

BASES FOR FUNCTIONS BEYOND THE FIRST BAIRE CLASS

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Abstract. We provide a finite basis for the class of Borel functions that are not in the first Baire class, as well as the class of Borel functions that are not σ -continuous with closed witnesses.

Introduction. A topological space is *analytic* if it is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$. A subset of a topological space is *Borel* if it is in the σ -algebra generated by open sets, F_{σ} if it is a union of countably-many closed sets, and G_{δ} if it is an intersection of countably-many open sets.

Suppose that X and Y are topological spaces. Given a family Γ of subsets of X , a function $\phi: X \rightarrow Y$ is Γ -*measurable* if $\phi^{-1}(V) \in \Gamma$ for every open set $V \subseteq Y$. A function is *Borel* if it is Borel-measurable, *Baire class one* if it is F_{σ} -measurable, and σ -*continuous with closed witnesses* if its domain is the union of countably-many closed sets on which it is continuous. A result of Jayne-Rogers (see [2, Theorem 1]) ensures that a function from an analytic metric space to a separable metric space has this property if and only if it is G_{δ} -measurable.

A *quasi-order* on a set Z is a reflexive transitive binary relation \leq on Z . A set $B \subseteq Z$ is a *basis* under \leq for Z if $\forall z \in Z \exists b \in B b \leq z$.

A *closed continuous embedding* of $\phi: X \rightarrow Y$ into $\phi': X' \rightarrow Y'$ consists of a pair of closed continuous embeddings $\pi_X: X \rightarrow X'$ and $\pi_Y: \overline{\phi(X)} \rightarrow \overline{\phi'(X')}$ such that $\phi' \circ \pi_X = \pi_Y \circ \phi$. Note that the existence of such a pair depends not only on the graphs of the functions ϕ and ϕ' , but on Y as well, since different choices of $Y \supseteq \phi(X)$ can lead to different values of $\overline{\phi(X)}$. Here we establish the following results.

THEOREM 1. *There is a 24-element basis under closed continuous embeddability for the class of non-Baire-class-one Borel functions between analytic metric spaces.*

THEOREM 2. *There is a 27-element basis under closed continuous embeddability for the class of non- σ -continuous-with-closed-witnesses Borel functions between analytic metric spaces.*

In Section 1, we discuss the compactification $\mathbb{N}_*^{\leq \mathbb{N}}$ of $\mathbb{N}^{\leq \mathbb{N}}$ underlying our arguments, as well as the corresponding compactification $\mathbb{N}_*^{\mathbb{N}}$ of $\mathbb{N}^{\mathbb{N}}$. In Section 2, we discuss the endomorphisms of $\mathbb{N}^{\leq \mathbb{N}}$ underlying our arguments. In Section 3, we provide a three-element basis for the class of Baire measurable functions from $\mathbb{N}^{\mathbb{N}}$ to

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separable metric spaces. In Section 4, we provide a three-element basis for the class of non- σ -continuous-with-closed-witnesses Baire-class-one functions from analytic metric spaces to separable metric spaces. In Section 5, we provide an eight-element basis for the class of all functions from $\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ to analytic metric spaces. And in Section 6, we establish Theorems 1 and 2.

§1. A compactification of $\mathbb{N}^{\leq \mathbb{N}}$. We use $s \frown t$ to denote the *concatenation* of sequences s and t , and we say that s is an *initial segment* of t , or $s \sqsubseteq t$, if there exists s' for which $t = s \frown s'$. Endow the set $\mathbb{N}_*^{\leq \mathbb{N}} = \mathbb{N}^{\leq \mathbb{N}} \cup \{t \frown (\infty) \mid t \in \mathbb{N}^{< \mathbb{N}}\}$ with the smallest topology with respect to which the sets of the form $\{t\}$ and $\mathcal{N}_t = \{c \in \mathbb{N}_*^{\leq \mathbb{N}} \mid t \sqsubseteq c\}$, where $t \in \mathbb{N}^{< \mathbb{N}}$, are clopen.

