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SYNCHRONIZATION AND FLUCTUATION THEOREMS FOR INTERACTING FRIEDMAN URNS

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Abstract

We consider a model of N interacting two-colour Friedman urns. The interaction model considered is such that the reinforcement of each urn depends on the fraction of balls of a particular colour in that urn as well as the overall fraction of balls of that colour in all the urns combined together. We show that the urns synchronize almost surely and that the fraction of balls of each colour converges to the deterministic limit of one-half, which matches with the limit known for a single Friedman urn. Furthermore, we use the notion of stable convergence to obtain limit theorems for fluctuations around the synchronization limit.

Keywords: Interacting urn model; Friedman urn; Pólya urn; synchronization; stable convergence; reinforcement; fluctuation theorem

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Secondary 60F05; 60B10; 60F99

1. Introduction

The classical Pólya urn scheme consists of an urn containing x balls of one colour and y balls of another colour. At time t, one ball is drawn randomly from the urn and its colour observed; it is then replaced in the urn, along with a ball of the same colour. This self-reinforcement is carried out repeatedly. Asymptotic properties of the Pólya urn process, and its generalizations and applications have been studied extensively (see [21] and [17], respectively). The Friedman urn model (proposed by Bernard Friedman in 1949 [13]) is a generalization of a Pólya urn, where the chosen ball is replaced with α balls of the same colour and β balls of another colour. It is known that, for a Pólya urn, the fraction of balls of either colour approaches, with probability 1, a random limit that is distributed according to a beta distribution. In the case of Friedman urns, this limit is deterministic and equal to $\frac{1}{2}$ with probability 1.

Interacting urn models have been an area of interest recently. For instance, in [5] and [18] lattice-based interacting Pólya urns were studied. In [2] a graph-based model, with urns at each vertex and pairwise interactions, was considered. In [16] interacting urns with exponential reinforcement were studied. In [19] the authors considered an interacting urn model in which a ball is sampled from each urn and then replaced in the urn along with a random number of balls of the same colour.

Building on the work of Dai Pra *et al.* [9] and Crimaldi *et al.* [7], we study the asymptotic properties of N (interacting) Friedman urns, each containing balls of two colours, say white and black. The reinforcement scheme is such that the probability of adding α white and β black balls to any urn at any time t depends not only on the fraction of balls of a given colour in that particular urn, but also on the fraction of that colour across all the N urns. We show that the urns

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synchronize almost surely and that the fraction of balls of each colour in every urn converges to the common deterministic limit of $\frac{1}{2}$. Rates of $\mathcal{L}^2$-convergence are obtained to illustrate the crucial difference between the synchronization phenomenon of interacting Friedman urns and the interacting Pólya urns of [9]. Furthermore, we study fluctuations in the fraction of balls of each colour as well as the overall fraction of balls of a colour around the limit $\frac{1}{2}$. The results obtained here are, in spirit, similar to those obtained in [12]. By that we mean that the scaling phenomenon resembles that observed in [12]; however, as one would expect, the regions for Gaussian and non-Gaussian behaviour depend on the interaction parameter.

The paper is organized as follows. We describe the model in detail in the next subsection. In Section 2 we show that the urns synchronize almost surely and obtain the rate of convergence. In Section 3 we prove limit theorems for fluctuations in the fraction of balls of each colour around the limit $\frac{1}{2}$, and fluctuations between the fraction of balls of a colour in one urn and the overall fraction of balls of that colour. Appendix A contains some relevant concepts and results pertaining to the notions of stable convergence and stochastic approximation used in the proofs.

1.1. Basic setup

Consider N urns denoted by $U(1), \dots, U(N)$ such that at time $t = 0$ each urn contains $W_0(i) > 0$ white and $B_0(i) > 0$ black balls. Let $N_0(i) = W_0(i) + B_0(i)$ denote the total number of balls in each urn at the beginning, and let $W_t(i)$ and $B_t(i)$ respectively denote the number of white and black balls in $U(i)$ at time t. Then

$$W_{t+1}(i) = W_t(i) + Y_{t+1}(i), \tag{1}$$

where $Y_{t+1}(i)$ denotes the number of white balls added to urn $U(i)$ at time $t+1$. We assume that $Y_t(i)$ for $i = 1, \dots, N$ are conditionally independent given the past. We denote the total number of balls in each urn at time t by $N_t(i)$. For notational simplicity, we start with the same number of balls (denoted by N_0) in each urn and add $\alpha + \beta$ balls with probability 1 at each time step. This assumption does not affect the asymptotic properties on the urn. Thus, $N_t = t(\alpha+\beta)+N_0$ for $t \geq 1$. Note that we have dropped the i since N_t is now independent of i.

Let $Z_t(i)$ be the fraction of white balls in $U(i)$ at time t, and let $Z_t = (1/NN_t)\sum_{i=1}^{N} W_t(i) = (1/N)\sum_i Z_t(i)$ be the overall fraction of white balls. Fix $\alpha, \beta \in \mathbb{N}$. Then consider the reinforcement model

$$\mathbb{P}(Y_{t+1}(i) = w \mid \mathcal{F}_t) = \begin{cases} pZ_t + (1-p)Z_t(i) & \text{for } w = \alpha, \\ 1 - pZ_t - (1-p)Z_t(i) & \text{for } w = \beta, \end{cases}$$

for some fixed $p \in [0, 1]$. The parameter p is called the interaction parameter. When $p = 0$, we obtain N-independent Friedman urns, each mimicking the classical single-urn Friedman model. While Z_t denotes the overall fraction of white balls throughout this paper, to avoid any confusion, we will explicitly mention whenever it represents the fraction of white balls for the single-urn model.

Remark 1. This model is inspired from the work carried out on interacting Pólya urns in [9]. The tools and concepts used to obtain results for the asymptotic behaviour of interacting Friedman urns are similar to those used in [7] and [9]. As and when necessary, we point out the differences and similarities between the two.

Before going into details, we define the setup rigorously. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider a family of Uniform[0, 1] random variables $U(t, i)$, $t, i \in \mathbb{N}$. Define by $\mathcal{F}_t$ the σ-field

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generated by $\{U(s,i); 0 \le s \le t,\ i \in \mathbb{N}\}$. Define

$$Y_{t+1}(i) = \begin{cases} \alpha & \text{if } U(t+1,i) \le pZ_t + (1-p)Z_t(i), \\ \beta & \text{otherwise.} \end{cases}$$

Then the $Y_t(i)$ are conditionally independent given $\mathcal{F}_t$.

For a classical Pólya urn, it is known that the limiting distribution of the fraction of balls of either colour is given by a beta distribution, whereas, for a classical Friedman urn, this limit is a deterministic quantity, namely, $\frac{1}{2}$. The following computation illustrates the underlying reason as to why the asymptotic behaviour of Pólya urns and Friedman urns is different:

$$\begin{aligned} \mathbb{E}[Z_{t+1}(i) \mid \mathcal{F}_t] &= \mathbb{E}\bigg[\frac{W_{t+1}(i)}{N_{t+1}} \;\bigg|\; \mathcal{F}_t\bigg] \\ &= \mathbb{E}\bigg[\frac{W_t(i) + Y_{t+1}(i)}{N_{t+1}} \;\bigg|\; \mathcal{F}_t\bigg] \\ &= \frac{N_t}{N_{t+1}} Z_t(i) + \frac{1}{N_{t+1}}[\alpha(pZ_t + (1-p)Z_t(i)) \\ &\qquad + \beta(1 - pZ_t - (1-p)Z_t(i))] \\ &= \frac{N_t + (\alpha-\beta)(1-p)}{N_{t+1}} Z_t(i) + \frac{(\alpha-\beta)p}{N_{t+1}} Z_t + \frac{\beta}{N_{t+1}}, \end{aligned} \tag{2}$$

$$\begin{aligned} \mathbb{E}[Z_{t+1} \mid \mathcal{F}_t] &= \mathbb{E}\bigg[\frac{1}{N}\sum_i Z_{t+1}(i) \;\bigg|\; \mathcal{F}_t\bigg] \\ &= \frac{1}{N}\sum_i \mathbb{E}[Z_{t+1}(i) \mid \mathcal{F}_t] \\ &= \frac{N_t + (\alpha-\beta)}{N_{t+1}} Z_t + \frac{\beta}{N_{t+1}}. \end{aligned} \tag{3}$$

In the case of Pólya urns, $\alpha = 1$ and $\beta = 0$. This means that in that case, Z_t is a martingale. Friedman urns do not satisfy the martingale property. This marks the essential difference between the two urn models. In the next section we prove an important synchronization result for interacting Friedman urns. We fix the following notation. By ρ we denote the ratio $(\alpha-\beta)/(\alpha+\beta)$ with $\alpha \ne \beta > 0$. Without loss of generality, we can assume that $\rho > 0$. We also assume that the interaction is nontrivial, that is, $0 < p < 1$.

