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Error bounds and exponential improvements for the asymptotic expansions of the gamma function and its reciprocal

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In 1994 Boyd derived a resurgence representation for the gamma function, exploiting the 1991 reformulation of the method of steepest descents by Berry and Howls. Using this representation, he was able to derive a number of properties of the asymptotic expansion for the gamma function, including explicit and realistic error bounds, the smooth transition of the Stokes discontinuities and asymptotics for the late coefficients. The main aim of this paper is to modify Boyd's resurgence formula, making it suitable for deriving better error estimates for the asymptotic expansions of the gamma function and its reciprocal. We also prove the exponentially improved versions of these expansions complete with error terms. Finally, we provide new (formal) asymptotic expansions for the coefficients appearing in the asymptotic series and compare their numerical efficacy with the results of earlier authors.

1. Introduction and main results

It is well known that, as $z \to \infty$ in the sector $|\arg z| \leqslant \pi - \delta < \pi$ for any $0 < \delta \leqslant \pi$, the gamma function and its reciprocal have the following asymptotic expansions:

$$\Gamma(z) \sim \sqrt{2\pi} z^{z-1/2} \mathrm{e}^{-z} \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_n}{z^n}, \tag{1.1}$$

$$\frac{1}{\Gamma(z)} \sim \frac{1}{\sqrt{2\pi}} z^{-z+1/2} \mathrm{e}^{z} \sum_{n=0}^{\infty} \frac{\gamma_n}{z^n}, \tag{1.2}$$

respectively. Here the γ_n are the so-called Stirling coefficients, the first few being $\gamma_0 = 1$ and

$$\gamma_1 = -\tfrac{1}{12}, \quad \gamma_2 = \tfrac{1}{288}, \quad \gamma_3 = \tfrac{139}{51\,840}, \quad \gamma_4 = -\tfrac{571}{2\,488\,320}.$$

For a detailed discussion of the computation of these coefficients, see the appendix. The first proof of the expansion (1.1) for $z > 0$ dates back to Laplace (see [10, p. 2]). Since the 20th century, these expansions have become standard textbook examples to illustrate various techniques, such as the method of Laplace itself or the method of steepest descents (see, for example, [10, pp. 53–58, 70–72], [25, pp. 24–28] and [33, pp. 60–62, 110–111]).

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Error bounds for the expansion (1.1) were derived by Olver [19, 20], although the application of these bounds requires the computation of extreme values of certain implicitly defined functions. It was not, however, until the end of the 20th century that simple, explicit error bounds for the asymptotic series (1.1) were found. Define for any $N \geqslant 1$ the remainder $R_N(z)$ by

$$\Gamma(z) = \sqrt{2\pi} z^{z-1/2} \mathrm{e}^{-z} \left(\sum_{n=0}^{N-1} (-1)^n \frac{\gamma_n}{z^n} + R_N(z) \right).$$

Boyd [6] (see also [23, p. 141]) showed that

$$|R_N(z)| \leqslant \frac{(1+\zeta(N))\Gamma(N)}{(2\pi)^{N+1}|z|^N} \frac{\min(\sec\theta, 2\sqrt{N}) + 1}{2} \quad \text{if } |\theta| \leqslant \frac{\pi}{2}, \tag{1.3}$$

where $\theta = \arg z$ and ζ denotes Riemann's zeta function. When $N = 1$, the quantity $\zeta(N)$ has to be replaced by 3.

Boyd's derivation of the error bound (1.3) is based on his resurgence formula for the gamma function coming from a general theory for complex Laplace-type integrals developed by Berry and Howls [2] (see also [5] and [25, pp. 94–99]). Boyd discussed not only this error bound but also the smooth transition of the Stokes discontinuities and the asymptotic behaviour of the coefficients γ_n by using the resurgence formula.

The main goal of this paper is to modify Boyd's resurgence formula, making it suitable for deriving better error estimates for both (1.1) and (1.2) when $\mathrm{Re}(z) > 0$. We also prove exponentially improved expansions for the gamma function and its reciprocal. Finally, we provide new (formal) asymptotic expansions for the Stirling coefficients and compare their numerical efficacy with the earlier results of Dingle and Boyd.

Similarly to $R_N(z)$, denote by $\tilde{R}_N(z)$ the relative remainder of the series (1.2) after $N \geqslant 1$ terms, so that the last retained term is $\gamma_{N-1} z^{1-N}$. In the following theorem, we give bounds for the error terms $R_N(z)$ and $\tilde{R}_N(z)$ when z is real and positive.

Theorem 1.1. *Suppose that $z > 0$ and $N \geqslant 1$. Then*

$$(-1)^N \gamma_{2N-1} \geqslant 0 \quad \text{and} \quad (-1)^{N+1} \gamma_{2N} \geqslant 0, \tag{1.4}$$

and

$$\begin{aligned}
(-1)^{N+1} R_{2N-1}(z) &= \Theta_1(z,N)(-1)^N \frac{\gamma_{2N-1}}{z^{2N-1}} + \Theta_2(z,N)(-1)^{N+1} \frac{\gamma_{2N}}{z^{2N}}, \\
(-1)^{N+1} R_{2N}(z) &= \Theta_2(z,N)(-1)^{N+1} \frac{\gamma_{2N}}{z^{2N}} - \Theta_3(z,N)(-1)^{N+1} \frac{\gamma_{2N+1}}{z^{2N+1}}, \\
(-1)^N \tilde{R}_{2N-1}(z) &= \Theta_1(z,N)(-1)^N \frac{\gamma_{2N-1}}{z^{2N-1}} - \Theta_2(z,N)(-1)^{N+1} \frac{\gamma_{2N}}{z^{2N}}, \\
(-1)^{N+1} \tilde{R}_{2N}(z) &= \Theta_2(z,N)(-1)^{N+1} \frac{\gamma_{2N}}{z^{2N}} + \Theta_3(z,N)(-1)^{N+1} \frac{\gamma_{2N+1}}{z^{2N+1}}.
\end{aligned} \tag{1.5}$$

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Here $0 < \Theta_i(z, N) < 1$ $(i = 1, 2, 3)$ *is a suitable number depending on* z *and* N*. In particular, we have*

$$|R_{2N-1}(z)| = (-1)^{N+1} R_{2N-1}(z) < (-1)^N \frac{\gamma_{2N-1}}{z^{2N-1}} + (-1)^{N+1} \frac{\gamma_{2N}}{z^{2N}},$$

$$|R_{2N}(z)| < \max\left((-1)^{N+1} \frac{\gamma_{2N}}{z^{2N}}, (-1)^{N+1} \frac{\gamma_{2N+1}}{z^{2N+1}} \right),$$

$$|\tilde{R}_{2N-1}(z)| < \max\left((-1)^{N} \frac{\gamma_{2N-1}}{z^{2N-1}}, (-1)^{N+1} \frac{\gamma_{2N}}{z^{2N}} \right)$$

and

$$|\tilde{R}_{2N}(z)| = (-1)^{N+1} \tilde{R}_{2N}(z) < (-1)^{N+1} \frac{\gamma_{2N}}{z^{2N}} + (-1)^{N+1} \frac{\gamma_{2N+1}}{z^{2N+1}}.$$

In the next theorem, we provide bounds for the remainders $R_N(z)$ and $\tilde{R}_N(z)$ assuming that $\operatorname{Re}(z) > 0$.

Theorem 1.2. *For any* $N \geqslant 1$*, we have*

$$|R_N(z)|, |\tilde{R}_N(z)| \leqslant \left(\frac{|\gamma_N|}{|z|^N} + \frac{|\gamma_{N+1}|}{|z|^{N+1}} \right) \times \begin{cases} |\csc(2\theta)| & \textit{if } \frac{1}{4}\pi < |\theta| < \frac{1}{2}\pi, \\ 1 & \textit{if } |\theta| \leqslant \frac{1}{4}\pi, \end{cases} \tag{1.6}$$

where $\theta = \arg z$*.*

An asymptotic series for the logarithm of the gamma function that is analogous to (1.1) is given by

$$\log \Gamma(z) \sim (z - \tfrac{1}{2}) \log z - z + \log \sqrt{2\pi} + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}} \tag{1.7}$$

as $z \to \infty$ in the sector $|\arg z| \leqslant \pi - \delta < \pi$ for any $0 < \delta \leqslant \pi$. Here B_n stands for the nth Bernoulli number. Denoting by $r_N(z)$ the remainder after $N - 1$ terms in this series, Lindelöf showed that

$$|r_N(z)| \leqslant \frac{|B_{2N}|}{2N(2N-1)|z|^{2N-1}} \times \begin{cases} |\csc(2\theta)| & \text{if } \frac{1}{4}\pi < |\theta| < \frac{1}{2}\pi, \\ 1 & \text{if } |\theta| \leqslant \frac{1}{4}\pi, \end{cases}$$

where $\theta = \arg z$ (see [28, p. 67]). Also, if $z > 0$ is real, then $r_N(z)$ is less than, but has the same sign as, the first neglected term (see, for example, [30, p. 65]). It is seen that our error bounds in theorem 1.1 and theorem 1.2 are the analogues of these results for the expansion (1.7).

From the above remark on $r_N(z)$, it follows that for any $z > 0$ we have $0 < r_1(z) < B_2/2z = 1/12z$, whence we obtain

$$1 < \frac{\Gamma(z)}{\sqrt{2\pi} z^{z-1/2} \mathrm{e}^{-z}} < \mathrm{e}^{1/12z} = 1 + \frac{1}{12z} + \frac{1}{288z^2} + \frac{1}{10\,368 z^3} + \cdots.$$

This is a well-known inequality (see, for example, [23, eqn 5.6.1, p. 138]). By theorem 1.1 we can improve the upper bound to

$$1 < \frac{\Gamma(z)}{\sqrt{2\pi} z^{z-1/2} \mathrm{e}^{-z}} < 1 + \frac{1}{12z} + \frac{1}{288z^2}$$

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for any $z > 0$; this is, as far as we know, a new identity. Thus, we have a simple estimate for the gamma function on the positive real line.

By the leading-order behaviour of the Stirling coefficients (see [26, p. 33]), we readily establish that the right-hand side of (1.6) is asymptotic to

$$\left(\frac{(1+\zeta(N))\Gamma(N)}{(2\pi)^{N+1}|z|^N} + \frac{\pi}{6N}\frac{(1+\zeta(N+1))\Gamma(N+1)}{(2\pi)^{N+2}|z|^{N+1}}\right)\begin{cases}|\csc(2\theta)| & \text{if } \frac{1}{4}\pi < |\theta| < \frac{1}{2}\pi,\\ 1 & \text{if } |\theta| \leqslant \frac{1}{4}\pi,\end{cases}$$

for odd N, and

$$\left(\frac{\pi}{6N}\frac{(1+\zeta(N))\Gamma(N)}{(2\pi)^{N+1}|z|^N} + \frac{(1+\zeta(N+1))\Gamma(N+1)}{(2\pi)^{N+2}|z|^{N+1}}\right)\begin{cases}|\csc(2\theta)| & \text{if } \frac{1}{4}\pi < |\theta| < \frac{1}{2}\pi,\\ 1 & \text{if } |\theta| \leqslant \frac{1}{4}\pi,\end{cases}$$

for even N. Since $1 < \frac{1}{2}(\sec\theta + 1)$ if $0 < |\theta| \leqslant \frac{1}{4}\pi$, and $|\csc(2\theta)| < \frac{1}{2}(\sec\theta + 1)$ if $\frac{1}{4}\pi < |\theta| < \frac{1}{2}\pi$, we infer that our bounds (1.6) are better than the bound (1.3) of Boyd if N and z are large and $\arg z$ is not too close to the imaginary axis.

When $|\arg z|$ is close to $\frac{1}{2}\pi$, the error bound (1.3) becomes

$$|R_1(z)| \leqslant \frac{3}{2\pi^2|z|} \quad \text{and} \quad |R_N(z)| \leqslant \frac{(1+\zeta(N))\Gamma(N)}{(2\pi)^{N+1}|z|^N}\frac{2\sqrt{N}+1}{2} \quad \text{for } N \geqslant 2. \tag{1.8}$$

Boyd does not actually prove these bounds, just mentions that the proof is similar to the proof of his bound for the error term of the large argument asymptotics of the Bessel function $K_\nu(z)$ given in an earlier paper of his [4]. Up to the first few steps we can indeed mimic the proof presented in [4], but at one point we need non-trivial estimates for the gamma function along certain rays of the complex plane. Nevertheless, we shall give a possible proof of (1.8) for $N \geqslant 2$. The case $N = 1$ remains unproved, though it is uninteresting for practical applications.

