THE CENTER OF SL$_2$ TILTING MODULES

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Abstract. In this note, we compute the centers of the categories of tilting modules for $G = \mathrm{SL}_2$ in prime characteristic, of tilting modules for the corresponding quantum group at a complex root of unity, and of projective $G_g T$-modules when $g = 1, 2$.

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1. Introduction. Let $\mathbb{K}$ denote an algebraically closed field and $\mathbf{Tilt} = \mathbf{Tilt}\big(\mathrm{SL}_2(\mathbb{K})\big)$ the additive $\mathbb{K}$-linear category of (left-)tilting modules for the algebraic group $\mathrm{SL}_2(\mathbb{K})$.

In this note, we compute the (categorical) center $\mathbf{Z}(\mathbf{Tilt})$ of $\mathbf{Tilt}$, using the explicit description of the Ringel dual of $\mathrm{SL}_2(\mathbb{K})$ from [**5**]. In characteristic zero, we have $\mathbf{Z}(\mathbf{Tilt}) \cong \prod_{\mathbb{N}} \mathbb{K}$ since $\mathbf{Tilt}$ is semisimple with simple objects indexed by $\mathbb{N}$. Hence, our main concern is the case of prime characteristic. Thus, for the duration, $\mathbb{K}$ is of characteristic $\mathsf{p} \geq 2$.

THEOREM A. *We have isomorphisms of $\mathbb{K}$-algebras:*

$$\mathbf{Z}(\mathbf{Tilt}) \cong \mathbb{K}[X_v \mid v \in \mathbb{N}]\Big/ \langle X_v X_w \mid v, w \in \mathbb{N}\rangle.$$

We will provide an explicit isomorphism in Theorem 4.6 and the discussion following it. For a possible interpretation of the central elements X_v via Donkin's tensor product theorem, see Section 4.3.

Finally, in Section 5, we compute the centers of the categories of tilting modules in the quantum group case, see Theorem 5.2, and of projective $G_g T$-modules for $g = 1, 2$, see Theorem 5.3, both for SL_2.

2. Preliminaries. Throughout, we fix a prime $\mathsf{p} \geq 2$. The main figure to keep in mind is Figure 1 which illustrates the underlying graph of the quiver algebra Z in the cases $\mathsf{p} = 3$, $\mathsf{p} = 5$, and $\mathsf{p} = 7$.

We also warn the reader that we will always have a ρ-shift of 1 and the crucial number will usually be "the highest weight plus 1," which we will denote by v, w, *etc.*

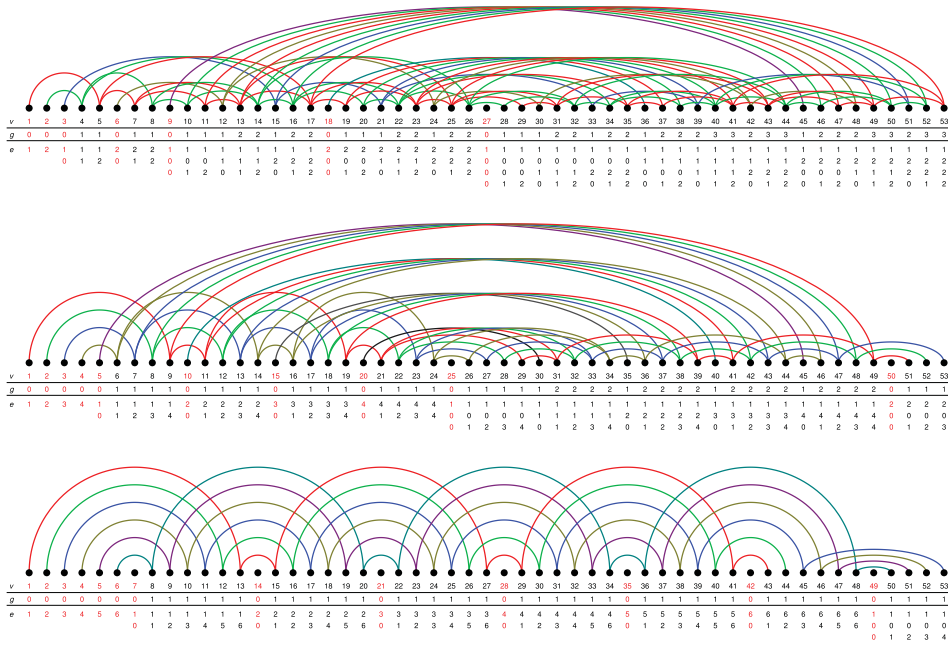In the following, we review basic notation related to Z. For more details, we refer to [**5**].

Figure 1. The full subquivers containing the first 53 vertices of the quiver underlying Z_p for $p \in \{3, 5, 7\}$, showing from top to bottom the numbers v of the vertices, the generation of v, and the p-adic expansion of v.

2.1. Some combinatorics of p-adic expansions.

DEFINITION 2.1. For any $v \in \mathbb{N}$, we write $[a_j, ..., a_0]_p := \sum_{i=0}^{j} a_i p^i = v$ for the p-adic expansion with digits $a_i \in \{0, ..., p-1\}$ and $a_j \neq 0$. We sometimes also write a_h for $h > j$, and then $a_h = 0$, by convention. Finally, the generation of v is the number of non-zero digits of v minus 1.

REMARK 2.2. Conversely, we will sometimes specify numbers by p-adic expansions with negative digits such as $[3, -1, -6, -5, 0, 5, -6]_7 = 320, 048$.

The generation of v is a complexity measure for the indecomposable tilting module of highest weight $v - 1$. For example, the indecomposables of generation zero are exactly the simple tilting modules. Moreover, to each non-simple indecomposable tilting module, one associates another tilting module of generation one lower, called its mother, whose highest weight (plus 1) is obtained by setting the lowest non-zero digit to zero. Tracing the matrilinear ancestry of an indecomposable tilting module through decreasing generation numbers, one arrives at a simple tilting module, called an eve.

DEFINITION 2.3. If $v = [a_j, ..., a_0]_p \in \mathbb{N}$ is of generation zero, then v is called an eve. The set of eves is denoted by Eve.

Note that $\text{Eve} = \text{Eve}^{<p} \cup \text{Eve}^{\geq p}$, with $\text{Eve}^{<p}$ and $\text{Eve}^{\geq p}$ having the evident meaning.

EXAMPLE 2.4. For $p = 7$, we have $\text{Eve}^{<p} = \{1, 2, 3, 4, 5, 6\}$ and

$$\text{Eve}^{\geq p} = \{[1, 0]_7, [2, 0]_7, ..., [6, 0]_7, [1, 0, 0]_7, [2, 0, 0]_7, ..., [6, 0, 0]_7, [1, 0, 0, 0]_7, ...\}.$$

DEFINITION 2.5. For two finite subsets $S, T \subset \mathbb{N}_0$ the distance between them is defined as $\mathsf{d}(S, T) = \min \left\{ |s - t| \mid s \in S, t \in T \right\}$, and we write $T > S$ to indicate the requirement that every element in T is strictly greater than every element in S.

DEFINITION 2.6. For $S \subset \mathbb{N}_0$ a finite set, we consider partitions $S = \bigsqcup_i S_i$ of S into subsets S_i of consecutive integers, which we call stretches. We fix the coarsest such partition.

The set S is called down-admissible for $v = [a_j, ..., a_0]_{\mathsf{p}}$ if:

(i) $a_{\min(S_i)} \neq 0$ for every i, and
(ii) if $s \in S$ and $a_{s+1} = 0$, then $s + 1 \in S$.

If $S \subset \mathbb{N}_0$ is down-admissible for $v = [a_j, ..., a_0]_{\mathsf{p}}$, then we define its downward reflection along S as:

$$v[S] := [a_j, \epsilon_{j-1} a_{j-1}, ..., \epsilon_0 a_0]_{\mathsf{p}}, \quad \epsilon_k = \begin{cases} 1 & \text{if } k \notin S, \\ -1 & \text{if } k \in S. \end{cases}$$

Conversely, S is up-admissible for $v = [a_j, ..., a_0]_{\mathsf{p}}$ if the following conditions are satisfied:

(i) $a_{\min(S_i)} \neq 0$ for every i, and
(ii) if $s \in S$ and $a_{s+1} = \mathsf{p} - 1$, then we also have $s + 1 \in S$.

If $S \subset \mathbb{N}_0$ is up-admissible for $v = [a_j, ..., a_0]_{\mathsf{p}}$, then we define its upward reflection along S as:

$$v(S) := [a'_{r(S)}, ..., a'_0]_{\mathsf{p}}, \quad a'_k = \begin{cases} a_k & \text{if } k \notin S, k - 1 \notin S, \\ a_k + 2 & \text{if } k \notin S, k - 1 \in S, \\ -a_k & \text{if } k \in S, \end{cases}$$

where we extend the digits of v by $a_h = 0$ for $h > j$ if necessary, and $r(S)$ is the biggest integer such that $a'_k \neq 0$.

EXAMPLE 2.7. Note that $S = \{7, 6\}$ is up-admissible for $v = [3, 1, 6, 5, 0, 5, 6]_7$. To compute $v(S)$ we let $v = [0, 3, 1, 6, 5, 0, 5, 6]_7$ and then we get

$$v(S) = [0, 3, 1, 6, 5, 0, 5, 6]_7 = [2, -3, 1, 6, 5, 0, 5, 6]_7 = [1, 4, 1, 6, 5, 0, 5, 6]_7.$$

The wave indicates the digits on which we apply S.

We tend to omit set brackets, for example, for singleton sets $\{i\}$ we also write $v[i]$ and $v(i)$ instead of $v[\{i\}]$ and $v(\{i\})$.

DEFINITION 2.8. If S is up-admissible, then we denote by $\overline{S} \subset \mathbb{N}_0$ the down-admissible hull of S, the smallest down-admissible set containing S, if it exists.

EXAMPLE 2.9. Let $\mathsf{p} = 7$ and $v = [3, 1, 6, 5, 0, 5, 6]_7$.

(a) The singleton sets $\{2\}$, $\{1\}$ and $\{i\}$ for $i \in \mathbb{N}_{>5}$ are not down-admissible for v. Of these, only $\{1\}$ has a down-admissible hull and it is $\overline{\{1\}} = \{2, 1\}$.
(b) Hence, the singleton sets $\{5\}$, $\{4\}$, $\{3\}$, and $\{0\}$ are minimal down-admissible for v, and we have

$$v[5] = [3, -1, 6, 5, 0, 5, 6]_7, \quad v[4] = [3, 1, -6, 5, 0, 5, 6]_7,$$
$$v[3] = [3, 1, 6, -5, 0, 5, 6]_7, \quad v[0] = [3, 1, 6, 5, 0, 5, -6]_7.$$

Together with $\{2, 1\}$, these are all minimal down-admissible sets for v.