2. Synchronization

From (1), we obtain

$$Z_{t+1}(i) = \frac{1}{N_{t+1}}[N_t Z_t(i) + Y_{t+1}(i)].$$

We already know that in the case of a single urn, the fraction of balls of each colour converges to $\frac{1}{2}$ almost surely (a.s.). In order to prove a similar result for interacting urns, we first show $\mathcal{L}^2$-synchronization, that is, we show that $\lim_{t\to\infty} \mathbb{E}[(Z_t(i) - Z_t)^2] = 0$. We also prove an almost-sure synchronization result.

The following theorem explicitly states the rates of convergence (to 0) of $\operatorname{var}(Z_t)$, $\operatorname{var}(Z_t(i))$ and $\operatorname{var}(Z_t - Z_t(i))$ for $i = 1, \dots, N$. While it is expected that all these quantities converge to 0, it is interesting to see that they do not always (i.e. in every regime determined by the interacting parameter p) go to 0 at the same rate. For two positive sequences a_t and b_t, we write $a_t \sim b_t$ if $0 < \liminf_{t\to\infty}(a_t/b_t) \le \limsup_{t\to\infty.}(a_t/b_t) < \infty$.

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2.1. $\mathcal{L}^2$-synchronization

Theorem 1. *Set* $\rho = (\alpha - \beta)/(\alpha + \beta) > 0$. *For every* $i \in \{1, \ldots, N\}$, *the following asymptotic estimates hold:*

$$\operatorname{var}(Z_t), \operatorname{var}(Z_t(i)) \sim \begin{cases} t^{2\rho-2} & \text{for } \rho > \frac{1}{2}, \\ t^{-1}\log t & \text{for } \rho = \frac{1}{2}, \\ t^{-1} & \text{for } \rho < \frac{1}{2}, \end{cases}$$

$$\operatorname{var}(Z_t(i) - Z_t) \sim \begin{cases} t^{2\rho-2\rho p-2} & \text{for } \rho > 1/2(1-p), \\ t^{-1}\log t & \text{for } \rho = 1/2(1-p), \\ t^{-1} & \text{for } \rho < 1/2(1-p). \end{cases}$$

Here $\operatorname{var}(X)$ *denotes the variance of a random variable* X.

Remark 2. The $\mathcal{L}^2$-synchronization result in [9] for the Pólya urns can be obtained by substituting $\rho = 1$.

Remark 3. (*Synchronization rate.*) Since $0 < p < 1, 1/2(1-p) > \frac{1}{2}$. So, the regime $\{\rho < \frac{1}{2}\}$ is a subset of $\{\rho < 1/[2(1-\rho)]\}$. Thus, for a given i, for $\rho < \frac{1}{2}$, the $\mathcal{L}^2$ rate of convergence of $\operatorname{var}(Z_t)$, $\operatorname{var}(Z_t(i))$, and $\operatorname{var}(Z_t - Z_t(i))$ are the same. However, in the interval $\frac{1}{2} < \rho < \frac{1}{2}(1-p)$, the $\operatorname{var}(Z_t - Z_t(i))$ converges to 0 faster than both $\operatorname{var}(Z_t)$ and $\operatorname{var}(Z_t(i))$. Indeed, the difference $\operatorname{var}(Z_t(i) - Z_t)$ converges to 0 at the rate $1/t$ for $\rho < 1/[2(1-p)]$, while both $\operatorname{var}(Z_t)$ and $\operatorname{var}(Z_t(i))$ converge at a rate $t^{2\rho-2}$ for $\frac{1}{2} < \rho < 1/[2(1-p)]$. Again, for $\rho > 1/[2(1-p)]$, $\operatorname{var}(Z_t(i) - Z_t)$ converges at a faster rate. This means that the $\mathcal{L}^2$-synchronization rate is faster than (or equal to) the rate at which $Z_t, Z_t(i) \to \frac{1}{2}$ in $\mathcal{L}^2$. This is a deviation from the behaviour observed in the interacting Pólya urns model.

To be able to prove Theorem 1, we first do some computations to obtain recurrence relations for $\operatorname{var}(Z_t)$ and $\operatorname{var}(Z_t(i))$, and the difference $\operatorname{var}(Z_t(i) - Z_t)$. The proof of the theorem then uses the same tools as those in [9]. In the following discussion, we denote $pZ_t + (1-p)Z_t(i)$ by a_i whenever convenient.

Let us first consider $\operatorname{var}(Z_{t+1}) = \mathbb{E}[\operatorname{var}(Z_{t+1} \mid \mathcal{F}_t)] + \operatorname{var}(\mathbb{E}[Z_{t+1} \mid \mathcal{F}_t])$. Then

$$\operatorname{var}(\mathbb{E}[Z_{t+1} \mid \mathcal{F}_t]) = \operatorname{var}\bigg(\frac{N_t + (\alpha-\beta)}{N_{t+1}} Z_t + \frac{\beta}{N_{t+1}}\bigg) = \bigg(\frac{N_t + (\alpha-\beta)}{N_{t+1}}\bigg)^2 \operatorname{var}(Z_t)$$

and

$$\begin{aligned} \mathbb{E}[\operatorname{var}(Z_{t+1} \mid \mathcal{F}_t)] &= \mathbb{E}\bigg[\frac{1}{N^2}\sum_i \operatorname{var}(Z_{t+1}(i) \mid \mathcal{F}_t)\bigg] \\ &= \mathbb{E}\bigg[\frac{1}{N^2}\sum_i \operatorname{var}\bigg(\frac{Y_{t+1}(i)}{N_{t+1}} \;\bigg|\; \mathcal{F}_t\bigg)\bigg] \\ &= \mathbb{E}\bigg[\frac{1}{N^2 N_{t+1}^2}\sum_i \alpha^2 a_i + \beta^2(1-a_i) - (\alpha a_i + \beta(1-a_i))^2\bigg] \\ &= \mathbb{E}\bigg[\frac{1}{N^2 N_{t+1}^2}\sum_i (\alpha-\beta)^2 a_i(1-a_i)\bigg] \\ &= \frac{1}{N^2 N_{t+1}^2}(\alpha-\beta)^2 \sum_i \mathbb{E}[a_i(1-a_i)] \end{aligned}$$

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$$= \frac{(\alpha-\beta)^2}{N^2 N_{t+1}^2} \sum_i \mathbb{E}[pZ_t + (1-p)Z_t(i) - p^2 Z_t^2 - (1-p)^2 Z_t^2(i) - 2p(1-p)Z_t Z_t(i)]$$

$$= \frac{(\alpha-\beta)^2}{N N_{t+1}^2} \mathbb{E}\bigg[Z_t - p(2-p)Z_t^2 - \frac{(1-p)^2}{N} \sum_i Z_t^2(i)\bigg]$$

$$= \frac{(\alpha-\beta)^2}{N N_{t+1}^2} \bigg[\frac{1}{2} - p(2-p)\bigg(\operatorname{var}(Z_t) + \frac{1}{4}\bigg) - \frac{(1-p)^2}{N} \sum_i \bigg(\operatorname{var}(Z_t(i)) + \frac{1}{4}\bigg)\bigg]$$

$$= \frac{(\alpha-\beta)^2}{N N_{t+1}^2} \bigg[\frac{1}{4} - p(2-p)\operatorname{var}(Z_t) - (1-p)^2 \operatorname{var}(Z_t(j))\bigg]$$

for any $j \in \{1, 2, \ldots, N\}$. So,

$$\operatorname{var}(Z_{t+1}) = \frac{(\alpha-\beta)^2}{N N_{t+1}^2} \bigg[\frac{1}{4} - p(2-p)\operatorname{var}(Z_t) - (1-p)^2 \operatorname{var}(Z_t(j))\bigg] + \bigg(\frac{N_t + (\alpha-\beta)}{N_{t+1}}\bigg)^2 \operatorname{var}(Z_t). \tag{4}$$