Theorem 1.3. *Suppose that* $N \geqslant 2$. *If* $|\arg z| \leqslant \frac{1}{2}\pi$, *then*

$$|R_N(z)|, |\tilde{R}_N(z)| \leqslant \frac{(1+\zeta(N))\Gamma(N)}{(2\pi)^{N+1}|z|^N}\frac{2\sqrt{N}+1}{2}.$$

Throughout this paper we shall use frequently the concept of the scaled gamma function $\Gamma^*(z)$, which is defined by

$$\Gamma^*(z) = \frac{\Gamma(z)}{\sqrt{2\pi}z^{z-1/2}\mathrm{e}^{-z}}$$

for $|\arg z| < \pi$. The asymptotic series (1.1), (1.2) and the error bounds can be extended to other sectors of the complex plane via the continuation formulae

$$\Gamma^*(z) = \frac{1}{1-\mathrm{e}^{\pm 2\pi \mathrm{i} z}}\frac{1}{\Gamma^*(z\mathrm{e}^{\mp\pi\mathrm{i}})} \quad \text{and} \quad \Gamma^*(z) = -\mathrm{e}^{\pm 2\pi\mathrm{i}z}\Gamma^*(z\mathrm{e}^{\pm 2\pi\mathrm{i}}). \tag{1.9}$$

Lines of the form $\arg z = (2m \pm \frac{1}{2})\pi$, where $m \in \mathbb{Z}$, are the Stokes lines for the gamma function and its reciprocal.

In the next theorem we give exponentially improved asymptotic expansions for the gamma function and its reciprocal. The expansion for the gamma function can be viewed as the mathematically rigorous form of the terminated expansion of

Dingle [13, pp. 461–462]. We express these expansions in terms of the terminant function $\hat{T}_p(w)$, whose definition and basic properties are given in § 3. Throughout this paper, empty sums are taken to be zero.

THEOREM 1.4. *Suppose that* $-\frac{3}{2}\pi \leqslant \arg z \leqslant \frac{3}{2}\pi$, $|z|$ *is large and* $N = 2\pi|z| + \rho$ *is a positive integer with* ρ *being bounded. Then*

$$R_N(z) = \mathrm{e}^{2\pi \mathrm{i} z} \sum_{m=0}^{M-1} (-1)^m \frac{\gamma_m}{z^m} \hat{T}_{N-m}(2\pi \mathrm{i} z) - \mathrm{e}^{-2\pi \mathrm{i} z} \sum_{m=0}^{M-1} (-1)^m \frac{\gamma_m}{z^m} \hat{T}_{N-m}(-2\pi \mathrm{i} z) + R_{N,M}(z) \tag{1.10}$$

and

$$\tilde{R}_N(z) = -\mathrm{e}^{2\pi \mathrm{i} z} \sum_{m=0}^{M-1} \frac{\gamma_m}{z^m} \hat{T}_{N-m}(2\pi \mathrm{i} z) + \mathrm{e}^{-2\pi \mathrm{i} z} \sum_{m=0}^{M-1} \frac{\gamma_m}{z^m} \hat{T}_{N-m}(-2\pi \mathrm{i} z) + \tilde{R}_{N,M}(z), \tag{1.11}$$

with $M \geqslant 0$ *being an arbitrary fixed integer, and*

$$R_{N,M}(z), \tilde{R}_{N,M}(z) = \mathcal{O}_{M,\rho}\left(\frac{\mathrm{e}^{-2\pi|z|}}{|z|^M} \right) \tag{1.12}$$

for $|\arg z| \leqslant \frac{1}{2}\pi$, *and*

$$R_{N,M}(z) = \mathcal{O}_{M,\rho}\left(\mathrm{e}^{\mp 2\pi \operatorname{Im}(z)} \left(\frac{1}{|1 - \mathrm{e}^{\mp 2\pi \mathrm{i} z}|} + \frac{1}{|z|^M} \right) \right)$$

and

$$\tilde{R}_{N,M}(z) = \mathcal{O}_{M,\rho}\left(\frac{\mathrm{e}^{\mp 2\pi \operatorname{Im}(z)}}{|z|^M} \right)$$

for $\frac{1}{2}\pi \leqslant \pm \arg z \leqslant \frac{3}{2}\pi$.

The expansion (1.10) without the error term $R_{N,M}(z)$ was also derived by Boyd, but he mistakenly gave the sign of the factor $\mathrm{e}^{-2\pi \mathrm{i} z}$ as positive.

For exponentially improved asymptotic expansions using Hadamard series, see [24] and [25, pp. 156–159].

While proving theorem 1.4 in § 3, we also obtain the following explicit bounds for the remainders in (1.10) and (1.11). Note that in this theorem N may not depend on z.

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Theorem 1.5. *For any integers $2 \leqslant M < N$, define the remainders $R_{N,M}(z)$ and $\tilde{R}_{N,M}(z)$ by (1.10) and (1.11), respectively. Then we have*

$$\begin{aligned}|R_{N,M}(z)|, |\tilde{R}_{N,M}(z)| \\ &\leqslant (6M+2)\frac{\zeta(M)\Gamma(M)\Gamma(N-M)}{(2\pi)^{N+2}|z|^N} + (2\sqrt{M}+1)\frac{\zeta(M)\Gamma(M)}{(2\pi)^{M+1}|z|^M} \\ &\quad \times (|\mathrm{e}^{2\pi\mathrm{i}z}\hat{T}_{N-M}(2\pi\mathrm{i}z)| + |\mathrm{e}^{-2\pi\mathrm{i}z}\hat{T}_{N-M}(-2\pi\mathrm{i}z)|), \qquad (1.13)\end{aligned}$$

provided that $|\arg z| \leqslant \frac{1}{2}\pi$.

The rest of the paper is organized as follows. In § 2, we prove the error bounds stated in theorems 1.1–1.3. In the first part of § 3, we prove the exponentially improved expansions given in theorem 1.4 and the error bounds given in theorem 1.5. In the second part, we reveal some interesting facts about the Stokes phenomenon for the gamma function and its reciprocal. We also discuss the smooth transition of the Stokes discontinuities. In § 4, we derive new asymptotic approximations for the Stirling coefficients γ_n and compare their numerical efficacy with the earlier results of Dingle and Boyd.

2. Proofs of the error bounds

Recall that for any $N \geqslant 1$ the remainder terms $R_N(z)$ and $\tilde{R}_N(z)$ are defined by

$$\Gamma^*(z) = \sum_{n=0}^{N-1}(-1)^n\frac{\gamma_n}{z^n} + R_N(z) \quad \text{and} \quad \frac{1}{\Gamma^*(z)} = \sum_{n=0}^{N-1}\frac{\gamma_n}{z^n} + \tilde{R}_N(z).$$

Suppose that $\mathrm{Re}(z) > 0$. Boyd's resurgence formulae [6, (2.14) and (4.2)] can be written in the form

$$\begin{aligned}R_N(z) &= \frac{1}{2\pi\mathrm{i}}\frac{\mathrm{i}^N}{z^N}\int_0^{+\infty}\frac{s^{N-1}\mathrm{e}^{-2\pi s}\Gamma^*(\mathrm{i}s)}{1-\mathrm{i}s/z}\,\mathrm{d}s \\ &\quad - \frac{1}{2\pi\mathrm{i}}\frac{(-\mathrm{i})^N}{z^N}\int_0^{+\infty}\frac{s^{N-1}\mathrm{e}^{-2\pi s}\Gamma^*(-\mathrm{i}s)}{1+\mathrm{i}s/z}\,\mathrm{d}s \qquad (2.1)\end{aligned}$$

and

$$\begin{aligned}\tilde{R}_N(z) &= \frac{1}{2\pi\mathrm{i}}\frac{(-\mathrm{i})^N}{z^N}\int_0^{+\infty}\frac{s^{N-1}\mathrm{e}^{-2\pi s}\Gamma^*(\mathrm{i}s)}{1+\mathrm{i}s/z}\,\mathrm{d}s \\ &\quad - \frac{1}{2\pi\mathrm{i}}\frac{\mathrm{i}^N}{z^N}\int_0^{+\infty}\frac{s^{N-1}\mathrm{e}^{-2\pi s}\Gamma^*(-\mathrm{i}s)}{1-\mathrm{i}s/z}\,\mathrm{d}s.\end{aligned}$$

We remark that he stated the formula $\tilde{R}_N(z)$ only for $N = 1$, but the formula for the general N follows easily from it. Using the property $\Gamma^*(\bar{z}) = \overline{\Gamma^*(z)}$, we deduce that

$$(-1)^{N+1}R_{2N-1}(z) = \frac{1}{\pi z^{2N-1}}\int_0^{+\infty}\frac{s^{2N-2}\mathrm{e}^{-2\pi s}\,\mathrm{Re}\,\Gamma^*(\mathrm{i}s)}{1+(s/z)^2}\,\mathrm{d}s - \frac{1}{\pi z^{2N}}\int_0^{+\infty}\frac{s^{2N-1}\mathrm{e}^{-2\pi s}\,\mathrm{Im}\,\Gamma^*(\mathrm{i}s)}{1+(s/z)^2}\,\mathrm{d}s, \tag{2.2}$$

$$(-1)^{N+1}R_{2N}(z) = -\frac{1}{\pi z^{2N}}\int_0^{+\infty}\frac{s^{2N-1}\mathrm{e}^{-2\pi s}\,\mathrm{Im}\,\Gamma^*(\mathrm{i}s)}{1+(s/z)^2}\,\mathrm{d}s - \frac{1}{\pi z^{2N+1}}\int_0^{+\infty}\frac{s^{2N}\mathrm{e}^{-2\pi s}\,\mathrm{Re}\,\Gamma^*(\mathrm{i}s)}{1+(s/z)^2}\,\mathrm{d}s,$$

$$(-1)^{N}\tilde{R}_{2N-1}(z) = \frac{1}{\pi z^{2N-1}}\int_0^{+\infty}\frac{s^{2N-2}\mathrm{e}^{-2\pi s}\,\mathrm{Re}\,\Gamma^*(\mathrm{i}s)}{1+(s/z)^2}\,\mathrm{d}s + \frac{1}{\pi z^{2N}}\int_0^{+\infty}\frac{s^{2N-1}\mathrm{e}^{-2\pi s}\,\mathrm{Im}\,\Gamma^*(\mathrm{i}s)}{1+(s/z)^2}\,\mathrm{d}s,$$

$$(-1)^{N+1}\tilde{R}_{2N}(z) = -\frac{1}{\pi z^{2N}}\int_0^{+\infty}\frac{s^{2N-1}\mathrm{e}^{-2\pi s}\,\mathrm{Im}\,\Gamma^*(\mathrm{i}s)}{1+(s/z)^2}\,\mathrm{d}s + \frac{1}{\pi z^{2N+1}}\int_0^{+\infty}\frac{s^{2N}\mathrm{e}^{-2\pi s}\,\mathrm{Re}\,\Gamma^*(\mathrm{i}s)}{1+(s/z)^2}\,\mathrm{d}s,$$

for any $N \geqslant 1$. These are the suitable forms of the remainders to obtain the realistic error bounds stated in theorems 1.1 and 1.2. From these and the formula $\gamma_N = z^N(\tilde{R}_N(z) - \tilde{R}_{N+1}(z))$, we infer that

$$(-1)^N\gamma_{2N-1} = \frac{1}{\pi}\int_0^{+\infty} s^{2N-2}\mathrm{e}^{-2\pi s}\,\mathrm{Re}\,\Gamma^*(\mathrm{i}s)\,\mathrm{d}s \tag{2.3}$$

and

$$(-1)^{N+1}\gamma_{2N} = -\frac{1}{\pi}\int_0^{+\infty} s^{2N-1}\mathrm{e}^{-2\pi s}\,\mathrm{Im}\,\Gamma^*(\mathrm{i}s)\,\mathrm{d}s, \tag{2.4}$$

for all $N \geqslant 1$. To complete the proof of theorems 1.1 and 1.2, we need the following lemma.

LEMMA 2.1. *For any* $s > 0$ *it holds that* $\mathrm{Re}\,\Gamma^*(\mathrm{i}s) \geqslant 0$ *and* $-\mathrm{Im}\,\Gamma^*(\mathrm{i}s) \geqslant 0$.