PROPOSITION 1.1. *The family \mathcal{B} of sets of the form $\{t\}$ and $\mathcal{N}_t \setminus (\{t\} \cup \bigcup_{j < i} \mathcal{N}_{t \frown (j)})$, where $i \in \mathbb{N}$ and $t \in \mathbb{N}^{< \mathbb{N}}$, is a clopen basis for $\mathbb{N}_*^{\leq \mathbb{N}}$.*

PROOF. Let τ be the topology generated by \mathcal{B} . As every set in \mathcal{B} is clearly clopen, it is sufficient to show that the sets $\{t\}$ and \mathcal{N}_t are τ -clopen for all $t \in \mathbb{N}^{< \mathbb{N}}$. As these sets are clearly τ -open, we need only show that they are τ -closed. As $\mathcal{N}_{t \frown (i)}$ is τ -closed in \mathcal{N}_t for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{< \mathbb{N}}$, a straightforward induction shows that \mathcal{N}_t is τ -closed for all $t \in \mathbb{N}^{< \mathbb{N}}$. As $\{t\}$ is τ -closed in \mathcal{N}_t for all $t \in \mathbb{N}^{< \mathbb{N}}$, it follows that $\{t\}$ is τ -closed for all $t \in \mathbb{N}^{< \mathbb{N}}$. ◻

PROPOSITION 1.2. *The space $\mathbb{N}_*^{\leq \mathbb{N}}$ is compact.*

PROOF. Suppose, towards a contradiction, that there is an open cover \mathcal{U} of $\mathbb{N}_*^{\leq \mathbb{N}}$ with no finite subcover.

LEMMA 1.3. *Suppose that $t \in \mathbb{N}^{< \mathbb{N}}$ and no finite set $\mathcal{V} \subseteq \mathcal{U}$ covers \mathcal{N}_t . Then there exists $j \in \mathbb{N}$ such that no finite set $\mathcal{V} \subseteq \mathcal{U}$ covers $\mathcal{N}_{t \frown (j)}$.* ◻

PROOF. Fix $U \in \mathcal{U}$ containing $t \frown (\infty)$. Proposition 1.1 then yields $i \in \mathbb{N}$ with $\mathcal{N}_t \setminus (\{t\} \cup \bigcup_{j < i} \mathcal{N}_{t \frown (j)}) \subseteq U$, in which case no finite set $\mathcal{V} \subseteq \mathcal{U}$ covers $\bigcup_{j < i} \mathcal{N}_{t \frown (j)}$, and it follows that there exists $j < i$ for which no finite set $\mathcal{V} \subseteq \mathcal{U}$ covers $\mathcal{N}_{t \frown (j)}$. ◻

By recursively applying Lemma 1.3, we obtain $b \in \mathbb{N}^{\mathbb{N}}$ such that for no $i \in \mathbb{N}$ is there a finite set $\mathcal{V} \subseteq \mathcal{U}$ covering $\mathcal{N}_{b \upharpoonright i}$. But Proposition 1.1 implies that every open neighborhood of b contains some $\mathcal{N}_{b \upharpoonright i}$.

Given a countable set I and a topological space X , we say that a sequence $(x_i)_{i \in I} \in X^I$ converges to a point $x \in X$, or $x_i \rightarrow x$, if for every open neighborhood U of x there are only finitely many $i \in I$ with $x_i \notin U$. We endow $\mathbb{N}^{< \mathbb{N}}$ with the partial order \sqsubseteq , and when I and X are equipped with partial orders \leq_I and \leq_X , we say that $(x_i)_{i \in I}$ is decreasing if $i \leq_I j \implies x_j \leq_X x_i$ for all $i, j \in I$.

