Canad. Math. Bull. Vol. **65** (4), 2022, pp. 825–844
http://dx.doi.org/10.4153/S0008439521000898

Maximal subgroups of nontorsion Grigorchuk–Gupta–Sidki groups

Dominik Francoeur and Anitha Thillaisundaram

Abstract. A Grigorchuk–Gupta–Sidki (GGS)-group is a subgroup of the automorphism group of the p-regular rooted tree for an odd prime p, generated by one rooted automorphism and one directed automorphism. Pervova proved that all torsion GGS-groups do not have maximal subgroups of infinite index. Here, we extend the result to nontorsion GGS-groups, which include the weakly regular branch, but not branch, GGS-group.

1 Introduction

The automorphism group of an infinite spherically homogeneous rooted tree is well established as a source of interesting finitely generated infinite groups, such as finitely generated groups of intermediate word growth, finitely generated infinite torsion groups, finitely generated amenable but not elementary amenable groups, and finitely generated just infinite groups. Early constructions were produced by Grigorchuk [10] and Gupta and Sidki [13] in the 1980s, which then led to a generalized family of so-called GGS-groups.

An important type of subgroup of the automorphism group of an infinite spherically homogeneous rooted tree is one having subnormal subgroup structure similar to the corresponding structure in the full group of automorphisms of the tree. These subgroups are termed branch groups (see Section 2 for the definition).

The study of maximal subgroups of finitely generated branch groups began with the work of Pervova [14, 15], who proved that the torsion Grigorchuk groups and the torsion GGS-groups do not contain maximal subgroups of infinite index. Bondarenko [3] gave the first example of a finitely generated branch group that does have maximal subgroups of infinite index. His method does not apply to groups acting on the binary and ternary rooted trees. However, recently Francoeur and Garrido [8] provided the first examples of finitely generated branch groups, acting on the binary rooted tree, with maximal subgroups of infinite index. Their examples are the nontorsion Šunić groups. For an extensive introduction to the subject of maximal subgroups of finitely generated branch groups, we refer the reader to [8].

Received by the editors December 14, 2020; revised October 20, 2021.
Published online on Cambridge Core November 2, 2021.
This research was supported by a London Mathematical Society Research in Pairs (Scheme 4) grant.
AMS subject classification: 20E08, 20E28.
Keywords: GGS-groups, branch groups, maximal subgroups.

Francoeur and Garrido (see [6]) have further shown that certain nontorsion GGS-groups, namely the generalized Fabrykowski–Gupta groups, one for each odd prime p, do not have maximal subgroups of infinite index. In this paper, we extend this result to all nontorsion GGS-groups. We recall that a GGS-group acts on the p-regular rooted tree (also called the p-adic tree) for p an odd prime, with generators a and b, where a cyclically permutes the p maximal subtrees rooted at the first-level vertices, whereas b fixes the first-level vertices pointwise and is recursively defined by the tuple $(a^{e_1}, \dots, a^{e_{p-1}}, b)$ which corresponds to the action of b on the maximal subtrees, for some exponents $e_1, \dots, e_{p-1} \in \mathbb{F}_p$.

Theorem 1.1 *Let G be a GGS-group acting on the p-regular rooted tree, for p an odd prime. Then, every maximal subgroup of G is of finite index. Furthermore, if G is branch, then every maximal subgroup is normal and of index p in G.*

There is only one GGS-group that is weakly branch but not branch; this is the GGS-group $\mathcal{G}$ where the automorphism b is defined by $(a, \dots, a, b)$. For the group $\mathcal{G}$, it turns out that there are maximal subgroups that are not normal, nor of index p in $\mathcal{G}$. Furthermore, there are infinitely many maximal subgroups (see Proposition 5.11).

We recall that two groups are *commensurable* if they have isomorphic subgroups of finite index. The following result is a consequence of the work of Grigorchuk and Wilson [12], and the proof follows exactly as in [1, Corollary 1.3].

Corollary 1.2 *Let G be a branch GGS-group acting on the p-regular rooted tree, for p an odd prime. If H is a group commensurable with G, then every maximal subgroup of H has finite index in H.*

1.1 Organization

Section 2 contains preliminary material on groups acting on the p-regular rooted tree. In Section 3, we formally define the GGS-groups and state some of their basic properties. In Section 4, we set up the scene to prove Theorem 1.1, whose proof we complete in Section 5.

1.2 Notation

Throughout, we use the convention $[x, y] = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$. For a group acting on a rooted tree, we always use the right action.

2 Preliminaries

In the present section, we recall the notion of branch groups and establish prerequisites for the rest of the paper. For more information, see [2, 11].

2.1 The p-regular rooted tree and its automorphisms

Let T be the p-regular rooted tree, for an odd prime p. Let $X = \{1, 2, \dots, p\}$ be an alphabet on p letters. The set of vertices of T can be identified with the free monoid

https://doi.org/10.4153/S0008439521000898 Published online by Cambridge University Press

X^*, and we will freely use this identification without special mention. The root of T corresponds to the empty word $\varnothing$, and for each word $v \in X^*$ and letter x, an edge connects v to vx. There is a natural length function on X^*, and the words w of length $|w| = n$, representing vertices that are at distance n from the root, form the *nth layer* of the tree. The *boundary* ∂T consisting of all infinite simple rooted paths is in one-to-one correspondence with the p-adic integers.

For u a vertex, we write T_u for the full rooted subtree of T that has its root at u and includes all vertices v with u a prefix of v. For any two vertices u and v, the subtrees T_u and T_v are isomorphic under the map that deletes the prefix u and replaces it by the prefix v. We refer to this identification as the *natural identification of subtrees* and write T_n to denote the subtree rooted at a generic vertex of level n.

We observe that every automorphism of T fixes the root and that the orbits of Aut T on the vertices of the tree T are precisely its layers. Let $f \in \mathrm{Aut}\, T$ be an automorphism of T. The image of a vertex u under f is denoted by u^f. For a vertex u, considered as a word over X, and a letter $x \in X$, we have $(ux)^f = u^f x'$ where $x' \in X$ is uniquely determined by u and f. This gives a permutation $f(u)$ of X, so that

$$(ux)^f = u^f x^{f(u)}.$$

The permutation $f(u)$ is called the *label* of f at u. The automorphism f is called *rooted* if $f(u) = 1$ for $u \neq \varnothing$. The automorphism f is called *directed*, with directed path ℓ for some $\ell \in \partial T$, if the support $\{u \mid f(u) \neq 1\}$ of its labelling is infinite and contains only vertices at distance 1 from ℓ.

The *section* of f at a vertex u is the unique automorphism f_u of $T \cong T_{|u|}$ given by the condition $(uv)^f = u^f v^{f_u}$ for $v \in X^*$.

2.2 Subgroups of Aut T

Let $G \leq \mathrm{Aut}\, T$. The *vertex stabilizer* $\mathrm{st}_G(u)$ is the subgroup consisting of elements in G that fix the vertex u. For $n \in \mathbb{N}$, the *nth level stabilizer* $\mathrm{St}_G(n) = \cap_{|v|=n} \mathrm{st}_G(v)$ is the subgroup of automorphisms that fix all vertices at level n. We emphasize the difference in the notation of vertex stabilizers and level stabilizers, which aims to avoid confusion from the fact that the first-level vertices of T are often identified with the elements of X. Note that elements in $\mathrm{St}_G(n)$ fix all vertices up to level n and that $\mathrm{St}_G(n)$ has finite index in G.

The full automorphism group Aut T is a profinite group:

$$\mathrm{Aut}\, T = \varprojlim_{n\to\infty} \mathrm{Aut}\, T_{[n]},$$

where $T_{[n]}$ denotes the subtree of T on the finitely many vertices up to level n. The topology of Aut T is defined by the open subgroups $\mathrm{St}_{\mathrm{Aut}}(n)$, for $n \in \mathbb{N}$. For $G \leq \mathrm{Aut}\, T$, we say that the subgroup G has the *congruence subgroup property* if for every subgroup H of finite index in G, there exists some $n \in \mathbb{N}$ such that $\mathrm{St}_G(n) \leq H$.

For $n \in \mathbb{N}$, every $g \in \mathrm{St}_{\mathrm{Aut}\, T}(n)$ can be identified with a collection $g_1, \ldots, g_{p^n}$ of elements of Aut T_n, where p^n is the number of vertices at level n. Denoting the vertices

https://doi.org/10.4153/S0008439521000898 Published online by Cambridge University Press

of T at level n by $u_1, \dots, u_{p^n}$, there is a natural isomorphism

$$\mathrm{St}_{\mathrm{Aut}\ T}(n) \cong \prod_{i=1}^{p^n} \mathrm{Aut}\ T_{u_i} \cong \mathrm{Aut}\ T_n \times \overset{p^n}{\cdots} \times \mathrm{Aut}\ T_n.$$

Recall that Aut T_n is isomorphic to Aut T via the natural identification of subtrees. Therefore, the decomposition $(g_1, \dots, g_{p^n})$ of g defines an embedding

$$\psi_n : \mathrm{St}_{\mathrm{Aut}\ T}(n) \to \prod_{i=1}^{p^n} \mathrm{Aut}\ T_{u_i} \cong \mathrm{Aut}\ T \times \overset{p^n}{\cdots} \times \mathrm{Aut}\ T.$$

For convenience, we will write $\psi = \psi_1$.

For $\omega \in X^*$, we further define

$$\varphi_\omega : \mathrm{st}_{\mathrm{Aut}\ T}(\omega) \to \mathrm{Aut}\ T_\omega \cong \mathrm{Aut}\ T$$

to be the natural restriction of $f \in \mathrm{st}_{\mathrm{Aut}\ T}(\omega)$ to its section f_ω.

