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Commutativity preserving extensions of groups

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In parallel to the classical theory of central extensions of groups, we develop a version for extensions that preserve commutativity. It is shown that the Bogomolov multiplier is a universal object parametrizing such extensions of a given group. Maximal and minimal extensions are inspected, and a connection with commuting probability is explored. Such considerations produce bounds for the exponent and rank of the Bogomolov multiplier.

Keywords: commutativity preserving extension; Bogomolov multiplier; commuting probability

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1. Introduction

Noether's problem [25] is one of the fundamental problems of invariant theory, and asks as to whether the field of Q-invariant functions $\mathbb{C}(V)^Q$ is purely transcendental over $\mathbb{C}$, where Q is a given finite group and V is a vector space over $\mathbb{C}$ (or, more generally, over an algebraically closed field of characteristic zero) equipped with a faithful linear generically free action of Q. Artin and Mumford [1] introduced an obstruction $\mathrm{H}^2_{\mathrm{nr}}(\mathbb{C}(V)^Q, \mathbb{Q}/\mathbb{Z})$ to this problem, called the *unramified Brauer group* of the field extension $\mathbb{C}(V)^Q/\mathbb{C}$. In his seminal work, Bogomolov [4] proved that $\mathrm{H}^2_{\mathrm{nr}}(\mathbb{C}(V)^Q, \mathbb{Q}/\mathbb{Z})$ is canonically isomorphic to the intersection of the kernels of restriction maps $\mathrm{H}^2(Q, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(A, \mathbb{Q}/\mathbb{Z})$, where A runs through all abelian subgroups of Q. A simplified description of $\mathrm{H}^2_{\mathrm{nr}}(\mathbb{C}(V)^Q, \mathbb{Q}/\mathbb{Z})$ was found in [21] by considering its dual $\mathrm{B}_0(Q)$. Following Kunyavskiĭ [18], we call the latter group the *Bogomolov multiplier* of Q. The description of B_0 is combinatorial and enables efficient explicit calculations. Furthermore, it relates Bogomolov multipliers to the commuting probability of a group [14], and shows that B_0 plays a role in describing the so-called *commutativity preserving central extensions* of groups, which are closely related to some problems in K-theory [21].

In this paper we develop a theory of commutativity preserving group extensions with abelian kernel. Specifically, let Q be a group and let N be a Q-module. Denote by $e = (\chi, G, \pi)$ the extension

$$1 \longrightarrow N \xrightarrow{\chi} G \xrightarrow{\pi} Q \longrightarrow 1$$

of N by Q. Following [21], we say that e is a *CP extension* if commuting pairs of elements of Q have commuting lifts in G. In the first part of the paper we define a subgroup $\mathrm{H}^2_{\mathrm{CP}}(Q, N)$ of the second cohomology group $\mathrm{H}^2(Q, N)$ that classifies CP extensions of N by Q up to equivalence. Then we focus on central CP extensions. We prove a variant of the universal coefficient theorem by showing that, given a trivial Q-module N, there is a split exact sequence

$$0 \longrightarrow \mathrm{Ext}(Q^{\mathrm{ab}}, N) \longrightarrow \mathrm{H}^2_{\mathrm{CP}}(Q, N) \longrightarrow \mathrm{Hom}(\mathrm{B}_0(Q), N) \longrightarrow 0.$$

In addition to that, we provide several characterizations of central CP extensions, and prove that these are closed under isoclinism of extensions. Subsequently, we show that the isoclinism classes of central CP extensions with a given factor group Q are in bijective correspondence with the orbits of the action of $\mathrm{Aut}\, Q$ upon the subgroups of $\mathrm{B}_0(Q)$.

In what follows, we consider stem central CP extensions of N by Q, where $|N| = |\mathrm{B}_0(Q)|$. We call such extensions *CP covers* of Q. These are analogues of the usual covers in the theory of Schur multipliers. We show that every finite group has a CP cover, and that all CP covers of isoclinic groups are isoclinic. Furthermore, we show how CP covers are, in a suitable sense, precisely the maximal central CP extensions of Q. In the succeeding section we then also consider minimal central CP extensions, i.e. those whose kernel is a cyclic group of prime order. Such extensions are parametrized by $\mathrm{H}^2_{\mathrm{CP}}(Q, \mathbb{F}_p)$. The main result in this direction is that this group is an elementary abelian p-group of rank $\mathrm{d}(Q) + \mathrm{d}(\mathrm{B}_0(Q))$.

Applying the theory of CP covers, we derive some bounds for the order, rank and exponent of the Bogomolov multiplier of a given finite group Q. We obtain bounds for $\mathrm{B}_0(Q)$ that correspond to those for Schur multipliers obtained by Jones and Wiegold [16] and Jones [15]. On the other hand, a special feature of B_0 is that it is closely related to the commuting probability of the group in question, that is, the probability that two randomly chosen elements of the group commute. This was already explored in [14], where we proved that if the commuting probability of Q is strictly greater than $1/4$, then $\mathrm{B}_0(Q)$ is trivial. Here we prove that if the commuting probability of a finite group Q is strictly greater than a fixed $\varepsilon > 0$, then the order of $\mathrm{B}_0(Q)$ can be bounded in terms of ε and the maximum of minimal numbers of generators of Sylow subgroups of Q. Furthermore, we show that $\exp B_0(Q)$ can be bounded in terms of ε only.

2. CP extensions

The purpose of this section is to establish a cohomological object that encodes all the information on CP extensions up to equivalence. We refer the reader to [5] for an account of the theory of group extensions.

Lemma 2.1. *The class of CP extensions is closed under equivalence of extensions.*

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Proof. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\mu_1} & G_1 & \xrightarrow{\varepsilon_1} & Q & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\theta} & & \| & & \\ 0 & \longrightarrow & N & \xrightarrow{\mu_2} & G_2 & \xrightarrow{\varepsilon_2} & Q & \longrightarrow & 1 \end{array}$$

be equivalent extensions with abelian kernel. Suppose that G_1 is a CP extension of N by Q. Choose $x_1, x_2 \in Q$ with $[x_1, x_2] = 1$. Then there exist $e_1, e_2 \in G_1$ such that $[e_1, e_2] = 1$ and $\varepsilon_1(e_i) = x_i$, $i = 1, 2$. Take $\bar{e}_i = \theta(e_i)$. Then $[\bar{e}_1, \bar{e}_2] = 1$ and $\varepsilon_2(\bar{e}_i) = x_i$. This proves that G_2 is a CP extension of N by Q. □

As in the classical setting of group extensions, CP extensions can be interpreted in a cohomological manner. Let Q and S be groups, and suppose that Q acts on S via $(x, y) \mapsto {}^x y$, where $x \in Q$ and $y \in S$. A map $\partial\colon Q \to S$ is a *derivation* (or *1-cocycle*) *from* Q *to* S if $\partial(xy) = {}^x\partial(y)\partial(x)$ for all $x, y \in Q$. Let N be a Q-module and fix $a \in N$. The map $\partial_a\colon Q \to N$, given by $\partial_a(g) = ga - a$, is a derivation. The maps of this type are called *inner derivations*.

A cocycle $\omega \in \mathrm{Z}^2(Q, N)$ is said to be a *CP cocycle* if for all commuting pairs $x_1, x_2 \in Q$ there exist $a_1, a_2 \in N$ such that

$$\omega(x_1, x_2) - \omega(x_2, x_1) = \partial_{a_1}(x_1) + \partial_{a_2}(x_2). \tag{2.1}$$

Denote by $\mathrm{Z}^2_{\mathrm{CP}}(Q, N)$ the set of all CP cocycles in $\mathrm{Z}^2(Q, N)$.