Proof. The proof is based on the following representation of $\Gamma^*(z)$ due to Stieltjes:

$$\Gamma^*(z) = \exp\left(\int_0^{+\infty}\frac{Q(t)}{(z+t)^2}\,\mathrm{d}t\right) \quad \text{for } |\arg z| < \pi, \tag{2.5}$$

where $Q(t) = \frac{1}{2}(t - t - (t-t)^2)$ (see, for example, [28, pp. 56–58]). We shall use the fact that $0 \leqslant Q(t) \leqslant \frac{1}{8}$. Substituting $z = \mathrm{i}s$ with $s > 0$ gives

$$\Gamma^*(\mathrm{i}s) = \exp\left(-\int_0^{+\infty}\frac{s^2-t^2}{(s^2+t^2)^2}Q(t)\,\mathrm{d}t - \mathrm{i}\int_0^{+\infty}\frac{2st}{(s^2+t^2)^2}Q(t)\,\mathrm{d}t\right),$$

whence we obtain

$$\operatorname{Re}\Gamma^*(\mathrm{i}s) = \exp\left(-\int_0^{+\infty} \frac{s^2-t^2}{(s^2+t^2)^2} Q(t)\,\mathrm{d}t\right) \cos\left(\int_0^{+\infty} \frac{2st}{(s^2+t^2)^2} Q(t)\,\mathrm{d}t\right)$$

and

$$-\operatorname{Im}\Gamma^*(\mathrm{i}s) = \exp\left(-\int_0^{+\infty} \frac{s^2-t^2}{(s^2+t^2)^2} Q(t)\,\mathrm{d}t\right) \sin\left(\int_0^{+\infty} \frac{2st}{(s^2+t^2)^2} Q(t)\,\mathrm{d}t\right).$$

To prove the lemma, it is enough to show that the integral under the trigonometric functions is non-negative and is at most $\frac{1}{2}\pi$ for any $s > 0$. As $Q(t)$ is non-negative, the integral is non-negative. On the other hand,

$$\int_0^1 \frac{2st}{(s^2+t^2)^2} Q(t)\,\mathrm{d}t = \frac{s}{2}\log\left(\frac{s^2}{s^2+1}\right) + \tfrac{1}{2}\arctan\left(\frac{1}{s}\right) \leqslant \frac{\pi}{4}$$

and

$$\int_1^{+\infty} \frac{2st}{(s^2+t^2)^2} Q(t)\,\mathrm{d}t \leqslant \int_1^{+\infty} \frac{2st}{(s^2+t^2)^2}\frac{1}{8}\,\mathrm{d}t = \frac{1}{8}\frac{s}{s^2+1} \leqslant \frac{1}{16},$$

whence we obtain

$$\int_0^{+\infty} \frac{2st}{(s^2+t^2)^2} Q(t)\,\mathrm{d}t \leqslant \frac{\pi}{4} + \frac{1}{16} < \frac{\pi}{2},$$

for any $s > 0$. □

The inequalities in (1.4) follow from the lemma and the representations (2.3) and (2.4). From theorem 1.1 we prove only the bound (1.5); the other results can be proved similarly. First, we note that

$$0 < \frac{1}{1+(s/z)^2} < 1$$

for any $s > 0$ and $z > 0$. Employing this inequality in (2.2) leads to

$$(-1)^{N+1} R_{2N-1}(z) = \frac{\Theta_1(z,N)}{\pi z^{2N-1}} \int_0^{+\infty} s^{2N-2}\mathrm{e}^{-2\pi s}\operatorname{Re}\Gamma^*(\mathrm{i}s)\,\mathrm{d}s - \frac{\Theta_2(z,N)}{\pi z^{2N}} \int_0^{+\infty} s^{2N-1}\mathrm{e}^{-2\pi s}\operatorname{Im}\Gamma^*(\mathrm{i}s)\,\mathrm{d}s,$$

where $\Theta_1(z,N)$ and $\Theta_2(z,N)$ are some functions of z and N that satisfy $0 < \Theta_1(z,N) < 1$ and $0 < \Theta_2(z,N) < 1$. Upon inserting (2.3) and (2.4) into this representation we obtain (1.5).

As for theorem 1.2, we prove the result for $R_{2N-1}(z)$; the proofs of the other bounds are similar. From (2.2) and lemma 2.1 it follows that

$$|R_{2N-1}(z)| \leqslant \frac{1}{\pi|z|^{2N-1}} \int_0^{+\infty} \frac{s^{2N-2}\mathrm{e}^{-2\pi s}\operatorname{Re}\Gamma^*(\mathrm{i}s)}{|1+(s/z)^2|}\,\mathrm{d}s - \frac{1}{\pi|z|^{2N}} \int_0^{+\infty} \frac{s^{2N-1}\mathrm{e}^{-2\pi s}\operatorname{Im}\Gamma^*(\mathrm{i}s)}{|1+(s/z)^2|}\,\mathrm{d}s. \quad (2.6)$$

It is easy to show that, for any $r > 0$,

$$\frac{1}{|1 + r\mathrm{e}^{-2\theta\mathrm{i}}|} \leqslant \begin{cases} |\csc(2\theta)| & \text{if } \frac{1}{4}\pi < |\theta| < \frac{1}{2}\pi, \\ 1 & \text{if } |\theta| \leqslant \frac{1}{4}\pi. \end{cases}$$

Applying this inequality and (2.3) and (2.4) to (2.6) proves the estimate (1.6) for the case of $R_{2N-1}(z)$.

To prove theorem 1.3, we shall use the lemma below.

Lemma 2.2. *For any $s > 0$ and $0 < \varphi < \frac{1}{2}\pi$, we have*

$$\left|\Gamma^*\left(\frac{\mathrm{i}s\mathrm{e}^{\mathrm{i}\varphi}}{\cos\varphi}\right)\right| \leqslant \frac{1}{\sqrt{1 - 2\mathrm{e}^{-2\pi s}\cos(2\pi s\tan\varphi) + \mathrm{e}^{-4\pi s}}} \leqslant \frac{1}{1 - \mathrm{e}^{-2\pi s}}. \tag{2.7}$$

Proof. An application of the reflection formula (1.9) for the gamma function and the relation $\overline{\Gamma^*(z)} = \Gamma^*(\bar{z})$ shows that

$$\begin{aligned} \log\left|\Gamma^*\left(\frac{\mathrm{i}s\mathrm{e}^{\mathrm{i}\varphi}}{\cos\varphi}\right)\right| &= -\tfrac{1}{2}\log(1 - 2\mathrm{e}^{-2\pi s}\cos(2\pi s\tan\varphi) + \mathrm{e}^{-4\pi s}) - \log\left|\Gamma^*\left(-\frac{\mathrm{i}s\mathrm{e}^{\mathrm{i}\varphi}}{\cos\varphi}\right)\right| \\ &= -\tfrac{1}{2}\log(1 - 2\mathrm{e}^{-2\pi s}\cos(2\pi s\tan\varphi) + \mathrm{e}^{-4\pi s}) - \log\left|\Gamma^*\left(\frac{\mathrm{i}s\mathrm{e}^{-\mathrm{i}\varphi}}{\cos\varphi}\right)\right|. \end{aligned}$$

From this, we infer that

$$\begin{aligned} \left|\Gamma^*\left(\frac{\mathrm{i}s\mathrm{e}^{\mathrm{i}\varphi}}{\cos\varphi}\right)\right| &= \frac{1}{\sqrt{1 - 2\mathrm{e}^{-2\pi s}\cos(2\pi s\tan\varphi) + \mathrm{e}^{-4\pi s}}}\left|\Gamma^*\left(\frac{\mathrm{i}s\mathrm{e}^{-\mathrm{i}\varphi}}{\cos\varphi}\right)\right|^{-1} \\ &\leqslant \frac{1}{1 - \mathrm{e}^{-2\pi s}}\left|\Gamma^*\left(\frac{\mathrm{i}s\mathrm{e}^{-\mathrm{i}\varphi}}{\cos\varphi}\right)\right|^{-1}. \end{aligned} \tag{2.8}$$

Let $z = x + \mathrm{i}y$ such that $|\arg z| < \frac{1}{2}\pi$. We show that $|1/\Gamma^*(z)|$ is bounded in the right half-plane. Indeed, if z is not too close to the origin, then by Stieltjes's formula, (2.5),

$$\begin{aligned} \left|\frac{1}{\Gamma^*(z)}\right| &\leqslant \exp\left(\frac{1}{8}\int_0^{+\infty}\frac{\mathrm{d}t}{|z+t|^2}\right) \\ &\leqslant \exp\left(\frac{1}{8\cos^2(\frac{1}{2}\theta)}\int_0^{+\infty}\frac{\mathrm{d}t}{(|z|+t)^2}\right) \\ &= \exp\left(\frac{1}{8|z|\cos^2(\frac{1}{2}\theta)}\right) \leqslant \exp\left(\frac{1}{4|z|}\right) \quad \text{with } \theta = \arg z. \end{aligned}$$

To see the boundedness near the origin, we note that

$$\begin{aligned} \left|\frac{1}{\Gamma^*(z)}\right| &= \left|\frac{z^{z+1/2}\sqrt{2\pi}}{\mathrm{e}^z}\right|\left|\frac{1}{\Gamma(z+1)}\right| \\ &= \sqrt{2\pi}\mathrm{e}^{-(\pi/2)|y| + y\arctan(x/y) - x}|z|^{x+1/2}\left|\frac{1}{\Gamma(z+1)}\right| \\ &\leqslant \sqrt{2\pi}\mathrm{e}^{-x}|z|^{x+1/2}\left|\frac{1}{\Gamma(z+1)}\right|, \end{aligned}$$

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and that the reciprocal gamma function is an entire function. Since $1/\Gamma^*(z)$ is holomorphic when $|\arg z| < \frac{1}{2}\pi$, continuous on its boundary and

$$\left|\frac{1}{\Gamma^*(\pm \mathrm{i}y)}\right| = \sqrt{1-\mathrm{e}^{-2\pi y}} \leqslant 1,$$

by the Phragmén–Lindelöf principle [31, p. 177],

$$\left|\frac{1}{\Gamma^*(z)}\right| \leqslant 1$$

holds for any z in the sector $|\arg z| \leqslant \frac{1}{2}\pi$. Employing this inequality with $z = \mathrm{i}s\mathrm{e}^{-\mathrm{i}\varphi}/\cos\varphi$ in (2.8) gives (2.7). □

We prove the claimed bound only for $R_N(z)$; the proof for $\tilde{R}_N(z)$ is completely analogous. Since $R_N(\bar{z}) = \overline{R_N(z)}$, we can assume that $0 \leqslant \theta = \arg z \leqslant \frac{1}{2}\pi$. The idea is to rotate the path of integration through an angle $0 < \varphi < \frac{1}{2}\pi$ in the first integral in (2.1), to find

$$R_N(z) = \frac{1}{2\pi\mathrm{i}}\frac{\mathrm{i}^N}{z^N}\int_0^{+\infty \mathrm{e}^{\mathrm{i}\varphi}} \frac{s^{N-1}\mathrm{e}^{-2\pi s}\Gamma^*(\mathrm{i}s)}{1-\mathrm{i}s/z}\,\mathrm{d}s - \frac{1}{2\pi\mathrm{i}}\frac{(-\mathrm{i})^N}{z^N}\int_0^{+\infty} \frac{s^{N-1}\mathrm{e}^{-2\pi s}\Gamma^*(-\mathrm{i}s)}{1+\mathrm{i}s/z}\,\mathrm{d}s.$$

By analytic continuation, this expression is certainly valid when $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Substituting $s = t\mathrm{e}^{\mathrm{i}\varphi}/\cos\varphi$ into the first integral and using the inequalities

$$\left|\frac{1}{1+\mathrm{i}s/z}\right| \leqslant 1 \quad \text{and} \quad \left|\frac{1}{1-\mathrm{i}t\mathrm{e}^{\mathrm{i}\varphi}/\cos\varphi z}\right| \leqslant \sec(\theta-\varphi),$$

we find

$$|R_N(z)| \leqslant \frac{\sec(\theta-\varphi)}{\cos^N\varphi}\frac{1}{|z|^N}\frac{1}{2\pi}\int_0^{+\infty} t^{N-1}\mathrm{e}^{-2\pi t}\left|\Gamma^*\left(\frac{\mathrm{i}t\mathrm{e}^{\mathrm{i}\varphi}}{\cos\varphi}\right)\right|\mathrm{d}t + \frac{1}{|z|^N}\frac{1}{2\pi}\int_0^{+\infty} s^{N-1}\mathrm{e}^{-2\pi s}|\Gamma^*(-\mathrm{i}s)|\,\mathrm{d}s.$$

The value $\varphi = \arctan(N^{-1/2})$ minimizes the function $\sec(\frac{1}{2}\pi-\varphi)\cos^{-N}\varphi$, and

$$\frac{\sec(\theta-\arctan(N^{-1/2}))}{\cos^N(\arctan(N^{-1/2}))} \leqslant \frac{\sec(\frac{1}{2}\pi-\arctan(N^{-1/2}))}{\cos^N(\arctan(N^{-1/2}))} = \left(1+\frac{1}{N}\right)^{(N+1)/2}\sqrt{N},$$

for any $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ with $N \geqslant 1$. Boyd [6, (3.9)] showed that

$$\frac{1}{2\pi}\int_0^{+\infty} s^{N-1}\mathrm{e}^{-2\pi s}|\Gamma^*(-\mathrm{i}s)|\,\mathrm{d}s \leqslant \frac{(1+\zeta(N))\Gamma(N)}{(2\pi)^{N+1}}\frac{1}{2} \quad \text{for } N \geqslant 2.$$