We write $G_u = \varphi_u(\mathrm{st}_G(u))$ for the restriction of the vertex stabilizer $\mathrm{st}_G(u)$ to the subtree rooted at a vertex u. We say that G is *self-similar* if G_u is contained in G for every vertex u, and we say that G is *fractal* if G_u equals G for every vertex u, after the natural identification of subtrees.

The subgroup $\mathrm{rist}_G(u)$, consisting of all automorphisms in G that fix all vertices v of T not having u as a prefix, is called the *rigid vertex stabilizer* of u in G. The *rigid nth level stabilizer* is the product

$$\mathrm{Rist}_G(n) = \prod_{i=1}^{p^n} \mathrm{rist}_G(u_i) \trianglelefteq G,$$

of the rigid vertex stabilizers of the vertices $u_1, \dots, u_{p^n}$ at level n.

Let G be a subgroup of Aut T acting *spherically transitively*, i.e., transitively on every layer of T. Then, G is a *weakly branch group* if $\mathrm{Rist}_G(n)$ is nontrivial for every $n \in \mathbb{N}$, and G is a *branch group* if $\mathrm{Rist}_G(n)$ has finite index in G for every $n \in \mathbb{N}$. If, in addition, the group G is self-similar and $1 \neq K \leq G$ with $K \times \cdots \times K \subseteq \psi(K \cap \mathrm{St}_G(1))$ and $|G : K| < \infty$, then G is said to be *regular branch over* K. If the condition $|G : K| < \infty$ in the previous definition is omitted, then G is said to be *weakly regular branch over* K.

3 The GGS-groups

By a, we denote the rooted automorphism, corresponding to the p-cycle $(12 \cdots p) \in \mathrm{Sym}(p)$, that cyclically permutes the vertices $u_1, \dots, u_p$ at the first level. Given a nonzero vector $\mathbf{e} = (e_1, e_2, \dots, e_{p-1}) \in (\mathbb{F}_p)^{p-1}$, we recursively define a directed automorphism $b \in \mathrm{St}_{\mathrm{Aut}\ T}(1)$ via

$$\psi(b) = (a^{e_1}, a^{e_2}, \dots, a^{e_{p-1}}, b).$$

We call the subgroup $G = G_{\mathbf{e}} = \langle a, b \rangle$ of Aut T the *GGS-group* associated with the defining vector $\mathbf{e}$. We observe that $\langle a \rangle \cong \langle b \rangle \cong C_p$ are cyclic groups of order p. Hence, whenever we refer to an exponent of a or of b, it goes without saying that it is an element of $\mathbb{F}_p$.

A GGS-group G acts spherically transitively on the tree T, and every section of every element of G is contained in G. Moreover, a GGS-group G is fractal.

If $G_{\mathbf{e}} = \langle a, b \rangle$ is a GGS-group corresponding to the defining vector $\mathbf{e}$, then G is an infinite p-group if and only if $\sum_{j=1}^{p-1} e_j = 0$ in $\mathbb{F}_p$ (cf. [11, 16]).

We also call $G = \langle a, b \rangle$ a *generalized Fabrykowski–Gupta group* if

$$\psi(b) = (a, 1, \dots, 1, b)$$

(see [6]). By [5, Theorem 2.16], it follows that a GGS-group whose defining vector contains only one nontrivial element is conjugate in Aut T to a generalized Fabrykowski–Gupta group.

We write $\mathcal{G} = \langle a, b \rangle$ with $\psi(b) = (a, \dots, a, b)$, for the GGS-group arising from the constant defining vector $(1, \dots, 1)$. It is known that $\mathcal{G}$ is weakly regular branch [5, Lemma 4.2] but not branch [4, Theorem 3.7].

For all other GGS-groups $G \neq \mathcal{G}$, we have from [5] that G is regular branch over $\gamma_3(G)$. Furthermore, from [4], a GGS-group G has the congruence subgroup property and is just infinite if and only if $G \neq \mathcal{G}$; we recall that an infinite group G is *just infinite* if all its proper quotients are finite.

The following result is useful.

Lemma 3.1 *For G a GGS-group, let $g \in G'$ and write $\psi(g) = (g_1, \dots, g_p)$. Then, $g_1 g_2 \dots g_p \in G'$.*

Proof It suffices to show the statement for $g = [a, b]$. As $[a, b] = (b^{-1})^a b$, we have that $\psi([a, b]) = (b^{-1} a^{e_1}, a^{e_2 - e_1}, \dots, a^{e_{p-1} - e_{p-2}}, a^{-e_{p-1}} b)$, and the result follows. ∎

As evident in the proof of the above result, all actions of group elements on the tree T will be taken on the right.

3.1 Length function

Here, we recall the following items from [15]: the abelianization G/G' of a GGS-group G and a natural length function on elements of G.

Let $G = \langle a, b \rangle$ be a GGS-group acting on the p-regular rooted tree T. We consider

$$H = \langle \hat{a}, \hat{b} \mid \hat{a}^p = \hat{b}^p = 1 \rangle,$$

the free product $\langle \hat{a} \rangle * \langle \hat{b} \rangle$ of cyclic groups $\langle \hat{a} \rangle \cong C_p$ and $\langle \hat{b} \rangle \cong C_p$. There is a unique epimorphism $\pi \colon H \to G$ such that $\hat{a} \mapsto a$ and $\hat{b} \mapsto b$, which induces an isomorphism from $H/H' \cong \langle \hat{a} \rangle \times \langle \hat{b} \rangle \cong C_p^2$ to G/G' (see [15]).

Let $h \in H$. Recall that h can be uniquely represented in the form

$$h = \hat{a}^{\alpha_1} \cdot \hat{b}^{\beta_1} \cdot \hat{a}^{\alpha_2} \cdot \hat{b}^{\beta_2} \cdots \hat{a}^{\alpha_m} \cdot \hat{b}^{\beta_m} \cdot \hat{a}^{\alpha_{m+1}},$$

where $m \in \mathbb{N} \cup \{0\}$, $\alpha_1, \dots, \alpha_{m+1} \in \mathbb{F}_p$ with $\alpha_i \neq 0$ for $i \in \{2, \dots, m\}$, and $\beta_1, \dots, \beta_m \in \mathbb{F}_p^*$.

We denote by $|h| = m$ the *length* of h, with respect to the factor $\langle \hat{b} \rangle$. Clearly, for $h_1, h_2 \in H$, we have

$$|h_1 h_2| \leq |h_1| + |h_2|. \tag{3.1}$$

https://doi.org/10.4153/S0008439521000898 Published online by Cambridge University Press

Then, for $G = \langle a, b\rangle$ a GGS-group, the *length* of $g \in G$ is

$$|g| = \min\{|h| \mid h \in \pi^{-1}(g)\}.$$

Based on (3.1), we deduce that for $g_1, g_2 \in G$,

$$|g_1 g_2| \le |g_1| + |g_2|. \tag{3.2}$$

We end this section with the following result from [1], specialized to the setting of GGS-groups:

Lemma 3.2 [1, Lemma 4.4] *Let G be a GGS-group, and $g \in St_G(1)$ with $\psi(g) = (g_1, \dots, g_p)$. Then, $\sum_{j=1}^{p} |g_j| \le |g|$, and $|g_j| \le \frac{|g|+1}{2}$ for each $j \in \{1, \dots, p\}$.*

In particular, if $|g| > 1$, then $|g_j| < |g|$ for every $j \in \{1, \dots, p\}$.

3.2 The GGS-group defined by the constant vector

Due to the group $\mathcal{G}$ not being branch, some further properties of $\mathcal{G}$ need to be established for the next section.

By [5, Lemma 4.2], the group $\mathcal{G}$ is weakly regular branch over K', where $K = \langle (ba^{-1})^{a^i} \mid i \in \{0, \dots, p-1\}\rangle^{\mathcal{G}}$.

Lemma 3.3 *Let $\mathcal{G}$ and K be as above. Then, K' is a subdirect product of $K \times \overset{p}{\cdots} \times K$ and $\psi(K'') \ge \gamma_3(K) \times \overset{p}{\cdots} \times \gamma_3(K)$.*

Proof This is similar to [9, Proof of Proposition 17]. We first show that K' is a subdirect product of $K \times \overset{p}{\cdots} \times K$. Indeed, writing $y_i = (ba^{-1})^{a^i}$ for $0 \le i \le p-1$, we have, for $0 \le i < j \le p-1$, that

$$\begin{aligned}
\psi([y_i, y_j]) &= \psi((b^{-1})^{a^{i-1}} (b^{-1})^{a^{j-2}} b^{a^{i-2}} b^{a^{j-1}}) \\
&= \begin{cases} (1, \overset{i-3}{\ldots}, 1, a^{-2}ba, b^{-1}a, 1, \overset{j-i-2}{\ldots}, 1, a^{-1}b^{-1}a^2, a^{-1}b, 1, \dots, 1) & \text{if } j > i+1, \\ (1, \overset{i-3}{\ldots}, 1, a^{-2}ba, b^{-2}a^2, a^{-1}b, 1, \dots, 1) & \text{if } j = i+1, \end{cases} \\
&= \begin{cases} (1, \overset{i-3}{\ldots}, 1, y_2, y_1^{-1}, 1, \overset{j-i-2}{\ldots}, 1, y_2^{-1}, y_1, 1, \dots, 1) & \text{if } j > i+1, \\ (1, \overset{i-3}{\ldots}, 1, y_2, (y_0^{-1} y_1^{-1})^a, y_1, 1, \dots, 1) & \text{if } j = i+1, \end{cases}
\end{aligned}$$

hence the result.