(c) The singleton sets which are up-admissible for v are $\{6\}$, $\{5\}$, $\{4\}$, $\{1\}$, and $\{0\}$.

(d) The set $S = \{5, 4, 3|0\}$ is down- and up-admissible for v, and $v(S)$ and $v[S]$ can be illustrated via

$$v[5, 4, 3|0] = [3, \underline{1, 6, 5}, 0, 5, \underline{6}]_7 = [3, -1, -6, -5, 0, 5, -6]_7,$$

$$v(5, 4, 3|0) = [3, \underset{\smile}{1, 6, 5}, 0, 5, \underset{\smile}{6}]_7 = [5, -1, -6, -5, 0, 7, -6]_7.$$

We have marked the digits where we reflect in the set S, either <u>down</u> or up̧.

2.2. The Ringel dual. Let $\mathbb{K} = \overline{\mathbb{K}}$ denote a field of characteristic p with prime field $\mathbb{F}_\mathsf{p}$.

DEFINITION 2.10. We define two functions $\mathtt{f}, \mathtt{g} \colon \mathbb{F}_\mathsf{p} \to \mathbb{F}_\mathsf{p}$ via

$$\mathtt{f}(a) = \begin{cases} (-1)^a \frac{2}{a} & \text{if } 1 \leq a \leq \mathsf{p} - 2, \\ 0 & \text{if } a = 0 \text{ or } a = \mathsf{p} - 1, \end{cases} \qquad \mathtt{g}(a) = \begin{cases} -(\frac{a+1}{a}) & \text{if } 1 \leq a \leq \mathsf{p} - 1, \\ -2 & \text{if } a = 0. \end{cases}$$

Note that $\mathtt{f}(\mathsf{p} - 1) = \mathtt{g}(\mathsf{p} - 1) = 0$ and $\mathtt{g}(a) = \mathtt{g}(\mathsf{p} - a - 1)^{-1}$ for $a \neq 0, \mathsf{p} - 1$.

Let **Tilt** denote the category of (left) tilting modules of $\mathrm{SL}_2(\mathbb{K})$, see for example [**6**, Section 1] for a concise summary of the main definitions and properties regarding **Tilt**. Let $\mathtt{T}(v - 1)$ denote (a choice of representative of) the indecomposable tilting module of highest weight $v - 1$.

DEFINITION 2.11. Define a $\mathbb{K}$-algebra:

$$\mathtt{Z} := \bigoplus\nolimits_{v, w \in \mathbb{N}} \mathrm{Hom}_{\textbf{Tilt}}\big(\mathtt{T}(v - 1), \mathtt{T}(w - 1)\big).$$

Let $\mathtt{e}_{v-1}$ be the idempotent in $\mathtt{Z}$ corresponding to $\mathtt{T}(v - 1)$.

DEFINITION 2.12. For each finite $S \subset \mathbb{N}_0$, we define scaling operators $\mathtt{f}_S, \mathtt{g}_S, \mathtt{h}_S \in \mathtt{Z}$ on $v = [a_j, ..., a_0]_\mathsf{p}$ as:

$$\mathtt{f}_S \mathtt{e}_{v-1} = \mathtt{f}(a_{\max(S)+1}) \mathtt{e}_{v-1}, \quad \mathtt{g}_S \mathtt{e}_{v-1} = \mathtt{g}(a_{\max(S)+1}) \mathtt{e}_{v-1}, \quad \mathtt{h}_S \mathtt{e}_{v-1} = \mathtt{g}(a_{\max(S)+1} - 1) \mathtt{e}_{v-1}.$$

EXAMPLE 2.13. Let again $\mathsf{p} = 7$, $v = [3, 1, 6, 5, 0, 5, 6]_7$ and $S = \{5, 4, 3|0\}$. Then,

$$\mathtt{f}_S \mathtt{e}_{v-1} = \mathtt{f}(3) \mathtt{e}_{v-1} = 4 \mathtt{e}_{v-1}, \quad \mathtt{g}_S \mathtt{e}_{v-1} = \mathtt{g}(3) \mathtt{e}_{v-1} = \mathtt{e}_{v-1}, \quad \mathtt{h}_S \mathtt{e}_{v-1} = \mathtt{g}(2) \mathtt{e}_{v-1} = 2 \mathtt{e}_{v-1}.$$

The following identifies $\mathtt{Z}$ explicitly and can be taken as an abstract definition of a quiver algebra isomorphic to $\mathtt{Z}$, see also Remark 2.15.

THEOREM 2.14 (See [**5**, Theorem 3.2]). *The algebra $\mathtt{Z}$ is generated by $\mathtt{e}_{v-1}$ for $v \in \mathbb{N}$, and elements $\mathtt{D}_S \mathtt{e}_{v-1}$ and $\mathtt{U}_{S'} \mathtt{e}_{v-1}$, where S and S' denote minimal down- and up-admissible stretches for v, respectively. These generators are subject to the following complete set of relations.*

(1) Idempotents.

$$\mathtt{e}_{v-1} \mathtt{e}_{w-1} = \delta_{v,w} \mathtt{e}_{v-1}, \quad \mathtt{e}_{v[S]-1} \mathtt{D}_S \mathtt{e}_{v-1} = \mathtt{D}_S \mathtt{e}_{v-1}, \quad \mathtt{e}_{v(S')-1} \mathtt{U}_{S'} \mathtt{e}_{v-1} = \mathtt{U}_{S'} \mathtt{e}_{v-1}.$$

(2) Containment. If $S' \subset S$, then we have

$$\mathtt{D}_{S'} \mathtt{D}_S \mathtt{e}_{v-1} = 0, \quad \mathtt{U}_S \mathtt{U}_{S'} \mathtt{e}_{v-1} = 0.$$

(3) Far-commutativity. If $\mathsf{d}(S, S') > 1$, then

$$\mathtt{D}_S \mathtt{D}_{S'} \mathtt{e}_{v-1} = \mathtt{D}_{S'} \mathtt{D}_S \mathtt{e}_{v-1}, \quad \mathtt{D}_S \mathtt{U}_{S'} \mathtt{e}_{v-1} = \mathtt{U}_{S'} \mathtt{D}_S \mathtt{e}_{v-1}, \quad \mathtt{U}_S \mathtt{U}_{S'} \mathtt{e}_{v-1} = \mathtt{U}_{S'} \mathtt{U}_S \mathtt{e}_{v-1}.$$

(4) Adjacency relations. If $\mathsf{d}(S, S') = 1$ *and* $S' > S$, *then*

$$\mathsf{D}_{S'}\mathsf{U}_S\mathsf{e}_{v-1} = \mathsf{D}_{S\cup S'}\mathsf{e}_{v-1}, \quad \mathsf{D}_S\mathsf{U}_{S'}\mathsf{e}_{v-1} = \mathsf{U}_{S'\cup S}\mathsf{e}_{v-1},$$

$$\mathsf{D}_{S'}\mathsf{D}_S\mathsf{e}_{v-1} = \mathsf{U}_S\mathsf{D}_{S'}\mathsf{h}_S\mathsf{e}_{v-1}, \quad \mathsf{U}_S\mathsf{U}_{S'}\mathsf{e}_{v-1} = \mathsf{h}_S\mathsf{U}_{S'}\mathsf{D}_S\mathsf{e}_{v-1}.$$

(5) Overlap relations. If $S' \geq S$ *with* $S' \cap S = \{s\}$ *and* $S' \not\subset S$, *then we have*

$$\mathsf{D}_{S'}\mathsf{D}_S\mathsf{e}_{v-1} = \mathsf{U}_{\{s\}}\mathsf{D}_S\mathsf{D}_{S'\setminus\{s\}}\mathsf{e}_{v-1}, \quad \mathsf{U}_S\mathsf{U}_{S'}\mathsf{e}_{v-1} = \mathsf{U}_{S'\setminus\{s\}}\mathsf{U}_S\mathsf{D}_{\{s\}}\mathsf{e}_{v-1}.$$

(6) Zigzag.

$$\mathsf{D}_S\mathsf{U}_S\mathsf{e}_{v-1} = \mathsf{U}_{\overline{S}}\mathsf{D}_{\overline{S}}\mathsf{g}_S\mathsf{e}_{v-1} + \mathsf{U}_T\mathsf{U}_{\overline{S}}\mathsf{D}_{\overline{S}}\mathsf{D}_T\mathsf{f}_S\mathsf{e}_{v-1}.$$

Here, if the down-admissible hull $\overline{S}$, *or the smallest minimal down-admissible stretch* T *with* $T > \overline{S}$ *does not exist, then the involved symbols are zero by definition.*

The elements of the form:

$$\mathsf{e}_{w-1}\mathsf{U}_{S'_{i_l}}\cdots\mathsf{U}_{S'_{i_0}}\mathsf{D}_{S_{i_0}}\cdots\mathsf{D}_{S_{i_k}}\mathsf{e}_{v-1}, \tag{Basis}$$

with $S'_{i_l} > \cdots > S'_{i_0}$, *and* $S_{i_0} < \cdots < S_{i_k}$, *form a basis for* $\mathsf{e}_{w-1}\mathsf{Z}\mathsf{e}_{v-1}$.
Finally, any word $\mathsf{e}_{w-1}\mathsf{F}\mathsf{e}_{v-1}$ *in the generators of* Z *expands in* (Basis) *via* (1)–(6). □

REMARK 2.15. In fact, as already mentioned above, we could alternatively define Z as the quiver algebra with underlying graphs as in Figure 1, using highest weights to index the vertices, and two arrows directed in opposite directions for each edge in this graph. The elements e_{v-1} are the vertex idempotents of this quiver algebra, and the elements D_-, called down arrows, U_-, called up arrows, are the arrows in this quiver, pointing left and right, respectively.

REMARK 2.16. Note that all appearing scalars in the presentation of Z in Theorem 2.14 are from $\mathbb{F}_\mathsf{p}$ rather than $\mathbb{K}$.

REMARK 2.17. In Theorem 2.14.(4) and (6), the right-hand sides of the shown relations feature morphisms indexed by admissible subsets that are not necessarily minimal. These morphisms are defined to be

$$\mathsf{D}_S\mathsf{e}_{v-1} := \mathsf{D}_{S_{i_1}}\cdots\mathsf{D}_{S_{i_k}}\mathsf{e}_{v-1}, \quad \mathsf{U}_{S'}\mathsf{e}_{v-1} := \mathsf{U}_{S'_{i_l}}\cdots\mathsf{U}_{S'_{i_1}}\mathsf{e}_{v-1}, \tag{2-1}$$

where the products are taken over the minimal down- and up-admissible stretches S_{i_j} and S'_{i_j}, respectively, such that $S = \bigsqcup_j S_{i_j}$ and $S' = \bigsqcup_j S'_{i_j}$, with $S_{i_1} < \cdots < S_{i_k}$ and $S'_{i_l} > \cdots > S'_{i_1}$.