From lemma 2.2, we obtain

$$\frac{1}{2\pi}\int_0^{+\infty} t^{N-1}\mathrm{e}^{-2\pi t}\left|\Gamma^*\left(\frac{\mathrm{i}t\mathrm{e}^{\mathrm{i}\varphi}}{\cos\varphi}\right)\right|\mathrm{d}t \leqslant \frac{1}{2\pi}\int_0^{+\infty}\frac{t^{N-1}\mathrm{e}^{-2\pi t}}{1-\mathrm{e}^{-2\pi t}}\,\mathrm{d}t = \frac{B_N\Gamma(N)}{(2\pi)^{N+1}}\frac{1}{2}$$

for $N \geqslant 3$ with $B_N = 2\zeta(N)$, using formula 25.5.1 in [23, p. 604]. This estimate also holds when $N = 2$, but for this case we derive a sharper bound using the first inequality in lemma 2.2:

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{+\infty} t^{2-1}\mathrm{e}^{-2\pi t}\left|\varGamma^*\left(\frac{\mathrm{i}t\mathrm{e}^{\mathrm{i}\varphi}}{\cos\varphi}\right)\right|\mathrm{d}t\\ &\quad\leqslant \frac{1}{2\pi}\int_0^{+\infty}\frac{t\mathrm{e}^{-2\pi t}}{\sqrt{1-2\mathrm{e}^{-2\pi t}\cos(2\pi t\tan(\arctan(2^{-1/2}))) + \mathrm{e}^{-4\pi t}}}\,\mathrm{d}t\\ &\quad= \frac{B_2\varGamma(2)}{(2\pi)^{2+1}}\frac{1}{2}\end{aligned}$$

with $B_2 = 2.81944984\dots < 2.82$. Therefore,

$$|R_N(z)| \leqslant \frac{1}{2}\left(\frac{B_N}{1+\zeta(N)}\left(1+\frac{1}{N}\right)^{(N+1)/2}\sqrt{N}+1\right)\frac{(1+\zeta(N))\varGamma(N)}{(2\pi)^{N+1}|z|^N}.$$

To complete the proof, we note that

$$\frac{B_N}{1+\zeta(N)}\left(1+\frac{1}{N}\right)^{(N+1)/2} < 2$$

for any $N \geqslant 2$.

3. Exponentially improved asymptotic expansions

We shall find it convenient to express our exponentially improved expansions in terms of the (scaled) terminant function, which is defined by

$$\hat{T}_p(w) = \frac{\mathrm{e}^{\pi\mathrm{i}p}w^{1-p}\mathrm{e}^{-w}}{2\pi\mathrm{i}}\int_0^{+\infty}\frac{t^{p-1}\mathrm{e}^{-t}}{w+t}\,\mathrm{d}t \quad \text{for } p>0 \quad\text{and}\quad |\arg w|<\pi,$$

and by analytic continuation elsewhere. Olver [22, (4.5) and (4.6)] showed that when $p \sim |w|$ and $w \to \infty$, we have

$$\hat{T}_p(w) = \begin{cases}\mathcal{O}(\mathrm{e}^{-w-|w|}) & \text{if } |\arg w| \leqslant \pi,\\ \mathcal{O}(1) & \text{if } -3\pi < \arg w \leqslant -\pi.\end{cases} \tag{3.1}$$

Concerning the smooth transition of the Stokes discontinuities, we shall use the more precise asymptotics

$$\hat{T}_p(w) = \tfrac{1}{2} + \tfrac{1}{2}\operatorname{erf}\left(c(\varphi)\sqrt{\tfrac{1}{2}|w|}\right) + \mathcal{O}\left(\frac{\mathrm{e}^{-|w|c^2(\varphi)/2}}{|w|^{1/2}}\right) \tag{3.2}$$

for $-\pi+\delta \leqslant \arg w \leqslant 3\pi-\delta$, $0<\delta\leqslant 2\pi$, and

$$\mathrm{e}^{-2\pi\mathrm{i}p}\hat{T}_p(w) = -\tfrac{1}{2} + \tfrac{1}{2}\operatorname{erf}\left(-\overline{c(-\varphi)}\sqrt{\tfrac{1}{2}|w|}\right) + \mathcal{O}\left(\frac{\mathrm{e}^{-|w|\overline{c^2(-\varphi)}/2}}{|w|^{1/2}}\right) \tag{3.3}$$

for $-3\pi+\delta \leqslant \arg w \leqslant \pi-\delta$, $0<\delta\leqslant 2\pi$. Here $\varphi = \arg w$ and erf denotes the error function. The quantity $c(\varphi)$ is defined implicitly by the equation

$$\tfrac{1}{2}c^2(\varphi) = 1 + \mathrm{i}(\varphi-\pi) - \mathrm{e}^{\mathrm{i}(\varphi-\pi)},$$

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and corresponds to the branch of $c(\varphi)$ that has the following expansion in the neighbourhood of $\varphi = \pi$:

$$c(\varphi) = (\varphi - \pi) + \tfrac{1}{6}\mathrm{i}(\varphi - \pi)^2 - \frac{1}{36}(\varphi - \pi)^3 - \frac{\mathrm{i}}{270}(\varphi - \pi)^4 + \cdots . \tag{3.4}$$

For complete asymptotic expansions, see [21]. We remark that Olver uses the alternative notation $F_p(w) = \mathrm{i}\mathrm{e}^{-\pi \mathrm{i} p}\hat{T}_p(w)$ for the terminant function and the other branch of the function $c(\varphi)$. For further properties of the terminant function, see, for example, [26, ch. 6].

3.1. Proof of the exponentially improved expansions

We start by proving the expansion (1.10) and the estimate (1.12) for the right half-plane. Let $0 \leqslant M < N$ be integers. First suppose, in addition, that $M \geqslant 2$. As in the proof of theorem 1.3, we rotate the path of integration by an angle $0 < \varphi < \frac{1}{2}\pi$ in the first integral of Boyd's resurgence formula (2.1) to find that, for any $s > 0$,

$$\Gamma^*(\mathrm{i}s) = \sum_{m=0}^{M-1} (-1)^m \frac{\gamma_m}{(\mathrm{i}s)^m} + R_M(\mathrm{i}s)$$

with

$$\begin{aligned} R_M(\mathrm{i}s) &= \frac{1}{2\pi\mathrm{i}}\frac{1}{s^M}\int_0^{+\infty \mathrm{e}^{\mathrm{i}\varphi}} \frac{t^{M-1}\mathrm{e}^{-2\pi t}\Gamma^*(\mathrm{i}t)}{1 - t/s}\,\mathrm{d}t \\ &\quad - \frac{1}{2\pi\mathrm{i}}\frac{1}{(-s)^M}\int_0^{+\infty} \frac{t^{M-1}\mathrm{e}^{-2\pi t}\Gamma^*(-\mathrm{i}t)}{1 + t/s}\,\mathrm{d}t \\ &= \frac{1}{2\pi\mathrm{i}}\frac{1}{(s\mathrm{e}^{-\mathrm{i}\varphi})^M}\int_0^{+\infty} \frac{t^{M-1}\exp(-2\pi t\mathrm{e}^{\mathrm{i}\varphi})\Gamma^*(\mathrm{i}t\mathrm{e}^{\mathrm{i}\varphi})}{1 - t\mathrm{e}^{\mathrm{i}\varphi}/s}\,\mathrm{d}t \\ &\quad - \frac{1}{2\pi\mathrm{i}}\frac{1}{(-s)^M}\int_0^{+\infty} \frac{t^{M-1}\mathrm{e}^{-2\pi t}\Gamma^*(-\mathrm{i}t)}{1 + t/s}\,\mathrm{d}t. \end{aligned} \tag{3.5}$$

A similar formula for $\Gamma^*(-\mathrm{i}s)$ can be obtained in the same way. First, we suppose that $|\arg z| < \frac{1}{2}\pi$. Substitution into (2.1) yields

$$\begin{aligned} R_N(z) &= \sum_{m=0}^{M-1} (-1)^m \frac{\gamma_m}{z^m}\frac{\mathrm{i}^{N-m}z^{m-N}}{2\pi\mathrm{i}}\int_0^{+\infty} \frac{s^{N-m-1}\mathrm{e}^{-2\pi s}}{1 - \mathrm{i}s/z}\,\mathrm{d}s \\ &\quad - \sum_{m=0}^{M-1} (-1)^m \frac{\gamma_m}{z^m}\frac{(-\mathrm{i})^{N-m}z^{m-N}}{2\pi\mathrm{i}}\int_0^{+\infty} \frac{s^{N-m-1}\mathrm{e}^{-2\pi s}}{1 + \mathrm{i}s/z}\,\mathrm{d}s + R_{N,M}(z) \end{aligned} \tag{3.6}$$

with

$$\begin{aligned} R_{N,M}(z) &= \frac{1}{2\pi\mathrm{i}}\frac{\mathrm{i}^N}{z^N}\int_0^{+\infty} \frac{s^{N-1}\mathrm{e}^{-2\pi s}R_M(\mathrm{i}s)}{1 - \mathrm{i}s/z}\,\mathrm{d}s \\ &\quad - \frac{1}{2\pi\mathrm{i}}\frac{(-\mathrm{i})^N}{z^N}\int_0^{+\infty} \frac{s^{N-1}\mathrm{e}^{-2\pi s}R_M(-\mathrm{i}s)}{1 + \mathrm{i}s/z}\,\mathrm{d}s. \end{aligned} \tag{3.7}$$

The integrals in (3.6) can be identified in terms of the terminant function since

$$\frac{(\pm\mathrm{i})^{N-m}z^{m-N}}{2\pi\mathrm{i}}\int_0^{+\infty}\frac{s^{N-m-1}\mathrm{e}^{-2\pi s}}{1\mp\mathrm{i}s/z}\,\mathrm{d}s=\mathrm{e}^{\pm2\pi\mathrm{i}z}\hat{T}_{N-m}(\pm2\pi\mathrm{i}z).$$

Therefore, we have the following expansion:

$$R_N(z)=\mathrm{e}^{2\pi\mathrm{i}z}\sum_{m=0}^{M-1}(-1)^m\frac{\gamma_m}{z^m}\hat{T}_{N-m}(2\pi\mathrm{i}z)\\ -\mathrm{e}^{-2\pi\mathrm{i}z}\sum_{m=0}^{M-1}(-1)^m\frac{\gamma_m}{z^m}\hat{T}_{N-m}(-2\pi\mathrm{i}z)+R_{N,M}(z). \tag{3.8}$$

Taking $z=r\mathrm{e}^{\mathrm{i}\theta}$, the representation (3.7) becomes

$$R_{N,M}(z)=\frac{1}{2\pi\mathrm{i}}\frac{1}{(-\mathrm{i}\mathrm{e}^{\mathrm{i}\theta})^N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2\pi r\tau}R_M(\mathrm{i}r\tau)}{1-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}\,\mathrm{d}\tau\\ -\frac{1}{2\pi\mathrm{i}}\frac{1}{(\mathrm{i}\mathrm{e}^{\mathrm{i}\theta})^N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2\pi r\tau}R_M(-\mathrm{i}r\tau)}{1+\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}\,\mathrm{d}\tau. \tag{3.9}$$