Thus, for every $k_1 \in K$, there is some $g_1 \in K'$ with $\psi(g_1) = (k_1, *, \dots, *)$. As $\mathcal{G}$ is weakly regular branch over K', for every $k_2 \in K'$, there is some $g_2 \in K'$ with $\psi(g_2) = (k_2, 1, \dots, 1)$. Therefore, $\psi([g_1, g_2]) = ([k_1, k_2], 1, \dots, 1)$, and the second part of the lemma follows. ∎

Proposition 3.4 *Let $\mathcal{G}$ be the GGS-group defined by the constant vector. Then, every proper quotient of $\mathcal{G}$ is virtually nilpotent and has maximal subgroups only of finite index.*

https://doi.org/10.4153/S0008439521000898 Published online by Cambridge University Press

Proof For the first statement, using [7, Theorem 4.10], it suffices to show that $\mathcal{G}/K''$ is virtually nilpotent. As $\mathcal{G}'/K''$ is of finite index in $\mathcal{G}/K''$, it suffices to show that $\mathcal{G}'/K''$ is nilpotent. Because $\psi(\mathcal{G}') \le K \times \overset{p}{\cdots} \times K$ by [5, Lemma 4.2(iii)], the result follows from the second part of the previous lemma.

The second statement is immediate, because every virtually nilpotent group has maximal subgroups only of finite index: indeed, a nilpotent group G has $\Phi(G)$ containing G' (see [17]). Then, every maximal subgroup of a nilpotent group G is normal and hence of finite index. As the property of having only maximal subgroups of finite index passes to finite extensions [6, Corollary 5.1.3], we have that virtually nilpotent groups have maximal subgroups only of finite index. ■

4 Prodense subgroups

In the present section, we lay out the strategy for proving Theorem 1.1, where it suffices to restrict to nontorsion GGS-groups in light of [15].

We recall that a subgroup H of G is *prodense* if $G = NH$ for every nontrivial normal subgroup N of G. Let G be a finitely generated group such that every proper quotient of G has maximal subgroups only of finite index. Then, by [7, Proposition 2.21], every maximal subgroup of infinite index in G is prodense, and every proper prodense subgroup is contained in a maximal subgroup of infinite index.

We now recall a key result concerning prodense subgroups of weakly branch groups.

Proposition 4.1 [7, Lemma 3.1 and Theorem 3.3] *Let T be a spherically homogeneous rooted tree and G a weakly branch group acting on T. Suppose that every proper quotient of G has maximal subgroups only of finite index. If H is a prodense subgroup of G, then H_u is a prodense subgroup of G_u, for every vertex u of T. Furthermore, if $M < G$ is a proper subgroup, then $M_u < G_u$ is a proper subgroup, and if $M < G$ is a maximal subgroup of G of infinite index, then M_u is a maximal subgroup of G_u, for every vertex u of T.*

Note that every proper quotient of a branch GGS-group has maximal subgroups only of finite index, because proper quotients of branch groups are virtually abelian (cf. [7, Proposition 2.22]). This, and Proposition 3.4, allows us to use the above proposition to show that nontorsion GGS-groups G do not possess proper prodense subgroups: for a prodense subgroup M of G, we will show that there exists a vertex u of T such that $M_u = G$. This then shows that M is not proper.

For G a nontorsion GGS-group and M a prodense subgroup of G, it remains to show that there exists a vertex u of T such that $M_u = G$. The following result was proved in a more general setting in [1], but it was stated for *dense* instead of prodense subgroups M. However, the proof, and hence the result, still holds for prodense subgroups.

Proposition 4.2 [1, Proposition 5.4] *Let $G = \langle a, b\rangle$ be a GGS-group that is not conjugate to a generalized Fabrykowski–Gupta group, and let M be a prodense subgroup of G. If $b \in M$, then there exists a vertex u of T such that $M_u = G$.*

https://doi.org/10.4153/S0008439521000898 Published online by Cambridge University Press

It follows from the above result that it suffices to show that $b \in M_u$ for M a prodense subgroup of G and u a vertex of T. This will be done in the next section.

5 Obtaining *b*

For $G = \langle a, b \rangle$ a nontorsion GGS-group with defining vector $\mathbf{e} = (e_1, \dots, e_{p-1})$, we write $\lambda := \sum_{i=1}^{p-1} e_i \neq 0$. Furthermore, we write $g \equiv_{G'} h$ when $gh^{-1} \in G'$, for elements $g, h \in G$.

Lemma 5.1 *Let $G = \langle a, b \rangle$ be a nontorsion GGS-group, and let $g \in G$ be such that $g \equiv_{G'} b^t$ for some $t \neq 0$. For $u \in X$, write $\varphi_u(g) \equiv_{G'} a^{n_u} b^{m_u}$ for some $n_u, m_u \in \mathbb{F}_p$. Then, exactly one of the following cases is true.*

(1) *There exists $u \in X$ such that $m_u \neq 0$ but $n_u = 0$.*
(2) *For all $u \in X$, if $m_u \neq 0$, then $n_u \neq 0$. Furthermore, there exist at least two vertices $u_1, u_2 \in X$ such that $n_{u_k} \neq \lambda m_{u_k}$ for $k \in \{1, 2\}$, and there exist at least two vertices $v_1, v_2 \in X$ such that $m_{v_1}, m_{v_2} \neq 0$.*

Proof As $g \equiv_{G'} b^t$, it follows that

$$g = (b^{j_1})^{a^{l_1}} (b^{j_2})^{a^{l_2}} \cdots (b^{j_n})^{a^{l_n}},$$

for some $n \in \mathbb{N}$, $l_1, \dots, l_n \in \mathbb{F}_p$ with $l_k \neq l_{k+1}$ for $1 \leq k \leq n-1$, and $j_1, \dots, j_n \in \mathbb{F}_p^*$ with $\sum_{k=1}^{n} j_k = t$ in $\mathbb{F}_p$.

Now, for $u \in X$, we have $\varphi_u(g) = \varphi_u(b^{a^{l_1}})^{j_1} \varphi_u(b^{a^{l_2}})^{j_2} \cdots \varphi_u(b^{a^{l_n}})^{j_n}$, and because

$$\varphi_u(b^{a^l}) = \begin{cases} a^{e_{u-l}} & \text{if } l \neq u, \\ b & \text{if } l = u, \end{cases}$$

we have $\varphi_u(g) \equiv_{G'} a^{n_u} b^{m_u}$ with

$$m_u = \sum j_k,$$

where the sum is taken over all k such that $l_k = u$, and

$$n_u = \sum_{\substack{i=0 \\ i \neq u}}^{p-1} e_{u-i}\, m_i = \sum_{j=1}^{p-1} e_j\, m_{u-j}. \tag{5.1}$$

It is clear that Cases 1 and 2 are mutually exclusive. Thus, it suffices to show that if Case 1 does not hold, then Case 2 does. So we assume now that Case 1 does not hold. Suppose that $n_u = \lambda m_u$ for all $u \in X$. Then,

$$\sum_{j=1}^{p-1} e_j\, m_{u-j} - \lambda m_u = 0 \quad \text{for all } u \in X.$$

https://doi.org/10.4153/S0008439521000898 Published online by Cambridge University Press

This is equivalent to

$$(e_1, \ldots, e_{p-1}, -\lambda) \cdot \begin{pmatrix} m_p \\ m_{p-1} \\ \vdots \\ m_2 \\ m_1 \end{pmatrix} = (e_1, \ldots, e_{p-1}, -\lambda) \cdot \begin{pmatrix} m_{p-1} \\ m_{p-2} \\ \vdots \\ m_1 \\ m_p \end{pmatrix} = \cdots$$

$$\cdots = (e_1, \ldots, e_{p-1}, -\lambda) \cdot \begin{pmatrix} m_1 \\ m_p \\ \vdots \\ m_3 \\ m_2 \end{pmatrix} = 0.$$

In other words,

$$\begin{pmatrix} m_p \\ m_{p-1} \\ \vdots \\ m_2 \\ m_1 \end{pmatrix}, \begin{pmatrix} m_{p-1} \\ m_{p-2} \\ \vdots \\ m_1 \\ m_p \end{pmatrix}, \ldots, \begin{pmatrix} m_1 \\ m_p \\ \vdots \\ m_3 \\ m_2 \end{pmatrix} \tag{5.2}$$

are all in the subspace orthogonal to $(e_1, \ldots, e_{p-1}, -\lambda)$. However, the vectors in (5.2) form the rows of a circulant matrix. The rank of this circulant matrix is less than p if and only if $\sum_{i=1}^{p} m_i = 0$ (cf. [5, Lemma 2.7(i)]). However, $\sum_{i=1}^{p} m_i = \sum_{k=1}^{n} j_k = t \neq 0$. Thus, there is at least one vertex $u_1 \in X$ such that $n_{u_1} \neq \lambda m_{u_1}$.

Suppose that there is only one such u_1. Then, $n_u = \lambda m_u$ for all $u \neq u_1$. Notice that we have

$$\sum_{u \in X} n_u = \sum_{u \in X} \sum_{j=1}^{p-1} e_j m_{u-j} = \sum_{j=1}^{p-1} \sum_{u \in X} e_j m_{u-j} = \sum_{j=1}^{p-1} e_j \sum_{u \in X} m_u = \lambda \sum_{u \in X} m_u.$$

Hence, we have

$$n_{u_1} + \lambda \sum_{u \neq u_1} m_u = \lambda \sum_{u \in X} m_u,$$

which yields $n_{u_1} = \lambda m_{u_1}$, a contradiction.

