $\lambda_c \geq 0$, called the decay parameter of the process X, such that, for all $i, j \in \mathbb{N}$, $\lambda_c = -\lim_{t\to\infty} (1/t) \log P_{ij}(t)$. To ensure the existence of quasi-stationary distributions, the decay parameter is usually required to be strictly greater than 0. On the existence and uniqueness of quasi-stationary distributions for birth–death processes, we have the following exact and detailed results.

Theorem 2.1. (van Doorn [16].) *Let X be a birth–death process satisfying $A = \infty$.*

(i) If $\lambda_c = 0$, then there is no quasi-stationary distribution.

(ii) If $S = \infty$ and $\lambda_c > 0$, then there is a one-parameter family of quasi-stationary distributions given by $\alpha_\lambda(i) = (\pi_i/d_1)\lambda Q_i(\lambda)$ for $\alpha_\lambda(i)$, $0 < \lambda \leq \lambda_c$, $i \in \mathbb{N}$.

(iii) If $S < \infty$ then $\lambda_c > 0$ and there is precisely one quasi-stationary distribution given by

$$\alpha = \left(\alpha_{\lambda_c}(i) = \frac{\pi_i \eta(i)}{\pi(\eta)} = \frac{\pi_i}{d_1} \lambda_c \eta(i),\ i \in \mathbb{N} \right).$$

Remark 2.1. According to [5], we know that $\lambda_c > 0$ if and only if

$$\delta := \sup_{n\geq 1} \sum_{i=1}^{n} \frac{1}{d_i \pi_i} \sum_{j=n}^{\infty} \pi_j < \infty.$$

The process X conditioned to never be absorbed, usually referred to as the Q-process, defined by $Y = (Y_t, t \geq 0)$, plays a key role in the proofs of our main results. If $\lambda_c > 0$, we know from [6, Proposition 5.9] that the Q-process Y, whose law starting from $i \in \mathbb{N}$ is given by $\mathbb{Q}_i(Y_{s_1} = i_1, \ldots, Y_{s_k} = i_k) = \lim_{t\to\infty} \mathbb{P}_i(X_{s_1} = i_1, \ldots, X_{s_k} = i_k \mid T_0 > t)$, is a Markov chain taking values in $\mathbb{N}$, with transition kernel, for all $i, j \in \mathbb{N}$,

$$\mathbb{Q}_i(Y_s = j) = \mathrm{e}^{\lambda_c s} \frac{\eta(j)}{\eta(i)} \mathbb{P}_i(X_s = j). \tag{2.2}$$

Let $(Q_t)_{t\geq 0}$ be the semigroup of the process Y under $\mathbb{Q}$. For all bounded and measurable functions f on $\mathbb{N}$ and $t \geq 0$, the equality (2.2) implies that, for all $i \in \mathbb{N}$,

$$Q_t f(i) = \frac{\mathrm{e}^{\lambda_c t}}{\eta(i)} P_t(\eta f)(i). \tag{2.3}$$

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From (2.3), we have, for all $i \in \mathbb{N}$,

$$P_t f(i) = \eta(i) \mathrm{e}^{-\lambda_c t} Q_t \left(\frac{f}{\eta} \right)(i). \tag{2.4}$$

According to [6, Chapter 5], we know that the process Y is still a birth–death process taking values in $\mathbb{N}$ with birth and death parameters given, for all $i \in \mathbb{N}$, by

$$\widetilde{b}_i = \frac{\eta(i+1)}{\eta(i)} b_i, \qquad \widetilde{d}_i = \frac{\eta(i-1)}{\eta(i)} d_i.$$

We can compute the coefficients $\widetilde{\pi} = (\widetilde{\pi}_i, i \in \mathbb{N})$ analogous to (1.1):

$$\widetilde{\pi}_1 = 1, \qquad \widetilde{\pi}_i = \frac{\widetilde{b}_1 \widetilde{b}_2 \cdots \widetilde{b}_{i-1}}{\widetilde{d}_2 \widetilde{d}_3 \cdots \widetilde{d}_i} = \eta^2(i) \pi_i, \qquad i \geq 2. \tag{2.5}$$

We define $\widetilde{P}_{ij}(t) = \mathbb{Q}_i(Y_t = j)$ for all $i, j \in \mathbb{N}$. Then, by (2.2) and (2.5), we get $\widetilde{\pi}_i \widetilde{P}_{ij}(t) = \widetilde{\pi}_j \widetilde{P}_{ji}(t)$ for all $i, j \in \mathbb{N}$. Namely, the process Y is reversible with respect to $\widetilde{\pi}$.