EXAMPLE 2.18. To be completely explicit with respect to Remark 2.15: if $\mathsf{p} = 3$, then the quiver of the idempotent truncation of Z supported on the vertices 0, 4, 6, 10, 12, and 16 would be
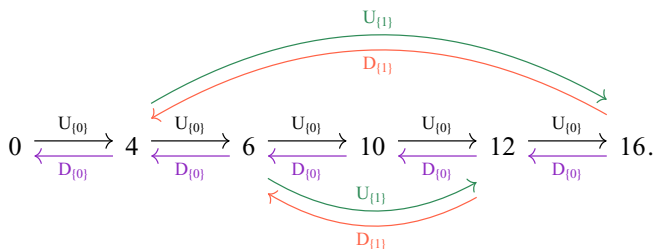
(The colors, here and below, are only to ease readability.) Here, the labeling is coming from, for example,

$$17 = [1, 2, 2]_3, 17[0] = [1, 2, \underline{2}]_3 = [1, 2, -2]_3 = 13, 17[1] = [1, \underline{2}, 2]_3 = [1, -2, 2]_3 = 5,$$

which give the two downward arrows from 16, namely $D_{\{0\}}e_{16} = e_{12}D_{\{0\}}$ and $D_{\{1\}}e_{16} = e_4 D_{\{1\}}$. We also have $D_{\{1,0\}}e_{16} = D_{\{0\}}D_{\{1\}}e_{16}$.

DEFINITION 2.19. For any down-admissible set S for v, we define the loop:

$$L^S_{v-1} := U_S D_S e_{v-1}.$$

LEMMA 2.20 (See [5, Lemma 3.23]). *Let $v \in \mathbb{N}$ with minimal down-admissible stretches $S_j, ..., S_0$. Then, we have the $\mathbb{K}$-algebra isomorphism:*

$$\mathrm{End}_{\mathbf{Tilt}}\big(\mathsf{T}(v-1)\big) \cong \mathbb{K}\big[L^{S_j}_{v-1}, ..., L^{S_0}_{v-1}\big]\Big/\big\langle (L^{S_j}_{v-1})^2, ..., (L^{S_0}_{v-1})^2\big\rangle,$$

and if S is down-admissible for v, then $L^S_{v-1} = \prod_{k|S_k \subset S} L^{S_k}_{v-1}$.   □

EXAMPLE 2.21. Again, let us consider $\mathsf{p} = 7$ and $v = [3, 1, 6, 5, 0, 5, 6]_7$. Recall that we have calculated the minimal down-admissible stretches of v in Example 2.9.(b). Hence, $\mathrm{End}_{\mathbf{Tilt}}\big(\mathsf{T}(v-1)\big)$ has generators $L^{\{5\}}_{v-1}, L^{\{4\}}_{v-1}, L^{\{3\}}_{v-1}$, and $L^{\{0\}}_{v-1}$ as well as $L^{\{2,1\}}_{v-1}$. The maximal loop $L^{\{5|4|3|2,1|0\}}_{v-1} = L^{\{5\}}_{v-1}L^{\{4\}}_{v-1}L^{\{3\}}_{v-1}L^{\{2,1\}}_{v-1}L^{\{0\}}_{v-1}$ can be thought of as a head-to-socle map on $\mathsf{T}(v-1)$.

For later use, we also recall

LEMMA 2.22 (See [5, Lemma 3.23]). *We have*

$$D_S U_S D_S e_{v-1} = 0, \quad e_{v-1}U_S D_S U_S = 0, \tag{2-2}$$

whenever S is down-admissible for v.

2.3. Closures of the algebra. Using Theorem 2.14, we get a sequence:

$$\mathsf{Z}^1 \twoheadleftarrow \mathsf{Z}^2 \twoheadleftarrow \mathsf{Z}^3 \twoheadleftarrow ... \twoheadleftarrow \varprojlim \mathsf{Z}^i =: \overline{\mathsf{Z}},$$

where Z^i is the quotient of Z obtained by the ideal generated by $\{e_{v-1} \mid v > i\}$. (Note that the category of $\mathbb{K}$-algebras is complete, that is, has all limits, so $\overline{\mathsf{Z}}$ is indeed a $\mathbb{K}$-algebra.

REMARK 2.23. Note that elements in Z are finite $\mathbb{K}$-linear combinations of elements from Theorem 2.14.(Basis), while elements in $\overline{\mathsf{Z}}$ can be (countably) infinite $\mathbb{K}$-linear combinations of these. In particular, the unit of $\overline{\mathsf{Z}}$ is

$$1 = \sum_{v\in\mathbb{N}} e_{v-1} \in \overline{\mathsf{Z}}, \tag{2-3}$$

while Z is only locally unital.

Below Z' will denote either Z or $\overline{\mathsf{Z}}$.

3. The center of the quiver algebra.   We will now compute the center of Z$'$.

3.1. Reduction to connected components.   Note that Theorem 2.14 implies that the $\mathbb{K}$-algebra $\overline{Z}$ decomposes as:

$$Z = \bigoplus_{e \in \mathrm{Eve}} Z_{e-1}, \quad Z_{e-1} := \bigoplus_{v, w \in (e)_{\mathsf{p}}} e_{w-1} Z e_{v-1},$$
$$\overline{Z} = \prod_{e \in \mathrm{Eve}} \overline{Z}_{e-1}, \quad \overline{Z}_{e-1} := \prod_{v, w \in (e)_{\mathsf{p}}} e_{w-1} \overline{Z} e_{v-1}.$$

Here, $(e)_{\mathsf{p}}$ denotes the set of natural numbers v such that the vertex $v - 1$ is in the connected component of the graph underlying Z which contains $e - 1$.

EXAMPLE 3.1. As can be seen in Figure 1, $(1)_3 = \{1, 5, 7, 11, 13, 17, ...\}$, see also Example 2.18.

Letting $Z(\_)$ denote the center of an algebra, the following is thus immediate.

LEMMA 3.2. *We have* $Z(Z) = \bigoplus_{e \in \mathrm{Eve}} Z(Z_{e-1})$ *and* $Z(\overline{Z}) = \prod_{e \in \mathrm{Eve}} Z(\overline{Z}_{e-1})$

The presentation of Z$'$ also immediately gives

LEMMA 3.3 (See [**5**, Proposition 5.3]). *There are isomorphisms of algebras* $Z'_{e-1} \cong Z'_{e'-1}$ *for all* e, $e' \in \mathrm{Eve}$ *with equal non-zero digits.*

REMARK 3.4. In fact, $Z'_{e-1} \cong Z'_{e'-1}$ for all e, $e' \in \mathrm{Eve}$, regardless of the digits. But the isomorphism is, in contrast to the one from Lemma 3.3, not immediate from Theorem A, the reason being the scalars f_S, g_S, and h_S appearing therein.

Thus, it suffices to compute $Z(Z'_{e-1})$ for $e \in \mathrm{Eve}^{<\mathsf{p}} \subset \mathrm{Eve}$.

3.2. The central elements.

DEFINITION 3.5. Let $v = [a_j, ..., a_0]_{\mathsf{p}} \in \mathbb{N}$. For $i \neq j$ and $a_i \neq 0$, we define elements in Z$'$ by

$$D_i e_{v-1} := D_{\overline{\{i\}}} e_{v-1}, \quad e_{v-1} U_i := e_{v-1} U_{\overline{\{i\}}}, \quad L_i e_{v-1} := (-1)^{a_i} a_i U_i D_i e_{v-1}.$$

Furthermore, we define

$$D(v) := \{i \in \mathbb{N}_0 \mid i < j, \, a_i \neq 0\}.$$

In words, $D(v)$ is the set of non-zero, non-leading digits of $v = [a_j, ..., a_0]_{\mathsf{p}}$. For $S \subset D(v)$, we consider

$$L_S e_{v-1} := \prod_{i \in S} L_i e_{v-1}. \tag{3-1}$$

Note that the factors in (3-1) commute by Lemma 2.20.

EXAMPLE 3.6. For $\mathsf{p} = 7$ and $v = [3, 1, 6, 5, 0, 5, 6]_7$, we have $D(v) = \{0, 1, 3, 4, 5\}$. Thus, for $S = D(v)$, we have $L_S e_{v-1} = L_5 L_4 L_3 L_1 L_0 e_{v-1}$.

DEFINITION 3.7. Let $S \subset \mathbb{N}_0$ be finite. We define an equivalence relation $\sim_S$ on $\mathbb{N}$ as follows. First, we let $v \sim_S v$ for all $v \in \mathbb{N}$. Further, for v, $w \in \mathbb{N}$ with $v \neq w$, we declare $v \sim_S w$ if $S \subset D(v) \cap D(w)$ and $v = w[k]$ or $w = v[k]$ for some $k \notin S$, Finally, we take the transitive closure.

In words, we have $v \sim_S w$ if and only if either $v = w$, or $v - 1$ and $w - 1$ are connected by a chain of arrows D_k or U_k for $k \notin S$ such that all vertices that are involved have S among their non-maximal and non-zero digits. We call such a chain an S-path.

DEFINITION 3.8. For $v \in (e)_p$, we introduce notation for the $\sim_{D(v)}$ equivalence class of v:

$$(v)_p^{\odot} := \{w \in (e)_p \mid v \sim_{D(v)} w\}.$$

EXAMPLE 3.9. Take again $p = 7$, $v = [3, 1, 6, 5, 0, 5, 6]_7$ and $S = D(v) = \{0, 1, 3, 4, 5\}$. Then:

(a) We have $w = v(5, 4, 3) = [4, 5, 0, 2, 0, 5, 6]_7 \notin (v)_p^{\odot}$, since $4 \in D(v)$, but $4 \notin D(w)$.
(b) Recall that the only up-admissible singleton set not in $D(v)$ is $\{6\}$, see Example 2.9.(c). Thus, the only direct neighbor of v in $(v)_p^{\odot}$ is $v_1 = v(6) = [1, 4, 1, 6, 5, 0, 5, 6]_7$.
(c) The set $(v)_p^{\odot}$ is an infinite set since $v_1 = v(6) = [1, 4, 1, 6, 5, 0, 5, 6]_7 \in (v)_p^{\odot}$, and recursively $v_k = v_{k-1}(5 + k) \in (v)_p^{\odot}$. Note that the first six digits of the v_k agree with those of v.

LEMMA 3.10. Let $v = [a_j, ..., a_0]_p \in \mathbb{N}$ and $w = [b_k, ..., b_0]_p \in \mathbb{N}$. If $w \in (v)_p^{\odot}$, then $b_i = a_i$ for $1 \le i < j$.