For the final statement, clearly $m_u \neq 0$ for at least one $u \in X$, because $\sum_{u \in X} m_u = t \neq 0$. Because we are not in Case 1, it follows that $n_u \neq 0$. From (5.1), the result follows. ∎

Lemma 5.2 *For G a nontorsion GGS-group, let $x \in G$ be such that $x \equiv_{G'} b^t$ for some $t \neq 0$, and write $\psi(x) = (x_1, x_2, \ldots, x_p)$. If x is in Case 2 of Lemma 5.1 and if $|x| = 2\mu$ for some $\mu \in \mathbb{N}$, then there exists $u \in X$ such that $x_u \neq 1$ and $|x_u| < \mu = \frac{|x|}{2}$.*

Proof Suppose for the sake of contradiction that this is not the case. By Lemma 3.2 and using the fact that x is in Case 2 of Lemma 5.1, there exist two vertices $v_1, v_2 \in X$ such that $|x_{v_1}| = |x_{v_2}| = \frac{|x|}{2}$ and $x_u = 1$ for all $u \in X \backslash \{v_1, v_2\}$. This means that

$$x = \left(b^{i_1}\right)^{a^{v_1}} \left(b^{j_1}\right)^{a^{v_2}} \cdots \left(b^{i_\mu}\right)^{a^{v_1}} \left(b^{j_\mu}\right)^{a^{v_2}},$$

https://doi.org/10.4153/S0008439521000898 Published online by Cambridge University Press

for some $i_1, \ldots, i_\mu, j_1, \ldots, j_\mu \in \mathbb{F}_p$. Because the result is true for x if and only if it is true for $x^{a^{-v_1}}$, we can suppose without loss of generality that

$$x = b^{i_1}\left(b^{j_1}\right)^{a^v} \cdots b^{i_\mu}\left(b^{j_\mu}\right)^{a^v},$$

for some $v \in X$. Let us set $i = \sum_{k=1}^{\mu} i_k$ and $j = \sum_{k=1}^{\mu} j_k$. Notice that $i + j = t$. Furthermore, because x is in Case 2 of Lemma 5.1, there must exist two vertices with nonzero powers of b, and our assumptions then imply that i and j are both nonzero.

For all $u \in X \backslash \{p, v\}$, the elements $\varphi_u(b)$ and $\varphi_u(b^{a^v})$ are both powers of a, and thus commute. Therefore, for $u \in X\backslash\{p, v\}$, we have

$$x_u = \varphi_u(x) = \varphi_u(b^i (b^j)^{a^v}) = a^{ie_u + je_{u-v}}.$$

Because $x_u = 1$ for all $u \in X\backslash\{p, v\}$, we have $ie_u + je_{u-v} = 0$ for all $u \in X\backslash\{p, v\}$. In other words, for all $2 \le k \le p-1$, we have $ie_{kv} + je_{(k-1)v} = 0$. By induction, we see that for all $1 \le k \le p-1$, we have $e_{kv} = r^{k-1} e_v$, where $r = -\frac{j}{i}$, which is well defined, because $i \neq 0$. Notice that $r \neq 0$, because $j \neq 0$, and $r \neq 1$, because $i + j = t \neq 0$. Therefore, we have

$$\lambda = \sum_{k=1}^{p-1} e_k = \sum_{k=1}^{p-1} e_{kv} = \sum_{k=1}^{p-1} r^{k-1} e_v = e_v \sum_{k=0}^{p-2} r^k = e_v \frac{r^{p-1} - 1}{r - 1} = 0,$$

a contradiction. The conclusion follows. ∎

Lemma 5.3 *Let $G = \langle a, b\rangle$ be a nontorsion GGS-group with $\lambda = \sum_{i=1}^{p-1} e_i$. Let $g \in G$ be such that $g \equiv_{G'} a^i b^j$ with $i \neq 0$ and $j \in \mathbb{F}_p$, and write $g = a^i \cdot \psi^{-1}((g_1, g_2, \ldots, g_p))$. Then, for all $u \in X$, we have $\varphi_u(g^p) \equiv_{G'} a^{\lambda j} b^j$, with $|\varphi_u(g^p)| \le \sum_{k=1}^{p} |g_k| \le |g|$.*

Proof This follows as in [6, Lemma 7.2.2]. ∎

We will now establish the following in several steps.

Proposition 5.4 *Let $G = \langle a, b\rangle$ be a nontorsion GGS-group, and let H be a prodense subgroup. Then, there exists $v_0 \in X^*$ such that $b \in H_{v_0}$.*

First, let us consider the set

$$U = \left\{ g \in \bigsqcup_{v \in X^*} H_v \mid g \equiv_{G'} a^{\lambda i} b^i \text{ for some } i \neq 0 \right\}.$$

Because H is a prodense subgroup of G and G' is a nontrivial normal subgroup in G, the set U is nonempty. Let $y \in U$ be an element of minimal length in U. Let $v \in X^*$ be such that $y \in H_v$. By Proposition 4.1, the subgroup H_v is prodense in $G_v = G$. Hence, if we prove the above proposition for H_v, we will also have proved it for H. Thus, without loss of generality, we may assume that v is the root of the tree X^*. Furthermore, in light of Lemma 5.3, we have "y," that is, an element of U of minimal length, everywhere.

Now, let us consider the set

$$V = \left\{ g \in \bigsqcup_{v \in X^*} H_v \mid g \equiv_{G'} b^i \text{ for some } i \neq 0 \right\},$$

and let $x \in V$ be an element of minimal length in V. Likewise, such an element exists as V is nonempty by the prodensity of H. Let $w \in X^*$ be such that $x \in H_w$. As before, we may assume that w is the root of the tree. Hence, using the fact that we get y everywhere, we have y and x at the root.

If $|x| = 1$, then $x = a^{-j} b^i a^j$ with $i \neq 0$ and $j \in \{1, \ldots, p\}$. Thus, we have that $\varphi_j(x) = b^i$. As i is invertible modulo p, we obtain $b \in H_j$.

We will now show that $|x|$ must be equal to 1. The case $|x| = 0$ is not possible, so we assume that $|x| > 1$. In this case, there cannot exist $u \in X$ such that $\varphi_u(x) \equiv_{G'} b^j$ with $j \neq 0$. Indeed, because $|\varphi_u(x)| < |x|$, this would contradict the minimality of x (cf. Lemma 3.2). Therefore, according to Lemma 5.1, there exists $u \in X$ such that $\varphi_u(x) \equiv_{G'} a^{i_u} b^{j_u}$ with $i_u \neq 0$ and $i_u \neq \lambda j_u$. For this $u \in X$, we write $x_u = \varphi_u(x)$.

Furthermore, also by Lemma 5.1, there exist two vertices $u_1, u_2 \in X$ such that $j_{u_1}, j_{u_2} \neq 0$ with $i_{u_1}, i_{u_2} \neq 0$. Now, by Lemma 5.3, we see that $\varphi_v(x_{u_i}^p) \in U$ for $i \in \{1, 2\}$ and $v \in X$, and thus

$$|y| \leq |\varphi_v(x_{u_i}^p)| \leq |x_{u_i}|.$$

Hence, using $\sum_{u \in X} |x_u| \leq |x|$, it follows that

$$2|y| \leq |x|. \tag{5.3}$$

Remark 5.5 We observe that if there is a $v \in X^*$ with $a^n \in H_v$ for some $n \neq 0$, then writing $y \equiv_{G'} a^{\lambda j} b^j$ for some $j \neq 0$, the product of y with $a^{-\lambda j}$ yields

$$|x| \leq |y|,$$

which contradicts (5.3), and so we are done in this case. Hence, in what follows, we will assume freely without special mention that $a \notin H_v$ for all $v \in X^*$.

Lemma 5.6 *In the set-up above, in particular, assuming that $|x| > 1$, let $v \in X^*$ be any vertex, and let $z \in H_v$. If $z \neq 1$, then $|z| \geq |y|$.*

Proof Without loss of generality, we may assume that $z \notin \mathrm{St}_G(1)$. Indeed, otherwise, because z is nontrivial, there must exist some $w \in X^*$ such that $z \in \mathrm{st}_G(w)$ and $\varphi_w(z) \notin \mathrm{St}_G(1)$. As we have $|z| \geq |\varphi_w(z)|$, it is sufficient to show that $|\varphi_w(z)| \geq |y|$. Thus, we may assume that $z \notin \mathrm{St}_G(1)$. Hence, we have $z \equiv_{G'} a^{i_z} b^{j_z}$ for some $i_z, j_z \in \mathbb{F}_p$ with $i_z \neq 0$.

Notice that for all $u \in X$, we have $\varphi_u(z^p) \equiv_{G'} a^{\lambda j_z} b^{j_z}$ by Lemma 5.3. In particular, if $j_z \neq 0$, by the minimality of $|y|$, we must have $|\varphi_u(z^p)| \geq |y|$. Because Lemma 5.3 also says that $|z| \geq |\varphi_u(z^p)|$, the conclusion follows in this case.

Thus, it only remains to treat the case where $j_z = 0$. Let us assume, for the sake of contradiction, that we have $|z| < |y|$, and let $j_y \neq 0$ be such that $y \equiv_{G'} a^{\lambda j_y} b^{j_y}$. Because $j_z = 0$, there exists $k \in \{1, 2, \ldots, p-1\}$ such that $z^k y \equiv_{G'} b^{j_y}$. By Lemmas 3.2 and 5.3 together with (3.2), for all $u \in X$, we have $|\varphi_u(z^k y)| \leq |z| + \frac{|y|+1}{2} < 2|y|$. If there were some $u \in X$ with $\varphi_u(z^k y) \equiv_{G'} b^l$ for some $l \neq 0$, by the minimality of $|x|$, we would have that $|x| \leq |\varphi_u(z^k y)| < 2|y|$, a contradiction to (5.3). Thus, by Lemma 5.1, there must exist two vertices $u_1, u_2 \in X$ such that $\varphi_{u_l}(z^k y) \equiv_{G'} a^{i_{u_l}} b^{j_{u_l}}$ with $i_{u_l}, j_{u_l} \neq 0$, for $l \in \{1, 2\}$.