For the birth–death process X satisfying $A = \infty$ and $S < \infty$, we know from [8, 9] that $\eta(i)$ is strictly increasing with $i \in \mathbb{N}$. When $i \in \mathbb{N}$, from (2.1), we see that $\eta(i)$ has the minimum value 1. Furthermore, we also have the following result.

Proposition 2.1. ([8], Lemma 3.4.) *Let X be a birth and death process satisfying $A = \infty$ and $S < \infty$. Then $\eta(\infty) := \lim_{i \to \infty} \eta(i) < \infty$.*

Proposition 2.1 plays a key role in the proofs of our main results. From Proposition 2.1 we get $\mu(\eta) < \infty$, so (1.5) is well-defined.

We can see that one of the main features of the Q-process Y is that it is an h-transform of the original absorbed process X. The equality (2.3) naturally suggests the use of the h-transform to deduce quasi-stationary properties. This general method has been used successfully in, for example, [7, 13, 14, 15]. Here, we also use the h-transform to study the quasi-stationarity of birth–death processes.

3. Proof of Theorem 1.1

We only need to show that (ii) and (iii) are equivalent. If (iii) holds, then there exists a unique quasi-stationary distribution and the distribution α defined in Theorem 2.1 is the unique quasi-stationary distribution. That is, (ii) holds.

If (ii) holds then $S < \infty$, so we know from the proof of [9, Theorem 3.1] that the Q-process Y is strongly ergodic, which means that $\lim_{t \to \infty} \sup_i \sum_{j \in \mathbb{N}} |\widetilde{P}_{ij}(t) - m_j| = 0$, where $m = (m_j = \pi_j \eta^2(j)/\pi(\eta^2), j \in \mathbb{N})$ is the unique stationary distribution of the process Y. It is well known (see, e.g., [1]) that strong ergodicity implies exponential ergodicity. So, if the process Y is strongly ergodic, then there exist two constants $C, \gamma > 0$ such that, for any $i \in \mathbb{N}$,

$$\|\mathbb{Q}_i(Y_t \in \cdot) - m\|_{\mathrm{TV}} \leq C\mathrm{e}^{-\gamma t}. \tag{3.1}$$

According to Proposition 2.1, we know that when $i \in \mathbb{N}$, $1 \leq \eta(i) \leq \eta(\infty) < \infty$. Therefore, if $f(i)$ is a bounded and measurable function on $\mathbb{N}$, then $f(i)/\eta(i)$ is also a bounded and measurable function on $\mathbb{N}$. Thus, from (2.4), for all $t \geq 0$, all probability measure μ on $\mathbb{N}$, and $f \in \mathcal{B}_1(\mathbb{N})$, we have

$$\mathbb{E}_\mu[f(X_t) \mid T_0 > t] = \frac{\mu(P_t f)}{\mu(P_t \mathbf{1})} = \frac{\mathrm{e}^{-\lambda_c t} \mu(\eta Q_t(f/\eta))}{\mathrm{e}^{-\lambda_c t} \mu(\eta Q_t(\mathbf{1}/\eta))} = \frac{(\eta \circ \mu) Q_t(f/\eta)}{(\eta \circ \mu) Q_t(\mathbf{1}/\eta)}.$$

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Note that

$$m(f/\eta) = \alpha(f)\frac{\pi(\eta)}{\pi(\eta^2)} = \frac{\alpha(f)}{\alpha(\eta)} \le \alpha(f).$$

So, for any $f \in \mathcal{B}_1(\mathbb{N})$, by (3.1) we get

$$\begin{aligned} &|(\eta \circ \mu)Q_t(f/\eta) - \alpha(f)| \le |(\eta \circ \mu)Q_t(f/\eta) - m(f/\eta)| \le C\mathrm{e}^{-\gamma t}, \\ &|(\eta \circ \mu)Q_t(\mathbf{1}/\eta) - 1| \le C\mathrm{e}^{-\gamma t}. \end{aligned} \tag{3.2}$$

Therefore, combining the inequalities in (3.2), for any $t > (\log C)/\gamma$ we have

$$\frac{\alpha(f) - C\mathrm{e}^{-\gamma t}}{1 + C\mathrm{e}^{-\gamma t}} \le \mathbb{E}_\mu[f(X_t) \mid T_0 > t] \le \frac{\alpha(f) + C\mathrm{e}^{-\gamma t}}{1 - C\mathrm{e}^{-\gamma t}}.$$