Proof. We prove by induction on the length ℓ of a $D(v)$-path that the first j digits of all elements $w \in (v)_p^{\odot}$ agree with those of v. For $\ell = 0$, there is nothing to prove. Now suppose w is at the end of a $D(v)$-path of length $\ell \ge 1$, whose last step is a morphism U_k or D_k from w' to w, with $k \notin D(v)$. Since w' is reached from v by a $D(v)$-path of length $\ell - 1$, the induction hypothesis implies that the first j digits of w' agree with those of v. In particular, each of the first j digits is either in $D(v)$ or zero, and so $k \ge j$. Then, $w = w'[k]$ and $w = w'(k)$ have the same first j digits as w' and also as v, respectively. $\square$

Lemma 3.10 implies that $(v)_p^{\odot}$ for $v = [a_j, ..., a_0]_p \in \mathbb{N}$ is a shifted copy of the block $(a_j)_p$, see [5, Proposition 5.2]. As a consequence, we get the following.

COROLLARY 3.11. *For every $v \in \mathbb{N}$, the set $(v)_p^{\odot}$ is infinite, and if $w \in (v)_p^{\odot}$, then $w \ge v$.*

The loops that we define now are only elements in $\overline{Z}$ but not in Z.

DEFINITION 3.12. For $v \in (e)_p$ and $v \ne e$, we define elements:

$$L_v := \sum_{w \in (v)_p^{\odot}} L_{D(v)} e_{w-1} \in \overline{Z}_{e-1}. \tag{3-2}$$

In words, L_v consists of the maximal loop at $v - 1$, together with the sum of all loops of the same type on $w - 1$ for all w in the shifted block $(v)_p^{\odot}$.

3.3. The center.   By Lemma 3.2, the following computes the center $Z(Z')$:

THEOREM 3.13. *For $e \in \text{Eve}^{<p}$, we have algebra isomorphisms:*

$$Z(Z_{e-1}) \cong \{0\}, \quad Z(\overline{Z}_{e-1}) \xrightarrow{\cong} \mathbb{K}[X_v \mid v \in (e)_p \setminus \{e\}] \big/ \langle X_v X_w \mid v, w \in (e)_p \setminus \{e\} \rangle, \quad L_v \mapsto X_v.$$

Proof. We first observe that we have $Z(Z'_{e-1}) \subset \prod_{v \in (e)_p} e_{v-1} Z' e_{v-1}$. Indeed, if $z \in Z(Z'_{e-1})$ and $v \in (e)_p$, then we have $ze_{v-1} = ze_{v-1}^2 = e_{v-1}ze_{v-1}$. The observation follows since z is by assumption a finite or infinite sum of terms ze_{v-1} for $v \in (e)_p$.

• Let us first consider $\overline{Z}_{e-1}$. Here, Lemma 3.17.(b) (proven below) shows that the infinite sums L_v defined above are in the center $Z(\overline{Z}_{e-1})$, while Lemma 3.19 (also proven below) shows that they, together with the identity, give a basis of $Z(\overline{Z}_{e-1})$. Thus,

$$Z(\overline{Z}_{e-1}) \cong \mathbb{K}\langle 1, L_v \mid v \in (e)_{\mathsf{p}} \setminus \{e\}\rangle.$$

The second isomorphism then follows because we already know the relations among the L_v, *cf.* Lemma 3.15.

• For the first isomorphism, note that we have an inclusion $Z(Z_{e-1}) \hookrightarrow Z(\overline{Z}_{e-1})$ of non-unital $\mathbb{K}$-algebras. However, Lemma 3.17 implies that all non-trivial elements of $Z(\overline{Z}_{e-1})$ are supported on infinitely many idempotents, so $Z(Z_{e-1}) = Z(\overline{Z}_{e-1}) \cap Z_{e-1} = \{0\}$. $\qquad\square$

3.4. Some lemmas for the proof of Theorem 3.13.   Next, we identify the relations among the L_v.

LEMMA 3.14. *For $e \in \mathrm{Eve}^{<\mathsf{p}}$ and $v \in (e)_{\mathsf{p}}$ with $v \neq e$, we have $0 \in \mathrm{D}(v)$.*

Proof. Any $v \in (e)_{\mathsf{p}}$ can be reached from e through a finite sequence of D_i and U_i. It is straightforward to check that under each such an arrow, the zeroth digit stays unchanged or is reflected to its negative. The zeroth digit of any v is, thus, either e or $\mathsf{p} - e$. Unless $v = e$, this implies $0 \in \mathrm{D}(v)$. $\qquad\square$

LEMMA 3.15. *For $e \in \mathrm{Eve}^{<\mathsf{p}}$ and $v, w \in (e)_{\mathsf{p}} \setminus \{e\}$, we have $L_v L_w = 0$.*

Proof. By Lemma 3.14, we have $0 \in \mathrm{D}(v)$ for any $v \in (e)_{\mathsf{p}}$ with $v \neq e$. If $(v)_{\mathsf{p}}^{\odot} \cap (w)_{\mathsf{p}}^{\odot} = \emptyset$, then we have $L_v L_w = 0$ trivially. Otherwise, the product $L_v L_w$ is supported on certain $z \in (v)_{\mathsf{p}}^{\odot} \cap (w)_{\mathsf{p}}^{\odot}$, but there it is a multiple of $L_0^2 e_{w-1} = 0$, see Lemma 2.20. $\qquad\square$

Next will be that the L_v are central, which needs

LEMMA 3.16. *Let $v \in (e)_{\mathsf{p}}$ and $i, k \in \mathrm{D}(v)$ with $i \neq k$ and $i \in \mathrm{D}(v[k])$. Then, we have*

$$e_{v-1} L_i U_k e_{v[k]-1} = e_{v-1} U_k L_i e_{v[k]-1} \neq 0, \tag{3-3}$$

$$e_{v-1} L_i U_i e_{v[i]-1} = e_{v-1} U_i L_i e_{v[i]-1} = 0. \tag{3-4}$$

In particular, a loop of type L_i can be transported along U_k, and thus, also along D_k, due to symmetry.

Proof. We rewrite both sides of the desired equations and introduce notation as follows. Here, we allow the case $k = i$, for which we have to prove that both sides are zero:

$$L_i U_k e_{v[k]-1} = (-1)^{a_i} a_i U_{S_i} D_{S_i} U_{S_k} e_{v[k]-1}$$

$$U_k L_i e_{v[k]-1} = (-1)^{b_i} b_i U_{S_k} U_{T_i} D_{T_i} e_{v[k]-1},$$

where $S_i = \overline{\{i\}}$ and $S_k = \overline{\{k\}}$ are the down-admissible hulls for v, $T_i := \overline{\{i\}}$ is the down-admissible hull for $v[k]$, and b_i denotes the ith digit of $v[k]$. In checking (3-3) and (3-4), there are four cases to consider.

• If S_i and S_k are distant, then $T_i = S_i$ and $a_i = b_i$, and (3-3) follows from far-commutativity Theorem 2.14.(3).

- If $S_k > S_i$ are adjacent, then we have $a_i = b_i$ and S_i is down-admissible for $v[k]$, which implies $S_i = T_i$. Now, we distinguish two cases. Suppose $a_k \neq 1$, then we compute

$$
\begin{aligned}
U_{S_i} D_{S_i} U_{S_k} e_{v[k]-1} &= U_{S_i} U_{S_k} U_{S_i} e_{v[k]-1} \\
&= g(a_k - 1) U_{S_k} D_{S_i} U_{S_i} e_{v[k]-1} \\
&= g(a_k - 1) U_{S_k} \Big(g(p - a_k) U_{S_i} D_{S_i} + f(p - a_k) U_{S_k} U_{S_i} D_{S_i} D_{S_k} \Big) e_{v[k]-1} \\
&= U_{S_k} U_{S_i} D_{S_i} e_{v[k]-1} = U_{S_k} U_{T_i} D_{T_i} e_{v[k]-1},
\end{aligned}
$$

as the f term gets killed by containment Theorem 2.14.(2) and we have $g(a_k - 1)$ $g(p - a_k) = 1$, because $a_k \neq 1$.

Now suppose $a_k = 1$. Then, we compute

$$
\begin{aligned}
U_{S_i} D_{S_i} U_{S_k} e_{v[k]-1} &= U_{S_i} U_{S_k \cup \{k-1\}} U_{S_i \setminus \{k-1\}} e_{v[k]-1} \\
&= U_{S_k} U_{S_i} D_{\{k-1\}} U_{S_i \setminus \{k-1\}} e_{v[k]-1} \\
&= U_{S_k} U_{S_i} D_{S_i} e_{v[k]-1},
\end{aligned}
$$

where we have used the overlap relation Theorem 2.14.(5). In either case, we deduce (3-3) since $a_i = b_i$.

- Suppose that $S_i > S_k$ are adjacent. Then, we have $a_i - 1 = b_i$. By the admissibility assumption, we have $b_i \neq 0$, which implies $T_i = S_i$, and we compute

$$
\begin{aligned}
U_{S_k} U_{T_i} D_{T_i} e_{v[k]-1} &= U_{S_k} U_{S_i} D_{S_i} e_{v[k]-1} \\
&= g(a_i - 1) U_{S_i} D_{S_k} D_{S_i} e_{v[k]-1} \\
&= g(a_i - 1) U_{S_i} D_{S_i} U_{S_k} e_{v[k]-1}.
\end{aligned}
$$

Here, we observe $g(a_i - 1) = -\frac{a_i}{a_i - 1} = \frac{(-1)^{a_i} a_i}{(-1)^{b_i} b_i}$, which verifies (3-3).

- Finally, if $S_i = S_k$, then we have $U_{S_i} D_{S_i} U_{S_i} e_{v[k]-1} = 0$ by (2-2). For the other side of the equation, we consider $U_{S_i} U_{T_i} D_{T_i} e_{v[k]-1}$. If $T_i = \{i\}$, then $S_i \supset T_i$ and the expression is zero by the containment relation Theorem 2.14.(2). Otherwise, we necessarily have $S_i = \{i\} \subset T_i$ and we use the overlap relation Theorem 2.14.(5) to get

$$
U_{S_i} U_{T_i} D_{T_i} e_{v[k]-1} = U_{T_i \setminus S_i} U_{S_i} D_{S_i} D_{T_i} e_{v[k]-1} = 0,
$$

where we have again used containment at the end. Hence, (3-4) holds. $\qquad\square$

LEMMA 3.17. *We have the following:*

(a) *If $z \in Z(\overline{Z}_{e-1})$ satisfies $z e_{v-1} = c L_v e_{v-1}$ for some $v \in (e)_p \setminus \{e\}$ and $c \in \mathbb{K}$, then $z e_{w-1}$ also contains $L_v e_{w-1}$ with coefficient c for any $w \in (v)_p^{\odot}$.*

(b) *For $v \in (e)_p \setminus \{e\}$, we have $L_v \in Z(\overline{Z}_{e-1})$.*

Proof. For the first part, expand L_v into a product of basic loops and observe that the $w \in (v)_p^{\odot}$ are precisely the vertices to which L_v can be transported using Lemma 3.16. In a central element, these loops L_v, therefore, have to appear with the same coefficient at any $w \in (v)_p^{\odot}$.