Let us write $y = a^{i_y} \cdot \psi^{-1}((y_1, \dots, y_p))$ and $z = a^{i_z} \cdot \psi^{-1}((z_1, \dots, z_p))$. For any $u \in X$, we have, remembering that we act on the tree T on the right,

$$\begin{aligned}\varphi_u(z^k y) &= \varphi_u\Big(\psi^{-1}\big((z_1, \dots, z_p)^{a^{-i_z}}(z_1, \dots, z_p)^{a^{-2i_z}} \cdots (z_1, \dots, z_p)^{a^{-ki_z}}(y_1, \dots, y_p)\big)\Big) \\ &= z_{u+i_z} z_{u+2i_z} \cdots z_{u+ki_z} y_u.\end{aligned}$$

We also have

$$\begin{aligned}\varphi_u(z^{k-p} y) &= \varphi_u\Big(\psi^{-1}\big((z_1^{-1}, \dots, z_p^{-1})(z_1^{-1}, \dots, z_p^{-1})^{a^{i_z}} \cdots (z_1^{-1}, \dots, z_p^{-1})^{a^{(p-k-1)i_z}}(y_1, \dots, y_p)\big)\Big) \\ &= z_u^{-1} z_{u-i_z}^{-1} \cdots z_{u+(k+1)i_z}^{-1} y_u.\end{aligned}$$

As $\varphi_u(z^p) \equiv_{G'} 1$, we have

$$\varphi_u(z^{k-p} y) \equiv_{G'} \varphi_u(z^p)\varphi_u(z^{k-p} y) \equiv_{G'} \varphi_u(z^k y).$$

In particular, this means that for $u \in \{u_1, u_2\}$, we have

$$\varphi_u(z^{k-p} y) \equiv_{G'} \varphi_u(z^k y) \equiv_{G'} a^{i_u} b^{j_u},$$

with $i_u, j_u \neq 0$. We have seen above that this implies that $|\varphi_u(z^{k-p} y)|, |\varphi_u(z^k y)| \geq |y|$.

Because $\sum_{n=1}^{p} |y_n| \leq |y|$, there must exist some $u \in \{u_1, u_2\}$ such that $|y_u| \leq \frac{|y|}{2}$. For this u, we have

$$|\varphi_u(z^k y)| \leq |y_u| + \sum_{n=1}^{k} |z_{u+ni_z}|,$$

and

$$|\varphi_u(z^{k-p} y)| \leq |y_u| + \sum_{n=0}^{p-k-1} |z_{u-ni_z}|.$$

Because $\sum_{n=0}^{p-1} |z_{u-ni_z}| \leq |z|$, we must have that one of $\sum_{n=1}^{k} |z_{u+ni_z}|$ or $\sum_{n=0}^{p-k-1} |z_{u-ni_z}|$ is at most $\frac{|z|}{2}$. Consequently, we either have

$$|\varphi_u(z^k y)| \leq |y_u| + \frac{|z|}{2} \leq \frac{|y| + |z|}{2} < |y|,$$

or $|\varphi_u(z^{k-p} y)| < |y|$. In either case, we get a contradiction, as required. ∎

The following result is essential. First, let us write $y = a^{i_y} \cdot \psi^{-1}((y_1, \dots, y_p)) \equiv_{G'} a^{\lambda j_y} b^{j_y}$, for some $j_y \neq 0$ and $i_y = \lambda j_y$. By abuse of notation, we will still write y for $\varphi_u(y^p)$ for any $u \in X$ (cf. Lemma 5.3). In addition, for notational convenience, we sometimes write z_v or $(z_1, \dots, z_p)_v$ for $\varphi_v(z)$, where $z = \psi^{-1}((z_1, \dots, z_p)) \in \mathrm{st}_G(v)$ and $v \in X^* \backslash \{\varnothing\}$.

Lemma 5.7 *In the set-up above, in particular assuming that $|x| > 1$, there is an element $w \in H_v$, for some $v \in X^*$, with $w \equiv_{G'} a^n b^m$ for $n \neq \lambda m$, $n \neq 0$ and $|w| = |y|$.*

Proof Let

$$W = \left\{ g \in \bigsqcup_{v \in X^*} H_v \mid g \equiv_{G'} a^n b^m \text{ for some } m, n \text{ with } n \neq 0, n \neq \lambda m \right\},$$

and let $w \in W$ be an element of minimal length. We need to show that $|w| = |y|$. Let us first remark that $|w| \leq \frac{|x|}{2}$. Indeed, because $|x| > 1$, it follows from Lemma 5.1 that there exist $u_1, u_2 \in X$ such that $x_{u_1}, x_{u_2} \in W$. Because $|x_{u_1}| + |x_{u_2}| \leq |x|$, we conclude that $|w| \leq \min\{|x_{u_1}|, |x_{u_2}|\} \leq \frac{|x|}{2}$.

Let us write $w = a^n \cdot \psi^{-1}((w_1, \ldots, w_p))$. There exists $k \in \{1, 2, \ldots, p-1\}$ with $y^k \equiv_{G'} y^{k-p} \equiv_{G'} a^n b^{kj_y}$. Using y, or more precisely the projection of some power of y to the vertex where w lives, to cancel a^n in w yields

$$\begin{aligned} |(y^{-k}w)_v| &= |((y_1^{-1}, \ldots, y_p^{-1})(y_1^{-1}, \ldots, y_p^{-1})^{a^{i_y}} \cdots (y_1^{-1}, \ldots, y_p^{-1})^{a^{(k-1)i_y}} (w_1, \ldots, w_p))_v| \\ &= |y_v^{-1} y_{v-i_y}^{-1} \cdots y_{v-(k-1)i_y}^{-1} w_v| \\ &\leq |y_v^{-1} y_{v-i_y}^{-1} \cdots y_{v-(k-1)i_y}^{-1}| + |w_v| \\ &\leq \sum_{d=0}^{k-1} |y_{v-di_y}| + |w_v|, \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} |(y^{p-k}w)_v| &= |y_{v-(p-1)i_y} \cdots y_{v-i_y} y_v y_v^{-1} y_{v-i_y}^{-1} \cdots y_{v-(k-1)i_y}^{-1} w_v| \\ &\leq |y_{v-(p-1)i_y} \cdots y_{v-ki_y}| + |w_v| \\ &\leq \sum_{d=k}^{p-1} |y_{v-di_y}| + |w_v|, \end{aligned} \tag{5.5}$$

for $v \in X$.

Because $\sum_{d=0}^{p-1} |y_{v-di}| \leq |y|$ for $v \in X$, we have

$$\min\{|(y^{-k}w)_v|, |(y^{p-k}w)_v|\} \leq \frac{|y|}{2} + |w_v| \leq \frac{|y|}{2} + \frac{|x|}{4} + \frac{1}{2} \leq \frac{|x|+1}{2},$$

and

$$\max\{|(y^{-k}w)_v|, |(y^{p-k}w)_v|\} \leq |y| + |w_v| \leq \frac{|x|}{2} + \frac{|x|}{4} + \frac{1}{2} \leq \frac{3|x|+2}{4},$$

making use of (5.3).

As we will see, this implies that there exists a $w_1 \in \{(y^{-k}w)_v, (y^{p-k}w)_v \mid v \in X\}$ such that $w_1 \in W$ and $|w_1| \leq \frac{|y|+|w|}{2}$.

Indeed, if $y^{-k}w$ is in Case 1 of Lemma 5.1, then there exists $v \in X$ such that $(y^{-k}w)_v \equiv_{G'} b^{t'}$ for some $t' \neq 0$. By the minimality of $|x|$ and the above inequalities, we must have $|x| \leq |(y^{-k}w)_v| \leq \frac{3|x|+2}{4}$. This is only possible if $|x| \leq 2$, and because we assume that $|x| > 1$, this implies $|x| = 2$, and so $|y| = 1$ by (5.3). Notice that because $|x| = 2$, we cannot have $|x| \leq |(y^{-k}w)_v| \leq \frac{|x|+1}{2}$, and thus, by the inequalities above, we must have $|(y^{p-k}w)_v| \leq \frac{|x|+1}{2} = \frac{3}{2}$. Because the length must be an integer, we have

https://doi.org/10.4153/S0008439521000898 Published online by Cambridge University Press

$|(y^{p-k}w)_v| \leq 1 \leq \frac{|y|+|w|}{2}$, where the last inequality follows from Lemma 5.6 applied to $w \neq 1$. Notice that $(y^{p-k}w)_v = (y^p)_v(y^{-k}w)_v \equiv_{G'} a^{\lambda j_y} b^{j_y+t'}$ with $\lambda j_y \neq 0$. Thus, in this case, $w_1 = (y^{p-k}w)_v$ has the required properties.

By symmetry, if $y^{p-k}w$ is in Case 1 of Lemma 5.1, then we can also find some w_1 of the required form. Thus, it only remains to check the case when both $y^{-k}w$ and $y^{p-k}w$ are in Case 2 of Lemma 5.1. In this case, there exist at least two vertices $v_1, v_2 \in X$ such that $(y^{-k}w)_{v_1}$ and $(y^{-k}w)_{v_2}$ have nonzero total a-exponent differing from λ times their total b-exponent, and it follows from Lemma 5.3 that $(y^{p-k}w)_{v_1}$ and $(y^{p-k}w)_{v_2}$ satisfy the same property. Because $|w_{v_1}| + |w_{v_2}| \leq |w|$, we can assume without loss of generality that $|w_{v_1}| \leq \frac{|w|}{2}$. Then, we have

$$\min\left\{|(y^{-k}w)_{v_1}|, |(y^{p-k}w)_{v_1}|\right\} \leq \frac{|y|}{2} + |w_{v_1}| \leq \frac{|y|+|w|}{2},$$

and we set w_1 as the smallest of these two elements.