Proposition 2.2. *$\mathrm{Z}^2_{\mathrm{CP}}(Q, N)$ is a subgroup of $\mathrm{Z}^2(Q, N)$ containing $\mathrm{B}^2(Q, N)$.*

Proof. It is clear that $\mathrm{Z}^2_{\mathrm{CP}}(Q, N)$ is a subgroup of $\mathrm{Z}^2(Q, N)$. Now let $\beta \in \mathrm{B}^2(Q, N)$. Then there exists a function $\phi\colon Q \to N$ such that

$$\beta(x_1, x_2) = x_1\phi(x_2) - \phi(x_1 x_2) + \phi(x_1)$$

for all $x_1, x_2 \in Q$. Suppose that these two elements commute. Then $\beta(x_1, x_2) - \beta(x_2, x_1) = \partial_{\phi(x_2)}(x_1) + \partial_{-\phi(x_1)}(x_2)$, and hence $\beta \in \mathrm{Z}^2_{\mathrm{CP}}(Q, N)$. □

Now define $\mathrm{H}^2_{\mathrm{CP}}(Q, N) = \mathrm{Z}^2_{\mathrm{CP}}(Q, N)/\,\mathrm{B}^2(Q, N)$. This is a subgroup of the ordinary cohomology group $\mathrm{H}^2(Q, N)$.

Example 2.3. Let Q be an abelian group and let N be a trivial Q-module. Then $\mathrm{H}^2_{\mathrm{CP}}(Q, N)$ coincides with $\mathrm{Ext}(Q, N)$.

Proposition 2.4. *Let N be a Q-module. Then the equivalence classes of CP extensions of N by Q are in bijective correspondence with the elements of $\mathrm{H}^2_{\mathrm{CP}}(Q, N)$.*

Proof. Let $e = (\chi, G, \pi)$ be an extension of N by Q. Let $\omega\colon Q \times Q \to N$ be a corresponding 2-cocycle. Then e is equivalent to the extension

$$1 \longrightarrow N \longrightarrow Q[\omega] \xrightarrow{\varepsilon} Q \longrightarrow 1,$$

where $Q[\omega]$ is, as a set, equal to $N \times Q$, the operation is given by $(a, x)(b, y) = (a + xb + \omega(x, y), xy)$, and $\varepsilon(a, x) = x$. By lemma 2.1 it suffices to show that the latter extension is CP if and only if $\omega \in \mathrm{Z}^2_{\mathrm{CP}}(Q, N)$. Let $x, y \in Q$ commute and let

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(a, x) and (b, y) be lifts of x and y in $Q[\omega]$. Then (a, x) and (b, y) commute if and only if $\omega(x, y) - \omega(y, x) = (y-1)a - (x-1)b = \partial_a(y) + \partial_{-b}(x)$. Thus the existence of commuting lifts of x and y is equivalent to $\omega \in \mathrm{Z}^2_{\mathrm{CP}}(Q, N)$. □

We now give some examples.

Example 2.5. Let Q be a group in which for every commuting pair x, y the subgroup $\langle x, y\rangle$ is cyclic. This is equivalent to Q having all abelian subgroups cyclic. In the case of finite groups, it is known [5, theorem VI.9.5] that such groups are precisely the groups with periodic cohomology, and this furthermore amounts to Q having cyclic Sylow p-subgroups for p odd, and cyclic or quaternion Sylow p-subgroups for $p = 2$. Infinite groups with this property include free products of cyclic groups (see [17]). Given such a group Q, it is clear that every commuting pair of elements in Q has a commuting lift. Thus every extension of Q is CP, and so $\mathrm{H}^2(Q, N) = \mathrm{H}^2_{\mathrm{CP}}(Q, N)$ for any Q-module N.

Example 2.6. Taking the simplest case $Q = C_p$ in the previous example, we see that every extension of a group by C_p is CP. Thus, in particular, every finite p-group can be viewed as being composed from a sequence of CP extensions.

Example 2.7. There are many examples of extensions that are not CP. One may simply take as G a group of nilpotency class 2 and factor by a subgroup generated by a non-trivial commutator. In fact, in the case in which the extension is central, it is more difficult to find examples of extensions that are CP. We will focus on inspecting central CP extensions in the following section. Consider now only extensions that are not central. Some small examples of extensions that fail to be CP are easily produced by taking a non-trivial action of a non-cyclic abelian group on an elementary abelian group. We give a concrete example. Take $Q = \langle x_1\rangle \times \langle x_2\rangle$ to be an elementary abelian p-group of rank 2, and let it act on $N = \langle a_1\rangle \times \langle a_2\rangle \times \langle a_3\rangle$, an elementary abelian p-group of rank 3, via the following rules:

$$a_1^{x_1} = a_1, \qquad a_2^{x_1} = a_2, \qquad a_3^{x_1} = a_3, \qquad a_1^{x_2} = a_2, \qquad a_2^{x_2} = a_1, \qquad a_3^{x_2} = a_3.$$

Thus N is a Q-module. Now construct an extension G corresponding to this action by specifying $x_2^{x_1} = x_2 a_3$. This extension is not CP because the commuting pair x_1, x_2 in Q does not have a commuting lift in G.

3. Central CP extensions

From now on we focus on a special type of CP extension, namely, those with central kernel. In terms of the cohomological interpretation, these correspond to the case in which the relevant module is trivial.

The fundamental result here is a CP version of the universal coefficient theorem. In other words, there exists a universal cohomological object that parametrizes all central CP extensions. We show below that this object is the Bogomolov multiplier. Let us first recall its definition in more detail [21]. Given a group Q, let $Q \wedge Q$ be the group generated by the symbols $x \wedge y$, where $x, y \in Q$, subject to the relations

$$xy \wedge z = (x^y \wedge z^y)(y \wedge z), \qquad x \wedge yz = (x \wedge z)(x^z \wedge y^z), \qquad x \wedge x = 1, \qquad (3.1)$$

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where $x, y, z \in Q$. The group $Q \wedge Q$ is said to be the *non-abelian exterior square* of Q, defined by Miller [20]. There is a surjective homomorphism $Q \wedge Q \to [Q, Q]$ given by $x \wedge y \mapsto [x, y]$. Miller [20] showed that the kernel $\mathrm{M}(Q)$ of this map is naturally isomorphic to the Schur multiplier $\mathrm{H}_2(Q, \mathbb{Z})$ of Q. Finally, define $\mathrm{B}_0(Q) = \mathrm{M}(Q)/\mathrm{M}_0(Q)$, where $\mathrm{M}_0(Q) = \langle x \wedge y \mid x, y \in Q,\ [x, y] = 1\rangle$; this is the *Bogomolov multiplier*. One can therefore consider $\mathrm{B}_0(Q)$ as the kernel of the induced commutator map from the *non-abelian curly exterior square* $Q \curlywedge Q = (Q \wedge Q)/\mathrm{M}_0(Q)$ to $[Q, Q]$. It was shown in [21] that $\mathrm{H}^2_{\mathrm{nr}}(\mathbb{C}(V)^Q, \mathbb{Q}/\mathbb{Z})$ is naturally isomorphic to $\mathrm{Hom}(\mathrm{B}_0(Q), \mathbb{Q}/\mathbb{Z})$.

Theorem 3.1. *Let N be a trivial Q-module. Then there is a split exact sequence*

$$0 \longrightarrow \mathrm{Ext}(Q^{\mathrm{ab}}, N) \xrightarrow{\psi} \mathrm{H}^2_{\mathrm{CP}}(Q, N) \xrightarrow{\tilde{\varphi}} \mathrm{Hom}(\mathrm{B}_0(Q), N) \longrightarrow 0, \tag{3.2}$$

where the maps ψ and $\tilde{\varphi}$ are induced by the universal coefficient theorem.