Now consider $\operatorname{var}(Z_{t+1}(i)) = \mathbb{E}[\operatorname{var}(Z_{t+1}(i) \mid \mathcal{F}_t)] + \operatorname{var}(\mathbb{E}[Z_{t+1}(i) \mid \mathcal{F}_t])$. From above, we already have

$$\operatorname{var}(Z_{t+1}(i) \mid \mathcal{F}_t) = \frac{(\alpha-\beta)^2}{N_{t+1}^2} a_i(1-a_i),$$

$$\mathbb{E}[\operatorname{var}(Z_{t+1}(i) \mid \mathcal{F}_t)] = \frac{(\alpha-\beta)^2}{N_{t+1}^2} \mathbb{E}[a_i(1-a_i)]$$

$$= \frac{(\alpha-\beta)^2}{N_{t+1}^2} [1 - p^2 \operatorname{var}(Z_t) - (1-p^2)\operatorname{var}(Z_t(i))].$$

Then

$$\operatorname{var}(Z_{t+1}(i)) = \frac{(\alpha-\beta)^2}{N_{t+1}^2} [1 - p^2 \operatorname{var}(Z_t) - (1-p^2)\operatorname{var}(Z_t(i))] + \frac{1}{N_{t+1}^2} \operatorname{var}([N_t + (\alpha-\beta)(1-p)]Z_t(i) + (\alpha-\beta)pZ_t)$$

$$= \frac{(\alpha-\beta)^2}{N_{t+1}^2} [1 - (1-p^2)\operatorname{var}(Z_t(i))] + \bigg(\frac{N_t + (\alpha-\beta)(1-p)}{N_{t+1}}\bigg)^2 \operatorname{var}(Z_t(i)) + \frac{2p(\alpha-\beta)(N_t + (\alpha-\beta)(1-p))}{N_{t+1}^2} \operatorname{cov}(Z_t, Z_t(i)). \tag{5}$$

Finally, let us consider the difference $\operatorname{var}(Z_t(i) - Z_t)$. We have

$$\operatorname{var}(Z_{t+1}(i) - Z_{t+1}) = \mathbb{E}[\operatorname{var}(Z_{t+1}(i) - Z_{t+1} \mid \mathcal{F}_t)] + \operatorname{var}(\mathbb{E}[Z_{t+1}(i) - Z_{t+1} \mid \mathcal{F}_t])$$

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and

$$\begin{aligned} Z_{t+1}(i) - Z_{t+1} &= \frac{W_{t+1}(i)}{N_{t+1}} - \frac{1}{N}\sum_i \frac{W_{t+1}(i)}{N_{t+1}} \\ &= \frac{N_t}{N_{t+1}} Z_t(i) \frac{Y_{t+1}(i)}{N_{t+1}} - \frac{1}{N}\sum_i \frac{N_t}{N_{t+1}} Z_t(i) \frac{Y_{t+1}(i)}{N_{t+1}} \\ &= \frac{N_t}{N_{t+1}}(Z_t(i) - Z_t) + \frac{1}{N_{t+1}}\left(1 - \frac{1}{N}\right) Y_{t+1}(i) - \frac{1}{N N_{t+1}} \sum_{j \neq i} Y_{t+1}(j). \end{aligned}$$

Then

$$\begin{aligned} \operatorname{var}(\mathbb{E}[Z_{t+1}(i) - Z_{t+1} \mid \mathcal{F}_t]) &= \operatorname{var}\left(\frac{N_t + (\alpha - \beta)(1-p)}{N_{t+1}}(Z_t(i) - Z_t)\right) \\ &= \left(\frac{N_t + (\alpha - \beta)(1-p)}{N_{t+1}}\right)^2 \operatorname{var}(Z_t(i) - Z_t) \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}[\operatorname{var}(Z_{t+1}(i) - Z_{t+1} \mid \mathcal{F}_t)] \\ &\quad = \mathbb{E}\left[\operatorname{var}\left(\frac{N-1}{N_{t+1}N} Y_{t+1}(i) - \frac{1}{N_{t+1}N}\sum_{j\neq i} Y_{t+1}(i) \,\middle|\, \mathcal{F}_t\right)\right] \\ &\quad = \mathbb{E}\left[\left(\frac{N-1}{N_{t+1}N}\right)^2 \operatorname{var}(Y_{t+1}(i) \mid \mathcal{F}_t) + \frac{1}{N_{t+1}^2 N^2}\sum_{j\neq i}\operatorname{var}(Y_{t+1}(j) \mid \mathcal{F}_t)\right] \\ &\quad = \mathbb{E}\left[\left(\frac{N-1}{N_{t+1}N}\right)^2 (\alpha-\beta)^2 a_i(1-a_i) + \frac{1}{N_{t+1}^2 N^2}(\alpha-\beta)^2 \sum_{j\neq i} a_j(1-a_j)\right] \\ &\quad = \frac{1}{N_{t+1}^2 N^2}(\alpha-\beta)^2 \mathbb{E}\left[(N^2 - 2N) a_i(1-a_i) + \sum_j a_j(1-a_j)\right] \\ &\quad = \frac{1}{N_{t+1}^2 N^2}(\alpha-\beta)^2 (N^2 - 2N)\left[\mathbb{E}[a_i(1-a_i)] + \sum_j \mathbb{E}[a_j(1-a_j)]\right] \\ &\quad = \frac{(\alpha-\beta)^2}{N_{t+1}^2}\left[\frac{N-2}{N}[1 - p^2 \operatorname{var}(Z_t) - (1-p^2)\operatorname{var}(Z_t(i))]\right] \\ &\qquad + \frac{(\alpha-\beta)^2}{N N_{t+1}^2}\left[\frac{1}{4} - 2(2-p)\operatorname{var}(Z_t) - (1-p)^2 \operatorname{var}(Z_t(i))\right]. \end{aligned}$$

So,

$$\begin{aligned} \operatorname{var}(Z_{t+1}(i) - Z_{t+1}) &= \left(\frac{N_t + (\alpha-\beta)(1-p)}{N_{t+1}}\right)^2 \operatorname{var}(Z_t(i) - Z_t) \\ &\quad + \frac{(\alpha-\beta)^2}{N_{t+1}^2}\left[\frac{N-2}{N}[1 - p^2 \operatorname{var}(Z_t) - (1-p^2)\operatorname{var}(Z_t(i))]\right] \\ &\quad + \frac{(\alpha-\beta)^2}{N N_{t+1}^2}\left[\frac{1}{4} - 2(2-p)\operatorname{var}(Z_t) - (1-p)^2 \operatorname{var}(Z_t(i))\right]. \end{aligned} \tag{6}$$

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2.2. Proof of Theorem 1

Suppose that $x_{t+1} = f(t)x_t + g(t)$ such that $0 < f(t) < 1$ for every $t \geq 0$ and $x_0 = 0$. If we set $y_0 = x_0$ and $y_t = x_t / \prod_{k=0}^{t-1} f(k)$, y_t satisfies

$$y_0 = 0 \quad \text{and} \quad y_{t+1} = y_t + F(t),$$

where $F(t) = g(t)/\prod_{k=0}^{t} f(k)$. Then we can verify that $y_t = \sum_{i=0}^{t-1} F(i)$ for $t \geq 1$. This leads to a solution for the original difference equation in x, which is given by

$$x_0 = 0 \quad \text{and} \quad x_t = \prod_{k=0}^{t-1} f(k) \sum_{i=0}^{t-1} \frac{g(i)}{\prod_{k=0}^{i} f(k)} \quad \text{for } t \geq 1.$$

Now define $x_{t+1} = \operatorname{var}(Z_{t+1})$. Then

$$x_{t+1} = f(t)x_t + g(t),$$

where $f(t) = ((N_t + (\alpha - \beta))/N_{t+1})^2$ and $g(t) \sim 1/t^2$. Then from (4) we obtain

$$\operatorname{var}(Z_t) \sim \begin{cases} t^{2\rho-2} & \text{for } \rho > \frac{1}{2}, \\ t^{-1} \log t & \text{for } \rho = \frac{1}{2}, \\ t^{-1} & \text{for } \rho < \frac{1}{2}. \end{cases}$$