We have thus found some $w_1 \in W$ with $|w_1| \leq \frac{|y|+|w|}{2}$. By the minimality of $|w|$, we have $|w| \leq |w_1|$, and so $|w| \leq |y|$. As we have $|y| \leq |w|$ from Lemma 5.6, we conclude that $|w| = |y|$. ∎

Finally, to prove Proposition 5.4, we return to our assumption that $|x| > 1$. We will now obtain a contradiction using the above results. Hence, $|x| = 1$, as required.

We will use w from Lemma 5.7 to show that there exists a $g \in \bigsqcup_{v \in X^*} H_v$ with $0 < |g| < |y|$, which will contradict Lemma 5.6.

As before, we write $y = a^{i_y} \cdot \psi^{-1}((y_1, \ldots, y_p)) \equiv_{G'} a^{\lambda j_y} b^{j_y}$ for some $j_y \neq 0$ and so $i_y = \lambda j_y$. From Lemma 5.7, we have $w \in H_v$ for some $v \in X^*$ with $w \equiv_{G'} a^n b^m$ for $n \neq \lambda m$, $n \neq 0$ and $|w| = |y|$. Let $k \in \{1, 2, \ldots, p-1\}$ be such that $y^k \equiv_{G'} a^n b^{k j_y}$. There must exist at least two vertices $v_1, v_2 \in X$ such that $(y^{-k}w)_{v_1}$ and $(y^{-k}w)_{v_2}$ have nonzero total a-exponent differing from λ times their total b-exponent. Indeed, as we saw in the proof of the previous lemma, both $y^{-k}w$ and $y^{p-k}w$ must be in Case 2 of Lemma 5.1, unless possibly when $|x| = 2$ and $|y| = 1$. As we will now see, this last case is impossible. Certainly, if $|x| = 2$, $|y| = 1$ and one of $y^{-k}w$ or $y^{p-k}w$ is in Case 1 of Lemma 5.1, then there exists $u \in X$ such that $(y^{-k}w)_u \equiv_{G'} b^{t'}$ or $(y^{p-k}w)_u \equiv_{G'} b^{t'}$ for some $t' \neq 0$. For concreteness, let us assume that $(y^{-k}w)_u \equiv_{G'} b^{t'}$; the other case is similar. Because $|y| = |w| = 1$, there exist $i_1, i_2, i_3, i_4 \in \{0, 1, \ldots, p-1\}$ such that $y = a^{i_1} b^{j_y} a^{i_2}$ and $w = a^{i_3} b^m a^{i_4}$ (in particular, notice that we have $i_1 + i_2 = i_y$ and $i_3 + i_4 = n$, where these equations are taken modulo p). A direct computation then yields that

$$\begin{aligned} y^{-k}w &= a^{-i_2}(b^{-j_y})(b^{-j_y})^{a^{i_y}} \cdots (b^{-j_y})^{a^{(k-1)i_y}} a^{i_2 - k i_y} a^{i_3} b^m a^{i_4} \\ &= (b^{-j_y})^{a^{i_2}} (b^{-j_y})^{a^{i_2+i_y}} \cdots (b^{-j_y})^{a^{i_2+(k-1)i_y}} (b^m)^{a^{n-i_3}}. \end{aligned}$$

Therefore, we have

$$(y^{-k}w)_u = (b^{-j_y})_{u-i_2}(b^{-j_y})_{u-i_2-i_y} \cdots (b^{-j_y})_{u-i_2-(k-1)i_y}(b^m)_{u-n+i_3}.$$

https://doi.org/10.4153/S0008439521000898 Published online by Cambridge University Press

Using the fact that for $v \in X$, we have

$$b_v = \begin{cases} a^{e_v} & \text{if } v \neq p, \\ b & \text{if } v = p, \end{cases}$$

and that $u - i_2, u - i_2 - i_y, \ldots, u - i_2 - (k-1)i_y$ are all different vertices, we see that there are only four different possible forms for $(y^{-kw})_u$: a^*, $a^* b^{-j_y} a^*$, $a^* b^m$, or $a^* b^{-j_y} a^* b^m$, where the stars represent unimportant (possibly zero) powers of a. Because we assumed that $(y^{-k}w)_u \equiv_{G'} b^{t'}$, we must have $2 = |x| \leq |(y^{-k}w)_u|$, and thus only the last form is possible, because the other forms yield elements of length at most one. In particular, notice that we have $t' = m - j_y$. Let us now consider the element $(y^{p-k}w^{1-p})_u = (y^p)_u(y^{-k}w)_u(w^{-p})_u$. Because $(y^p)_u \equiv_{G'} a^{\lambda j_y} b^{j_y}$ and $w^{-p} \equiv_{G'} a^{-\lambda m} b^{-m}$, we have $(y^{p-k}w^{1-p})_u \equiv_{G'} a^{\lambda(j_y - m)} b^{t' + j_y - m} = a^{-\lambda t'}$. Considerations of length allow us to conclude that in fact, we must have $(y^{p-k}w^{1-p})_u = a^{-\lambda t'}$. Indeed, in exactly the same way as above, we find that

$$(y^{p-k}w^{1-p})_u = \prod_{d=1}^{p-k} (b^{j_y})_{u - i_2 + d i_y} \prod_{v \in X \setminus \{u - n + i_3\}} (b^{-m})_v.$$

As the sets $\{u - i_2 + i_y, u - i_2 + 2i_y, \ldots, u - i_2 + (p-k)i_y\}$ and $X \setminus \{u - n + i_3\}$ are disjoint from $\{u - i_2, u - i_2 - i_y, \ldots, u - i_2 - (k-1)i_y\}$ and $\{u - n + i_3\}$, respectively, and that the latter two sets both contain p by previous considerations, we conclude that all the elements in the above product are powers of a, and thus that $(y^{p-k}w^{1-p})_u = a^{-\lambda t'}$. As $t' \neq 0$, this element is nontrivial, but it is of length 0, a contradiction with Lemma 5.6. A similar argument yields a contradiction if we assume that $y^{p-k}w$ is in Case 1 of Lemma 5.1. We conclude that both $y^{-k}w$ and $y^{p-k}w$ must be in Case 2 of Lemma 5.1.

Let $u \in \{v_1, v_2\}$. We may assume that $|w_u| = \frac{|w|}{2}$, else we are done. Indeed, otherwise, one of $|w_{v_1}|$ or $|w_{v_2}|$ must be strictly smaller than $\frac{|w|}{2}$. Let us suppose without loss of generality that $|w_{v_2}| < \frac{|w|}{2}$. Then, either $|(y^{-k}w)_{v_2}| < \frac{|y|}{2} + \frac{|w|}{2} = |y|$ or $|(y^{p-k}w)_{v_2}| < \frac{|y|}{2} + \frac{|w|}{2} = |y|$. As this contradicts Lemma 5.6, we have that

$$|w_{v_1}| = |w_{v_2}| = \frac{|w|}{2}, \tag{5.6}$$

and hence $|y| = |w| = 2\mu$ for some $\mu \in \mathbb{N}$.

By symmetry, that is, by considering $w^{k'}y$ for some k', it likewise follows that only two first-level sections of y are of nonzero length and that the sum of their length must be $|y|$. Let $i_1, i_2 \in X$ be such that

$$|y_{i_1}| = |y_{i_2}| = \frac{|y|}{2} = \mu.$$

If $k = 1$, then $|x| \leq |y^{-1}w| \leq |y| + |w| = 2|y|$ and hence $|y^{-1}w| = 2|y|$ by (5.3). Therefore, it follows from Lemma 5.2 that there exists $u \in X$ such that $(y^{-1}w)_u \neq 1$ and $|(y^{-1}w)_u| < |y|$. As this contradicts Lemma 5.6, we conclude that the case $k = 1$

https://doi.org/10.4153/S0008439521000898 Published online by Cambridge University Press

is impossible. Likewise, we see that the case $k = p-1$ is impossible by looking at $y^{p-(p-1)}w = yw$.

Let us now assume that $1 < k < p-1$. For $v \in X$, recall that we have

$$|(y^{-k}w)_v| = |y_v^{-1}y_{v-i_y}^{-1}\cdots y_{v-(k-1)i_y}^{-1}w_v| \le \sum_{d=0}^{k-1}|y_{v-di_y}| + |w_v|,$$

and

$$|(y^{p-k}w)_v| = |(y_{v-ki_y}^{-1}y_{v-(k+1)i_y}^{-1}\cdots y_{v-(p-1)i_y}^{-1})^{-1}w_v| \le \sum_{d=k}^{p-1}|y_{v-di_y}| + |w_v|.$$

Because we assume that $|w_{v_1}| = |w_{v_2}| = \frac{|y|}{2}$, it follows that unless $\sum_{d=0}^{k-1}|y_{v_l-di_y}| = \sum_{d=k}^{p-1}|y_{v_l-di_y}| = \frac{|y|}{2}$ for $l \in \{1,2\}$, then one of $|(y^{-k}w)_{v_l}|$ or $|(y^{p-k}w)_{v_l}|$ is strictly smaller than $|y|$, which is impossible.