Proof. By the universal coefficient theorem, we have a split exact sequence

$$0 \longrightarrow \mathrm{Ext}(Q^{\mathrm{ab}}, N) \xrightarrow{\psi} \mathrm{H}^2(Q, N) \xrightarrow{\varphi} \mathrm{Hom}(\mathrm{M}(Q), N) \longrightarrow 0. \tag{3.3}$$

Let $[\omega]$ belong to $\mathrm{Ext}(Q^{\mathrm{ab}}, N)$. Then [2] the map ψ can be described as $\psi([\omega]) = [\omega \circ (\mathrm{ab} \times \mathrm{ab})]$, where $\mathrm{ab}\colon Q \to Q^{\mathrm{ab}}$. If $x, y \in Q$ commute, then

$$\psi([\omega])(x, y) = \omega(x[Q, Q], y[Q, Q]) = \omega(y[Q, Q], x[Q, Q]) = \psi([\omega])(y, x),$$

and therefore ψ maps the group $\mathrm{Ext}(Q^{\mathrm{ab}}, N)$ into $\mathrm{H}^2_{\mathrm{CP}}(Q, N)$. The map φ can be described as follows. Suppose that $[\omega] \in \mathrm{H}^2(Q, N)$ represents a central extension

$$0 \longrightarrow N \longrightarrow \tilde{Q} \xrightarrow{\pi} Q \longrightarrow 1. \tag{3.4}$$

Let $z = \prod_i (x_i \wedge y_i) \in \mathrm{M}(Q)$, that is, $\prod_i [x_i, y_i] = 1$. Choose $\tilde{x}_i, \tilde{y}_i \in \tilde{Q}$ such that $\pi(\tilde{x}_i) = x_i$ and $\pi(\tilde{y}_i) = y_i$. Define $\tilde{z} = \prod_i [\tilde{x}_i, \tilde{y}_i]$. Clearly, $\tilde{z} \in N$, and it can be verified that the map φ is well defined by the rule $\varphi([\omega]) = (z \mapsto \tilde{z})$.

Suppose now that $[\omega]$ belongs to $\mathrm{H}^2_{\mathrm{CP}}(Q, N)$. Let z belong to $\mathrm{M}_0(Q)$. Then z can be written as $z = \prod_i (x_i \wedge y_i)$, where $[x_i, y_i] = 1$ for all i. Since the extension (3.4) is a central CP extension, we can choose commuting lifts $(\tilde{x}_i, \tilde{y}_i)$ of the commuting pairs (x_i, y_i). By the above definition, $\tilde{z} = 0$, and hence φ is trivial when restricted to $\mathrm{M}_0(Q)$. Thus φ induces an epimorphism $\tilde{\varphi}\colon\ \mathrm{H}^2_{\mathrm{CP}}(Q, N) \to \mathrm{Hom}(\mathrm{B}_0(Q), N)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^2(Q, N) & \xrightarrow{\varphi} & \mathrm{Hom}(\mathrm{M}(Q), N) \\ \uparrow \iota & & \uparrow \rho^* \\ \mathrm{H}^2_{\mathrm{CP}}(Q, N) & \xrightarrow{\tilde{\varphi}} & \mathrm{Hom}(\mathrm{B}_0(Q), N) \end{array}$$

Here the map ρ^* is induced by the canonical epimorphism $\rho\colon\ \mathrm{M}(Q) \to \mathrm{B}_0(Q)$. Therefore, it follows that $\ker \tilde{\varphi} = \ker \varphi|_{\mathrm{im}\,\iota} = \mathrm{im}\,\psi$. This shows that the sequence (3.2) is exact. Furthermore, the splitting of the sequence (3.3) yields that the sequence (3.2) is also split. This proves the result. □

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We offer a sample application of the above theorem. Recall that Schur's theory of covering groups originally arose in the context of projective representations (see [27]). Schur showed that there is a natural correspondence between the elements of $\mathrm{H}^2(Q,\mathbb{C}^\times)$ and projective representations of Q. To every projective representation $\rho\colon Q \to \mathrm{GL}(V)$ one can associate a cocycle $\alpha \in \mathrm{Z}^2(Q,\mathbb{C}^\times)$ via the rule $\rho(x)\rho(y) = \alpha(x,y)\rho(xy)$ for every $x,y \in Q$. Projectively equivalent representations induce cohomologous cocycles, and a cocycle is a coboundary if and only if the representation is equivalent to a linear representation. It is readily verified that CP extensions integrate well into this setting.

Proposition 3.2. *Projective representations $\rho\colon Q \to \mathrm{GL}(V)$ with the property that $[\rho(x_1),\rho(x_2)] = 1$ whenever $[x_1,x_2] = 1$ correspond to cohomological classes of CP cocycles $\alpha \in \mathrm{Z}^2(Q,\mathbb{C}^\times)$, i.e. elements of $\mathrm{B}_0(Q)$.*

In particular, if $\mathrm{B}_0(Q)$ is trivial, every projective representation of Q that preserves commutativity is similar to a linear representation.

The maps that preserve commutativity have been studied in detail in other algebraic structures; see [29] for a survey. A connection between certain commutativity preservers in groups and central CP extensions can be made along the following lines. Let $\rho\colon Q \to S$ be a set-theoretical map from Q to a group S such that $\rho(1) = 1$ and the induced map $\rho\colon Q \to S/Z(S)$ is a homomorphism. We may thus write $\rho(x)\rho(y) = \alpha(x,y)\rho(xy)$ for some function $\alpha\colon Q \times Q \to Z(S)$. In view of the associativity of multiplication, α is in fact a $Z(S)$-valued 2-cocycle. As above, such maps ρ that preserve commutativity correspond to the elements of $\mathrm{H}^2_{\mathrm{CP}}(Q, Z(S))$.

Given a group G, we define $\mathrm{K}(G) = \{[x,y] \mid x,y \in G\}$ to be the set of commutators in G. Next, we give a simple criterion for determining whether or not a given central extension is CP. This result will later be used repeatedly.

Proposition 3.3. *Let*

$$e\colon\ 1 \longrightarrow N \xrightarrow{\ \chi\ } G \xrightarrow{\ \pi\ } Q \longrightarrow 1$$

be a central extension. Then e is a CP extension if and only if $\chi(N) \cap \mathrm{K}(G) = 1$.

Proof. Define $M = \chi(N)$. Suppose that $M \cap \mathrm{K}(G) = 1$. Choose $x,y \in Q$ with $[x,y] = 1$. We have $x = \pi(g)$ and $y = \pi(h)$ for some $g,h \in G$. Then $\pi([g,h]) = 1$, and hence $[g,h] \in M \cap \mathrm{K}(G) = 1$. Thus g and h are commuting lifts of x and y, respectively.

Conversely, suppose that e is a CP central extension. Choose $[g,h] \in M \cap K(G)$. By assumption, there exists a commuting lift $(g_1,h_1) \in G \times G$ of the commuting pair $(\pi(g),\pi(h))$. We can thus write $g_1 = ga$, $h_1 = hb$, where $a,b \in M$. It follows that $1 = [g_1,h_1] = [ga,hb] = [g,h]$, and hence M is a CP subgroup of G. □

It is clear from the proof above that the implication from right to left also holds for non-central extensions. In the general case, however, the equivalence fails. For example, when Q is a cyclic group and G non-abelian, we certainly have $\chi(N) \cap \mathrm{K}(G) = \mathrm{K}(G) > 1$, and the extension is CP.

We proceed with some further characterizations of central CP extensions. We say that a normal abelian subgroup N of a group G is a *CP subgroup* of G if the

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extension

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1$$

is a CP extension. In the case in which N is central in G, proposition 3.3 implies that N is a CP subgroup if and only $N \cap \mathrm{K}(G) = 1$. The following lemma will be needed.

LEMMA 3.4. *Let N be a central CP subgroup of G. Then the sequences*

$$0 \to \mathrm{B}_0(G) \to \mathrm{B}_0(G/N) \to N \cap G' \to 0$$

and

$$N \otimes G^{\mathrm{ab}} \to \mathrm{M}_0(G) \to \mathrm{M}_0(G/N) \to 0$$

are exact.