Setting $x_{t+1} = \operatorname{var}(Z_{t+1}(i))$, we obtain $g(t) \sim 1/t$. Then from (5) we have

$$\operatorname{var}(Z_t(i)) \sim \begin{cases} t^{2\rho-2} & \text{for } \rho > \frac{1}{2}, \\ t^{-1} \log t & \text{for } \rho = \frac{1}{2}, \\ t^{-1} & \text{for } \rho < \frac{1}{2}. \end{cases}$$

Again, setting $x_{t+1} = \operatorname{var}(Z_{t+1}(i) - Z_{t+1})$, we obtain $f(t) = ((N_t + (\alpha - \beta)(1 - p))/N_{t+1})^2$ and $g(t) \sim 1/t^2$. Then from (6) we obtain

$$\operatorname{var}(Z_t(i) - Z_t) \sim \begin{cases} t^{2\rho-2\rho p-2} & \text{for } \rho > 1/2(1-p), \\ t^{-1} \log t & \text{for } \rho = 1/2(1-p), \\ t^{-1} & \text{for } \rho < 1/2(1-p). \end{cases}$$

2.3. Almost-sure synchronization

Since we already have $\mathcal{L}^2$-synchronization, it is sufficient to show that the almost-sure limits $\lim_{t\to\infty} Z_t$ and $\lim_{t\to\infty} Z_t(i)$ exist. This follows from the next proposition.

Proposition 1. *Suppose that $Z_t(i)$ and Z_t are quasimartingales (for details, see [22]). That is, $\sum_{t=0}^{\infty} \mathbb{E}[|\mathbb{E}[Z_{t+1}(i) \mid \mathcal{F}_t] - Z_t(i)|] < \infty$ and $\sum_{t=0}^{\infty} \mathbb{E}[|\mathbb{E}[Z_{t+1} \mid \mathcal{F}_t] - Z_t|] < \infty$.*

Proof. From the computations carried out in the previous section, we have

$$\mathbb{E}[Z_{t+1} \mid \mathcal{F}_t] = \frac{N_t + (\alpha - \beta)}{N_{t+1}} Z_t + \frac{\beta}{N_{t+1}}.$$

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Then

$$\begin{aligned}\sum_{t=0}^{\infty}\mathbb{E}[\mathbb{E}[Z_{t+1}\mid\mathcal{F}_t]-Z_t] &= \sum_{t=0}^{\infty}\mathbb{E}\bigg[\bigg|\frac{N_t+(\alpha-\beta)}{N_{t+1}}Z_t+\frac{\beta}{N_{t+1}}-Z_t\bigg|\bigg]\\ &= \sum_{t=0}^{\infty}\mathbb{E}\bigg[\frac{N_t+(\alpha-\beta)-N_{t+1}}{N_{t+1}}Z_t+\frac{\beta}{N_{t+1}}\bigg]\\ &= \sum_{t=0}^{\infty}\mathbb{E}\bigg[\frac{-2\beta}{N_{t+1}}Z_t+\frac{\beta}{N_{t+1}}\bigg]\\ &= \sum_{t=0}^{\infty}\frac{-2\beta}{N_{t+1}}\mathbb{E}\bigg[Z_t-\frac{1}{2}\bigg].\end{aligned}$$

Similarly,

$$\mathbb{E}[Z_{t+1}(i)\mid\mathcal{F}_t]=\frac{N_t+(\alpha-\beta)(1-p)}{N_{t+1}}Z_t(i)+\frac{(\alpha-\beta)p}{N_{t+1}}Z_t+\frac{\beta}{N_{t+1}}.$$

So, we obtain

$$\begin{aligned}&\mathbb{E}[|\mathbb{E}[Z_{t+1}(i)\mid\mathcal{F}_t]-Z_t(i)|]\\ &\quad=\mathbb{E}\bigg[\bigg|\frac{N_t+(\alpha-\beta)(1-p)}{N_{t+1}}Z_t(i)+\frac{(\alpha-\beta)p}{N_{t+1}}Z_t+\frac{\beta}{N_{t+1}}-Z_t(i)\bigg|\bigg]\\ &\quad=\mathbb{E}\bigg[\bigg|\frac{N_t+(\alpha-\beta)-N_{t+1}}{N_{t+1}}Z_t(i)+\frac{\beta}{N_{t+1}}+\frac{(\alpha-\beta)p}{N_{t+1}}Z_t-\frac{(\alpha-\beta)p}{N_{t+1}}Z_t(i)\bigg|\bigg]\\ &\quad=\mathbb{E}\bigg[\bigg|\frac{-2\beta}{N_{t+1}}\bigg(Z_t(i)-\frac{1}{2}\bigg)+\frac{(\alpha-\beta)p}{N_{t+1}}(Z_t-Z_t(i))\bigg|\bigg]\\ &\quad\le\mathbb{E}\bigg[\bigg|\frac{-2\beta}{N_{t+1}}\bigg(Z_t(i)-\frac{1}{2}\bigg)\bigg|+\bigg|\frac{(\alpha-\beta)p}{N_{t+1}}(Z_t-Z_t(i))\bigg|\bigg].\end{aligned}$$

The proposition now follows from Theorem 1. □

Using this proposition and the fact that a bounded quasimartingale (see [11] and [22]) has an almost-sure limit, yields the following theorem.

Theorem 2. *Let Z_t and $Z_t(i)$ be as defined above. Then, for every $i\in\{1,2,\dots,N\}$,*

$$\lim_{t\to\infty}(Z_t-Z_t(i))=0\quad a.s.$$

Thus, as $t\to\infty$, the $Z_t(i)$ synchronize a.s. and converge to the same limit as $\lim_{t\to\infty}Z_t$.

Taking expectations on both sides of (2) and (3), it follows that $\mathbb{E}[Z_t(i)],\mathbb{E}[Z_t]\to\frac{1}{2}$ as $t\to\infty$. Thus, we have the following result.

Corollary 1. *Let Z_t and $Z_t(i)$ be as defined above. Then, for every $i\in\{1,2,\dots,N\}$, $Z_t, Z_t(i)\xrightarrow{\text{a.s.}}\frac{1}{2}$.*

3. Fluctuations

In his paper titled 'Bernard Freidman's urn' [12], Freedman proved some crucial results for fluctuations in the fraction of a particular colour of ball in a Friedman (two-colour) urn around the limit $\frac{1}{2}$.

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The results obtained here reduce to the fluctuation results in [12] when $N = 1$. In the case of interacting urns we need to look at fluctuations in each urn around the limit as well as the fluctuation of the overall fraction, i.e. Z_t. We use the notion of stable convergence (see Appendix A as well as [7]) to obtain limit theorems for fluctuations of $Z_t - Z_t(i)$ for a fixed i. This gives us a stronger form of convergence than the convergence in distribution. The fluctuation results are stated in Theorems 3, 4, and 5 below. Using the Cramér–Wold device, this canonically extends to a limit theorem for the vector $(Z_t - Z_t(1), \dots, Z_t - Z_t(N))$. The tools used here are similar to those used in [7]; however, the behaviour of Friedman urns is different from that of Pólya urns. We first state the results and then proceed to prove them.

Theorem 3. *Let $\rho = (\alpha - \beta)/(\alpha + \beta)$. Then the following statements hold.*

- *For* $0 < \rho < \frac{1}{2}$,

$$\sqrt{t}\left(Z_t - \frac{1}{2}\right) \xrightarrow{\text{stably}} \mathcal{N}\left(0, \frac{\rho^2}{4N(1-2\rho)}\right).$$

- *For* $\rho = \frac{1}{2}$,

$$\frac{\sqrt{t}}{\sqrt{\log t}}\left(Z_t - \frac{1}{2}\right) \xrightarrow{\text{stably}} \mathcal{N}\left(0, \frac{\rho^2}{4N}\right).$$

Theorem 4. *Let $\rho = (\alpha - \beta)/(\alpha + \beta)$. Then the following statements hold.*

- *For* $0 < \rho < 1/2(1-p)$,

$$\sqrt{t}(Z_t - Z_t(i)) \xrightarrow{\text{stably}} \mathcal{N}\left(0, \frac{(1-1/N)\rho^2}{4[1-2\rho(1-p)]}\right).$$

- *For* $\rho = 1/2(1-p)$,

$$\frac{\sqrt{t}}{\sqrt{\log t}}(Z_t - Z_t(i)) \xrightarrow{\text{stably}} \mathcal{N}\left(0, \left(1 - \frac{1}{N}\right)\frac{\rho^2}{4}\right).$$

Theorem 5. *Let $\rho = (\alpha - \beta)/(\alpha + \beta)$. Then the following statements hold.*

- *For $\rho > \frac{1}{2}$ and $u = 1 - \rho$,*

$$t^u\left(Z_t - \tfrac{1}{2}\right) \xrightarrow{\text{a.s.}/\mathcal{L}^1} \widetilde{V}$$

for some real random variable $\widetilde{V}$ such that $\mathbb{P}(\widetilde{V} \neq 0) > 0$.