Notice also that if there exists $v \in X\backslash\{v_1,v_2\}$ such that $0 < |\prod_{d=0}^{k-1}y_{v-di_y}^{-1}| < |y|$, then $(y^{-k}w)_v$ is a nontrivial element of length strictly smaller than $|y|$, because we must have $|w_v| = 0$ when v is different from v_1 or v_2. As this contradicts Lemma 5.6, for all $v \in X\backslash\{v_1,v_2\}$, we must have either $|\prod_{d=0}^{k-1}y_{v-di_y}^{-1}| = 0$ or $|\prod_{d=0}^{k-1}y_{v-di_y}^{-1}| = |y|$. Likewise, we must also have either $|\prod_{d=k}^{p-1}y_{v-di_y}^{-1}| = 0$ or $|\prod_{d=k}^{p-1}y_{v-di_y}^{-1}| = |y|$.

This implies strong restrictions on the form of y. Recall from above that there exist $i_1, i_2 \in X$ such that $|y_{i_1}| = |y_{i_2}| = \frac{|y|}{2}$. This implies that, up to renaming i_1 and i_2, we must have

$$y = a^{i_y}(b^{s_1})^{a^{i_1}}(b^{s_2})^{a^{i_2}}\cdots(b^{s_{2\mu-1}})^{a^{i_1}}(b^{s_{2\mu}})^{a^{i_2}}, \tag{5.7}$$

where $s_1 + \cdots + s_{2\mu} = j_y$. In particular, for all $u \in X\backslash\{i_1,i_2\}$, we have $|y_u| = 0$. This implies that if, for some $v \in X$, the set $\{v - di_y \mid 0 \le d \le k-1\}$ contains exactly one of either i_1 or i_2, then $|\prod_{d=0}^{k-1}y_{v-di}^{-1}| = \frac{|y|}{2}$. By the above considerations, there can be only two such sets, namely when $v = v_1$ or $v = v_2$. As the next lemma shows, this can only be the case if $i_2 = i_1 + i_y$ or $i_2 = i_1 - i_y$.

Lemma 5.8 *Let $i, k, i_1, i_2 \in \mathbb{F}_p$ be four elements of $\mathbb{F}_p$, with $i \neq 0$, $i_1 \neq i_2$, and $1 < k < p-1$. For all $v \in \mathbb{F}_p$, let*

$$I_v = \{v - di \mid 0 \le d \le k-1\}.$$

If there exist two elements $v_1, v_2 \in \mathbb{F}_p$ such that

$$|I_v \cap \{i_1,i_2\}| = 1 \iff v \in \{v_1,v_2\},$$

then either $i_2 = i_1 + i$ or $i_2 = i_1 - i$.

Proof Let $f:\mathbb{F}_p \to \mathbb{F}_p$ be the map defined by $f(x) = i^{-1}(x - i_1)$. As f is a bijection, we have $|f(I_v) \cap \{f(i_1), f(i_2)\}| = |I_v \cap \{i_1,i_2\}|$, and because we have

$$f(I_v) = \{f(v) - d \mid 0 \le d \le k-1\},$$

for all $v \in \mathbb{F}_p$, it suffices to prove the result for $i = 1$ and $i_1 = 0$.

https://doi.org/10.4153/S0008439521000898 Published online by Cambridge University Press

Suppose for the sake of contradiction that $i_2 \neq 1, -1$, and let $v_1 \in \{k, k+1, \dots, p-1\}$ be the smallest element, with respect to the standard order on $\{k, k+1, \dots, p-1\}$, such that $i_2 \in I_{v_1}$. Note that v_1 exists, because $\bigcup_{v=k}^{p-1} I_v = \{1, 2, \dots, p-1\}$. Note also that because $v_1 \in \{k, k+1, \dots, p-1\}$, we know that $0 \notin I_{v_1}$. We claim that $i_2 \in I_{v_1+1}$ and that $0 \notin I_{v_1+1}$. Indeed, we have $I_{v_1+1} \backslash I_{v_1} = \{v_1+1\}$. Thus, if we had $0 \in I_{v_1+1}$, this would imply that $v_1 = p-1$. Because $k < p-1$, by the minimality of v_1, this would then imply that $i_2 \notin I_{p-2}$, and therefore $i_2 = p-1$, a contradiction. Similarly, because $I_{v_1} \backslash I_{v_1+1} = \{v_1 - k + 1\}$, if we had $i_2 \notin I_{v_1+1}$, this would imply that $i_2 = v_1 - k + 1$. Because $k > 1$, we have $i_2 < v_1$, and so $i_2 \in I_{v_1-1}$. By the minimality of v_1, this implies that $v_1 = k$. However, this then means that $i_2 = 1$, a contradiction.

We conclude that $i_2 \in I_{v_1}, I_{v_1+1}$ and that $0 \notin I_{v_1}, I_{v_1+1}$. Likewise, we can also find $v_2 \in \{i_2+k, i_2+k+1, \dots, i_2+p-1\}$ such that $0 \in I_{v_2}, I_{v_2+1}$ and $i_2 \notin I_{v_2}, I_{v_2+1}$. Clearly, v_1, v_1+1, v_2, v_2+1 are four different elements, and we have $|I_v \cap \{0, i_2\}| = 1$ for all $v \in \{v_1, v_1+1, v_2, v_2+1\}$, which contradicts our assumptions. The result follows. ∎

Coming back to our considerations on the form of y, we have

$$y = a^{i_y} (b^{s_1})^{a^{i_1}} (b^{s_2})^{a^{i_2}} \cdots (b^{s_{2\mu-1}})^{a^{i_1}} (b^{s_{2\mu}})^{a^{i_2}},$$

where $i_2 = i_1 + i_y$ or $i_2 = i_1 - i_y$ by the previous lemma. If we had $i_2 = i_1 - i_y$, then we would have

$$\varphi_{i_2 - i_y}(y^p) = y_{i_2} y_{i_1} y_{i_2 + 2i_y} \cdots y_{i_2 + (p-1)i_y}.$$

From the form of y above, we see that y_{i_2} ends with $b^{s_{2\mu}}$ and y_{i_1} begins with b^{s_1}, which implies that $|y_{i_2} y_{i_1}| \leq |y_{i_2}| + |y_{i_1}| - 1$. Consequently, we have $|\varphi_{i_2 - i_y}(y^p)| < |y|$, a contradiction to the minimality of y by Lemma 5.3. We conclude that we must have $i_2 = i_1 + i_y$.

To finish the proof, it suffices to show that if y is of the above form, then one of $(y^p)_{v_1}$ or $(y^p)_{v_2}$ is not. Indeed, we can then repeat the whole argument above with $y' = (y^p)_v$ and $w' = (y^{-k} w)_v$ for the corresponding $v \in \{v_1, v_2\}$ and we will reach a contradiction.

Let us then prove this last claim. Suppose that

$$y = a^{i_y} (b^{s_1})^{a^{i_1}} (b^{s_2})^{a^{i_1 + i_y}} \cdots (b^{s_{2\mu-1}})^{a^{i_1}} (b^{s_{2\mu}})^{a^{i_1 + i_y}}. \tag{5.8}$$

For $u \in \{v_1, v_2\}$, we have

$$(y^p)_u = y_{u+i_y} y_{u+2i_y} \cdots y_{u+pi_y},$$

and because we have supposed above that $\sum_{d=0}^{k-1} |y_{u-di_y}| = \frac{|y|}{2}$ and that $|y_{i_1}| = |y_{i_1+i_y}| = \frac{|y|}{2}$, we conclude that either $u = i_1$ or $u - ki_y = i_1$. Thus, up to renaming v_1 and v_2, we can assume that $v_1 = i_1 + ki_y$ and $v_2 = i_1$. It follows that

$$(y^p)_{v_1} = a^{k_1} y_{i_1} y_{i_1+i_y} a^{k_2}, \tag{5.9}$$

where $a^{k_1} = y_{v_1+i_y} \cdots y_{v_1+(p-k-1)i_y}$ and $a^{k_2} = y_{v_1+(p-k+2)i_y} \cdots y_{v_1+pi_y}$, and that

$$(y^p)_{v_2} = y_{i_1+i_y} a^{k_2+k_1} y_{i_1}. \tag{5.10}$$

https://doi.org/10.4153/S0008439521000898 Published online by Cambridge University Press

From (5.8) and (5.9), we see that

$$(y^p)_{v_1} = a^{k_1} b^{t_1} a^{r_1} b^{t_2} a^{r_2} \cdots a^{r_{2\mu-1}} b^{t_{2\mu}} a^{k_2}$$
$$= a^{k_1+\sum_{d=1}^{2\mu-1} r_d + k_2} (b^{t_1})^{a^{\sum_{d=1}^{2\mu-1} r_d + k_2}} (b^{t_2})^{a^{\sum_{d=2}^{2\mu-1} r_d + k_2}} \cdots (b^{t_{2\mu-1}})^{a^{r_{2\mu-1}+k_2}} (b^{t_{2\mu}})^{a^{k_2}},$$

for some $t_1, \ldots, t_{2\mu}$ and $r_1, \ldots, r_{2\mu-1}$ in $\mathbb{F}_p$. Suppose that $(y^p)_{v_1}$ is of the same form as y, namely that there exists $i_3 \in X$ such that

$$(y^p)_{v_1} = a^{i_y} (b^{t_1})^{a^{i_3}} (b^{t_2})^{a^{i_3+i_y}} \cdots (b^{t_{2\mu-1}})^{a^{i_3}} (b^{t_{2\mu}})^{a^{i_3+i_y}}.$$

This implies that $r_d = -i_y$ if d is odd and $r_d = i_y$ if d is even. Therefore, we obtain $\sum_{d=1}^{2\mu-1} r_d = -i_y$. Because we know from Lemma 5.3 that $k_1 + k_2 + \sum_{d=1}^{2\mu-1} r_d = i_y$, we conclude that $k_1 + k_2 = 2i_y$.