Proof. Let G and N be given via free presentations, that is, $G = F/R$ and $N = S/R$. The fact that N is a central CP subgroup of G is then equivalent to $\langle \mathrm{K}(F) \cap S\rangle \leqslant R$. This immediately implies that $\langle \mathrm{K}(F) \cap S\rangle = \langle \mathrm{K}(F) \cap R\rangle$. With the above identifications and Hopf's formula for the Bogomolov multiplier [21] we have that $\mathrm{B}_0(G) = (F' \cap R)/\langle \mathrm{K}(F) \cap R\rangle$, $\mathrm{B}_0(G/N) = (F' \cap S)/\langle \mathrm{K}(F) \cap S\rangle$, $\mathrm{M}_0(G) = \langle \mathrm{K}(F) \cap R\rangle/[F, R]$, and $\mathrm{M}_0(G/N) = \langle \mathrm{K}(F) \cap S\rangle/[F, S]$. By [2, p. 41] there is a Ganea map $N \otimes G^{\mathrm{ab}} \to \mathrm{M}(G)$ whose image can be identified with $[F, S]/[F, R]$. As $[F, S] \leqslant \langle \mathrm{K}(F) \cap R\rangle$, the Ganea map actually maps $N \otimes G^{\mathrm{ab}}$ into $\mathrm{M}_0(G)$. The rest of the proof is now straightforward. □

PROPOSITION 3.5. *Let N be a central subgroup of a group G. The following are equivalent.*

(a) *N is a CP subgroup of G.*

(b) *The canonical map $\mathrm{M}_0(G) \to \mathrm{M}_0(G/N)$ is surjective.*

(c) *The canonical map $\varphi\colon G \curlywedge G \to G/N \curlywedge G/N$ is an isomorphism.*

Proof. Let $G = F/R$ and $N = S/R$ be free presentations of G and N. Then the image of the map $\mathrm{M}_0(G) \to \mathrm{M}_0(G/N)$ can be identified with $\langle \mathrm{K}(F) \cap R\rangle/[F, S]$. Thus the above map is surjective if and only if $\langle \mathrm{K}(F) \cap R\rangle = \langle \mathrm{K}(F) \cap S\rangle$. In particular, $\langle \mathrm{K}(F) \cap S\rangle \leqslant R$, and therefore N is a CP subgroup of G. This, together with lemma 3.4, shows that (a) and (b) are equivalent. Furthermore, from [21] it follows that $\ker \varphi = \langle x \curlywedge y \mid [x, y] \in N\rangle$. Hence φ is injective if and only if $\mathrm{K}(G) \cap N = 1$; hence (a) and (c) are equivalent. □

We now discuss comparing different extensions. Let

$$e_1\colon\ 1 \longrightarrow N_1 \xrightarrow{\chi_1} G_1 \xrightarrow{\pi_1} Q_1 \longrightarrow 1$$

and

$$e_2\colon\ 1 \longrightarrow N_2 \xrightarrow{\chi_2} G_2 \xrightarrow{\pi_2} Q_2 \longrightarrow 1$$

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be central extensions. Following [2], we say that e_1 and e_2 are *isoclinic* if there exist isomorphisms $\eta\colon Q_1 \to Q_2$ and $\xi\colon G_1' \to G_2'$ such that the diagram

$$\begin{array}{ccc} Q_1 \times Q_1 & \xrightarrow{c_1} & G_1' \\ \downarrow{\scriptstyle \eta\times\eta} & & \downarrow{\scriptstyle \xi} \\ Q_2 \times Q_2 & \xrightarrow{c_2} & G_2' \end{array}$$

commutes, where the maps c_i, $i = 1, 2$, are defined by the rules $c_i(\pi_i(x), \pi_i(y)) = [x, y]$. Note that these are well defined, since the extensions are central.

PROPOSITION 3.6. *Let e_1 and e_2 be isoclinic central extensions. If e_1 is a CP extension, then so is e_2.*

Proof. We use the same notation as above. Choose a commuting pair (x_2, y_2) of elements of Q_2. Define $x_2 = \eta(x_1)$ and $y_2 = \eta(y_1)$, where $x_1, y_1 \in Q_1$. Clearly, $[x_1, y_1] = 1$. As e_1 is a CP central extension, we can choose commuting lifts $g_1, h_1 \in G_1$ of x_1 and y_1, respectively. We can write $x_2 = \pi_2(g_2)$ and $y_2 = \pi_2(h_2)$ for some $g_2, h_2 \in G_2$. By definition, $1 = \xi([g_1, h_1]) = [g_2, h_2]$, and hence g_2 and h_2 are commuting lifts in G_2 of x_2 and y_2, respectively. □

We now show how CP extensions up to isoclinism of a given group can be obtained from an action of its Bogomolov multiplier.

LEMMA 3.7 (Moravec [21]). *Let N be a normal subgroup of a group G. Then the sequence of groups*

$$\mathrm{B}_0(G) \to \mathrm{B}_0(G/N) \to \frac{N}{\langle N \cap \mathrm{K}(G) \rangle} \to G^{\mathrm{ab}} \to (G/N)^{\mathrm{ab}} \to 0$$

with canonical maps is exact.

THEOREM 3.8. *The isoclinism classes of central CP extensions with factor group isomorphic to Q correspond to the orbits of the action of $\operatorname{Aut} Q$ on the subgroups of $\mathrm{B}_0(Q)$ given by $(\varphi, U) \mapsto \mathrm{B}_0(\varphi)U$, where $\varphi \in \operatorname{Aut} Q$ and $U \leqslant \mathrm{B}_0(Q)$.*

Proof. Let

$$e\colon\ 1 \longrightarrow N \xrightarrow{\chi} G \xrightarrow{\pi} Q \longrightarrow 1$$

be a central CP extension. As $\chi(N) \cap K(G) = 1$, it follows from [2] and lemma 3.7 that we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \mathrm{B}_0(G) & \xrightarrow{\mathrm{B}_0(\pi)} & \mathrm{B}_0(Q) & \xrightarrow{\tilde{\theta}(e)} & N & \xrightarrow{\tau} & G^{\mathrm{ab}} & \longrightarrow & Q^{\mathrm{ab}} & \longrightarrow & 0 \\ & & \twoheaduparrow & & \twoheaduparrow & & \| & & \| & & \| & & \\ & & \mathrm{M}(G) & \xrightarrow{\mathrm{M}(\pi)} & \mathrm{M}(Q) & \xrightarrow{\theta_*(e)} & N & \xrightarrow{\tau} & G^{\mathrm{ab}} & \longrightarrow & Q^{\mathrm{ab}} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & & & & & & & \\ & & \mathrm{M}_0(G) & \longrightarrow & \mathrm{M}_0(Q) & & & & & & & & \end{array}$$

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By the exactness we have that the image of $\chi\tilde{\theta}(e)$ is equal to $\chi(N) \cap G'$, which is equal to the image of $\chi\theta_*(e)$. Since χ is injective, it follows that $\tilde{\theta}(e)$ and $\theta_*(e)$ have the same image. Furthermore, we claim that $\ker \tilde{\theta}(e) = \ker \theta_*(e) / \mathrm{M}_0(Q)$. To this end, consider free presentations $G = F/R$, $N = S/R$ and $Q = F/S$. Since the extension e is CP, it follows that $\langle \mathrm{K}(F) \cap S \rangle \leqslant R$. With the above identifications we have that $\ker \theta_*(e) = (F' \cap R)/[F, S]$ and $\ker \tilde{\theta}(e) = (F' \cap R)/\langle \mathrm{K}(F) \cap S \rangle$. As $\mathrm{M}_0(Q) = \langle \mathrm{K}(F) \cap S \rangle / [F, S]$, the equality follows.

Now let

$$e_i\colon\ 1 \longrightarrow N_i \xrightarrow{\chi_i} G_i \xrightarrow{\pi_i} Q_i \longrightarrow 1 \quad (i = 1, 2),$$

by central CP extensions, and let $\eta\colon Q_1 \to Q_2$ be an isomorphism of groups. By [2, proposition III.2.3] we have that η induces isoclinism between e_1 and e_2 if and only if $\mathrm{M}(\eta) \ker \theta_*(e_1) = \ker \theta_*(e_2)$. By the above, this is equivalent to $\mathrm{B}_0(\eta) \ker \tilde{\theta}(e_1) = \ker \tilde{\theta}(e_2)$. The proof of [2, proposition III.2.6] can now be suitably modified to obtain the result; we skip the details. □

4. Maximal CP extensions

In this section, we deal with studying maximal central CP extensions of a given group. Maximal here refers to the size of the kernel in a suitable representative extension under isoclinism. Recall that an extension

$$1 \longrightarrow N \xrightarrow{\chi} G \longrightarrow Q \longrightarrow 1$$

is termed to be *stem* whenever $\chi(N) \leqslant [G, G]$. The motivation comes from the following lemma.