- *For $\rho > 1/2(1-p)$ and $u = 1 - (1-p)\rho$,*

$$t^u(Z_t - Z_t(i)) \xrightarrow{\text{a.s.}/\mathcal{L}^1} \widetilde{X}$$

for some real random variable $\widetilde{X}$ such that $\mathbb{P}(\widetilde{X} \neq 0) > 0$.

We now prove the above theorems, starting with Theorem 4. We write $x_t \approx y_t$ if $\lim_{t\to\infty} x_t = \lim_{t\to\infty} y_t$.

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Proof of Theorem 4. Define $X_k = Z_k - Z_k(i)$. Set

$$\begin{aligned}
L_0 &= X_0,\\
L_t &= X_t - \sum_{k=0}^{t-1}(\mathbb{E}[X_{k+1} \mid \mathcal{F}_k] - X_k)\\
&= X_t - \sum_{k=0}^{t-1}[\mathbb{E}[Z_{k+1} - Z_{k+1}(i) \mid \mathcal{F}_k] - (Z_k - Z_k(i))]\\
&= X_t - \sum_{k=0}^{t-1}\bigg[\frac{N_k + (\alpha-\beta)(1-p)}{N_{k+1}}(Z_k - Z_k(i)) - (Z_k - Z_k(i))\bigg]\\
&= X_t - \sum_{k=0}^{t-1}\bigg[\frac{N_k + (\alpha-\beta)(1-p)}{N_{k+1}} - 1\bigg](Z_k - Z_k(i))\\
&= X_t - \sum_{k=0}^{t-1}\frac{(\alpha-\beta)(1-p) - (\alpha+\beta)}{N_{k+1}}(Z_k - Z_k(i))\\
&= X_t - \sum_{k=0}^{t-1}\frac{(\alpha+\beta)(\rho(1-p)-1)}{N_{k+1}}X_k.
\end{aligned}$$

Then

$$L_{t+1} - L_t = X_{t+1} - X_t + \frac{(\alpha+\beta)(\rho(1-p)-1)}{N_{t+1}}X_t.$$

That is,

$$X_{t+1} = \bigg[1 - \frac{(\alpha+\beta)(1-\rho(1-p))}{N_{t+1}}\bigg]X_t + \Delta L_{t+1}.$$

Note that L_t is an $\mathcal{F}$-martingale by construction. Iterating the above relation, we can write

$$X_{t+1} = c_{1,t}X_1 + \sum_{k=1}^{t} c_{k+1,t}\Delta L_{k+1},$$

where $c_{t+1,t} = 1$ and $c_{k,t} = \prod_{h=k}^{t}[1 - (\alpha+\beta)(1-\rho(1-p))/N_{h+1}]$ for $k \le t$. Note that

$$\begin{aligned}
c_{1,t} &= \prod_{h=1}^{t}\bigg(1 - \frac{(\alpha+\beta)(\rho(1-p)-1)}{N_{h+1}}\bigg)\\
&= \exp\bigg[\sum_{h=1}^{t}\ln\bigg(1 - \frac{(\alpha+\beta)(\rho(1-p)-1)}{N_{h+1}}\bigg)\bigg]\\
&\sim \exp\bigg[-(1-\rho(1-p))\sum_{h=1}^{t}\frac{(\alpha+\beta)}{N_{h+1}}\bigg]\\
&\sim \exp[-(1-\rho(1-p))(\alpha+\beta)\ln(N_{t+1})]\\
&\sim t^{-(1-\rho(1-p))}.
\end{aligned}$$

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Then $\sqrt{t}c_{1,t} \sim t^{-(1-\rho(1-p))+1/2} \to 0$ for $0 < \rho < 1/2(1-p)$. Observe that the same argument gives

$$\lim_{k\to\infty} \sup_{t\geq k} \left| \frac{c_{k,t}}{(k/t)^{1-\rho(1-p)}} - 1 \right| = 0. \tag{7}$$

So, it is enough to prove that $\sqrt{t}\sum_k c_{k+1,t}\Delta L_{k+1} \to \mathcal{N}(0, \rho^2/4[1-2\rho(1-p)])$.

This can be proved using Theorem 7 (see Appendix A) by verifying the following conditions for $U_{t,k+1} = \sqrt{t}\sum_k c_{k+1,t}\Delta L_{k+1}$.

(a) $\max_{1\leq k\leq t} |U_{t,k}| \to 0$.

(b) $\mathbb{E}[\max_{1\leq k\leq t} U_{t,k}^2]$ is bounded in t.

(c) $\sum_{k=1}^t U_{t,k}^2 \to \rho^2/4[1-2\rho(1-p)]$.

We will now verify these conditions. For (a), since $\Delta L_{k+1} = X_{t+1} - X_t + (\alpha+\beta)(1-\rho(1-p))X_t/N_{t+1}$, $|\Delta L_{k+1}| \sim O(k^{-1})$.

For (b), use (7) and (a) to obtain

$$\begin{aligned}
\mathbb{E}\Big[\max_{1\leq k\leq t} U_{t,k}^2\Big] &\leq \mathbb{E}\bigg[\sum_{k=1}^t U_{t,k}^2\bigg] \\
&= t\sum_{k=1}^t c_{k+1,t}^2 \mathbb{E}[Y_{k+1}^2] \\
&\approx t\sum_{k=1}^{t-1}\bigg(\frac{k}{t}\bigg)^{2(1-\rho(1-p))} O(k^{-2}) + tO(k^{-2}) \\
&= \frac{1}{t^{1-2\rho(1-p)}}\sum_{k=1}^{t-1}\frac{k^2 O(k^{-2})}{k^{2\rho(1-p)}} + \frac{t^2 O(k^{-2})}{t}.
\end{aligned}$$

It follows that $\mathbb{E}[\max_{1\leq k\leq t} U_{t,k}^2]$ is bounded in t.

Let us now consider (c). We have

$$\sum_{k=1}^t U_{t,k}^2 = t\sum_{k=1}^t c_{k+1,t}^2 (\Delta L_{k+1})^2 \approx \frac{1}{t^{1-2\rho(1-p)}}\sum_{k=1}^{t-1}\frac{k^2(\Delta L_{k+1})^2}{k^{2\rho(1-p)}} + \frac{t^2(\Delta L_{t+1})^2}{t}.$$

From (a) we obtain

$$\begin{aligned}
(\Delta L_{k+1})^2 &= \bigg(X_{k+1} - X_k + \frac{(\alpha+\beta)(1-\rho(1-p))}{N_{k+1}}X_k\bigg)^2 \\
&= \bigg((Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i)) + \frac{(\alpha+\beta)(1-\rho(1-p))}{N_{k+1}}(Z_k - Z_k(i))\bigg)^2 \\
&\approx \bigg((Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i)) + \frac{1}{k}(Z_k - Z_k(i))\bigg)^2 \\
&\approx [(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))]^2 + \frac{1}{k^2}(Z_k - Z_k(i))^2 \\
&\quad + \frac{1}{k}(Z_k - Z_k(i))[(Z_{k+1} - Z_k) - (Z_{k+1}(i) - Z_k(i))].
\end{aligned}$$

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Since we know that $Z_t \xrightarrow{\text{a.s}} Z_t(i)$ and $X_k^2 \sim O(k^{-2})$, we can get rid of some the terms above. We obtain

$$\sum_{k=1}^{t} U_{t,k}^2 \xrightarrow{\text{a.s.}} t \sum_{k=1}^{t} c_{k+1,t}^2 [(Z_{k+1}-Z_k)^2 + (Z_{k+1}(i)-Z_k(i))^2 - 2(Z_{k+1}-Z_k)(Z_{k+1}(i)-Z_k(i))].$$