For the second part, we write $v = [a_j, ..., a_0]_\mathsf{p}$ and observe that every arrow U_k or D_k in the quiver, for which we write Y_k, is one of the following:

(1) Y_k is not adjacent to any $w \in (v)^\odot_\mathsf{p}$, in which case it trivially commutes

$$Y_k \mathrm{L}_v = 0, \quad \mathrm{L}_v Y_k = 0.$$

(2) Y_k is adjacent to an $w \in (v)^\odot_\mathsf{p}$, but not a generator for the equivalence relation $\sim_{\mathrm{D}(v)}$. In this case, Lemma 3.10 implies $k \leq j$ and commutation follows from (3-4).

(3) Y_k is a generating arrow of the equivalence relation $\sim_{\mathrm{D}(v)}$, in which case it commutes with the loops of type $\mathrm{L}_{\mathrm{D}(v)}$ by (3-3).   □

The following example shows that the assumption $i \in \mathrm{D}(v[k])$ in Lemma 3.16 is necessary.

EXAMPLE 3.18. Let $\mathsf{p} = 3$. Then, there is no element in $z \in Z(\overline{\mathrm{Z}}_0)$ such that $z\mathrm{e}_{12} = \mathrm{U}_1\mathrm{D}_1\mathrm{e}_{12}$. Indeed, we can compute

$$z\mathrm{e}_{10}\mathrm{D}_0\mathrm{e}_{12} = \mathrm{D}_0 z\mathrm{e}_{12} = \mathrm{D}_0\mathrm{U}_1\mathrm{D}_1\mathrm{e}_{12} = \mathrm{U}_{1,0}\mathrm{D}_1\mathrm{e}_{12}$$

However, $\mathrm{U}_{1,0}\mathrm{D}_1\mathrm{e}_{12} \neq z\mathrm{e}_{10}\mathrm{D}_0\mathrm{e}_{12}$ for any $z\mathrm{e}_{10}$, as is easily verified since the endomorphisms of e_{10} are spanned by e_{10} and $\mathrm{U}_0\mathrm{D}_0\mathrm{e}_{10}$.

Neither is there $z \in Z(\overline{\mathrm{Z}}_0)$ such that $z\mathrm{e}_{16} = \mathrm{U}_1\mathrm{D}_1\mathrm{e}_{16}$, as such a loop, if central, could be transported to e_{12}.

Finally, we show that L_v, together with the unit, form a basis of the center.

LEMMA 3.19. *The center $Z(\overline{\mathrm{Z}}_{e-1})$ has a basis given by the unit (cf. (2-3)) and the L_v for $v \in (e)_\mathsf{p}$ with $v \neq e$.*

Proof. To see linear independence, first note that no linear combination of the (nilpotent) L_v can be the unit. Moreover, assuming that $\sum_{i=1}^r a_i \mathrm{L}_{v_i} = 0$ for $a_i \in \mathbb{K}$ and $v_i < v_j$ for $i < j$, we can multiply this equation with e_{v_1-1} and get $a_1 \mathrm{L}_{\mathrm{D}(v_1)}\mathrm{e}_{v_1-1} = a_1 \mathrm{L}_{v_1}\mathrm{e}_{v_1-1} = \sum_{i=1}^r a_i \mathrm{L}_{v_i}\mathrm{e}_{v_1-1} = 0$. Hence, $a_1 = 0$, since $\mathrm{L}_{\mathrm{D}(v_1)}\mathrm{e}_{v_1-1}$ is a basis element in the corresponding endomorphism ring. We can repeat this process to show that all a_i are zero.

Suppose now we are given $z \in Z(\overline{\mathrm{Z}}_{e-1})$. Let $v \in (e)_\mathsf{p}$ be minimal with $z\mathrm{e}_{v-1} \neq 0$. If $v = e$, then $z\mathrm{e}_{v-1} = c \cdot \mathrm{e}_{v-1}$ for some non-zero scalar $c \in \mathbb{K}$. In this case, we proceed with the central element $z - c \cdot 1$ in place of z.

Now we assume that $v \neq e$ and claim that $z\mathrm{e}_{v-1} = c \cdot \mathrm{L}_v\mathrm{e}_{v-1}$ for some non-zero scalar $c \in \mathbb{K}$ since otherwise we could find a smaller v.

To see this, suppose that $z\mathrm{e}_{v-1}$ contains a summand $d \cdot \mathrm{L}_S\mathrm{e}_{v-1}$ and suppose there is an $i \in \mathrm{D}(v) \setminus S$. We write $S_i := \overline{\{i\}}$ for the admissible hull and split $S = S_{>i} \cup S_{<i}$ into subsets of indices greater and smaller than i, respectively. Then, $z\mathrm{D}_i\mathrm{e}_{v-1} = \mathrm{D}_i z\mathrm{e}_{v-1}$ contains a summand:

$$d \cdot \mathrm{D}_{S_i}\mathrm{L}_S\mathrm{e}_{v-1} = d \cdot \mathrm{D}_{S_i}\mathrm{U}_{S_{>i}}\mathrm{U}_{S_{<i}}\mathrm{D}_{S_{<i}}\mathrm{D}_{S_{>i}}\mathrm{e}_{v-1}$$

$$= \begin{cases} d \cdot \mathrm{U}_{S_{>i}\cup S_i}\mathrm{U}_{S_{<i}}\mathrm{D}_{S_{<i}}\mathrm{D}_{S_{>i}}\mathrm{e}_{v-1} & \text{if } S_{>i} \text{ is adjacent to } i, \\ d \cdot \mathrm{U}_{S_{>i}}\mathrm{U}_{S_{<i}}\mathrm{D}_{S_{<i}}\mathrm{D}_i\mathrm{D}_{S_{>i}}\mathrm{e}_{v-1} & \text{if } S_{>i} \text{ is distant to } i, \end{cases}$$

where we have used the adjacency relation Theorem 2.14.(4) in the first case, the far-commutativity relation Theorem 2.14.(3) in the second case, and also that D_{S_i} commutes with $\mathrm{U}_{S_{<i}}\mathrm{D}_{S_{<i}}$, as verified in the proof of Lemma 3.16. In either case, the result is non-zero.

A similar argument now shows that this term cannot be canceled by other summands of $D_i z e_{v-1}$. Thus, we have $z D_i e_{v-1} \neq 0$, in contradiction to the minimality of v.

Now define $z' = z - c L_v$. By Lemma 3.17, we have $z' \in Z(\overline{Z}_{e-1})$ and by Corollary 3.11 we know that z' is supported at vertices $w > v$. Now proceed by induction. $\qquad \square$

4. The center of the tilting category.   We now explain how $Z(\overline{Z})$ and $\mathbf{Z}(\mathbf{Tilt})$ are related.

4.1. Some general (and well-known) facts.   Let $\mathbb{K}$ denote any commutative unital ring. For an additive $\mathbb{K}$-linear category $\mathbf{C}$, we let $\mathbf{End}(\mathbf{C})$ denote its category of $\mathbb{K}$-linear endofunctors (which are thus automatically additive) and natural transformations. Within this category, we have the $\mathbb{K}$-algebra of endomorphism of any endofunctor F denoted by $\mathrm{End}_{\mathbf{End}(\mathbf{C})}(F)$. Recall the following definition.

DEFINITION 4.1. Let $\mathbf{C}$ be an additive $\mathbb{K}$-linear category. Then, its center $\mathbf{Z}(\mathbf{C})$ is defined as:

$$\mathbf{Z}(\mathbf{C}) = \mathrm{End}_{\mathbf{End}(\mathbf{C})}(\mathrm{Id}_{\mathbf{C}}),$$

that is, the natural transformations of the identity functor $\mathrm{Id}_{\mathbf{C}}$ of $\mathbf{C}$.

LEMMA 4.2.   *We have the following.*

(a) *The space $\mathbf{Z}(\mathbf{C})$ is a commutative $\mathbb{K}$-algebra.*
(b) *If $\mathbf{C}$ and $\mathbf{D}$ are equivalent additive $\mathbb{K}$-linear categories, then $\mathbf{Z}(\mathbf{C}) \cong \mathbf{Z}(\mathbf{D})$ as commutative $\mathbb{K}$-algebras.*
(c) *If $\mathbf{C} \cong \prod_{i \in I} \mathbf{C}_i$ as additive $\mathbb{K}$-linear categories with trivial hom spaces between $\mathbf{C}_i$ and $\mathbf{C}_j$ unless $i = j$, then $\mathbf{Z}(\mathbf{C}) \cong \prod_{i \in I} \mathbf{Z}(\mathbf{C}_i)$ as commutative $\mathbb{K}$-algebras.*

Proof. The first claim is evident. For the second claim we observe that, for any additive $\mathbb{K}$-linear full subcategory $\mathbf{D} \subset \mathbf{C}$, restriction defines a $\mathbb{K}$-algebra homomorphism $r \colon \mathbf{Z}(\mathbf{C}) \to \mathbf{Z}(\mathbf{D})$. If $\mathbf{C}$ and $\mathbf{D}$ are equivalent, then one can check that r is an isomorphism. The final claim can be proven *mutatis mutandis* as the second claim. $\qquad \square$

Let us consider the example $p\mathbf{Mod}\text{-}A$ (projective right modules of some $\mathbb{K}$-algebra A).

LEMMA 4.3.   *Let A be a $\mathbb{K}$-algebra.*

(a) *We have an inclusion $Z(A) \hookrightarrow \mathbf{Z}(p\mathbf{Mod}\text{-}A)$ of commutative $\mathbb{K}$-algebras.*
(b) *If A is unital, then we have an isomorphism $Z(A) \xrightarrow{\cong} \mathbf{Z}(p\mathbf{Mod}\text{-}A)$ of commutative $\mathbb{K}$-algebras.*

Proof. We claim that the map:

$$Z(A) \ni c \mapsto \chi_c := \left\{ \chi_P \colon P \to P, m \mapsto mc \mid P \in p\mathbf{Mod}\text{-}A \right\}$$

gives the required inclusion or isomorphism, respectively.

Indeed, it is easy to see that χ_c is actually in $\mathbf{Z}(p\mathbf{Mod}\text{-}A)$ and that this map is a well-defined inclusion of $\mathbb{K}$-algebras.