Lemma 4.1. *Every central CP extension is isoclinic to a stem central CP extension.*

Proof. The argument follows along the lines of [2, proposition III.2.6]. Let

$$e\colon\ 1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be a central CP extension. Put $U = \ker \tilde{\theta}(e)$, where $\tilde{\theta}(e)$ is the homomorphism $\mathrm{B}_0(Q) \to N$ from the five-term exact sequence in lemma 3.7. The subgroup U of $\mathrm{B}_0(Q)$ determines a central CP extension $\bar{e}$ of $\mathrm{B}_0(Q)/U$ by Q via theorem 3.1 applied to the epimorphism $\mathrm{B}_0(Q) \to \mathrm{B}_0(Q)/U$. Thus $\tilde{\theta}(\bar{e})$ corresponds to the natural projection $\mathrm{B}_0(Q) \to \mathrm{B}_0(Q)/U$. Note that $\bar{e}$ is a stem central CP extension isoclinic to e (see the proof of theorem 3.8). The kernel of the extension $\bar{e}$ is precisely $\mathrm{B}_0(Q)/U \cong \mathrm{im}\, \tilde{\theta}(e) \cong \ker(N \to G/[G, G]) = N \cap [G, G]$. □

Up to isoclinism of extensions, it therefore suffices to consider stem central CP extensions.

Given a group Q, any stem central CP extension of a group N by Q with $|N| = |\mathrm{B}_0(Q)|$ is called a *CP cover* of Q. The following theorem justifies the terminology.

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THEOREM 4.2. *Let Q be a finite group given via a free presentation $Q = F/R$. Set $H = F/\langle \mathrm{K}(F) \cap R\rangle$ and $A = R/\langle K(F) \cap R\rangle$.*

(1) *A is a finitely generated central subgroup of H and its torsion subgroup is $T(A) = ([F,F] \cap R)/\langle K(F) \cap R\rangle \cong \mathrm{B}_0(Q)$.*

(2) *Let C be a complement to $T(A)$ in A. Then H/C is a CP cover of Q.*

(3) *Let G be a stem central CP extension of a group N by Q. Then G is a homomorphic image of H and in particular N is a homomorphic image of $\mathrm{B}_0(Q)$.*

(4) *Let G be a CP cover of Q with kernel N. Then $N \cong \mathrm{B}_0(Q)$ and G is isomorphic to a quotient of H by a complement of $T(A)$ in A.*

(5) *CP covers of Q are precisely the stem central CP extensions of Q of maximal order.*

(6) *CP covers of Q are represented by the cocycles $\tilde{\varphi}^{-1}(1_{\mathrm{B}_0(Q)})$ in $\mathrm{H}^2(Q, \mathrm{B}_0(Q))$, where $\tilde{\varphi}$ is the mapping induced by the universal coefficients theorem 3.1.*

Proof. This all follows from the arguments in [10, Hauptsatz V.23.5] in combination with the Hopf formula for the Bogomolov multiplier from [21]. □

Using theorem 4.2, a fast algorithm for computing the Schur covering groups as developed in [24] may be combined with an algorithm for determining $F/\langle \mathrm{K}(F) \cap R\rangle$ from [13] to effectively determine the CP covers of a given group. It is straightforward to combine the two implementations in GAP [28].

COROLLARY 4.3. *The number of CP covers of a group Q is at most $|\mathrm{Ext}(Q^{\mathrm{ab}}, \mathrm{B}_0(Q))|$. In particular, perfect groups have a unique CP cover.*

EXAMPLE 4.4. Let Q be a 4- or 12-cover of $\mathrm{PSL}(3,4)$. The group Q is a quasi-simple group and it is shown in [18] that $\mathrm{B}_0(Q) \cong C_2$, so Q has a unique proper CP cover.

We stress an important difference between Schur covering groups and CP covers, indicating a more intimate connection of the latter with the theory of (universal) covering spaces from algebraic topology [9].

THEOREM 4.5. *The Bogomolov multiplier of a CP cover is trivial.*

Proof. Let G be a CP cover of Q with kernel $N \cong \mathrm{B}_0(Q)$ satisfying $N \leqslant Z(G) \cap [G,G]$ and $N \cap \mathrm{K}(G) = 1$. Consider a CP cover $H \xrightarrow{\pi} G$ with kernel $M \cong \mathrm{B}_0(G)$ satisfying $M \leqslant Z(H) \cap [H,H]$ and $M \cap \mathrm{K}(H) = 1$. The group H is a central extension of $L = \pi^{-1}(N)$ by Q, since π preserves commutativity. Moreover, we have $L \leqslant \pi^{-1}([G,G]) = [H,H]$ since $M \leqslant [H,H]$, and $L \cap \mathrm{K}(H) \leqslant \pi^{-1}(N \cap \mathrm{K}(G)) \cap \mathrm{K}(H) \leqslant M \cap \mathrm{K}(H) = 1$. We conclude that H is a stem central CP extension of L by Q; therefore $|L| \leqslant |\mathrm{B}_0(Q)|$ by theorem 4.2, and so $L \cong \mathrm{B}_0(Q)$. This implies that $M = \mathrm{B}_0(G) = 1$, as required. □

Note that a similar proof gives that the Bogomolov multiplier of a Schur covering group is also trivial; see [8, lemma 2.4.1].

For further use of theorem 4.5, we record a straightforward corollary of lemma 3.4.

Lemma 4.6. *Whenever N is a central CP subgroup of a group G with $\mathrm{B}_0(G) = 0$, we have $\mathrm{B}_0(G/N) \cong N \cap [G, G]$. If in addition $N \leqslant [G, G]$, then the group G is a CP cover of G/N with kernel $N \cong \mathrm{B}_0(G/N)$.*

It follows readily that central CP extensions behave much as topological covering spaces.

Corollary 4.7. *Let Q be a group and let G be a CP cover of Q. For every filtration of subgroups $1 = N_0 \leqslant N_1 \leqslant \cdots \leqslant N_n = \mathrm{B}_0(Q)$ there is a corresponding sequence of groups $G_i = G/N_i$, where G_i is a central CP extension of G_j with kernel $N_j/N_i \cong \mathrm{B}_0(G_j)/\mathrm{B}_0(G_i)$ whenever $i \leqslant j$.*

We now explore CP covers with respect to isoclinism. At first we list some auxiliary results.

Lemma 4.8. *Let*

$$e\colon\ 1 \longrightarrow N \xrightarrow{\ \chi\ } G \xrightarrow{\ \pi\ } Q \longrightarrow 1$$

be a central CP extension. Then $\pi(Z(G)) = Z(Q)$ and $Z(G) \cong N \times Z(Q)$.

Proof. It is straightforward to see that if

$$e_i\colon\ 1 \longrightarrow N \xrightarrow{\ \chi_i\ } G_i \xrightarrow{\ \pi_i\ } Q \longrightarrow 1$$

are equivalent central extensions for $i = 1, 2$, then $\pi_1(Z(G_1)) = \pi_2(Z(G_2))$. Thus we may replace the extension e by the extension

$$1 \longrightarrow N \longrightarrow G[\omega] \xrightarrow{\ \varepsilon\ } Q \longrightarrow 1,$$

which is obtained similarly to as in the proof of proposition 2.4. As $\omega \in \mathrm{Z}^2_{\mathrm{CP}}(Q, N)$, the condition that $(n, q) \in Z(G[\omega])$ is equivalent to $q \in Z(Q)$. Hence, $\varepsilon(Z(G[\omega])) = Z(Q)$. □

Lemma 4.9. *Let G be a CP cover of Q. Then $Z(G) \cong Z(Q) \times \mathrm{B}_0(Q)$, and G is stem if and only if Q is stem.*

Proof. The first part follows from lemma 4.8. The second part then follows from the first and the fact that $\mathrm{B}_0(Q) \leqslant [G, G]$. □

It follows from the latter lemma that the central quotient of a CP cover is naturally isomorphic to the central quotient of the base group, and so the nilpotency class of a CP cover does not exceed that of the base group. This is all a special case of the following observation.