To conclude, we use Lemma 1 given in Appendix A with $b_k = k^{1-2\rho(1-p)}$ and $a_k = k^{2\rho(1-p)}$. Let $U_k = k^2[(Z_{k+1} - Z_k)^2 + (Z_{k+1}(i) - Z_k(i))^2 - 2(Z_{k+1} - Z_k)(Z_{k+1}(i) - Z_k(i))]$. Then it is easy to verify that $\sum_{k=1}^{\infty} \mathbb{E}[U_k^2]/a_k b_k^2 < \infty$ as $b_k \to \infty$. We have

$$\frac{1}{b_t} \sum_{k=1}^{t} \frac{1}{a_k} \to \frac{1}{1 - 2\rho(1-p)}.$$

This implies (by Lemma 1) that $\sum_{k=1}^{t} U_k^2$ converges to $U/(1 - 2\rho(1-p))$ a.s., where U is such that $\mathbb{E}[U_k \mid \mathcal{F}_k] \to U$. We now verify that U is in fact deterministic. Note that

$$\begin{aligned}
\mathbb{E}[k^2(Z_{k+1}(i) - Z_k(i))^2 \mid \mathcal{F}_k] &= \frac{k^2}{N_{k+1}^2} \mathbb{E}[(Y_{k+1}(i) - (\alpha+\beta)Z_k(i))^2 \mid \mathcal{F}_k] \\
&= \frac{k^2(\alpha+\beta)^2}{N_{k+1}^2} \mathbb{E}\bigg[\bigg(\frac{Y_{k+1}(i)}{\alpha+\beta} - Z_k(i)\bigg)^2 \;\bigg|\; \mathcal{F}_k\bigg] \\
&\xrightarrow{\text{a.s.}} \frac{\rho^2}{4}.
\end{aligned}$$

Similarly,

$$\mathbb{E}[k^2(Z_{k+1} - Z_k)^2 \mid \mathcal{F}_k] \xrightarrow{\text{a.s.}} \frac{\alpha^2+\beta^2}{2N(\alpha+\beta)^2} + \frac{N-1}{4N} - \frac{1}{4} = \frac{\rho^2}{4N}$$

and

$$\begin{aligned}
&\mathbb{E}[k^2(Z_{k+1}(i) - Z_k(i))(Z_{k+1} - Z_k) \mid \mathcal{F}_k] \\
&\quad = \frac{k^2(\alpha+\beta)^2}{N_{k+1}^2} \mathbb{E}\bigg[\bigg(\frac{Y_{k+1}(i)}{\alpha+\beta} - Z_k(i)\bigg)\bigg(\frac{\sum_{i=1}^{N} Y_{k+1}(i)}{N(\alpha+\beta)} - Z_k\bigg) \;\bigg|\; \mathcal{F}_k\bigg] \\
&\quad \xrightarrow{\text{a.s}} \frac{\rho^2}{4N}.
\end{aligned}$$

Thus, we obtain $U_k \xrightarrow{\text{a.s.}} (\rho^2/4)(1 - 1/N)$. This concludes the proof of (1). The proof for the second case with $\rho = 1/2(1-p)$ is essentially the same. □

The proof of Theorem 3 follows along the same lines as above. We sketch the essential argument below.

Proof of Theorem 3. Let $V_k = Z_k - \frac{1}{2}$. Denote by L_t the martingale

$$L_0 = V_0, \qquad L_t = V_t + \sum_{k=0}^{t-1} (\mathbb{E}[V_{k+1} \mid \mathcal{F}_k] - V_k).$$

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Then

$$V_{t+1} = \Delta L_{t+1} + \left(1 + \frac{(\alpha+\beta)(1-\rho)}{N_{t+1}}\right) V_t$$

or

$$\Delta L_{t+1} = V_{t+1} - \left(1 + \frac{(\alpha+\beta)(1-\rho)}{N_{t+1}}\right) V_t. \tag{8}$$

Writing

$$V_{t+1} = c_{1,t} V_1 + \sum_{k=1}^{t} c_{k+1,t} \Delta L_{k+1},$$

we have $c_{1,t} \sim t^{-(1-\rho)}$. Thus, $\sqrt{t} c_{1,t} \to 0$ for $\rho < \frac{1}{2}$. Following the same steps as in the above proof, and using (8), it can be verified that $\lim_{k\to\infty} \sup_{t\geq k} |c_{k,t}/(k/t)^{1-\rho} - 1| = 0$, and that (a) and (b) (as in proof of Theorem 4) hold. It boils down to showing that $\sum_{k=1}^{t} U_{t,k} = \sum_{k=1}^{t} t c_{k+1,t}^2 (\Delta L_{k+1})^2 \to \mathcal{N}(0, \rho^2/4N(2\rho-1))$. We have

$$\begin{aligned}
(\Delta L_{k+1})^2 &= \left(V_{k+1} - \left(1 + \frac{(\alpha+\beta)(1-\rho)}{N_{k+1}}\right) V_k\right)^2 \\
&= \left(Z_{k+1} - Z_k - \frac{(\alpha+\beta)(1-\rho)}{N_{k+1}} V_k\right)^2 \\
&\approx \left(Z_{k+1} - Z_k - \frac{(1-\rho)}{k} V_k\right)^2.
\end{aligned}$$

Thus, since we know that $Z_t \xrightarrow{\text{a.s.}} \frac{1}{2}$, the only relevant term is $Z_{k+1} - Z_k$. That is, $\sum_{k=1}^{t} U_{t,k}^2 \xrightarrow{\text{a.s.}} t \sum_{k=1}^{t} c_{k+1,t}^2 (Z_{k+1} - Z_k)^2$. We can now use Lemma 1 given in Appendix A with $b_k = k^{1-2\rho}$ and $a_k = k^{2\rho}$. Let $U_k = k^2 (Z_{k+1} - Z_k)^2$. Then $\sum_{k=1}^{t} U_k^2$ converges to $U/(2\rho - 1)$ a.s., where U is a such that $\mathbb{E}[U_k \mid \mathcal{F}_k] \to U$. Since we completed this computation in the proof of Theorem 4 above, we know that in this case $U = \rho^2/4N$. This concludes the proof. The proof for the case $\rho = \frac{1}{2}$ follows similarly. □

We now prove Theorem 5.

Proof of Theorem 5. We first prove the second part. Define $\widetilde{X}_k = t^u (Z_k - Z_k(i))$. It follows from the computations carried out in the proof of Theorem 4 that $\mathbb{E}[\widetilde{X}_t^2] < \infty$. It is therefore enough to show that $\widetilde{X}_t$ is a quasimartingale. Indeed, we have

$$\begin{aligned}
\sum_k \mathbb{E}[|\mathbb{E}[\widetilde{X}_{k+1} \mid \mathcal{F}_k] - \widetilde{X}_k|] &= \sum_k \mathbb{E}\left[\left|(k+1)^u \frac{N_k + (\alpha-\beta)(1-p)}{N_{k+1}} X_k - \widetilde{X}_k\right|\right] \\
&= \sum_k \mathbb{E}\left[\left|\left(1 + \frac{1}{k}\right)^u \frac{N_k + (\alpha-\beta)(1-p)}{N_{k+1}} \widetilde{X}_k - \widetilde{X}_k\right|\right] \\
&= \sum_k O\left(\frac{1}{t^2}\right) u \mathbb{E}[|\widetilde{X}_k|] \\
&< \infty.
\end{aligned}$$

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Now the almost-sure convergence follows from the fact that a bounded quasimartingale converges a.s. and in $\mathcal{L}^1$. It remains to prove that $\mathbb{P}(\widetilde{X} \neq 0) > 0$. For this, it suffices to prove that $\widetilde{X}_t^2$ is bounded in $\mathcal{L}^p$ for a suitable $p > 1$. Indeed, this implies that $\mathbb{E}[\widetilde{X}^2] = \lim_{t\to\infty} \mathbb{E}[\widetilde{X}_t^2] = \lim_{t\to\infty} t^{2u}\mathbb{E}[X_t^2] > 0$.