If we now turn our attention to $(y^p)_{v_2}$, it follows from (5.8) and (5.10) that

$$(y^p)_{v_2} = a^{r'_1} b^{t'_1} \cdots a^{r'_\mu} b^{t'_\mu} a^{k_1+k_2} b^{t'_{\mu+1}} a^{r'_{\mu+1}} \cdots b^{t'_{2\mu}} a^{r'_{2\mu}},$$

for some $t'_1, \ldots, t'_{2\mu}$ and $r'_1, \ldots, r'_{2\mu}$ in $\mathbb{F}_p$. Using the same reasoning as above, for $(y^p)_{v_2}$ to be of the same form as y, we would need either $k_1 + k_2 = i_y$ or $k_1 + k_2 = -i_y$, depending on the parity of μ.

We conclude that $(y^p)_{v_1}$ and $(y^p)_{v_2}$ cannot both be in the same form as y. Indeed, this would imply that either $i_y = 2i_y$ or $-i_y = 2i_y$. Because $i_y \neq 0$, the first equation is impossible, and the second can only be satisfied if $p = 3$. However, we assumed that there existed some $1 < k < p - 1$, which is impossible if $p = 3$.

Therefore, there is some $u \in \{v_1, v_2\}$ such that $y' = (y^p)_u$ is not in the form of (5.8). Setting $w' = (y^{-k} w)_u$ and repeating the whole argument above with y' and w' will thus necessarily yield a contradiction. We conclude that $|x| = 1$, and the result follows.

Theorem 5.9 *Let G be a GGS-group acting on the p-regular rooted tree, for p an odd prime. Then, G does not contain any proper prodense subgroups.*

Proof By [15], it suffices to consider the nontorsion GGS-groups G. Furthermore, we may suppose that G is not conjugate to a generalized Fabrykowski–Gupta group, as otherwise the result follows by [6, Theorem 7.2.7]. Suppose on the contrary that M is a proper prodense subgroup of G. By Proposition 4.1, for every vertex $u \in X^*$, we have M_u is properly contained in G_u. However, by Propositions 4.2 and 5.4, there exists $v \in X^*$ such that the subgroup M_v is all of G. This gives the required contradiction. ∎

The first statement of Theorem 1.1 is now proved. We show the second.

Proposition 5.10 *Let G be a branch GGS-group acting on the p-regular rooted tree, for an odd prime p. Then, every maximal subgroup of G is normal and of index p.*

Proof Let M be a maximal subgroup of G. From the previous result, it follows that M has finite index in G. By [4], the group G has the congruence subgroup property, so there exists an $n \in \mathbb{N}$ such that $\mathrm{St}_G(n) \leq M$. As $G/\mathrm{St}_G(n)$ is a finite p-group, it follows that $G' \leq M \trianglelefteq_p G$. Hence the result. ∎

For the constant GGS-group $\mathcal{G}$, the situation is different.

Proposition 5.11 *Let $\mathcal{G}$ be the weakly branch, but not branch, GGS-group acting on the p-regular rooted tree, for an odd prime p. Then, there are infinitely many maximal subgroups. In particular, the group $\mathcal{G}$ has maximal subgroups that are neither normal, nor of index p.*

Proof By [4, Proposition 3.4] and using the notation of Section 3.2, we have $\mathcal{G}/K' \cong (\mathbb{Z}/p\mathbb{Z}) \ltimes \mathbb{Z}^{p-1}$ where the action of $\mathbb{Z}/p\mathbb{Z}$ on $\mathbb{Z}^{p-1}$ is given by the matrix

$$A = \left(\begin{array}{c|c} 0_{1\times p-2} & -1 \\ \hline I_{p-2\times p-2} & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \end{array}\right).$$

We see that for all $q \in \mathbb{N}$, the subgroup $(q\mathbb{Z})^{p-1} \leq \mathbb{Z}^{p-1}$ is invariant under the action of $\mathbb{Z}/p\mathbb{Z}$. Therefore, for all $q \in \mathbb{N}$ different from 0 or 1, we can consider the subgroup $(\mathbb{Z}/p\mathbb{Z}) \ltimes (q\mathbb{Z})^{p-1}$, which is nontrivial and proper in $(\mathbb{Z}/p\mathbb{Z}) \ltimes \mathbb{Z}^{p-1}$. As $(\mathbb{Z}/p\mathbb{Z}) \ltimes \mathbb{Z}^{p-1}$ is finitely generated, this subgroup is contained in a maximal subgroup $M_q \leq (\mathbb{Z}/p\mathbb{Z}) \ltimes \mathbb{Z}^{p-1}$.

We notice that if $q_1, q_2 \in \mathbb{N}$ are two different prime numbers, then $M_{q_1} \neq M_{q_2}$. Indeed, otherwise, we would have $(q_1\mathbb{Z})^{p-1}, (q_2\mathbb{Z})^{p-1} \leq M_{q_1}$, which would imply $\mathbb{Z}^{p-1} \leq M_{q_1}$, because q_1 and q_2 are coprime. As we already had $(\mathbb{Z}/p\mathbb{Z}) \ltimes (q_1\mathbb{Z})^{p-1} \leq M_{q_1}$, this means that $M_{q_1} = (\mathbb{Z}/p\mathbb{Z}) \ltimes \mathbb{Z}^{p-1}$, which is absurd.

We have thus shown that we have an infinite number of maximal subgroups of $\mathcal{G}/K'$ which are all pairwise distinct. By the correspondence theorem, the same is true for $\mathcal{G}$. This immediately implies that $\mathcal{G}$ has maximal subgroups that are not of index p, because there can only be finitely many such subgroups. It also implies that $\mathcal{G}$ admits maximal subgroups that are not normal. Indeed, otherwise, the Frattini subgroup of $\mathcal{G}$ would contain $\mathcal{G}'$, because the quotient of any group by a normal maximal subgroup must be a cyclic group of prime order, and thus abelian, but $\mathcal{G}/\mathcal{G}'$ is a p-group, which would imply that every maximal subgroup is of index p. ∎

Acknowledgment We thank the referees for suggesting valuable improvements to the exposition of the paper.

References

[1] T. Alexoudas, B. Klopsch, and A. Thillaisundaram, *Maximal subgroups of multi-edge spinal groups*. Groups Geom. Dyn. **10**(2016), 619–648.

[2] L. Bartsholdi, R. I. Grigorchuk, and Z. Šunik, *Handbook of algebra. Vol.* 3, North-Holland, Amsterdam, 2003.

[3] I. V. Bondarenko, *Finite generation of iterated wreath products*. Arch. Math. (Basel) **95**(2010), no. 4, 301–308.

[4] G. A. Fernández-Alcober, A. Garrido, and J. Uria-Albizuri, *On the congruence subgroup property for GGS-groups*. Proc. Amer. Math. Soc. **145**(2017), no. 8, 3311–3322.

[5] G. A. Fernández-Alcober and A. Zugadi-Reizabal, *GGS-groups: order of congruence quotients and Hausdorff dimension*. Trans. Amer. Math. Soc. **366**(2014), 1993–2017.

[6] D. Francoeur, *On maximal subgroups and other aspects of branch groups*. Ph.D. thesis, University of Geneva, 2019.
[7] D. Francoeur, *On maximal subgroups of infinite index in branch and weakly branch groups*. J. Algebra **560**(2020), 818–851.
[8] D. Francoeur and A. Garrido, *Maximal subgroups of groups of intermediate growth*. Adv. Math. **340**(2018), 1067–1107.
[9] A. Garrido and J. Uria-Albizuri, *Pro-$\mathcal{C}$ congruence properties for groups of rooted tree automorphisms*. Arch. Math. (Basel) **112**(2019), no. 2, 123–137.
[10] R. I. Grigorchuk, *On Burnside's problem on periodic groups*. Funktsional. Anal. i Prilozhen **14**(1980), no. 1, 53–54.
[11] R. I. Grigorchuk, *Just infinite branch groups*. In: New horizons in pro-p groups, Birkhäuser, Boston, 2000.
[12] R. I. Grigorchuk and J. S. Wilson, *A structural property concerning abstract commensurability of subgroups*. J. Lond. Math. Soc. **68**(2003), no. 2, 671–682.
[13] N. Gupta and S. Sidki, *On the Burnside problem for periodic groups*. Math. Z. **182**(1983), no. 3, 385–388.
[14] E. L. Pervova, *Everywhere dense subgroups of a group of tree automorphisms*. Tr. Mat. Inst. Steklova **231**(2000), 356–367 (Din. Sist., Avtom. i. Beskon. Gruppy).
[15] E. L. Pervova, *Maximal subgroups of some non-locally finite p-groups*. Internat. J. Algebra Comput. **15**(2005), nos. 5–6, 1129–1150.
[16] T. Vovkivsky, *Infinite torsion groups arising as generalizations of the second Grigorchuk group*. In: Algebra (Moscow, 1998), de Gruyter, Berlin, 2000.
[17] H. Zassenhaus, *Lehrbuch der Gruppentheorie*, Chelsea, New York, 1958.

Instituto de Ciencias Matemáticas, Calle Nicolás Cabrera, no. 13-15, Campus Cantoblanco, Universidad Autónoma de Madrid, 28049 Madrid, Spain
e-mail: dominik.francoeur@icmat.es

Centre for Mathematical Sciences, Lund University, 223 62 Lund, Sweden
e-mail: anitha.t@cantab.net

https://doi.org/10.4153/S0008439521000898 Published online by Cambridge University Press