Proposition 4.10. *CP covers of isoclinic groups are isoclinic.*

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Proof. Let G_1 be a CP cover of a group Q_1 with the covering projection $p_1\colon G_1 \to Q_1$ and let Q_2 be isoclinic to Q_1 via the compatible pair of isomorphisms

$$\alpha\colon Q_2/Z(Q_2) \to Q_1/Z(Q_1) \quad \text{and} \quad \beta\colon [Q_2, Q_2] \to [Q_1, Q_1].$$

Let G_2 be a CP cover of Q_2 with the covering projection $p_2\colon G_2 \to Q_2$. We show that G_2 is isoclinic to G_1. To this end, let $\bar{p}_i\colon G_i/Z(G_i) \to Q_i/Z(Q_i)$ be the natural homomorphisms induced by p_is. Lemma 4.1 implies that $\bar{p}_i$ is in fact an isomorphism. Define $\tilde{\alpha}\colon G_2/Z(G_2) \to G_1/Z(G_1)$ as $\tilde{\alpha} = (\bar{p}_1)^{-1}\alpha\bar{p}_2$. This is clearly an isomorphism. Next, observe that theorem 4.2 shows that the covering projections p_i also induce isomorphisms $p_i \curlywedge p_i\colon [G_i, G_i] \to Q_i \curlywedge Q_i$ defined as $[x, y] \mapsto p_i(x) \curlywedge p_i(y)$. Furthermore, it was shown in [22] that α induces an isomorphism $\alpha^{\curlywedge}\colon Q_2 \curlywedge Q_2 \to Q_1 \curlywedge Q_1$ via $\alpha^{\curlywedge}(x_1 \curlywedge x_2) = y_1 \curlywedge y_2$, where $y_i Z(Q_1) = \alpha(x_i Z(Q_2))$. Now define $\tilde{\beta}\colon [G_2, G_2] \to [G_1, G_1]$ as $\tilde{\beta} = (p_1 \curlywedge p_1)^{-1}\alpha^{\curlywedge}(p_2 \curlywedge p_2)$. This is clearly an isomorphism, and it readily follows from the compatibility relations between α and β that the isomorphisms $\tilde{\alpha}$ and $\tilde{\beta}$ are also compatible. These induce an isoclinism between the CP covers G_1 and G_2. □

As a corollary, the derived subgroup of a CP cover is uniquely determined. Note that given a group Q and its CP cover G, we have $[G, G] \cong Q \curlywedge Q$ by theorem 4.2. In particular, groups belonging to the same isoclinism family have naturally isomorphic curly exterior squares, and therefore also Bogomolov multipliers.

Let Φ be an isoclinism family of finite groups, referred to as the *base family*, and let G be an arbitrary group in Φ. By proposition 4.10, CP covers of G all belong to the same isoclinism family. We denote this family by $\tilde{\Phi}$ and call it the *covering family* of Φ.

PROPOSITION 4.11. *Every group in a covering family is a CP cover of a group in the base family.*

Proof. Let G_1 be a CP cover of a group Q_1 with the covering projection $p_1\colon G_1 \to Q_1$ and let G_2 be isoclinic to G_1 via the compatible pair of isomorphisms

$$\alpha\colon G_2/Z(G_2) \to G_1/Z(G_1) \quad \text{and} \quad \beta\colon [G_2, G_2] \to [G_1, G_1].$$

By theorem 4.5, we have $\mathrm{B}_0(G_1) = 0$, and so $\mathrm{B}_0(G_2) = 0$ by proposition 4.10. The commutator homomorphism $\kappa_i\colon G_i \curlywedge G_i \to [G_i, G_i]$ is therefore an isomorphism, and we implicitly identify the two groups. Consider the group $N = \beta^{-1}\,\mathrm{B}_0(Q_1) \leqslant [G_2, G_2]$. Note that N is central in G_2. Furthermore, whenever $[x_1, x_2] \in N$ for some $x_1, x_2 \in G_2$ with $\alpha(x_i Z(G_2)) = y_i Z(G_1)$, we have $[y_1, y_2] = \beta([x_1, x_2]) \in \mathrm{B}_0(Q_1)$, and so $[x_1, x_2] = \beta^{-1}([y_1, y_2]) = 1$ since the covering projection $G_1 \to Q_1$ is commutativity preserving. Now put $Q_2 = G_2/N$. By lemma 4.6, the group G_2 is a CP cover of Q_2 with kernel $N \cong \mathrm{B}_0(Q_2)$. Finally, it is straightforward that the isomorphisms α and β naturally induce an isoclinism between the groups $Q_2 = G_2/\beta^{-1}(\mathrm{B}_0(Q_1))$ and $G_1/\mathrm{B}_0(Q_1) \cong Q_1$. □

Note that lemma 4.9 now implies that CP covers of the stem of the base family form the stem of the covering family.

The following examples show that a given isoclinism family can be a covering family for more than one base family. Moreover, a group in a covering family can be a CP cover of non-isomorphic groups belonging to the same base family.

Example 4.12. Consider the isoclinism family that contains groups of smallest possible order having non-trivial Bogomolov multipliers [6]. This is the family Φ_{16} of [11]. Its stem groups are of order 64, and its covering family $\tilde{\Phi}_{16}$ is precisely the isoclinism family Φ_{36} of [11], whose stem groups are of order 128.

Example 4.13. Let G be a Schur covering group of the abelian group C_4^4, generated by g_1, g_2, g_3, g_4. Put $w = [g_1, g_2][g_3, g_4]$ and set $G_1 = G/\langle w\rangle$, $G_2 = G/\langle w^2\rangle$. It is readily verified that neither w nor w^2 is a commutator in G. Since $\mathrm{B}_0(G) = 0$, it follows that G is a CP cover of both G_1 and of G_2. Applying lemma 3.7 gives $\mathrm{B}_0(G_1) \cong C_4$ and $\mathrm{B}_0(G_2) \cong C_2$, so G_1 and G_2 do not belong to the same isoclinism family.

Example 4.14. Let Q be a stem group in the family Φ_{30} of [11] and let G be a CP cover of Q. It is shown in [13] that $\mathrm{B}_0(Q) = \langle w_1\rangle \times \langle w_2\rangle \cong C_2 \times C_2$ for some $w_1, w_2 \in G$. Set $G_1 = G/\langle w_1\rangle$ and $G_2 = G/\langle w_2\rangle$. The groups G_1 and G_2 are isoclinic and non-isomorphic groups of order 256, and G is a CP cover of both of them. Is can be verified using the algorithm for computing CP covers that the groups G_1, G_2 in fact have exactly two non-isomorphic CP covers in common.

It is well known that Schur covering groups of a given group are all isoclinic; see, for example, [10, Satz V.23.6]. Neither proposition 4.10 nor proposition 4.11, however, have a counterpart in the theory of Schur covering groups, as the following simple example shows.

Example 4.15. Let Φ be the isoclinism family of all finite abelian groups. We plainly have $\tilde{\Phi} = \Phi$. Let p be an arbitrary prime. The Schur cover of C_{p^2} is C_{p^2}, and the Schur cover of $C_p \times C_p$ is isomorphic to the unitriangular group $\mathrm{UT}_3(p)$. The two covers are not isoclinic. Note also that the group $C_p \times C_p$ is not a Schur covering group of any group.

5. Minimal CP extensions

In this section, we focus on central CP extensions of a cyclic group of prime order by some given group Q. We call such extensions *minimal* CP extensions. By corollary 4.7, every central CP extension is built from a sequence of such minimal extensions. As in the classical theory of central extensions, this corresponds to considering $\mathbb{F}_p$-cohomology. We thus set $\mathrm{H}^2_{\mathrm{CP}}(Q) = \mathrm{H}^2_{\mathrm{CP}}(Q, \mathbb{F}_p)$, the action of Q on $\mathbb{F}_p$ being trivial. Relying on theorem 4.2, the heart of the matter here is relating a given presentation of Q with the object $\mathrm{H}^2_{\mathrm{CP}}(Q)$. The following result is obtained.

Theorem 5.1. *The group* $\mathrm{H}^2_{\mathrm{CP}}(Q)$ *is elementary abelian of rank* $\mathrm{d}(Q) + \mathrm{d}(\mathrm{B}_0(Q))$.