We consider the first integral. Using the integral formula (3.5), $R_M(\mathrm{i}r\tau)$ can be written in the form

$$\begin{aligned}R_M(\mathrm{i}r\tau)&=\frac{1}{2\pi\mathrm{i}}\frac{1}{(r\tau\mathrm{e}^{-\mathrm{i}\varphi})^M}\int_0^{+\infty}\frac{t^{M-1}\mathrm{e}^{-2\pi t\mathrm{e}^{\mathrm{i}\varphi}}\Gamma^*(\mathrm{i}t\mathrm{e}^{\mathrm{i}\varphi})}{1-t\mathrm{e}^{\mathrm{i}\varphi}/r\tau}\,\mathrm{d}t\\ &\quad-\frac{1}{2\pi\mathrm{i}}\frac{1}{(-r\tau)^M}\int_0^{+\infty}\frac{t^{M-1}\mathrm{e}^{-2\pi t}\Gamma^*(-\mathrm{i}t)}{1+t/r\tau}\,\mathrm{d}t\\ &=\frac{1}{2\pi\mathrm{i}}\frac{1}{(r\tau\mathrm{e}^{-\mathrm{i}\varphi})^M}\biggl(\int_0^{+\infty}\frac{t^{M-1}\mathrm{e}^{-2\pi t\mathrm{e}^{\mathrm{i}\varphi}}\Gamma^*(\mathrm{i}t\mathrm{e}^{\mathrm{i}\varphi})}{1-t\mathrm{e}^{\mathrm{i}\varphi}/r}\,\mathrm{d}t\\ &\qquad+(\tau-1)\int_0^{+\infty}\frac{t^{M-1}\exp(-2\pi t\mathrm{e}^{\mathrm{i}\varphi})\Gamma^*(\mathrm{i}t\mathrm{e}^{\mathrm{i}\varphi})}{(1-r\tau/t\mathrm{e}^{\mathrm{i}\varphi})(1-t\mathrm{e}^{\mathrm{i}\varphi}/r)}\,\mathrm{d}t\biggr)\\ &\quad-\frac{1}{2\pi\mathrm{i}}\frac{1}{(-r\tau)^M}\biggl(\int_0^{+\infty}\frac{t^{M-1}\mathrm{e}^{-2\pi t}\Gamma^*(-\mathrm{i}t)}{1+t/r}\,\mathrm{d}t\\ &\qquad+(\tau-1)\int_0^{+\infty}\frac{t^{M-1}\mathrm{e}^{-2\pi t}\Gamma^*(-\mathrm{i}t)}{(1+r\tau/t)(1+t/r)}\,\mathrm{d}t\biggr).\end{aligned}$$

Therefore, the first integral in (3.9) can be estimated as follows

$$\begin{aligned}&\left|\frac{1}{2\pi\mathrm{i}}\frac{1}{(-\mathrm{i}\mathrm{e}^{\mathrm{i}\theta})^N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2\pi r\tau}R_M(\mathrm{i}r\tau)}{1-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}\,\mathrm{d}\tau\right|\\ &\quad\leqslant\frac{1}{2\pi r^M}\left|\int_0^{+\infty}\frac{t^{M-1}\exp(-2\pi t\mathrm{e}^{\mathrm{i}\varphi})\Gamma^*(\mathrm{i}t\mathrm{e}^{\mathrm{i}\varphi})}{1-t\mathrm{e}^{\mathrm{i}\varphi}/r}\,\mathrm{d}t\right|\left|\frac{1}{2\pi}\int_0^{+\infty}\frac{\tau^{N-M-1}\mathrm{e}^{-2\pi r\tau}}{1-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}\,\mathrm{d}\tau\right|\\ &\qquad+\frac{1}{2\pi r^M}\int_0^{+\infty}\tau^{N-M-1}\mathrm{e}^{-2\pi r\tau}\left|\frac{\tau-1}{\tau+\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}}\right|\\ &\qquad\qquad\times\left|\frac{1}{2\pi}\int_0^{+\infty}\frac{t^{M-1}\exp(-2\pi t\mathrm{e}^{\mathrm{i}\varphi})\Gamma^*(\mathrm{i}t\mathrm{e}^{\mathrm{i}\varphi})}{(1-r\tau/t\mathrm{e}^{\mathrm{i}\varphi})(1-t\mathrm{e}^{\mathrm{i}\varphi}/r)}\,\mathrm{d}t\right|\mathrm{d}\tau\end{aligned}$$

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$$
\begin{aligned}
&+\frac{1}{2\pi r^M}\left|\int_0^{+\infty}\frac{t^{M-1}\mathrm{e}^{-2\pi t}\Gamma^*(-\mathrm{i}t)}{1+t/r}\,\mathrm{d}t\right|\left|\frac{1}{2\pi}\int_0^{+\infty}\frac{\tau^{N-M-1}\mathrm{e}^{-2\pi r\tau}}{1-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}\,\mathrm{d}\tau\right|\\
&+\frac{1}{2\pi r^M}\int_0^{+\infty}\tau^{N-M-1}\mathrm{e}^{-2\pi r\tau}\left|\frac{\tau-1}{\tau+\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}}\right|\\
&\qquad\times\left|\frac{1}{2\pi}\int_0^{+\infty}\frac{t^{M-1}\mathrm{e}^{-2\pi t}\Gamma^*(-\mathrm{i}t)}{(1+r\tau/t)(1+t/r)}\,\mathrm{d}t\right|\mathrm{d}\tau.
\end{aligned}
$$

Noting that

$$
\left|\frac{\tau-1}{\tau+\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}}\right|\leqslant 1,\quad \left|\frac{1}{1+t/r}\right|\leqslant 1,\quad \left|\frac{1}{(1+r\tau/t)(1+t/r)}\right|\leqslant 1
$$

and

$$
\left|\frac{1}{1-t\mathrm{e}^{\mathrm{i}\varphi}/r}\right|\leqslant \csc\varphi,\quad \left|\frac{1}{(1-r\tau/t\mathrm{e}^{\mathrm{i}\varphi})(1-t\mathrm{e}^{\mathrm{i}\varphi}/r)}\right|\leqslant \csc^2\varphi
$$

for any positive r, t and τ, we deduce the upper bound

$$
\begin{aligned}
&\left|\frac{1}{2\pi\mathrm{i}}\frac{1}{(-\mathrm{i}\mathrm{e}^{\mathrm{i}\theta})^N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2\pi r\tau}R_M(\mathrm{i}r\tau)}{1-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}\,\mathrm{d}\tau\right|\\
&\quad\leqslant\frac{\csc\varphi}{2\pi r^M}\int_0^{+\infty}t^{M-1}|\exp(-2\pi t\mathrm{e}^{\mathrm{i}\varphi})\Gamma^*(\mathrm{i}t\mathrm{e}^{\mathrm{i}\varphi})|\,\mathrm{d}t\left|\frac{1}{2\pi}\int_0^{+\infty}\frac{\tau^{N-M-1}\mathrm{e}^{-2\pi r\tau}}{1-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}\,\mathrm{d}\tau\right|\\
&\qquad+\frac{\csc^2\varphi}{2\pi}\int_0^{+\infty}t^{M-1}|\exp(-2\pi t\mathrm{e}^{\mathrm{i}\varphi})\Gamma^*(\mathrm{i}t\mathrm{e}^{\mathrm{i}\varphi})|\,\mathrm{d}t\\
&\qquad\quad\times\frac{1}{2\pi r^M}\int_0^{+\infty}\tau^{N-M-1}\mathrm{e}^{-2\pi r\tau}\,\mathrm{d}\tau\\
&\qquad+\frac{1}{2\pi r^M}\int_0^{+\infty}t^{M-1}\mathrm{e}^{-2\pi t}|\Gamma^*(-\mathrm{i}t)|\,\mathrm{d}t\left|\frac{1}{2\pi}\int_0^{+\infty}\frac{\tau^{N-M-1}\mathrm{e}^{-2\pi r\tau}}{1-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}\,\mathrm{d}\tau\right|\\
&\qquad+\frac{1}{2\pi}\int_0^{+\infty}t^{M-1}\mathrm{e}^{-2\pi t}|\Gamma^*(-\mathrm{i}t)|\,\mathrm{d}t\,\frac{1}{2\pi r^M}\int_0^{+\infty}\tau^{N-M-1}\mathrm{e}^{-2\pi r\tau}\,\mathrm{d}\tau.
\end{aligned}
$$

A straightforward computation shows that this upper bound simplifies to

$$
\begin{aligned}
&\left|\frac{1}{2\pi\mathrm{i}}\frac{1}{(-\mathrm{i}\mathrm{e}^{\mathrm{i}\theta})^N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2\pi r\tau}R_M(\mathrm{i}r\tau)}{1-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}\,\mathrm{d}\tau\right|\\
&\quad\leqslant\left(\frac{\csc\varphi}{\cos^M\varphi}\frac{1}{2\pi}\int_0^{+\infty}t^{M-1}\mathrm{e}^{-2\pi t}\left|\Gamma^*\left(\frac{\mathrm{i}t\mathrm{e}^{\mathrm{i}\varphi}}{\cos\varphi}\right)\right|\mathrm{d}t\right.\\
&\qquad\qquad\left.+\frac{1}{2\pi}\int_0^{+\infty}t^{M-1}\mathrm{e}^{-2\pi t}|\Gamma^*(-\mathrm{i}t)|\,\mathrm{d}t\right)\frac{|\mathrm{e}^{2\pi\mathrm{i}z}\hat{T}_{N-M}(2\pi\mathrm{i}z)|}{|z|^M}\\
&\qquad+\left(\frac{\csc^2\varphi}{\cos^M\varphi}\frac{1}{2\pi}\int_0^{+\infty}t^{M-1}\mathrm{e}^{-2\pi t}\left|\Gamma^*\left(\frac{\mathrm{i}t\mathrm{e}^{\mathrm{i}\varphi}}{\cos\varphi}\right)\right|\mathrm{d}t\right.\\
&\qquad\qquad\left.+\frac{1}{2\pi}\int_0^{+\infty}t^{M-1}\mathrm{e}^{-2\pi t}|\Gamma^*(-\mathrm{i}t)|\,\mathrm{d}t\right)\frac{\Gamma(N-M)}{(2\pi)^{N-M+1}|z|^N}.
\end{aligned}
$$

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With the choice $\varphi = \arctan(M^{-1/2})$, we have

$$\frac{\csc\varphi}{\cos^M\varphi} = \left(\frac{M+1}{M}\right)^{(M+1)/2}\sqrt{M} < 2\sqrt{M}$$

and

$$\frac{\csc^2\varphi}{\cos^M\varphi} = \left(\frac{M+1}{M}\right)^{M/2+1} M < 3M.$$

By lemma 2.2, we obtain the estimate

$$\frac{1}{2\pi}\int_0^{+\infty} t^{M-1}\mathrm{e}^{-2\pi t}\left|\Gamma^*\left(\frac{\mathrm{i}t\mathrm{e}^{\mathrm{i}\varphi}}{\cos\varphi}\right)\right|\mathrm{d}t \leqslant \frac{1}{2\pi}\int_0^{+\infty}\frac{t^{M-1}\mathrm{e}^{-2\pi t}}{1-\mathrm{e}^{-2\pi t}}\,\mathrm{d}t = \frac{\zeta(M)\Gamma(M)}{(2\pi)^{M+1}}.$$

The other type of integral can be bounded by the same quantity since

$$\frac{1}{2\pi}\int_0^{+\infty} t^{M-1}\mathrm{e}^{-2\pi t}|\Gamma^*(-\mathrm{i}t)|\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{+\infty}\frac{t^{M-1}\mathrm{e}^{-2\pi t}}{\sqrt{1-\mathrm{e}^{-2\pi t}}}\,\mathrm{d}t < \frac{\zeta(M)\Gamma(M)}{(2\pi)^{M+1}}.$$

Therefore, we find

$$\begin{aligned}\left|\frac{1}{2\pi\mathrm{i}}\frac{1}{(-\mathrm{i}\mathrm{e}^{\mathrm{i}\theta})^N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2\pi r\tau}R_M(\mathrm{i}r\tau)}{1-\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}\,\mathrm{d}\tau\right| &\leqslant (2\sqrt{M}+1)\frac{\zeta(M)\Gamma(M)}{(2\pi)^{M+1}|z|^M}|\mathrm{e}^{2\pi\mathrm{i}z}\hat{T}_{N-M}(2\pi\mathrm{i}z)| \\ &\quad + (3M+1)\frac{\zeta(M)\Gamma(M)\Gamma(N-M)}{(2\pi)^{N+2}|z|^N}.\end{aligned}$$

Similarly, we have the following upper bound for the second integral in (3.9):

$$\begin{aligned}\left|\frac{1}{2\pi\mathrm{i}}\frac{1}{(\mathrm{i}\mathrm{e}^{\mathrm{i}\theta})^N}\int_0^{+\infty}\frac{\tau^{N-1}\mathrm{e}^{-2\pi r\tau}R_M(-\mathrm{i}r\tau)}{1+\mathrm{i}\tau\mathrm{e}^{-\mathrm{i}\theta}}\,\mathrm{d}\tau\right| &\leqslant (2\sqrt{M}+1)\frac{\zeta(M)\Gamma(M)}{(2\pi)^{M+1}|z|^M}|\mathrm{e}^{-2\pi\mathrm{i}z}\hat{T}_{N-M}(-2\pi\mathrm{i}z)| \\ &\quad + (3M+1)\frac{\zeta(M)\Gamma(M)\Gamma(N-M)}{(2\pi)^{N+2}|z|^N}.\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}|R_{N,M}(z)| \leqslant (2\sqrt{M}+1)&\frac{\zeta(M)\Gamma(M)}{(2\pi)^{M+1}|z|^M} \\ &\times(|\mathrm{e}^{2\pi\mathrm{i}z}\hat{T}_{N-M}(2\pi\mathrm{i}z)| + |\mathrm{e}^{-2\pi\mathrm{i}z}\hat{T}_{N-M}(-2\pi\mathrm{i}z)|) \\ &+ (6M+2)\frac{\zeta(M)\Gamma(M)\Gamma(N-M)}{(2\pi)^{N+2}|z|^N}.\end{aligned}$$