To prove surjectivity under the assumption of unitality of A, given $\chi \in \mathbf{Z}(p\mathbf{Mod}\text{-}A)$, its value on the projective right A-module A is by definition a map of right A-modules $\chi_A \colon A \to A$. Let $c := \chi_A(1) \in A$, which actually belongs to $Z(A)$. Next, for any

$P \in p\mathbf{Mod}\text{-}A$ and any $m \in P$, there is a unique $f \in \mathrm{Hom}_{p\mathbf{Mod}\text{-}A}(A, P)$ with $f(1) = m$. Using this and naturality, we get

$$\chi_P(m) = \chi_P\big(f(1)\big) = f\big(\chi_A(1)\big) = f(c) = mc,$$

which proves surjectivity. □

4.2. The tilting center. Let us modify Lemma 4.3, which we need to do since Z is non-unital:

LEMMA 4.4. *We have an isomorphism of commutative $\mathbb{K}$-algebras given by:*

$$Z(\overline{Z}) \xrightarrow{\cong} \mathbf{Z}(p\mathbf{Mod}\text{-}Z), \quad Z(\overline{Z}) \ni c \mapsto \chi_c := \big\{ \chi_P \colon P \to P, m \mapsto mc \mid P \in p\mathbf{Mod}\text{-}Z \big\}.$$

Proof. Defining $c := \chi_{\overline{Z}}(\sum_{v \in \mathbb{N}} e_{v-1})$, the argument is the same as in Lemma 4.3. □

By the well-understood block decomposition:

$$\mathbf{Tilt} = \bigoplus_{e \in \mathrm{Eve}} \mathbf{Tilt}_{e-1}, \quad \mathbf{Tilt}_{e-1} = \big\{ T(v-1) \mid v \in (e)_p \big\},$$

where hom spaces between $\mathbf{Tilt}_{e-1}$ and $\mathbf{Tilt}_{e'-1}$ are trivial unless $e = e'$.

DEFINITION 4.5. *For $v \in (e)_p \setminus \{e\}$, we define the natural transformation:*

$$\chi_v := \big\{ \chi_T \colon T \to T, \chi_T = L_v \mid T \in \mathbf{Tilt}_{e-1} \big\}, \tag{4-1}$$

where L_v was defined in Definition 3.12.

Thus, we can finally prove Theorem A, which is a consequence of:

THEOREM 4.6. *For $e \in \mathrm{Eve}^{<p}$, we have an $\mathbb{K}$-algebra isomorphism:*

$$\mathbf{Z}(\mathbf{Tilt}_{e-1}) \xrightarrow{\cong} \mathbb{K}\big[X_v \mid v \in (e)_p \setminus \{e\}\big] \big/ \langle X_v X_w \mid v, w \in (e)_p \setminus \{e\} \rangle, \quad \chi_v \mapsto X_v.$$

Proof. By Theorem 2.14 and Lemma 4.2.(b), we need to compute $\mathbf{Z}(p\mathbf{Mod}\text{-}Z_{e-1})$. This in turn, by Lemma 4.4, reduces to compute the algebra center of $\overline{Z}_{e-1}$, which is Theorem 3.13. □

4.3. The center and Donkin's tensor product theorem. We expect that the central elements L_v admit an interpretation via Donkin's tensor product theorem, which, in our notation, takes the following form.

PROPOSITION 4.7 (See [4, Proposition 2.1]). *For $v = [a_j, ..., a_0]_p$, we have an isomorphism of (left) $SL_2(\mathbb{K})$-modules:*

$$T(v-1) \cong T(a_j - 1)^j \otimes \big(\bigotimes_{i=0}^{j-1} T(a_i + p - 1)^i\big),$$

where the superscripts indicate Frobenius twist.

LEMMA 4.8. *Let $v = [a_j, ..., a_0]_p$ and $v' = [1, a_{j-1}, ..., a_0]_p$. For all $w = [b_k, ..., b_j, b_{j-1}, ..., b_0]_p \in (v)_p^{\circleddash}$, define $w' = [b_k, ..., b_j]_p$. We have an isomorphism of (left) $SL_2(\mathbb{K})$-modules:*

$$T(w-1) \cong T(w'-1)^j \otimes T(v'-1).$$

Proof. First, note that Lemma 3.10 gives $w = [b_k, ..., b_{j+1}, b_j, a_{j-1}, ..., a_0]_p$. Thus, applying Proposition 4.7 proves the claim. □

EXAMPLE 4.9. As before, let $\mathsf{p} = 7$, $v = [3, 1, 6, 5, 0, 5, 6]_7$ with $v' = [1, 1, 6, 5, 0, 5, 6]_7$. Then, $w = [1, 4, 1, 6, 5, 0, 5, 6]_7 \in (v)_\mathsf{p}^\odot$ and $w' = [1, 4]_7$. Proposition 4.7 and Lemma 4.8 give

$$\mathrm{T}(v' - 1) \cong \mathrm{T}(7)^5 \otimes \mathrm{T}(12)^4 \otimes \mathrm{T}(11)^3 \otimes \mathrm{T}(6)^2 \otimes \mathrm{T}(11)^1 \otimes \mathrm{T}(12)^0,$$

$$\mathrm{T}(v - 1) \cong \mathrm{T}(2)^6 \otimes \mathrm{T}(v' - 1),$$

$$\mathrm{T}(w - 1) \cong \mathrm{T}(w' - 1)^6 \otimes \mathrm{T}(v' - 1) \cong \mathrm{T}(0)^7 \otimes \mathrm{T}(10)^6 \otimes \mathrm{T}(v' - 1).$$

In the situation of Lemma 4.8 where

$$\mathrm{T}(w - 1) \cong \mathrm{T}(w' - 1)^j \otimes \mathrm{T}(v' - 1)$$

it seems plausible that $\mathrm{L}_v \mathsf{e}_{w-1} = c \cdot \mathrm{id}_{\mathrm{T}(w'-1)^j} \otimes \mathrm{L}_{v'}$ for some scalar $c \in \mathbb{K}$. One approach to proving this statement involves giving a diagrammatic interpretation of Frobenius twists in the Temperley–Lieb category. Alternatively, it would also be interesting to have a proof that such elements are indeed central and span the center, which does not use the explicit presentation of **Tilt**, for example, using the abstract basis given in [**2**].

4.4. Blocks via the center.   Let us discuss a classical application of the center of a category, mimicking the well-understood case of category O. Note that $\mathbf{Z}(\mathbf{C})$ acts naturally on all objects X of $\mathbf{C}$ via

$$\chi \cdot \mathrm{f} := \mathrm{f} \circ \chi_\mathrm{X} = \chi_\mathrm{X} \circ \mathrm{f}, \quad \chi \in \mathbf{Z}(\mathbf{C}), \mathrm{f} \in \mathrm{End}_\mathbf{C}(\mathrm{X}).$$

In particular, for any character of the center $\chi_s \colon \mathbf{Z}(\mathbf{C}) \to \mathbb{K}$, we can fiber the category $\mathbf{C}$ by defining full subcategories $\mathbf{C}_s \subset \mathbf{C}$ consisting of all objects where $\mathbf{Z}(\mathbf{C})$ acts with character χ_s.

By Theorem A, the $\mathbf{C}_s$ for $\mathbf{C} = \mathbf{Tilt}$ are exactly the blocks $\mathbf{Tilt}_{e-1}$, which follows because each $\mathbb{K}$-algebra $\mathbf{Z}(\mathbf{Tilt}_{e-1})$, which has only one non-nilpotent basis element, has exactly one simple.

4.5. The Casimir element.   Let $C \colon A \to \mathbb{K}$ denote the Casimir element of $\mathrm{SL}_2(\mathbb{K})$, which is an element in its distribution algebra, that is, a map from the coordinate ring A of $\mathrm{SL}_2(\mathbb{K})$ to $\mathbb{K}$. Recall that C acts on every $\mathrm{SL}_2(\mathbb{K})$ module M using the coaction Δ_M of A on M:

$$C \colon \mathrm{M} \xrightarrow{\Delta_\mathrm{M}} \mathrm{M} \otimes_\mathbb{K} A \xrightarrow{1 \otimes C} \mathrm{M} \otimes_\mathbb{K} \mathbb{K} \xrightarrow{\cong} \mathrm{M}.$$

Let $\mathrm{L}(v - 1)$ denote the simple $\mathrm{SL}_2(\mathbb{K})$ module of highest weight $v - 1$. We choose $C = (h + 1)^2 + 4fe$ and see that C acts on $\mathrm{L}(v - 1)$ as $v^2 \mathrm{id}$. The proof of Lemma 3.14 implies that C acts on $\mathrm{L}(v - 1)$ as multiplication by $v^2 = a_0^2 = (\mathsf{p} - a_0)^2 = e^2 \in \mathbb{F}_\mathsf{p}$ for every $v \in (e)_\mathsf{p}$. Since all $L(v - 1)$ that appear in composition series of objects of $\mathbf{Tilt}_{e-1}$ have $v \in (e)_\mathsf{p}$, we see that C acts by this scaling on all objects in $\mathbf{Tilt}_{e-1}$. Hence, in the blocks $Z(\overline{\mathsf{Z}}_{e-1})$ of the Ringel dual, we have $C = e^2 \cdot 1$.

5. Two other cases.

REMARK 5.1. The two theorems below can be proven, *mutatis mutandis*, as for **Tilt**, but the computations are much simpler. Thus, we decided to keep the proofs short.

5.1. A variation: the quantum case (generation 1). In the quantum group case, the same calculations, using the Ringel dual computed in [3], work, but are much simpler, and so is the result. Let **Tilt**^{q} denote the quantum analog of **Tilt** with quantum parameter q, where we use Lusztig's divided power quantum group. Throughout this part, [3] serves as our reference for statements about **Tilt**^{q}. However, we use the notation from this paper. In particular, we write $[a_j, ..., a_0]_k := \sum_{i=0}^{j} a_i k^i = v$ for the k-adic expansion with digits $0 \leq a_i < k$.

We will work over a field $\mathbb{C}(q)$ and distinguish the following cases:

- The quantum parameter q is a formal parameter, or $q \in \mathbb{C}$ is $q = \pm 1$ or not a root of unity. Here, we set $k = \infty$.
- The quantum parameter $q \in \mathbb{C}$ is a root of unity $q \neq \pm 1$. Here, k denotes the order of q^2.

We also let

$$\chi_v := \left\{ \chi_T \colon T \to T, \, \chi_T = L_v^{\{0\}} \mid T \in \textbf{Tilt} \right\}, \quad \delta_v := \left\{ \delta_T \colon T \to T, \, \delta_T = \delta_v \mid T \in \textbf{Tilt} \right\}, \quad (5\text{-}1)$$

where $L_v^{\{0\}}$ and δ_v are the maps that act as non-zero only on summands $T(v-1)$, on which they are a head-to-socle map and the identity, respectively.

THEOREM 5.2. *We have the following.*

(a) For $k = \infty$, there is an isomorphism of $\mathbb{C}(q)$-algebras:

$$\textbf{Z}(\textbf{Tilt}^q) \xrightarrow{\cong} \prod_{\mathbb{N}} \mathbb{C}(q), \quad \delta_v \mapsto 1_v.$$

(b) For the root of unity case, there is an isomorphism of $\mathbb{C}$-algebras:

$$\textbf{Z}(\textbf{Tilt}^q) \xrightarrow{\cong} \bigoplus_{i=-1}^{k-2} \texttt{X}(i), \quad \texttt{X}(i) = \begin{cases} \prod_{\mathbb{N}} \mathbb{C} & \text{if } i = -1, \\ \mathbb{C}[X_v \mid v \in \mathbb{N}] \Big/ \langle X_v X_w \mid v, w \in \mathbb{N} \rangle & \text{otherwise}, \end{cases}$$
$$(5\text{-}2)$$

where $\chi_w \mapsto X_v$, and $\delta_x \mapsto 1_v$ for $w, x \in \mathbb{N}$ as explained in the proof.