Proof. Let $Q = F/R$ be a presentation of Q. Consider first the canonical central CP extension $H = F/\langle \mathrm{K}(F) \cap R\rangle$ of Q. The kernel of this extension is the group $A = R/\langle \mathrm{K}(F) \cap R\rangle$.

We first claim that $\mathrm{H}^2_{\mathrm{CP}}(H) = 0$. By lemma 3.7, we have $\mathrm{B}_0(H) = 0$, and it then follows from theorem 3.1 that $\mathrm{H}^2_{\mathrm{CP}}(H) = \mathrm{Ext}(H^{\mathrm{ab}}, \mathbb{F}_p) = 0$.

Next we show that the minimal CP extensions are precisely the kernel of the inflation map from Q to H:

$$\mathrm{H}^2_{\mathrm{CP}}(Q) = \ker(\inf_Q^H \colon \mathrm{H}^2(Q) \to \mathrm{H}^2(H)).$$

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Indeed, it follows from the above claim that $\mathrm{H}^2_{\mathrm{CP}}(Q) \leqslant \ker \inf^H_Q$. Conversely, let $\omega \in \ker \inf^H_Q$. Hence there is a function $\phi\colon H \to \mathbb{F}_p$ such that $\inf^H_Q(\omega)(x_1, x_2) = \phi(x_1) + \phi(x_2) - \phi(x_1 x_2)$. Pick any commuting pair $u, v \in Q$. Then there exists a commuting lift $\tilde{u}, \tilde{v} \in H$ of these elements. Therefore, $\omega(u, v) = \inf^H_Q(\omega)(\tilde{u}, \tilde{v}) = \inf^H_Q(\omega)(\tilde{v}, \tilde{u}) = \omega(v, u)$, and so $\omega \in \mathrm{H}^2_{\mathrm{CP}}(Q)$.

Let us now restrict ourselves to choosing the presentation $Q = F/R$ to be minimal in the sense that $\mathrm{d}(Q) = \mathrm{d}(F)$. In this case, we invoke the inflation-restriction cohomological exact sequence for the surjection $H \to Q$ with kernel A. Together with the above, it immediately follows that $\mathrm{H}^2_{\mathrm{CP}}(Q) \cong \mathrm{Hom}(A, \mathbb{F}_p)$. Finally, we have by theorem 4.2 that the torsion $T(A) \cong \mathrm{B}_0(Q)$ in A has a free complement of rank $\mathrm{d}(F) = \mathrm{d}(Q)$. The proof is complete. □

We present a corollary of the above proof.

Corollary 5.2. *Let $Q = F/R$ be a presentation with $\mathrm{d}(Q) = \mathrm{d}(F)$. Let $\mathrm{r}(F, R)$ be the minimal number of relators in R that generate R as a normal subgroup of F, and let $\mathrm{r}_{\mathrm{K}}(F, R)$ be the number of relators among these that belong to $\mathrm{K}(F)$. Then $\mathrm{d}(\mathrm{B}_0(Q)) \leqslant \mathrm{r}(F, R) - \mathrm{r}_{\mathrm{K}}(F, R) - \mathrm{d}(Q)$.*

Proof. Going back to the proof of theorem 5.1, it is clear that $\mathrm{rank}\, A \leqslant \mathrm{r}(F, R) - \mathrm{r}_{\mathrm{K}}(F, R)$. The claim follows immediately. □

The corollary may be applied to show that the Bogomolov multiplier of a group is trivial. This works with classes of groups that may be given by a presentation with many simple commutators among relators. As an example, the group of unitriangular matrices $\mathrm{UT}_n(p)$ has a presentation in which all relators are commutators (see [3]), whence immediately $\mathrm{B}_0(\mathrm{UT}_n(p)) = 0$. The same holds for lower central quotients of $\mathrm{UT}_n(p)$. This was already proved in [19]; see also [12]. Another example is the braid group B_n with $n - 1$ generators and $n - 2$ braid relators that are not commutators, thereby again $\mathrm{B}_0(B_n) = 0$.

6. Bounds for $\mathbf{B_0}$

Using the theory of CP covers, we now show how one can produce bounds on the number of isoclinism classes of central CP extensions in terms of the internal structure of the given group. Equivalently, we bound the size of the Bogomolov multiplier. The first result is an adaptation of the argument from [15].

Proposition 6.1. *Let Q be a finite group and let S be a normal subgroup such that Q/S is cyclic. Then $|\mathrm{B}_0(Q)|$ divides $|\mathrm{B}_0(S)| \cdot |S^{\mathrm{ab}}|$, and $\mathrm{d}(\mathrm{B}_0(Q)) \leqslant \mathrm{d}(\mathrm{B}_0(S)) + \mathrm{d}(S^{\mathrm{ab}})$.*

Proof. Let G be a CP cover of Q. Thus G contains a subgroup $N \leqslant [G, G] \cap Z(G)$ such that $G/N \cong Q$ and $N \cong \mathrm{B}_0(Q)$. Choose X in G such that $X/N \cong S$. We may write $G = \langle u, X\rangle$ for some u. There is thus an epimorphism $\theta\colon X \to [G, G]/[X, X]$ given by $\theta(x) = [u, x][X, X]$. Therefore, $|\mathrm{B}_0(Q)| = |N| = |N/(N \cap [X, X])| \cdot |N \cap [X, X]|$. Now, since $NX' \leqslant \ker \theta$, it follows that $|N/(N \cap [X, X])| \leqslant |[G, G]/[X, X]| \leqslant |X/N[X, X]| = |S^{\mathrm{ab}}|$. Observe that the CP covering extension G of Q induces a central CP extension X of S with kernel N. Whence, by lemma 4.1, we have that $N \cap [X,X]$ is the kernel of the associated stem extension. It now follows from theorem 4.2 that $|N \cap [X,X]| \leqslant |\mathrm{B}_0(Q)|$. This completes the proof of the first claim. For the second one, we similarly have $\mathrm{d}(\mathrm{B}_0(Q)) = \mathrm{d}(N) \leqslant \mathrm{d}(N/(N \cap [X,X])) + \mathrm{d}(N \cap [X,X])$. The result follows from $\mathrm{d}(N/(N \cap [X,X])) \leqslant \mathrm{d}([G,G]/[X,X]) \leqslant \mathrm{d}(S^{\mathrm{ab}})$. □

Next, we also provide a bound for the exponent. This is an analogy of [16].

PROPOSITION 6.2. *Let Q be a finite group and let S be a subgroup. Then $\mathrm{B}_0(Q)^{|Q:S|}$ embeds into $\mathrm{B}_0(S)$.*

Proof. Let G be a CP cover of Q. Again, G contains a subgroup $N \leqslant [G,G] \cap Z(G)$ such that $G/N \cong Q$ and $N \cong \mathrm{B}_0(Q)$. Choose X in G such that $X/N \cong S$. Consider the transfer map $\theta\colon G \to X/[X,X]$. Since N is central in G, we have $\theta(n) = n^{|Q:S|}[X,X]$ for all $n \in N$. But as $N \leqslant [G,G]$, we must also have that $N \leqslant \ker\theta$. Therefore, $N^{|Q:S|} \leqslant N \cap [X,X]$. As in the proof of the previous proposition, we have that $N \cap [X,X]$ embeds into $\mathrm{B}_0(S)$. This completes the proof. □

These results may be applied in various ways, depending on the structural properties of the group in question, to provide some absolute bounds on the order, rank or exponent of the Bogomolov multiplier. As an example, consider a p-group Q that has a maximal subgroup M with $\mathrm{B}_0(M) = 0$. The above propositions imply that for such groups, $\mathrm{B}_0(Q)$ is elementary abelian of rank at most $\mathrm{d}(M)$. Such groups include B_0-minimal groups, the building blocks of groups with non-trivial Bogomolov multipliers, and were inspected to some extent in [13]. It was shown that every B_0-minimal group can be generated by at most 4 elements. Schreier's index formula $\mathrm{d}(M) - 1 \leqslant |Q:M|(\mathrm{d}(Q) - 1)$ (see [26, theorem 6.1.8]) then gives an absolute upper bound on the number of generators of a maximal subgroup M. Whence, we have the following corollary.