By continuity, this bound holds in the closed sector $|\arg z| \leqslant \frac{1}{2}\pi$. Suppose that $N = 2\pi|z| + \rho$, where ρ is a bounded quantity. Employing Stirling's formula, we find that

$$(6M+2)\frac{\zeta(M)\Gamma(M)\Gamma(N-M)}{(2\pi)^{N+2}|z|^N} = \mathcal{O}_{M,\rho}\left(\frac{\mathrm{e}^{-2\pi|z|}}{|z|^{M+1/2}}\right)$$

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as $z \to \infty$. Olver's estimation (3.1) shows that

$$(2\sqrt{M}+1)\frac{\zeta(M)\Gamma(M)}{(2\pi)^{M+1}|z|^M}(|\mathrm{e}^{2\pi\mathrm{i}z}\hat{T}_{N-M}(2\pi\mathrm{i}z)| + |\mathrm{e}^{-2\pi\mathrm{i}z}\hat{T}_{N-M}(-2\pi\mathrm{i}z)|) = \mathcal{O}_{M,\rho}\left(\frac{\mathrm{e}^{-2\pi|z|}}{|z|^M}\right)$$

for large z. Therefore, we have that

$$R_{N,M}(z) = \mathcal{O}_{M,\rho}\left(\frac{\mathrm{e}^{-2\pi|z|}}{|z|^M}\right) \tag{3.10}$$

as $z \to \infty$ in the sector $|\arg z| \leqslant \frac{1}{2}\pi$. If $M = 0$ or 1, we define $R_{N,M}(z)$ by (3.8). Consequently,

$$R_{N,1}(z) = \frac{\mathrm{e}^{-2\pi\mathrm{i}z}\hat{T}_{N-1}(-2\pi\mathrm{i}z) - \mathrm{e}^{2\pi\mathrm{i}z}\hat{T}_{N-1}(2\pi\mathrm{i}z)}{12z} + R_{N,2}(z), \tag{3.11}$$

$$R_{N,0}(z) = \mathrm{e}^{2\pi\mathrm{i}z}\hat{T}_N(2\pi\mathrm{i}z) - \mathrm{e}^{-2\pi\mathrm{i}z}\hat{T}_N(-2\pi\mathrm{i}z) + R_{N,1}(z). \tag{3.12}$$

The proof of (1.12) for the cases $M = 0, 1$ now follows from these representations, the bound (3.10) we have established and Olver's estimate, (3.1).

The proof of the expansion (1.11) and the estimates (1.12), (1.13) for the remainder $\tilde{R}_{N,M}(z)$ is similar.

Consider now the sector $\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$. Assume again that $M \geqslant 2$. When z enters the sector $\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$, the pole in the first integral in (3.7) crosses the integration path. According to the residue theorem, we obtain

$$\begin{aligned} R_{N,M}(z) &= \mathrm{e}^{2\pi\mathrm{i}z}R_M(z) + \frac{1}{2\pi\mathrm{i}}\frac{\mathrm{i}^N}{z^N}\int_0^{+\infty}\frac{s^{N-1}\mathrm{e}^{-2\pi s}R_M(\mathrm{i}s)}{1-\mathrm{i}s/z}\,\mathrm{d}s \\ &\quad - \frac{1}{2\pi\mathrm{i}}\frac{(-\mathrm{i})^N}{z^N}\int_0^{+\infty}\frac{s^{N-1}\mathrm{e}^{-2\pi s}R_M(-\mathrm{i}s)}{1+\mathrm{i}s/z}\,\mathrm{d}s \\ &= \mathrm{e}^{2\pi\mathrm{i}z}R_M(z) + \tilde{R}_{N,M}(z\mathrm{e}^{-\pi\mathrm{i}}) \end{aligned}$$

when $\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$. To find the analytic continuation of $R_M(z)$ to this sector, we apply the same argument but for the integral representation (2.1) to deduce

$$\begin{aligned} R_M(z) &= \mathrm{e}^{2\pi\mathrm{i}z}\Gamma^*(z) + \frac{1}{2\pi\mathrm{i}}\frac{\mathrm{i}^M}{z^M}\int_0^{+\infty}\frac{s^{M-1}\mathrm{e}^{-2\pi s}\Gamma^*(\mathrm{i}s)}{1-\mathrm{i}s/z}\,\mathrm{d}s \\ &\quad - \frac{1}{2\pi\mathrm{i}}\frac{(-\mathrm{i})^M}{z^M}\int_0^{+\infty}\frac{s^{M-1}\mathrm{e}^{-2\pi s}\Gamma^*(-\mathrm{i}s)}{1+\mathrm{i}s/z}\,\mathrm{d}s \\ &= \mathrm{e}^{2\pi\mathrm{i}z}\Gamma^*(z) + \tilde{R}_M(z\mathrm{e}^{-\pi\mathrm{i}}) \\ &= \frac{1}{\mathrm{e}^{-2\pi\mathrm{i}z}-1}\frac{1}{\Gamma^*(z\mathrm{e}^{-\pi\mathrm{i}})} + \tilde{R}_M(z\mathrm{e}^{-\pi\mathrm{i}}). \end{aligned}$$

Here we have made use of the connection formula (1.9). Therefore, the analytic continuation of the expansion (3.8) to the sector $\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$ can be obtained by setting

$$R_{N,M}(z) = \frac{e^{2\pi i z}}{e^{-2\pi i z} - 1}\frac{1}{\Gamma^*(ze^{-\pi i})} + e^{2\pi i z}\tilde{R}_M(ze^{-\pi i}) + \tilde{R}_{N,M}(ze^{-\pi i}).$$

In the proof of lemma 2.2 we showed that the reciprocal scaled gamma function is bounded in the right half-plane; hence, by theorem 1.3 and the estimate (1.12) we infer that

$$\begin{aligned} R_{N,M}(z) &= \mathcal{O}\left(\frac{e^{-2\pi \operatorname{Im}(z)}}{|1 - e^{-2\pi i z}|}\right) + \mathcal{O}_M\left(\frac{e^{-2\pi \operatorname{Im}(z)}}{|z|^M}\right) + \mathcal{O}_{M,\rho}\left(\frac{e^{-2\pi|z|}}{|z|^M}\right) \\ &= \mathcal{O}_{M,\rho}\left(e^{-2\pi \operatorname{Im}(z)}\left(\frac{1}{|1 - e^{-2\pi i z}|} + \frac{1}{|z|^M}\right)\right) \end{aligned}$$

as $z \to \infty$ in the closed sector $\frac{1}{2}\pi \leqslant \arg z \leqslant \frac{3}{2}\pi$. The extension to the cases $M = 0, 1$ follows from (3.1), (3.11) and (3.12). Similarly, we find that

$$\tilde{R}_{N,M}(z) = -e^{2\pi i z}R_M(ze^{-\pi i}) + R_{N,M}(ze^{-\pi i})$$

for $\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$, and therefore

$$\tilde{R}_{N,M}(z) = \mathcal{O}_M\left(\frac{e^{-2\pi \operatorname{Im}(z)}}{|z|^M}\right) + \mathcal{O}_{M,\rho}\left(\frac{e^{-2\pi|z|}}{|z|^M}\right) = \mathcal{O}_{M,\rho}\left(\frac{e^{-2\pi \operatorname{Im}(z)}}{|z|^M}\right)$$

as $z \to \infty$ in the sector $\frac{1}{2}\pi \leqslant \arg z \leqslant \frac{3}{2}\pi$. The proof of the corresponding estimates for the sector $-\frac{3}{2}\pi \leqslant \arg z \leqslant -\frac{1}{2}\pi$ is completely analogous.

3.2. Stokes phenomenon and Berry's transition

It was shown by Paris and Wood [27, (3.2) and (3.4)] that the asymptotic expansions

$$\log \Gamma^*(z) \sim \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}} - \begin{cases} 0 & \text{if } |\arg z| < \frac{1}{2}\pi, \\ \frac{1}{2}\log(1 - e^{\pm 2\pi i z}) & \text{if } \arg z = \pm\frac{1}{2}\pi, \\ \log(1 - e^{\pm 2\pi i z}) & \text{if } \frac{1}{2}\pi < \pm \arg z < \pi, \end{cases} \tag{3.13}$$

hold as $z \to \infty$. Expanding the logarithm into its Taylor series yields

$$\log \Gamma^*(z) \sim \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}} + \sum_{k=1}^{\infty} S_k(\theta)e^{\pm 2\pi i k z} \tag{3.14}$$

as $z \to \infty$ in the sector $|\arg z| \leqslant \pi - \delta < \pi$ for any $0 < \delta \leqslant \pi$. Here

$$S_k(\theta) = \begin{cases} \dfrac{1}{k} & \text{if } \frac{1}{2}\pi < |\theta| < \pi, \\ \dfrac{1}{2k} & \text{if } \theta = \pm\frac{1}{2}\pi, \\ 0 & \text{if } |\theta| < \frac{1}{2}\pi, \end{cases} \tag{3.15}$$

are the Stokes multipliers and $\theta = \arg z$. The upper or lower sign is taken in (3.14) and (3.15) according to whether z is in the upper or lower half-plane. Taking the

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exponential of both sides in (3.14), we arrive at the expansions

$$\Gamma^*(z) \sim \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_n}{z^n} + \mathscr{S}_1(\theta)\mathrm{e}^{\pm 2\pi \mathrm{i} z} \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_n}{z^n} + \cdots + \mathscr{S}_k(\theta)\mathrm{e}^{\pm 2\pi \mathrm{i} k z} \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_n}{z^n} + \cdots \tag{3.16}$$

and

$$\frac{1}{\Gamma^*(z)} \sim \sum_{n=0}^{\infty} \frac{\gamma_n}{z^n} + \tilde{\mathscr{S}}_1(\theta)\mathrm{e}^{\pm 2\pi \mathrm{i} z} \sum_{n=0}^{\infty} \frac{\gamma_n}{z^n} + \cdots + \tilde{\mathscr{S}}_k(\theta)\mathrm{e}^{\pm 2\pi \mathrm{i} k z} \sum_{n=0}^{\infty} \frac{\gamma_n}{z^n} + \cdots \tag{3.17}$$

as $z \to \infty$ in the sector $|\arg z| \leqslant \pi - \delta < \pi$, $0 < \delta \leqslant \pi$, with the Stokes multipliers

$$\mathscr{S}_k(\theta) = \begin{cases} 1 & \text{if } \frac{1}{2}\pi < |\theta| < \pi, \\ \dfrac{1}{k!}\left(\dfrac{1}{2}\right)_k & \text{if } \theta = \pm\frac{1}{2}\pi, \\ 0 & \text{if } |\theta| < \frac{1}{2}\pi, \end{cases}$$

and

$$\tilde{\mathscr{S}}_1(\theta) = \begin{cases} -1 & \text{if } \frac{1}{2}\pi < |\theta| < \pi, \\ -\frac{1}{2} & \text{if } \theta = \pm\frac{1}{2}\pi, \\ 0 & \text{if } |\theta| < \frac{1}{2}\pi, \end{cases}$$

$$\tilde{\mathscr{S}}_k(\theta) = \begin{cases} 0 & \text{if } \frac{1}{2}\pi < |\theta| < \pi, \\ \dfrac{1}{k!}\left(-\dfrac{1}{2}\right)_k & \text{if } \theta = \pm\frac{1}{2}\pi, \\ 0 & \text{if } |\theta| < \frac{1}{2}\pi \end{cases} \qquad \text{for } k \geqslant 2.$$

Here $(x)_k = \Gamma(x+k)/\Gamma(x)$ stands for the Pochhammer symbol [23, p. 136]. It is seen that there is a discontinuous change in the coefficients of the exponential terms when $\arg z$ changes continuously across $\arg z = \pm\frac{1}{2}\pi$. We have encountered a Stokes phenomenon with Stokes lines $\arg z = \pm\frac{1}{2}\pi$. The formulae for $\mathscr{S}_1(\theta)$ and $\tilde{\mathscr{S}}_1(\theta)$ are in agreement with Dingle's non-rigorous 'final main rule' in his theory of terminants [13, p. 414], namely that half the discontinuity occurs on reaching the Stokes ray, and half on leaving it the other side. However, for the higher-order Stokes multipliers this rule is no longer valid.