Proof. (a). This claim is clear as **Tilt**^{q} is semisimple and its simples are indexed by $\mathbb{N}$. (b). For the root of unity case, we recall that we have equivalences of additive $\mathbb{K}$-linear categories:

$$\textbf{Tilt}^q \cong \textbf{Tilt}^q_{St} \oplus \textbf{Tilt}^q_0 \oplus ... \oplus \textbf{Tilt}^q_{k-2},$$
$$\textbf{Tilt}^q_0 \cong ... \cong \textbf{Tilt}^q_{k-2} \cong p\textbf{Mod-}\texttt{Z}^q,$$

where **Tilt**$^q_{St}$ is semisimple with simple objects being the indecomposable quantum tilting modules whose highest weight v satisfies $a_0 = 0$, which is the first case in (5-2).

Moreover, the Ringel dual $\texttt{Z}^q$ of **Tilt**$^q_0$ is a zigzag algebra with a boundary condition on the vertex set $\mathbb{N}$, that is,

$$(v_0 - 1) \underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}} (v_1 - 1) \underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}} (v_2 - 1) \underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}} (v_3 - 1) \underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}} ... \, ,$$

where $v_0 = 1$ and $v_{i+1} = v_i(0)$ for $i \geq 0$, subject to the relations

$$D_{\{0\}}D_{\{0\}}e_{v-1} = 0, \ U_{\{0\}}U_{\{0\}}e_{v-1} = 0, \ D_{\{0\}}U_{\{0\}}e_{v-1} = U_{\{0\}}D_{\{0\}}e_{v-1} \text{ for } v \neq 1, \ D_{\{0\}}U_{\{0\}}e_0 = 0.$$

(Note the boundary relation $D_{\{0\}}U_{\{0\}}e_0 = 0$.) Formally, the algebra Z^q is generated by e_{v-1} for $v \in \mathbb{N}$ being of the form $v = 1(0)...(0)$, and elements $D_{\{0\}}e_{v-1}$ and $U_{\{0\}}e_{v-1}$, modulo the relations shown above, which are the analogs of the relations in Theorem 2.14.

Observe that we have central elements in Z^q of the form:

$$L_v^{\{0\}} = D_{\{0\}}U_{\{0\}}e_{v-1} = U_{\{0\}}D_{\{0\}}e_{v-1}, \ \text{ for } v \neq 1.$$

These are central since they are annihilated by any element except their idempotent. Moreover, no other element is central, implying the second case in (5-2), using arguments as in Section 4.1 and the quantum analog of Lemma 4.4. In particular, $L_v^{\{0\}}$ corresponds to the natural transformation in (5-1) and satisfies the relations of X_v in $\mathbb{C}[X_v \mid v \in \mathbb{N}]\big/\langle X_v X_w \mid v, w \in \mathbb{N}\rangle$. $\qquad\square$

5.2. A variation: the $G_g T$ case for $g = 1, 2$ (generations 1, 2). Recall that $[a_j, ..., a_0]_{\mathsf{p}} = \sum_{i=0}^{j} a_i \mathsf{p}^i = v$. Using the same methods, in the case of projective $G_g T$-modules for $SL_2(\mathbb{K})$ and $g = 1, 2$, one can also calculate the center of the corresponding additive $\mathbb{K}$-linear category $G_g T$-$p\mathbf{Mod}$. The corresponding Ringel duals were computed in [1], which is also our reference for statements about $G_g T$-$p\mathbf{Mod}$. Again, the resulting algebras are much simpler than for $\mathbf{Tilt}$.

Before we can state the theorem, let us define natural transformations $L_v^{\{0\}}$ and δ_v *mutatis mutandis* as in (5-1). We also need

$$\chi_v^X := \big\{\chi_P^X \colon P \to P, \ \chi_P^X = L_v^{\{0\}}L_v^{\{1\}} \mid P \in G_2 T\text{-}p\mathbf{Mod}\big\},$$

$$\chi_x^Y := \big\{\chi_P^Y \colon P \to P, \ \chi_P^Y = L_x^{\{0\}} \mid P \in G_2 T\text{-}p\mathbf{Mod}\big\},$$

where both, $L_v^{\{0\}}L_v^{\{1\}}$ and $L_x^{\{0\}}$, are non-zero only on corresponding indecomposable projective $G_2 T$-modules $P(v-1)$ and $P(x-1)$ for $v \in \mathbb{Z} \setminus \mathsf{p}\mathbb{Z}$ and $x \in \mathsf{p}\mathbb{Z}$, where they are the head-to-socle maps. We further need an analog of (4-1), namely

$$\chi_i^Z := \big\{\chi_P^Z \colon P \to P, \ \chi_P^Z = L_i \mid P \in G_2 T\text{-}p\mathbf{Mod}\big\},$$

where L_i is only zero on all indecomposable summands $P(j-1)$ for $j \in r(i)$ (the row of i, cf. (5-5)). On such $P(j-1)$, it is a map that factors through the highest weights of its horizontal neighbors (on the grid as in (5-5)), which is unique up to scalars.

THEOREM 5.3. *We have the following.*

(a) For $g = 1$, we have $\mathbb{K}$-algebra isomorphisms:

$$\mathbf{Z}(G_1 T\text{-}p\mathbf{Mod}) \xrightarrow{\cong} \bigoplus_{i=-1}^{k-2} \mathbf{X}(i), \ \mathbf{X}(i) = \begin{cases} \prod_{\mathbb{Z}} \mathbb{K} & \text{if } i = -1, \\ \mathbb{K}[X_v \mid v \in \mathbb{Z}]\big/\langle X_v X_w \mid v, w \in \mathbb{Z}\rangle & \text{otherwise,} \end{cases}$$
$$(5\text{-}3)$$

where $\chi_w \mapsto X_v$, and $\delta_x \mapsto 1_v$ for $w, x \in \mathbb{Z}$ as explained in the proof.

(b) For $g = 2$, we have $\mathbb{K}$-algebra isomorphisms:

$$\mathbf{Z}\big(G_2 T\text{-}p\mathbf{Mod}\big) \xrightarrow{\cong} \prod_{i=0}^{\mathsf{p}-2} \mathtt{X}(i),$$

$$\mathtt{X}(i) \cong \begin{cases} \mathbf{Z}\big(G_1 T\text{-}p\mathbf{Mod}\big) & \text{if } i = -1, \\ \mathbb{K}[X_v, Y_x, Z_i \mid v, x \in \mathbb{Z}, i \in \mathrm{Eve}^{<\mathsf{p}}]\Big/ \mathrm{I} & \text{otherwise}, \end{cases} \tag{5-4}$$

where

$$\mathrm{I} = \left\langle \begin{array}{l} X_v X_w, X_v Y_y, X_v Z_j, \\ Y_x X_w, Y_x Y_y, Y_x Z_j, \\ Z_i X_w, Z_i Y_y, Z_i Z_j, \end{array} \middle| \begin{array}{l} v, w \in \mathbb{Z}, \\ x, y \in \mathbb{Z}, \\ i, j \in \mathrm{Eve}^{<\mathsf{p}} \end{array} \right\rangle,$$

and $\mathrm{L}_w^X \mapsto X_v$, and $\mathrm{L}_y^Y \mapsto Y_x$ for $w, y \in \mathbb{Z}$, and $\mathrm{L}_j^Z \mapsto Z_i$ for $j \in \mathrm{Eve}^{<\mathsf{p}}$ as explained in the proof.

Proof. (a). Very similar as in the quantum case we have a decomposition of additive $\mathbb{K}$-linear categories:

$$G_1 T\text{-}p\mathbf{Mod} \cong G_1 T\text{-}p\mathbf{Mod}_{St} \oplus G_1 T\text{-}p\mathbf{Mod}_0 \oplus \ldots \oplus G_1 T\text{-}p\mathbf{Mod}_{\mathsf{p}-2},$$

$$G_1 T\text{-}p\mathbf{Mod}_0 \cong \ldots \cong G_1 T\text{-}p\mathbf{Mod}_{\mathsf{p}-2} \cong p\mathbf{Mod}\text{-}\mathsf{Z}^{G_1 T},$$

where $G_2 T\text{-}p\mathbf{Mod}_{St}$ is semisimple with simple objects being projective $G_1 T$-modules indexed as in the first case in (5-3). The Ringel dual quiver algebra $\mathsf{Z}^{G_1 T}$ of $G_1 T\text{-}p\mathbf{Mod}_0$ in this case is a zigzag algebra on the vertex set $\mathbb{Z}$, that is,

$$\ldots \underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}} (v_{-1} - 1) \underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}} (v_0 - 1) \underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}} (v_1 - 1) \underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}} (v_2 - 1) \underset{\mathrm{D}_{\{0\}}}{\overset{\mathrm{U}_{\{0\}}}{\rightleftarrows}} \ldots ,$$

where $v_0 = 1$ and $v_{i+1} = v_i(0)$ for $i \in \mathbb{N}_0$, $v_{-i} = -v_i + 2$ for i even, and $v_{-i} = -v_i + 2\mathsf{p} - 2$ for i odd. These are the numbers with p-adic expansion $[n_1, a_0]_{\mathsf{p}}$ where $a_0 \in \{0, \ldots, \mathsf{p} - 1\}$ and $n_1 \in \mathbb{Z}$, which can be reached from 1 by successive upward or downward reflection in the zeroth digit. The generators $\mathrm{D}_{\{0\}}$ and $\mathrm{U}_{\{0\}}$ are subject to the relations:

$$\mathrm{D}_{\{0\}}\mathrm{D}_{\{0\}}\mathrm{e}_{v-1} = 0, \; \mathrm{U}_{\{0\}}\mathrm{U}_{\{0\}}\mathrm{e}_{v-1} = 0, \; \mathrm{D}_{\{0\}}\mathrm{U}_{\{0\}}\mathrm{e}_{v-1} = \mathrm{U}_{\{0\}}\mathrm{D}_{\{0\}}\mathrm{e}_{v-1}.$$

Formally, the algebra $\mathsf{Z}^{G_1 T}$ is generated by e_{v-1} for $v \in \mathbb{Z}$ being of the form $v = \pm\big(1(0)\ldots(0)\big)$, and elements $\mathrm{D}_{\{0\}}\mathrm{e}_{v-1}$ and $\mathrm{U}_{\{0\}}\mathrm{e}_{v-1}$. The relations are the ones above together with the usual idempotent relations. Thus, the situation is analogous to the quantum case. In particular, the same arguments, *mutatis mutandis*, as for the quantum group give the second case in (5-3).