COROLLARY 6.3. *The Bogomolov multiplier of a B_0-minimal p-group is an elementary abelian group of rank at most $3p+1$.*

Another direct application is to consider any abelian subgroup A of a given group Q. Since $\mathrm{B}_0(A) = 0$, we have the following.

COROLLARY 6.4. *Let Q be a finite group and let A be an abelian subgroup. Then $\exp \mathrm{B}_0(Q)$ divides $|Q:A|$.*

7. Commuting probability

A probabilistic approach to the study of Bogomolov multipliers was been undertaken in [14], where the impact of the commuting probability on the Bogomolov multiplier was explored. Here the *commuting probability* $\mathrm{cp}(G)$ of a finite group G is defined to be the probability that two randomly chosen elements of G commute, and is equal to $\mathrm{cp}(G) = |\{(x,y) \in G \times G \mid [x,y] = 1\}|/|G|^2$. It turns out that CP extensions provide a natural setting for both commuting probability and Bogomolov multipliers. This is based on the following observation.

PROPOSITION 7.1. *An extension $N \longrightarrow G \xrightarrow{\pi} Q$ is a central CP extension if and only if $\mathrm{cp}(G) = \mathrm{cp}(Q)$.*

Proof. Consider the homomorphism $\pi^2\colon G\times G\to Q\times Q$. Note that commuting pairs in G map to commuting pairs in Q, and hence

$$(\pi^2)^{-1}(\{(x,y)\in Q\times Q \mid [x,y]=1\}) \supseteq \{(x,y)\in G\times G \mid [x,y]=1\}. \qquad (7.1)$$

The containment (7.1) is an equality if and only if the extension is CP and N is a central subgroup of G. On the other hand, notice that the fibres of π^2 are of order $|N|^2$, and therefore $\mathrm{cp}(G)=|\{(x,y)\in G\times G \mid [x,y]=1\}|/|G|^2 \leqslant |N|^2|\{(x,y)\in Q\times Q \mid [x,y]=1\}|/|G|^2=\mathrm{cp}(Q)$ with equality precisely when (7.1) is an equality. This completes the proof. □

Remark 7.2. Consider a central extension $\langle z\rangle \longrightarrow G \xrightarrow{\pi} Q$. It follows from the above proof that this extension is a CP extension if and only if all conjugacy classes of Q lift with respect to π to exactly p different conjugacy classes in G.

The study of central CP extensions is thus equivalent to the study of extensions that preserve commuting probability. This may be exploited in providing a connection between the Bogomolov multiplier and commuting probability based on CP extensions. We give a simple example illustrating this.

Corollary 7.3. *For every number p in the range of the commuting probability function, there exists a group G with $\mathrm{cp}(G)=p$ and $\mathrm{B}_0(G)=0$.*

Proof. Let Q be an arbitrary group with $\mathrm{cp}(Q)=p$, and let G be a CP cover of Q. Then $\mathrm{cp}(G)=p$ by proposition 7.1 and $\mathrm{B}_0(G)=0$ by theorem 4.5. □

Another way to look at this relation is on the level of isoclinism families. As a direct consequence of corollary 4.7, we have that for every isoclinism family Φ and every subgroup N of $\mathrm{B}_0(\Phi)$, there is a family Φ' with $\mathrm{cp}(\Phi')=\mathrm{cp}(\Phi)$ and $\mathrm{B}_0(\Phi')=N$.

Example 7.4. Consider the isoclinism family Φ_{16} as given in example 4.12. We have $\mathrm{cp}(\Phi_{16})=\mathrm{cp}(\Phi_{36})=1/4$, while $\mathrm{B}_0(\Phi_{16})\cong C_2$ and $\mathrm{B}_0(\Phi_{36})=0$.

This connection also sheds new light on the results of [14]. There, we observed the structure of the Bogomolov multiplier while fixing a large commuting probability. First of all, those results can be applied in the context of CP extensions.

Corollary 7.5. *Let Q be a finite group with $\mathrm{cp}(Q)>1/4$. Then every central CP extension of Q is isoclinic to an extension with a trivial kernel.*

Proof. The Bogomolov multiplier of Q is trivial by [14, corollary 1.2]. Every central CP extension of Q is isoclinic to a stem extension by lemma 4.1, and the kernel of the latter extension must be trivial by theorem 4.2. □

Secondly, the bounds for the Bogomolov multiplier from § 6 can be applied in the setting of commuting probability. This is, in a way, a non-absolute version of the main result of [14].

Theorem 7.6. *Let $\varepsilon>0$, and let Q be a group with $\mathrm{cp}(Q)>\varepsilon$. Then $|\mathrm{B}_0(Q)|$ can be bounded in terms of a function of ε and $\max\{\mathrm{d}(S) \mid S$ a Sylow subgroup of $Q\}$. Moreover, $\exp \mathrm{B}_0(Q)$ can be bounded in terms of a function of ε.*

Proof. Since the p-part of $\mathrm{B}_0(Q)$ embeds into the Bogomolov multiplier of a p-Sylow subgroup of Q, we are immediately reduced to considering only p-groups. It follows from [7, 23] that Q has a subgroup K of nilpotency class 2 with $|Q : K|$ and $|[K,K]|$ both bounded by a function of ε. Applying proposition 6.1 repeatedly on a sequence of subgroups from Q to K, each of index p in the previous one, it follows that $\mathrm{d}(\mathrm{B}_0(Q))$ can be bounded in terms of ε and $\mathrm{d}(\mathrm{B}_0(K))$. Now, $\mathrm{d}(\mathrm{B}_0(K)) \leqslant \mathrm{d}(M(K))$, and we can use the Ganea map $[K,K] \otimes K/[K,K] \to M(K)$, whose cokernel embeds into $M(K/[K,K])$. Note that $\mathrm{d}([K,K] \otimes K/[K,K]) \leqslant \mathrm{d}(K)^2$ and $\mathrm{d}(M(K/[K,K])) \leqslant \binom{\mathrm{d}(K)}{2}$. Whence we obtain a bound for $\mathrm{d}(\mathrm{B}_0(Q))$ in terms of ε and $\mathrm{d}(Q)$. For the exponent, use proposition 6.2 to bound $\exp \mathrm{B}_0(Q)$ by a function of $|Q : K|$ and $\exp \mathrm{B}_0(K)$. If K is abelian, then we are done. If not, then choose a commutator z in K. Set $J_z = \langle x \curlywedge y \mid [x,y] = z \rangle \leqslant Q \curlywedge Q$, and denote by X the kernel of the map $\mathrm{B}_0(K) \to \mathrm{B}_0(K/\langle z \rangle)$. Then it follows from [14] that there is a commutative diagram as follows:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & X & \longrightarrow & \mathrm{B}_0(K) & \longrightarrow & \mathrm{B}_0(K/\langle z \rangle) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & J_z & \longrightarrow & K \curlywedge K & \longrightarrow & K/\langle z \rangle \curlywedge K/\langle z \rangle & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \langle z \rangle & \longrightarrow & [K,K] & \longrightarrow & [K,K]/\langle z \rangle & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

Observe that $\exp J_z = p$, and so $\exp X = p$. It then follows that $\exp \mathrm{B}_0(K)$ is at most $p \cdot \exp \mathrm{B}_0(K/\langle z \rangle)$. Repeating this process with $K/\langle z \rangle$ instead of z until we reach an abelian group, we conclude that $\exp \mathrm{B}_0(K)$ divides $|[K,K]|$. The latter is bounded in terms of ε alone. The proof is now complete. □

We end with an intriguing corollary concerning the exponent of the Schur multiplier.

Corollary 7.7. *Given $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that for every group Q with $\mathrm{cp}(Q) > \varepsilon$ we have $\exp M(Q) \leqslant C \cdot \exp Q$.*

Proof. We have that $\exp M(Q) \leqslant \exp \mathrm{B}_0(Q) \cdot \exp M_0(Q)$ and $\exp M_0(Q) \leqslant \exp Q$. Now apply theorem 7.6. □

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