From (3.2) we have relations of the type $\mathbb{E}_\mu[f(X_t) \mid T_0 > t] = a(t)/b(t)$, with $a(t) = \alpha(f) + \delta(t)$, $b(t) = 1 + \epsilon(t)$, and $\max\{|\delta(t)|, |\epsilon(t)|\} \le C\mathrm{e}^{-\gamma t}$. Then, it suffices to use the expansion

$$\frac{1}{1+\epsilon(t)} = 1 - \epsilon(t) + \frac{\epsilon^2(t)}{1+\epsilon(t)}$$

to get that $|\mathbb{E}_\mu[f(X_t) \mid T_0 > t] - \alpha(f)|$ is bounded by $C'\mathrm{e}^{-\gamma t}$ for some finite constant C', and the result follows straightforwardly.

4. Proof of Theorem 1.2

In this section we give the proof of Theorem 1.2, which is similar to [14, Theorem 2.1] where the author considered the exponential convergence of conditioned distributions to a quasi-stationary distribution in total variation and in 1-Wasserstein distance for general Markov processes under several difficult-to-check conditions. For birth–death processes we have a much simpler and explicit condition. Our more restricted context enables us to obtain a more detailed result.

We only consider initial measures μ on $\mathbb{N}$ such that $\|(\mathrm{d}(\eta \circ \mu)/\mathrm{d}m) - 1\|_2 < +\infty$, since if $\|(\mathrm{d}(\eta \circ \mu)/\mathrm{d}m) - 1\|_2 = +\infty$ then Theorem 1.2 is trivially satisfied. Recall that if the birth–death process X satisfies $A = \infty$ and $S < \infty$, then the Q-process Y is strongly ergodic. Thus, we know from [4, Theorem 1.1] that $(Q_t)_{t\ge 0}$ converges exponentially in the $\mathbb{L}^2(m)$-norm, i.e. there is a positive ε such that, for all $f \in \mathbb{L}^2(m)$ and $t \ge 0$,

$$\|Q_t f - m(f)\|_2 \le \|f - m(f)\|_2 \mathrm{e}^{-\varepsilon t}. \tag{4.1}$$

Note that η is bounded on $\mathbb{N}$ and has the minimum value 1, so if f is a measurable function on $\mathbb{N}$ such that $|f| \le \psi$ and $\alpha(\psi^2) < +\infty$, then f/η is also a measurable function on $\mathbb{N}$ and belongs to $\mathbb{L}^2(m)$. From Section 2, we know that the process Y is reversible with respect to $\widetilde{\pi}$, which implies reversibility with respect to m. Thus, by (4.1) and the Cauchy–Schwarz inequality, for

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any probability measure μ on $\mathbb{N}$ and any measurable function f on $\mathbb{N}$ such that $|f| \le \psi$, we have

$$\begin{aligned}
\sup_{|f|\le\psi}\left|\mu Q_t\left(\frac{f}{\eta}\right)-\alpha(f)\right| &\le \sup_{|f|\le\psi}\left|\mu Q_t\left(\frac{f}{\eta}\right)-m\left(\frac{f}{\eta}\right)\right| \\
&= \sup_{|f|\le\psi}\left|m\left(\frac{\mathrm{d}\mu}{\mathrm{d}m}Q_t\left(\frac{f}{\eta}\right)-\frac{f}{\eta}\right)\right| \\
&= \sup_{|f|\le\psi}\left|m\left(\frac{f}{\eta}Q_t\left(\frac{\mathrm{d}\mu}{\mathrm{d}m}\right)-\frac{f}{\eta}\right)\right| \\
&= \sup_{|f|\le\psi}\left|m\left[\frac{f}{\eta}\left(Q_t\left(\frac{\mathrm{d}\mu}{\mathrm{d}m}-1\right)\right)\right]\right| \\
&\le \left[m\left(\frac{\psi^2}{\eta^2}\right)\right]^{\frac{1}{2}}\left\|\frac{\mathrm{d}\mu}{\mathrm{d}m}-1\right\|_2 \mathrm{e}^{-\varepsilon t} \\
&\le \left[\alpha\left(\frac{\psi^2}{\eta}\right)\right]^{\frac{1}{2}}\left\|\frac{\mathrm{d}\mu}{\mathrm{d}m}-1\right\|_2 \mathrm{e}^{-\varepsilon t}.
\end{aligned}$$