(c). We start by recalling that

$$G_2 T\text{-}p\mathbf{Mod} \cong G_2 T\text{-}p\mathbf{Mod}_{St} \oplus G_2 T\text{-}p\mathbf{Mod}_0 \oplus \ldots \oplus G_2 T\text{-}p\mathbf{Mod}_{\mathsf{p}-2},$$

$$G_2 T\text{-}p\mathbf{Mod}_{St} \cong G_1 T\text{-}p\mathbf{Mod}, \quad G_2 T\text{-}p\mathbf{Mod}_0 \cong \ldots \cong G_2 T\text{-}p\mathbf{Mod}_{\mathsf{p}-2} \cong p\mathbf{Mod}\text{-}\mathsf{Z}^{G_2 T},$$

where $G_2 T\text{-}p\mathbf{Mod}_{St}$ is no longer semisimple, but rather equivalent to $G_1 T\text{-}p\mathbf{Mod}$. This gives us the first summand in (5-4). The Ringel dual quiver algebra $\mathsf{Z}^{G_2 T}$ still has vertex set $\mathbb{Z}$, but arranged on a grid, for example, if $\mathsf{p} = 5$, then, as explained in [**1**, Section 6.3]:

$$(5\text{-}5)$$
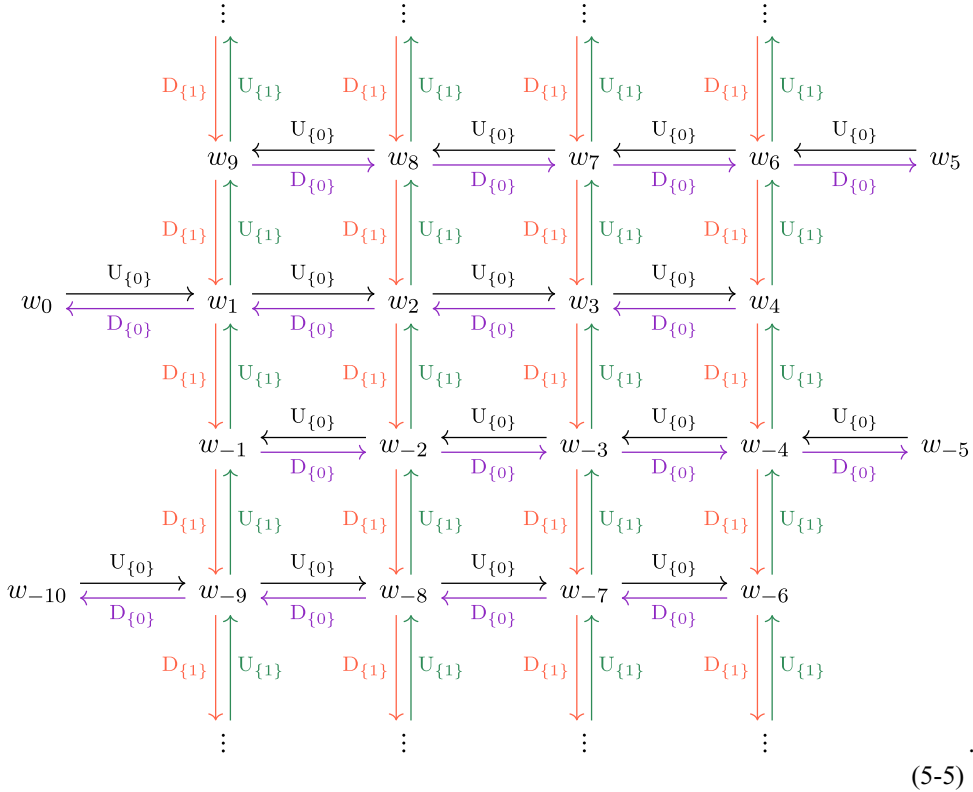
The indexing of the positively labeled vertices $w_i = v_i - 1$ hereby works as follows. $v_0 = [1]_5 (= 1)$, and each horizontal step is (0), for example, $v_1 = [1]_5(0) = [2, -1]_5 = [1, 4]_5 (= 9)$, while each vertical step is (1), for example, $v_9 = [1, 4]_5(1) = [2, -1, 4]_5 (= 49)$. Moreover, $w_{-i} = -w_i$, if $i \geq 0$ is even, $w_{-i} = -w_i + 2\mathsf{p} - 4$, if $i > 0$ is odd. The v_i are exactly the integers with p-adic expansion $[n_2, a_1, a_0]_\mathsf{p}$ where $a_0, a_1 \in \{0, ..., \mathsf{p} - 1\}$ and $n_2 \in \mathbb{Z}$, which can be reached from 1 by reflection in the first two digits.

Precisely, the algebra $Z^{G_2 T}$ is generated by e_{v-1} for $v \in \mathbb{Z}$ being as above, and elements $D_{\{0\}} e_{v-1}$ and $U_{\{0\}} e_{v-1}$ as well as $D_{\{1\}} e_{v-1}$ and $U_{\{1\}} e_{v-1}$. The relations are such that each column and each row is a zigzag algebra, and all squares commute, that is,

(1) Each row is a zigzag algebra, *cf.* Theorem 2.14.(2) and (6), that is,

$$U_{\{0\}} U_{\{0\}} e_{v-1} = 0 = D_{\{0\}} D_{\{0\}} e_{v-1}, \quad D_{\{0\}} U_{\{0\}} e_{v-1} = U_{\{0\}} D_{\{0\}} e_{v-1}.$$

(No boundary condition.)

(2) Each column is a zigzag algebra, *cf.* Theorem 2.14.(2) and (6), that is,
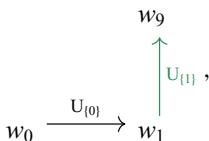
$$U_{\{1\}} U_{\{1\}} e_{v-1} = 0 = D_{\{1\}} D_{\{1\}} e_{v-1}, \quad D_{\{1\}} U_{\{1\}} e_{v-1} = U_{\{1\}} D_{\{1\}} e_{v-1}.$$

(3) All squares commute, *cf.* Theorem 2.14.(4), that is,

$$D_{\{1\}} U_{\{0\}} e_{v-1} = D_{\{0\}} D_{\{1\}} e_{v-1}, \quad D_{\{0\}} U_{\{1\}} e_{v-1} = U_{\{1\}} U_{\{0\}} e_{v-1},$$
$$D_{\{1\}} D_{\{0\}} e_{v-1} = D_{\{0\}} U_{\{1\}} e_{v-1}, \quad D_{\{0\}} D_{\{1\}} e_{v-1} = D_{\{1\}} U_{\{0\}} e_{v-1}.$$

These hold only for complete squares, that is,

$$
\begin{array}{ccc}
 & & w_9 \\
 & & \uparrow \\
 & & \Big| U_{\{1\}} \\
w_0 & \xrightarrow{\ U_{\{0\}}\ } & w_1
\end{array}
\ ,
$$

as in (5-5), does not satisfy any relation, and is in particular, not zero.

Hence, we get row and column loops:

$$
L_i := L_{\{0\}} e_{v_i-1} = D_{\{0\}} U_{\{0\}} e_{v_i-1} = U_{\{0\}} D_{\{0\}} e_{v_i-1},
$$

$$
L'_i := L_{\{1\}} e_{v_i-1} = D_{\{1\}} U_{\{1\}} e_{v_i-1} = U_{\{1\}} D_{\{1\}} e_{v_i-1}.
$$

(Here, as in (5-5), only one expression for $L_{\{0\}} e_{v_i-1}$ makes sense if $\mathsf{p}|i$. The above is just a shorthand notation.) The relations imply that
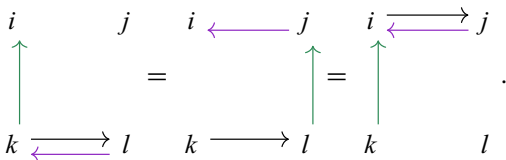
$$
e_{v_i-1} \mathsf{Z}^{G_2 T} e_{v_i-1} \cong
\begin{cases}
\mathbb{K}[L_i] \big/ \langle L_i^2 \rangle & \text{if } \mathsf{p}|i, \\
\mathbb{K}[L_i, L'_i] \big/ \langle L_i^2, (L'_i)^2 \rangle & \text{otherwise.}
\end{cases}
\tag{5-6}
$$

Note that these have bases $\{e_{v_i-1}, L_i\}$ and $\{e_{v_i-1}, L_i, L'_i, L_i L'_i\}$, respectively. Also, the loops L_i, for $\mathsf{p}|i$, and $L_i L'_i$, otherwise, are central since only the idempotent e_{v_i-1} does not annihilate them.

But there are more central elements. To define them, let $c(j)$ denote the set of indexes of the jth column, read left to right. For example, $c(2) = \{..., -8, -2, 2, 8, ...\}$ for $\mathsf{p} = 5$, *cf.* (5-5). Then, we sum row loops over their column $r(j)$, that is:

$$
L_{c(j)} = \sum_{i \in c(j)} L_i \in \overline{\mathsf{Z}}^{G_2 T}.
$$

We have $L_{c(j)} \in Z(\overline{\mathsf{Z}}^{G_2 T})$: that $L'_{c(j)}$ commutes with all idempotents is clear. Moreover, each vertical arrow annihilates $L'_{c(j)}$ from both sides and each horizontal arrow transports a loop L_i to its neighbor, illustrated as:

$$
\begin{array}{ccccccc}
i & \quad & j & \quad i \xleftarrow{\quad} j & \quad i \xrightarrow{\quad} j & \\
\uparrow & = & & \uparrow \ = \ \uparrow & \quad \xleftarrow{\quad} & \\
\Big| & & & \Big| & \uparrow & \uparrow \\
k \xrightarrow{\quad} l & & & k \xrightarrow{\quad} l & \quad k \qquad l &
\end{array}
\ .
$$

Furthermore, a bit more thought (using arguments as in Section 3) proves that the central elements which we have identified, that is,

$$
L_i, \text{ for } \mathsf{p}|i, \quad L_i L'_i, \text{ otherwise}, \quad L'_{r(j)}, \text{ for } j \in \{1, ..., \mathsf{p} - 1\},
$$

form a basis of $Z(\overline{\mathsf{Z}}^{G_2 T})$, and it remains to calculate the relations among these. We already know the relations among the elements L_i and $L_i L'_i$, see (5-6). Further, we have

$$
L_{c(j)} L_{c(k)} = 0, \quad L_{c(j)} L_i = 0 = L_i L_{c(j)}, \text{ for } \mathsf{p}|i, \quad L_{c(j)} L_i L'_i = 0 = L_i L'_i L_{c(j)}.
$$

This concludes the proof. $\qquad\qquad\square$

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