The proof is similar to that of Theorem 3.5 of [7]. We only sketch the proof and illustrate the important steps for the sake of completion. The idea is to again obtain a recurrence relation as follows. For $\varepsilon > 0$, let $x_t = \mathbb{E}[|X_t|^{2+\varepsilon}]$. We know that

$$X_{t+1} = \frac{N_t}{N_{t+1}} X_t + \frac{1}{N_{t+1}}\bigg(\frac{\sum_{j=1}^N Y_{t+1}(j)}{N} - Y_{t+1}(i)\bigg).$$

So, using the binomial expansion and collecting higher-order terms,

$$\begin{aligned} x_{t+1} = {} & \bigg(\frac{N_t}{N_{t+1}}\bigg)^{2+\varepsilon} \mathbb{E}[|X_t|^{2+\varepsilon}] \\ & + (2+\varepsilon)\bigg(\frac{N_t}{N_{t+1}}\bigg)^{1+\varepsilon} \frac{1}{N_{t+1}} \mathbb{E}\bigg[|X_t|^{1+\varepsilon}\operatorname{sgn}(X_t)\bigg(\frac{\sum_{j=1}^N Y_{t+1}(j)}{N} - Y_{t+1}(i)\bigg)\bigg] \\ & + O\bigg(\frac{1}{t^2}\bigg). \end{aligned}$$

Recall that $u = 1 - \rho(1-p)$. This gives

$$\begin{aligned} x_{t+1} &= \bigg(1 - \frac{(\alpha-\beta)(2+\varepsilon)(1-\rho(1-p))}{N_{t+1}}\bigg)x_t + g(t) \\ &= \bigg(1 - \frac{(\alpha-\beta)(2+\varepsilon)u}{N_{t+1}}\bigg)x_t + g(t), \end{aligned}$$

where $x_0 = 0$ and $g(t) = O(1/t^2)$. Then, for sufficiently small $\varepsilon > 0$, using the method used in the proof of Theorem 1, we obtain $\mathbb{E}[|X_t|^{2+\varepsilon}] = O(1/t^{(2+\varepsilon)u})$. This implies that $\widetilde{X}^2$ is bounded in $\mathcal{L}^{1+\varepsilon/2}$.

Now, for the first part, from Proposition 1, we already know that $Z_t - \frac{1}{2}$ is a quasimartingale. Define $\widetilde{V}_k = t^u(Z_k - \frac{1}{2})$ for $u = 1-\rho$. Then $\mathbb{E}[\widetilde{V}_t^2] < \infty$ and $\widetilde{V}_t$ is a quasimartingale. This implies that it converges a.s. and in $\mathcal{L}^1$ to some real random variable $\widetilde{V}$. To prove that $\mathbb{P}(\widetilde{V} \neq 0) > 0$, following a similar computation as above, for $x_{t+1} = \mathbb{E}[|Z_t - \frac{1}{2}|^{2+\varepsilon}]$, we obtain

$$x_{t+1} = 1 - \frac{(2+\varepsilon)u}{N_{t+1}/(\alpha+\beta)},$$

where $u = 1 - \rho$. Thus, we have $\mathbb{E}[|V_t|^{2+\varepsilon}] = O(1/t^{(2+\varepsilon)u})$ for $u = 1-\rho$. This concludes the proof. □

Note that we have not said anything about the fluctuations in the fraction of white balls in each urn, i.e. $Z_t(i)$, around the limit $\frac{1}{2}$. Since in the case of Pólya urns, Z_t is a martingale, it is possible to prove a stronger fluctuation result for Z_t (see Theorem 3.1 of [7]). This leads to a stable convergence result for $Z_t(i) - \frac{1}{2}$. However, in the case of Friedman urns we do not have the martingale property at our disposal. We, therefore, prove a convergence in distribution fluctuation result for $Z_t(i)$ using stochastic approximation techniques.

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3.1. Stochastic approximation approach

Stochastic approximation schemes are heavily used in optimization problems and reinforcement learning (see [4]). Naturally, the method is very well suited to handling urn model problems as well. We reformulate the problem in terms of stochastic approximation theory. Define $\widetilde{Z}_t = (Z_1(i), \dots, Z_N(i))$ and $\widetilde{Y}_t = (Y_t(1), \dots, Y_t(N))$. Then, according to our reinforcement model, we have

$$\begin{aligned}\widetilde{Z}_{t+1} &= \frac{N_t}{N_{t+1}}\widetilde{Z}_t + \frac{1}{N_{t+1}}\widetilde{Y}_{t+1} \\ &\approx \widetilde{Z}_t + \frac{1}{t+1}\bigg(-\widetilde{Z}_t + \frac{\widetilde{Y}_{t+1}}{(\alpha+\beta)}\bigg) \\ &\approx \widetilde{Z}_t - \frac{1}{t+1}(\widetilde{Z}_t - g(\widetilde{Z}_t)) + \frac{1}{t+1}\bigg(\frac{\widetilde{Y}_{t+1}}{(\alpha+\beta)} - g(\widetilde{Z}_t)\bigg),\end{aligned} \tag{9}$$

where $g(\widetilde{Z}_t) = \mathbb{E}[\widetilde{Y}_{t+1}/(\alpha+\beta) \mid \mathcal{F}_t] = (1-\rho)/2 + \rho(pZ_t + (1-p)\widetilde{Z}_t)$. Recall that Z_t denotes the overall fraction of white balls given by $(1/N)\sum_i Z_t(i)$. Equation (9) corresponds to the classical stochastic approximation scheme,

$$x_{t+1} = x_t - a(t)h(x_t) + a(t)M_{t+1}, \tag{10}$$

where $a(t) = 1/(t+1)$ and $M_{t+1} = \widetilde{Y}_{t+1} - g(\widetilde{Z}_t)$ is an N-dimensional martingale difference sequence. It can be easily verified that $h(x_t) = \widetilde{Z}_t - g(\widetilde{Z}_t)$ is a Lipschitz function from $\mathbb{R}^N$ to $\mathbb{R}^N$. Conditions (C1)–(C4) given in Section A.2 can also be verified. According to the 'ordinary differential equation (ODE) approach' of the theory of stochastic approximation (see [4]), the scheme in (10) will a.s. converge to the equilibria of the limiting ODE given by $\dot{x} = h(x(t))$. In other words, $\widetilde{Z}_t$ converges to the set $\{h = 0\}$. It is easy to check that this implies that $Z_t(i) \xrightarrow{\text{a.s.}} \frac{1}{2}$ for every $i \in \{1, \dots, N\}$.

Furthermore, using concepts from [15], it is possible to obtain fluctuation results too. Let $B^\top$ denote the transpose of a matrix B. Using Theorem A.2 of [15], we can prove the following result.

Theorem 6. *Let $\delta_0 = (\frac{1}{2}, \dots, \frac{1}{2})$ be an N-dimensional vector, and let A denote the $N \times N$ matrix given by*

$$\begin{pmatrix} \frac{1}{2} - \rho(1-p) - \rho p/N & -\rho p/N & \dots & -\rho p/N \\ -\rho p/N & \frac{1}{2} - \rho(1-p) - \rho p/N & \dots & -\rho p/N \\ \vdots & \vdots & \ddots & \vdots \\ -\rho p/N & -\rho p/N & \dots & \frac{1}{2} - \rho(1-p) - \rho p/N \end{pmatrix}.$$

Then the following statements hold.