Note that

$$\mathbb{E}_\mu[f(X_t) \mid T_0 > t] = \frac{(\eta\circ\mu)Q_t(f/\eta)}{(\eta\circ\mu)Q_t(\mathbf{1}/\eta)},$$

so, for any $t > \{\log[(\alpha(\psi^2/\eta))^{1/2}\|(\mathrm{d}(\eta\circ\mu)/\mathrm{d}m) - 1\|_2]\}/\varepsilon$, we get

$$\begin{aligned}
&\frac{\alpha(f) - (\alpha(\psi^2/\eta))^{1/2}\|(\mathrm{d}(\eta\circ\mu)/\mathrm{d}m) - 1\|_2 \mathrm{e}^{-\varepsilon t}}{1 + (\alpha(\psi^2/\eta))^{1/2}\|(\mathrm{d}(\eta\circ\mu)/\mathrm{d}m) - 1\|_2 \mathrm{e}^{-\varepsilon t}} \\
&\qquad \le \mathbb{E}_\mu[f(X_t) \mid T_0 > t] \le \frac{\alpha(f) + (\alpha(\psi^2/\eta))^{1/2}\|(\mathrm{d}(\eta\circ\mu)/\mathrm{d}m) - 1\|_2 \mathrm{e}^{-\varepsilon t}}{1 - (\alpha(\psi^2/\eta))^{1/2}\|(\mathrm{d}(\eta\circ\mu)\mathrm{d}m) - 1\|_2 \mathrm{e}^{-\varepsilon t}}. \qquad (4.2)
\end{aligned}$$

Since $\alpha(\psi^2) < +\infty$, by the Cauchy–Schwarz inequality we have $\alpha(\psi) < +\infty$. Thus, by (4.2), for any $t > \{\log((\alpha(\psi^2/\eta))^{1/2}\|(\mathrm{d}(\eta\circ\mu)/\mathrm{d}m) - 1\|_2)\}/\varepsilon$, we obtain

$$\sup_{|f|\le\psi} |\mathbb{E}_\mu[f(X_t) \mid T_0 > t] - \alpha(f)| \le \max\{C_1, C_2\}\left(\alpha\left(\frac{\psi^2}{\eta}\right)\right)^{\frac{1}{2}}\left\|\frac{\mathrm{d}(\eta\circ\mu)}{\mathrm{d}m}-1\right\|_2 \mathrm{e}^{-\varepsilon t},$$

where

$$C_1 := \left(1 + \frac{1+\alpha(\psi)}{1-b}\right), \qquad C_2 := 2 + \alpha(\psi),$$

and b is a constant on (0, 1).

Set $\phi_t(\mu) := \mathbb{P}_\mu(X_t \in \cdot | T_0 > t)$. For any $t \ge 0$ and any probability measure μ on $\mathbb{N}$, we know from [14, Lemma 2.7] that

$$\eta\circ\phi_t(\mu) = (\eta\circ\mu)Q_t. \qquad (4.3)$$

There exists $t_\mu \ge 0$ such that, for any $t \ge t_\mu$,

$$\left(\alpha\left(\frac{\psi^2}{\eta}\right)\right)^{\frac{1}{2}}\left\|\frac{\mathrm{d}(\eta\circ\phi_t(\mu))}{\mathrm{d}m}-1\right\|_2 \mathrm{e}^{-\varepsilon t} < b.$$

Hence, by (4.1), (4.3), and the above result, for any $t \geq t_\mu$, we get

$$\sup_{|f|\leq\psi} |\mathbb{E}_\mu[f(X_t) \mid T_0 > t] - \alpha(f)| \leq \max\{C_1, C_2\}\left(\alpha\left(\frac{\psi^2}{\eta}\right)\right)^{\frac{1}{2}} \left\| \frac{\mathrm{d}(\eta \circ \phi_{t_\mu}(\mu))}{\mathrm{d}m} - 1 \right\|_2 \mathrm{e}^{-\varepsilon(t-t_\mu)}$$
$$\leq \max\{C_1, C_2\}\left(\alpha\left(\frac{\psi^2}{\eta}\right)\right)^{\frac{1}{2}} \left\| \frac{\mathrm{d}(\eta \circ \mu)}{\mathrm{d}m} - 1 \right\|_2 \mathrm{e}^{-\varepsilon t}.$$

This ends the proof of Theorem 1.2.

Acknowledgements

The authors would like to thank two anonymous referees for carefully reading this paper, and for various helpful remarks and suggestions.

Funding information

This work was supported by the National Natural Science Foundation of China (Grant No. 12001184) and Outstanding Youth Project of Education Department of Hunan Province (Grant No. 19B307).

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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