The interesting behaviour of the asymptotic series for the reciprocal gamma function is worth noting. On the Stokes lines $\arg z = \pm\frac{1}{2}\pi$, infinitely many subdominant exponential terms appear in the expansion and, as $\arg z$ passes through the values $\pm\frac{1}{2}\pi$, all but one of them disappear.

In the important paper [1], Berry provided a new interpretation of the Stokes phenomenon; he found that, assuming optimal truncation, the transition between compound asymptotic expansions is of error-function type, thus yielding a smooth, although very rapid, transition as a Stokes line is crossed.

Using the exponentially improved expansions given in theorem 1.4, we show that the asymptotic expansions of the gamma function and its reciprocal exhibit the Berry transition between the two asymptotic series across the Stokes lines $\arg z = \pm\frac{1}{2}\pi$. More precisely, we shall find that the first few terms of the series in (3.16) and (3.17), corresponding to the subdominant exponentials $\mathrm{e}^{\pm 2\pi \mathrm{i}z}$, 'emerge' in a rapid and smooth way as $\arg z$ passes through $\pm\frac{1}{2}\pi$.

From theorem 1.4, we conclude that if $N \approx 2\pi|z|$, then for large z, $|\arg z| < \pi$, we have

$$\Gamma^*(z) \approx \sum_{n=0}^{N-1} (-1)^n \frac{\gamma_n}{z^n} + \mathrm{e}^{2\pi\mathrm{i}z} \sum_{m=0} (-1)^m \frac{\gamma_m}{z^m} \hat{T}_{N-m}(2\pi\mathrm{i}z) - \mathrm{e}^{-2\pi\mathrm{i}z} \sum_{m=0} (-1)^m \frac{\gamma_m}{z^m} \hat{T}_{N-m}(-2\pi\mathrm{i}z)$$

and

$$\frac{1}{\Gamma^*(z)} \approx \sum_{n=0}^{N-1} \frac{\gamma_n}{z^n} - \mathrm{e}^{2\pi\mathrm{i}z} \sum_{m=0} \frac{\gamma_m}{z^m} \hat{T}_{N-m}(2\pi\mathrm{i}z) + \mathrm{e}^{-2\pi\mathrm{i}z} \sum_{m=0} \frac{\gamma_m}{z^m} \hat{T}_{N-m}(-2\pi\mathrm{i}z),$$

where $\sum_{m=0}$ means that the sum is restricted to the first few terms of the series. In the upper half-plane the terms involving $\hat{T}_{N-m}(-2\pi\mathrm{i}z)$ are exponentially small, the dominant contribution comes from the terms involving $\hat{T}_{N-m}(2\pi\mathrm{i}z)$. Under the above assumptions on N, from (3.2) and (3.4), the terminant functions have the asymptotic behaviour

$$\hat{T}_{N-m}(2\pi\mathrm{i}z) \sim \tfrac{1}{2} + \tfrac{1}{2}\operatorname{erf}((\theta - \tfrac{1}{2}\pi)\sqrt{\pi|z|})$$

provided that $\arg z = \theta$ is close to $\frac{1}{2}\pi$, z is large and m is small in comparison to N. Therefore, when $\theta < \frac{1}{2}\pi$, the terminant functions are exponentially small; for $\theta = \frac{1}{2}\pi$, they are asymptotically $\frac{1}{2}$ up to an exponentially small error; when $\theta > \frac{1}{2}\pi$, the terminant functions are asymptotic to 1 with an exponentially small error. Thus, the transition across the Stokes line $\arg z = \frac{1}{2}\pi$ is effected rapidly and smoothly. Similarly, in the lower half-plane, the dominant contribution is controlled by the terms involving $\hat{T}_{N-m}(-2\pi\mathrm{i}z)$. From (3.3) and (3.4), we have

$$\hat{T}_{N-m}(-2\pi\mathrm{i}z) \sim -\tfrac{1}{2} + \tfrac{1}{2}\operatorname{erf}((\theta + \tfrac{1}{2}\pi)\sqrt{\pi|z|})$$

under the assumptions that $\arg z = \theta$ is close to $-\frac{1}{2}\pi$, z is large and m is small in comparison with $N \approx 2\pi|z|$. Thus, when $\theta > -\frac{1}{2}\pi$, the terminant functions are exponentially small; for $\theta = -\frac{1}{2}\pi$, they are asymptotic to $-\frac{1}{2}$ with an exponentially small error; and when $\theta < -\frac{1}{2}\pi$, the terminant functions are asymptotically -1 up to an exponentially small error. Therefore, the transition through the Stokes line $\arg z = -\frac{1}{2}\pi$ is carried out rapidly and smoothly.

We remark that the smooth transition of the subdominant exponential $\mathrm{e}^{2\pi\mathrm{i}z}$ was also discussed by Boyd [6], though he used a slightly different approximation for the terminant functions.

4. The asymptotics of the late coefficients

The asymptotic form of the Stirling coefficients γ_n is well known. Their leading-order behaviour was investigated by Watson [32] using the method of Darboux, and by Diekmann [12] using the method of steepest descents. Murnaghan and Wrench [16] gave higher approximations by employing Darboux's method. Complete asymptotic expansions were derived by Dingle [13, pp. 158–159], though his results were obtained by methods that were formal and interpretive, rather than rigorous. His expansions may be written, in our notation, as

$$\gamma_{2n-1} \approx \frac{(-1)^n 2}{(2\pi)^{2n}} \sum_{k=0}^{\infty} (-1)^k \gamma_{2k} (2\pi)^{2k} \Gamma(2n-2k-1)\zeta(2n-2k) \tag{4.1}$$

and

$$\gamma_{2n} \approx \frac{(-1)^n 2}{(2\pi)^{2n}} \sum_{k=0}^{\infty} (-1)^k \gamma_{2k+1} (2\pi)^{2k} \Gamma(2n-2k-1)\zeta(2n-2k). \tag{4.2}$$

Finally, Boyd [6] gave expansions similar to Dingle's, complete with error bounds, using truncated forms of the approximations

$$\Gamma^*(z) \sim \sum_{k=0}^{\infty} (-1)^k \frac{\gamma_k}{z^k} \quad \text{and} \quad \Gamma^*(z) \sim \frac{1}{1-\mathrm{e}^{2\pi \mathrm{i} z}} \sum_{k=0}^{\infty} (-1)^k \frac{\gamma_k}{z^k}$$

with $z = \mathrm{i}s$ $(s > 0)$ and his resurgence formula for the gamma function. Although both expansions are valid along the positive imaginary axis in Poincaré's sense, from (3.13) it is seen that the first one is more suitable when $|\arg z| < \frac{1}{2}\pi$ and the second one is more suitable when $\frac{1}{2}\pi < \arg z < \pi$. In our notation, Boyd's results can be written as

$$\gamma_{2n-1} = \frac{(-1)^n 2}{(2\pi)^{2n}} \sum_{k=0}^{K-1} (-1)^k \gamma_{2k} (2\pi)^{2k} \Gamma(2n-2k-1) + M_K(2n-1), \tag{4.3}$$

$$\gamma_{2n} = \frac{(-1)^n 2}{(2\pi)^{2n}} \sum_{k=0}^{K-1} (-1)^k \gamma_{2k+1} (2\pi)^{2k} \Gamma(2n-2k-1) + M_K(2n) \tag{4.4}$$

and

$$\gamma_{2n-1} = \frac{(-1)^n 2}{(2\pi)^{2n}} \sum_{k=0}^{K-1} (-1)^k \gamma_{2k} (2\pi)^{2k} \Gamma(2n-2k-1)\zeta(2n-2k-1) + \hat{M}_K(2n-1), \tag{4.5}$$

$$\gamma_{2n} = \frac{(-1)^n 2}{(2\pi)^{2n}} \sum_{k=0}^{K-1} (-1)^k \gamma_{2k+1} (2\pi)^{2k} \Gamma(2n-2k-1)\zeta(2n-2k-1) + \hat{M}_K(2n). \tag{4.6}$$

Here $1 \leqslant K < n$ and the truncation errors M_K and $\hat{M}_K$ can be bounded explicitly and realistically.

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Boyd observed that estimating the error term $M_K(2n-1)$ in (4.3) via the exponentially improved expansion of the gamma function (1.10) along the imaginary axis leads to an improved version of the late coefficient formula, (4.3). His improved expansion [6, (3.42)] also shed some light on the idea behind Dingle's formula, (4.1), especially on the appearance of Riemann's zeta function in the approximation and its numerical superiority over Boyd's formula (4.3).

The main goal of this section is to derive new asymptotic series for the Stirling coefficients using the representations (2.3) and (2.4) and an exponentially improved asymptotic expansion for the gamma function. These new expansions use all the exponentially small terms in (3.16) and provide a full explanation of the remarkable accuracy of Dingle's series (4.1) and (4.2). From (3.13), we see that

$$\Gamma^*(\mathrm{i}s) \sim \frac{1}{\sqrt{1-\mathrm{e}^{-2\pi s}}} \sum_{k=0}^{\infty} (-1)^k \frac{\gamma_k}{(\mathrm{i}s)^k} \tag{4.7}$$

as $s \to +\infty$. Consequently, we have

$$\operatorname{Re} \Gamma^*(\mathrm{i}s) \sim \frac{1}{\sqrt{1-\mathrm{e}^{-2\pi s}}} \sum_{k=0}^{\infty} (-1)^k \frac{\gamma_{2k}}{s^{2k}}$$

and

$$\operatorname{Im} \Gamma^*(\mathrm{i}s) \sim \frac{1}{\sqrt{1-\mathrm{e}^{-2\pi s}}} \sum_{k=0}^{\infty} (-1)^k \frac{\gamma_{2k+1}}{s^{2k+1}}$$

as $s \to +\infty$. Substituting these expansions into (2.3) and (2.4) yields the formal asymptotic series

$$\gamma_{2n-1} \approx \frac{(-1)^n 2}{(2\pi)^{2n}} \sum_{k=0}^{\infty} (-1)^k \gamma_{2k} (2\pi)^{2k} \Gamma(2n-2k-1) \xi(2n-2k-1) \tag{4.8}$$

and

$$\gamma_{2n} \approx \frac{(-1)^n 2}{(2\pi)^{2n}} \sum_{k=0}^{\infty} (-1)^k \gamma_{2k+1} (2\pi)^{2k} \Gamma(2n-2k-1) \xi(2n-2k-1) \tag{4.9}$$

for large n. Here, the function $\xi(r)$ is given by the Dirichlet series

$$\begin{aligned} \xi(r) &= \frac{(2\pi)^r}{\Gamma(r)} \int_0^{+\infty} \frac{s^{r-1}\mathrm{e}^{-2\pi s}}{\sqrt{1-\mathrm{e}^{-2\pi s}}}\,\mathrm{d}s \\ &= \sum_{m=0}^{\infty} \frac{(1/2)_m}{m!(m+1)^r} \\ &= 1 + \frac{1}{2}\frac{1}{2^r} + \frac{3}{8}\frac{1}{3^r} + \frac{5}{16}\frac{1}{4^r} + \frac{35}{128}\frac{1}{5^r} + \cdots, \end{aligned}$$

provided that $r > \frac{1}{2}$. The formal expansions in (4.8) and (4.9) can be turned into exact results by constructing error bounds for the series (4.7), but we do not pursue the details here. We shall assume that optimal truncation of these series provides good approximations for the Stirling coefficients γ_n.

Table 1. *Approximations for γ_{101}, using optimal truncation.*

Values of n and K	$n = 51$, $K = 26$
Exact numerical value of γ_{2n-1}	$-0.718920823005286472090671337669485196245 \times 10^{77}$
Dingle's approximation (4.1) to γ_{2n-1}	$-0.718920823005286472090671337669485196372 \times 10^{77}$
Error	0.127×10^{41}
Boyd's approximation (4.3) to γ_{2n-1}	$-0.718920823005286472090671337669343420137 \times 10^{77}$
Error	$-0.141776108 \times 10^{47}$
Boyd's approximation (4.5) to γ_{2n-1}	$-0.718920823005286472090671337669626972607 \times 10^{77}$
Error	$0.141776362 \times 10^{47}$
Approximation (4.8) to γ_{2n-1}	$-0.718920823005286472090671337669485196372 \times 10^{77}$
Error	0.127×10^{41}

For large n and fixed k, we have

$$\zeta(2n-2k) \approx 1 + \frac{1}{2^{2n-2k}} + \frac{1}{3^{2n-2k}},$$
$$\zeta(2n-2k-1) \approx 1 + 2\frac{1}{2^{2n-2k}} + 3\frac{1}{3^{2n-2k}}$$

and

$$\xi(2n-2k-1) \approx 1 + \frac{1}{2^{2n-2k}} + \frac{9}{8}\frac{1}{3^{2n-2k}}.$$

These approximations explain Boyd's observation, namely that, assuming optimal truncation, Dingle's expansions provide better approximations than Boyd's original series. We also get a numerical explanation of the appearance of Riemann's zeta function in Dingle's expansions.