- *For $\rho < \frac{1}{2}$, $\sqrt{n}(\widetilde{Z}_t - \delta_0) \xrightarrow{\text{D}} \mathcal{N}(0, A^{-1}\rho^2/8(1-2\rho))$.*
- *For $\rho = \frac{1}{2}$, $\sqrt{n}(\widetilde{Z}_t - \delta_0)/\sqrt{\log n} \xrightarrow{\text{D}} \mathcal{N}(0, \Sigma)$ for some $N \times N$ positive semidefinite matrix Σ.*
- *For $\rho > \frac{1}{2}$, $n^{1-\rho}(\widetilde{Z}_t - \delta_0)$ converges a.s. and in $\mathcal{L}^1$ towards a finite random variable.*

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Proof. We can write the stochastic approximation scheme as

$$\widetilde{Z}_{t+1} = \widetilde{Z}_t - \frac{1}{t+1} h(\widetilde{Z}_t) + \frac{1}{t+1} M_{t+1},$$

where $h(\widetilde{Z}_t) = [1 - \rho(1-p)]\widetilde{Z}_t - (1-\rho)/2 - \rho p Z_t$ and M_{t+1} is an $\mathcal{F}_t$-adapted martingale difference sequence. As mentioned above, conditions (C1)–(C4) given in Section A.2 can be easily verified. Note that the Jacobian of h, Dh, is a constant diagonalizable matrix given by

$$\begin{pmatrix} 1-\rho(1-p)-\rho p/N & -\rho p/N & \dots & -\rho p/N \\ -\rho p/N & 1-\rho(1-p)-\rho p/N & \dots & -\rho p/N \\ \vdots & \vdots & \ddots & \vdots \\ -\rho p/N & -\rho p/N & \dots & 1-\rho(1-p)-\rho p/N \end{pmatrix}.$$

Thus, $Dh = [1-\rho(1-p)]I - (\rho p/N)J$, where I denotes an $N \times N$ identity matrix and J denotes an $N \times N$ matrix of 1s. It is easy to see that the smallest eigenvalue of Dh is given by $\lambda_{\min} = 1 - \rho$. Then the first case, $\rho < \frac{1}{2}$, directly follows from Theorem A.2 of [15], such that the variance is given by $\Sigma/(2\lambda_{\min} - 1)$, where

$$\Sigma = \int_0^\infty (\mathrm{e}^{-Au})^\top \Gamma \mathrm{e}^{-Au} \, \mathrm{d}u$$

with A as defined in the statement of the theorem ($A = Dh - I/2$) and Γ the almost-sure limit of $\mathbb{E}[M_{t+1}^\top M_{t+1} \mid \mathcal{F}_t]$ as $t \to \infty$. It is easy to verify that $\Gamma = \rho^2/4$. For the case in which $\rho = \frac{1}{2}$, the matrix Σ cannot be simplified in terms of A, as A is not invertible for $\rho = \frac{1}{2}$. Finally, the case in which $\rho > \frac{1}{2}$ also follows from Theorem A.2 of [15]. This concludes the proof. □

Remark 4. The matrix A is not invertible for $\rho = 1/2(1-p)$ as well. However, since, $1/2(1-p) > \frac{1}{2}$, this is not relevant in the regime $\{\rho < \frac{1}{2}\}$.

Convergence in distribution versions of Theorems 3, 4, and 5 can also be proved using the stochastic approximation method.

Appendix A

We state some auxiliary results.

Lemma 1. (Lemma A.1 of [7].) *Let G be an (increasing) filtration, and let (Y_k) be a G-adapted sequence of real random variables such that $\mathbb{E}[Y_k \mid G_{k-1}] \to Y$ a.s. for some real random variable Y. Moreover, let (a_k) and (b_k) be two sequences of strictly positive real numbers such that*

$$b_k \to \infty, \qquad \sum_{k=1}^{\infty} \frac{\mathbb{E}[Y_k^2]}{a_k b_k^2} < \infty.$$

Then

- *if $(1/b_t)\sum_{k=1}^t 1/a_k \to \gamma$ for a constant γ, $(1/b_t)\sum_{k=1}^t Y_k/a_k \to \gamma Y$;*
- *if $b_t \sum_{k\ge t} 1/a_k b_k^2 \to \gamma$ for a constant γ, $b_t \sum_{k\ge t} Y_k/a_k b_k^2 \to \gamma Y$.*

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A.1. Stable convergence

The notion of stable convergence was introduced by Rényi [23] and is in a sense a stronger version of convergence in distribution. There has been quite a lot of work on the concept and application of the notion of stable convergence. We only state some results concerning stable convergence from [3] and [14] that are relevant to this paper. The interested reader can refer to these and [1], [8], [10], and [20] for more details. For generalizations such as the strong form of stable convergence and almost-sure conditional convergence, we refer the reader to [6] and [8], respectively. For sake of completeness, we first define the concept of stable convergence as in [7].

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let S be a Polish space, endowed with its Borel σ-field. A *kernel* on S, or a random probability measure on S, is a collection $K = \{K(\omega)\colon \omega \in \Omega\}$ of probability measures on the Borel σ-field of S such that, for each bounded Borel real function f on S, the map

$$\omega \mapsto K(f)(\omega) = \int f(x)\, K(\omega)(\mathrm{d}x)$$

is $\mathcal{A}$-measurable.

On $(\Omega, \mathcal{A}, \mathbb{P})$, let (Y_t) be a sequence of S-valued random variables and let K be a kernel on S. Then we say that Y_t converges *stably* to K, and we write $Y_t \overset{\text{stably}}{\longrightarrow} K$, if

$$\mathbb{P}(Y_t \in \cdot \mid H) \overset{\text{weakly}}{\longrightarrow} \mathbb{E}[K(\cdot) \mid H] \quad \text{for all } H \in \mathcal{A} \text{ with } \mathbb{P}(H) > 0.$$

Clearly, if $Y_t \overset{\text{stably}}{\longrightarrow} K$ then Y_t converges in distribution to the probability distribution $\mathbb{E}[K(\cdot)]$. In fact, if $\{Y_n\}$ is a sequence of random variables on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ converging in distribution to Y, then we say that this convergence is stable if, for all continuity points y of Y and all events $A \in \mathcal{A}$, the limit $\lim_{n\to\infty} \mathbb{P}(\{Y_n \le y\} \cup A) = Q_y(A)$ exists and $Q_y(A) \to \mathbb{P}(A)$ as $y \to \infty$.

Moreover, the convergence in probability of Y_t to a random variable Y is equivalent to the stable convergence of Y_t to a special kernel, which is the Dirac kernel $K = \delta_Y$ (see Corollary 4 of [8]).

Theorem 7. (Theorem 3.2 of [14].) *Let $\{S_{n,k}, \mathcal{F}_{n,k}\colon 1 \le k \le k_n, n \ge 1\}$ be a zero-mean, square-integrable martingale array with differences $Y_{n,k}$, and let η^2 be an a.s. finite random variable. Suppose that*

- $\max_{1\le k\le k_n} |Y_{n,k}| \to 0$*;*
- $\mathbb{E}[\max_{1\le k\le k_n}]$ *is bounded in n;*
- $\sum_{k=1}^{k_n} Y_{n,k}^2 \to \eta^2$,

and that the σ-fields are nested, i.e. $\mathcal{F}_{n,k} \subseteq \mathcal{F}_{n+1,k}$ for $1 \le k \le k_n$, $n \ge 1$. Then $S_{n,k}$ converges stably to a random variable with characteristic function $\phi(u) = \mathbb{E}[\exp(-\eta^2 u^2/2)]$, i.e. to the Gaussian kernel $\mathcal{N}(0, \eta^2)$. (Here, the notation $\mathcal{N}(0, 0)$ denotes the Dirac distribution ε_0.)

A.2. Stochastic approximation

The martingale method, the method of moments, and stochastic approximation are all popular methods for analysing random processes with reinforcement (see [21]). Stochastic approximation has been used extensively in urn models. A classical stochastic approximation scheme in

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$\mathbb{R}^k$ is given by

$$x_{n+1} = x_n - a(n)h(x_n) + a(n)M_{n+1} \quad (11)$$

with the initial point x_0. The solutions of the above scheme asymptotically track the solutions of the ODE given by

$$\dot{x}(t) = -h(x(t))$$

under the following conditions.

(C1) The map $h\colon \mathbb{R}^k \to \mathbb{R}^k$ is Lipschitz.

(C2) Step sizes $\{a(n)\}$ are positive, satisfying $\sum_n a(n) = \infty$ and $\sum_n a(n)^2 < \infty$.

(C3) $\{M_n\}$ is a martingale difference sequence with respect to $\mathcal{F}_n = \sigma(x_m, M_m; m \leq n)$. Furthermore, the M_n are square integrable with $\mathbb{E}[\|M_{n+1}\|^2 \mid \mathcal{F}_n] \leq K(1 + \|x\|^2)$ a.s. for $n \geq 0$ and some constant $K > 0$.

(C4) The trajectories (or the iterates) of (11) remain bounded a.s., i.e. $\sup_n \|x_n\| < \infty$ a.s.

A detailed analysis of convergence and stability can be found in [4].

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