We remark that Boyd's improved series [6, (3.42)] for γ_{2n-1} is (4.8) with the approximate values

$$\xi(2n-2k-1) \approx \begin{cases} 1 + 2^{-2n} & \text{if } k = 0, \\ 1 & \text{if } k > 0. \end{cases}$$

In our calculations we truncated the expansions of Dingle (like Boyd did) and our expansions at $k = K - 1$ and chose the value of K optimally. Optimality here means that we choose K in terms of n such that the last term of the remaining series is asymptotically the smallest in absolute value for large n. It can be shown that the optimal choice of K for both the expansions of γ_{2n-1} and γ_{2n} is $K = \lceil \frac{1}{2}n \rceil$, i.e. in the sums k runs from 0 to $\lceil \frac{1}{2}n \rceil - 1$. Tables 1 and 2 display the numerical results obtained for the coefficients γ_{101} and γ_{100} by using the optimally truncated approximations of Dingle, Boyd and ourselves.

It can be seen from the numerical computations that our expansions provide better approximations than that of Boyd's and are comparable with the expansions

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Table 2. *Approximations for γ_{100}, using optimal truncation.*

Values of n and K	$n = 50$, $K = 25$
Exact numerical value of γ_{2n}	$-0.23893978966159359567744753712975301 2\times 10^{74}$
Dingle's approximation (4.2) to γ_{2n}	$-0.238939789661593595677447537129753175 \times 10^{74}$
Error	0.163×10^{41}
Boyd's approximation (4.4) to γ_{2n}	$-0.238939789661593595677447537129564608 \times 10^{74}$
Error	-0.188403×10^{44}
Boyd's approximation (4.6) to γ_{2n}	$-0.238939789661593595677447537129941741 \times 10^{74}$
Error	0.188729×10^{44}
Approximation (4.9) to γ_{2n}	$-0.238939789661593595677447537129753175 \times 10^{74}$
Error	0.163×10^{41}

of Dingle. We remark that the approximate numerical value of γ_{100} arising from Boyd's formula (4.6) was given incorrectly in [6, table 4].

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Appendix A. Computation of the Stirling coefficients

In this appendix we have collected some known recurrence representations of the Stirling coefficients, γ_n. The exact values of γ_n up to γ_{30} can be found in [29, 34]. Explicit formulae for the Stirling coefficients are given by Boyd [7], Brassesco and Méndez [8], Comtet [9, p. 267], De Angelis [11], López *et al*. [15], Nemes [17] and Wrench [34].

A.1. Recurrence relations

Based on the Lagrange inversion formula, Brassesco and Méndez [8] find recursive formulae for the computation of the Stirling coefficients. Define the sequence b_n by the recurrence relation

$$b_n = \frac{2-n}{3n+3} b_{n-1} - \frac{1}{n+1} \sum_{k=2}^{n-3} (k+1) b_{k+1} b_{n-k} \tag{A 1}$$

for $n \geqslant 4$ with $b_1 = 1$, $b_3 = \frac{1}{36}$. Then the coefficients γ_n can be computed as

$$\gamma_n = (-1)^n \frac{(2n+1)!}{2^n n!} b_{2n+1}.$$

This recurrence was also given by Borwein and Corless [3].

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Upon replacing k by $n-k-1$ in the sum, we see that the recurrence relation (A 1) may be written in the form

$$b_n = \frac{2-n}{3n+3} b_{n-1} - \frac{1}{2} \sum_{k=2}^{n-3} b_{k+1} b_{n-k}.$$

This formula was also found by Copson [10, p. 56].

Wrench [34] found recurrence formulae in terms of the Bernoulli numbers B_k, since

$$\gamma_{2n-1} = -\frac{1}{2n-1} \sum_{k=1}^{n} \frac{B_{2k}}{2k} \gamma_{2n-2k} \quad \text{and} \quad \gamma_{2n} = -\frac{1}{2n} \sum_{k=1}^{n} \frac{B_{2k}}{2k} \gamma_{2n-2k+1}$$

for $n \geqslant 1$ with $\gamma_0 = 1$. To derive these results, he used the formal generating function

$$\exp\left(\sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} x^{2n-1} \right) = \exp\left(\sum_{n=1}^{\infty} \frac{B_{n+1}}{n(n+1)} x^n \right) = \sum_{n=0}^{\infty} (-1)^n \gamma_n x^n, \quad \text{(A 2)}$$

which follows from (1.7). We derive here another type of recurrence formula using the generating function (A 2). Differentiating both sides of (A 2) with respect to x and dividing each side by the exponential expression on the left-hand side of (A 2), we find

$$\sum_{n=1}^{\infty} \frac{B_{2n}}{2n} x^{2n-2} = \exp\left(-\sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} x^{2n-1} \right) \sum_{n=1}^{\infty} (-1)^n n \gamma_n x^{n-1}.$$

Noting that

$$\exp\left(-\sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} x^{2n-1} \right) = \exp\left(\sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} (-x)^{2n-1} \right) = \sum_{n=0}^{\infty} \gamma_n x^n \quad \text{(A 3)}$$

and equating the coefficients of equal powers of x, we deduce the recursive formulae

$$\left. \begin{aligned} \gamma_{2n-1} &= -\frac{B_{2n}}{2n(2n-1)} + \frac{1}{2n-1} \sum_{k=1}^{2n-2} (-1)^k k \gamma_k \gamma_{2n-k-1}, \\ \gamma_{2n} &= -\frac{1}{2n} \sum_{k=1}^{2n-1} (-1)^k k \gamma_k \gamma_{2n-k} \end{aligned} \right\} \quad \text{(A 4)}$$

for $n \geqslant 1$ with $\gamma_0 = 1$. From (A 2) and (A 3) we can immediately obtain the known expression

$$\sum_{k=0}^{n} (-1)^k \gamma_k \gamma_{n-k} = 0$$

for $n \geqslant 1$ (see, for example, [26, p. 33]). When n is odd, this is a simple identity; for $n \geqslant 2$ even it gives

$$\gamma_n = -\frac{1}{2} \sum_{k=1}^{n-1} (-1)^k \gamma_k \gamma_{n-k},$$

which is equivalent to the second recurrence in (A 4).

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A.2. Representations using polynomial sequences

In 1952, Lauwerier [14] showed that the coefficients in asymptotic expansions of Laplace-type integrals can be calculated by means of linear recurrence relations. As an illustration of his method, he considered, *inter alia*, the Stirling coefficients γ_n. Define the sequence of polynomials $P_0(x), P_1(x), P_2(x), \dots$ via the recurrence

$$P_n(x) = -P_{n-1}(x) + \tfrac{1}{2}x^{-n/2}\int_0^x t^{n/2}P_{n-1}(t)\,\mathrm{d}t$$

for $n \geqslant 1$ with $P_0(x) = 1$. Then the coefficients γ_n can be recovered from the formula

$$\gamma_n = \frac{(-1)^n}{\sqrt{2\pi}}\int_0^{+\infty} \mathrm{e}^{-t/2}t^{n-1/2}P_{2n}(t)\,\mathrm{d}t.$$

It is known that the Stirling coefficients are related to certain polynomials $U_n(x)$ appearing in the uniform asymptotic expansions of the modified Bessel functions. These polynomials satisfy the following recurrence:

$$U_n(x) = \tfrac{1}{2}x^2(1-x^2)U'_{n-1}(x) + \frac{1}{8}\int_0^x (1-5t^2)U_{n-1}(t)\,\mathrm{d}t$$

for $n \geqslant 1$ with $U_0(x) = 1$. The coefficients γ_n are then given by $\gamma_n = U_n(1)$ (see, for example, [18]).

References

1 M. V. Berry. Uniform asymptotic smoothing of Stokes's discontinuities. *Proc. R. Soc. Lond.* A **422** (1989), 7–21.

2 M. V. Berry and C. J. Howls. Hyperasymptotics for integrals with saddles. *Proc. R. Soc. Lond.* A **434** (1991), 657–675.

3 J. M. Borwein and R. M. Corless. Emerging tools for experimental mathematics. *Am. Math. Mon.* **106** (1999), 899–909.

4 W. G. C. Boyd. Stieltjes transforms and the Stokes phenomenon. *Proc. R. Soc. Lond.* A **429** (1990), 227–246.

5 W. G. C. Boyd. Error bounds for the method of steepest descents. *Proc. R. Soc. Lond.* A **440** (1993), 493–518.

6 W. G. C. Boyd. Gamma function asymptotics by an extension of the method of steepest descents. *Proc. R. Soc. Lond.* A **447** (1994), 609–630.

7 W. G. C. Boyd. Approximations for the late coefficients in asymptotic expansions arising in the method of steepest descents. *Meth. Applic. Analysis* **2** (1995), 475–489.

8 S. Brassesco and M. A. Méndez. The asymptotic expansion for $n!$ and the Lagrange inversion formula. *Ramanujan J.* **24** (2011), 219–234.

9 L. Comtet. *Advanced combinatorics* (Dordrecht: Reidel, 1974).

10 E. T. Copson. Asymptotic expansions (Cambridge University Press, 1965).

11 V. De Angelis. Asymptotic expansions and positivity of coefficients for large powers of analytic functions. *Int. J. Math. Math. Sci.* **2003** (2003), 1003–1025.

12 O. Diekmann. *Asymptotic expansion of certain numbers related to the gamma function* (Amsterdam: Mathematisch Centrum, 1975).

13 R. B. Dingle. *Asymptotic expansions: their derivation and interpretation* (Academic, 1973).

14 H. A. Lauwerier. The calculation of the coefficients of certain asymptotic series by means of linear recurrent relations. *Appl. Sci. Res.* B **2** (1952), 77–84.

15 J. L. López, P. Pagola and E. Pérez Sinusía. A simplification of Laplace's method: applications to the gamma function and Gauss hypergeometric function. *J. Approx. Theory* **161** (2009), 280–291.

16 F. D. Murnaghan and J. W. Wrench Jr. *The converging factor for the exponential integral*. David Taylor Model Basin Report 1535 (1963).

17 G. Nemes. An explicit formula for the coefficients in Laplace's method. *Constr. Approx.* **38** (2013), 471–487.

18 F. W. J. Olver. The asymptotic expansion of Bessel functions of large order. *Phil. Trans. R. Soc. Lond.* A **247** (1954), 328–368.

19 F. W. J. Olver. Error bounds for the Laplace approximation for definite integrals. *J. Approx. Theory* **1** (1968), 293–313.

20 F. W. J. Olver. Why steepest descents? *SIAM Rev.* **12** (1970), 228–247.

21 F. W. J. Olver. Uniform, exponentially improved, asymptotic expansions for the generalized exponential integral. *SIAM J. Math. Analysis* **22** (1991), 1460–1474.

22 F. W. J. Olver. Uniform, exponentially improved, asymptotic expansions for the confluent hypergeometric function and other integral transforms. *SIAM J. Math. Analysis* **22** (1991), 1475–1489.

23 F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (eds). *NIST handbook of mathematical functions* (Cambridge University Press, 2010).

24 R. B. Paris. On the use of Hadamard expansions in hyperasymptotic evaluation of Laplace-type integrals. I. Real variable. *J. Computat. Appl. Math.* **167** (2004), 293–319.

25 R. B. Paris. *Hadamard expansions and hyperasymptotic evaluation: an extension of the method of steepest descents* (Cambridge University Press, 2011).

26 R. B. Paris and D. Kaminski. *Asymptotics and Mellin–Barnes integrals* (Cambridge University Press, 2001).

27 R. B. Paris and A. D. Wood. Exponentially-improved asymptotics for the gamma function. *J. Computat. Appl. Math.* **41** (1992), 135–143.

28 R. Remmert and L. D. Kay. *Classical topics in complex function theory* (Springer, 1997).

29 R. Spira. Calculation of the gamma function by Stirling's formula. *Math. Comp.* **25** (1971), 317–322.

30 N. M. Temme. *Special functions: an introduction to the classical functions of mathematical physics* (Wiley, 1996).

31 E. C. Titchmarsh. *The theory of functions*, 2nd edn (Oxford University Press, 1976).

32 G. N. Watson. Theorems stated by Ramanujan (V): approximations connected with e^x. *Proc. Lond. Math. Soc.* **29** (1929), 293–308.

33 R. Wong. *Asymptotic approximations of integrals* (Philadelphia, PA: SIAM, 2001).

34 J. W. Wrench Jr. Concerning two series for the gamma function. *Math. Comput.* **22** (1968), 617–626.

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