PROPOSITION 1.4. *The space $\mathbb{N}_*^{\leq \mathbb{N}}$ has a compatible ultrametric.*

PROOF. Fix a decreasing sequence $(\varepsilon_t)_{t \in \mathbb{N}^{< \mathbb{N}}}$ of positive real numbers converging to zero. Set $d(a, a) = 0$ for all $a \in \mathbb{N}_*^{\leq \mathbb{N}}$, as well as $d(a, b) = \max\{\varepsilon_t \mid t \in \{a \upharpoonright \min(|a|, i(a, b)), b \upharpoonright \min(|b|, i(a, b))\} \cap \mathbb{N}^{< \mathbb{N}}\}$ for all distinct $a, b \in \mathbb{N}_*^{\leq \mathbb{N}}$, where $i(a, b) = \min\{i \in \mathbb{N} \mid a \upharpoonright i \neq b \upharpoonright i\}$.

To see that d is an ultrametric, suppose that $a, b, c \in \mathbb{N}_*^{\leq \mathbb{N}}$ are pairwise distinct. Observe that if $i(a, c) < \max\{i(a, b), i(b, c)\}$, then $d(a, c) \in \{d(b, c), d(a, b)\}$, so $d(a, c) \leq \max\{d(a, b), d(b, c)\}$. And if $i(a, c) = \max\{i(a, b), i(b, c)\}$, then setting $i = i(a, b) = i(a, c) = i(b, c)$, it follows that

$$\begin{aligned} d(a, c) &= \max\{\varepsilon_t \mid t \in \{a \upharpoonright i, c \upharpoonright i\} \cap \mathbb{N}^{< \mathbb{N}}\} \\ &\leq \max\{\varepsilon_t \mid t \in \{a \upharpoonright i, b \upharpoonright i, c \upharpoonright i\} \cap \mathbb{N}^{< \mathbb{N}}\} \\ &= \max\{d(a, b), d(b, c)\}. \end{aligned}$$

And if $i(a, c) > \max\{i(a, b), i(b, c)\}$, then setting $\varepsilon = d(a, b) = d(b, c)$ and $t = a \upharpoonright i(a, b) = c \upharpoonright i(b, c)$, it follows that $d(a, c) \leq \varepsilon_t \leq \varepsilon$, and therefore $d(a, c) \leq \max\{d(a, b), d(b, c)\}$.

As $\{t\} = \mathcal{B}(t, \varepsilon_t)$ and $\mathcal{N}_i \setminus \{t\} = \mathcal{B}(\mathcal{N}_i \setminus \{t\}, \varepsilon_t)$ for all $t \in \mathbb{N}^{< \mathbb{N}}$, and $\mathcal{N}_i \setminus (\{t\} \cup \bigcup_{j \leq i} \mathcal{N}_{t \smallfrown (j)}) = \mathcal{B}(\mathcal{N}_i \setminus (\{t\} \cup \bigcup_{j \leq i} \mathcal{N}_{t \smallfrown (j)}), \min\{\varepsilon_{t \smallfrown (j)} \mid j \leq i\})$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{< \mathbb{N}}$, Proposition 1.1 ensures that every open subset of $\mathbb{N}_*^{\leq \mathbb{N}}$ is d -open.

Given $b \in \mathbb{N}^{\mathbb{N}}$ and $\varepsilon > 0$, fix $i \in \mathbb{N}$ with $\varepsilon_{b \upharpoonright i} < \varepsilon$, set $t = b \upharpoonright i$, and note that $\mathcal{N}_i \subseteq \mathcal{B}(b, \varepsilon)$. Given $t \in \mathbb{N}^{< \mathbb{N}}$ and $\varepsilon > 0$, fix $i \in \mathbb{N}$ with $\varepsilon_{t \smallfrown (j)} < \varepsilon$ for all $j \geq i$, and observe that $\mathcal{N}_i \setminus (\{t\} \cup \bigcup_{j < i} \mathcal{N}_{t \smallfrown (j)}) \subseteq \mathcal{B}(t \smallfrown (\infty), \varepsilon)$. Thus every d -open subset of $\mathbb{N}_*^{\leq \mathbb{N}}$ is open. \dashv

It follows that $\mathbb{N}_*^{\leq \mathbb{N}}$ is Polish. As the space $\mathbb{N}_*^{\mathbb{N}} = \mathbb{N}_*^{\leq \mathbb{N}} \setminus \mathbb{N}^{< \mathbb{N}}$ is a perfect subset of $\mathbb{N}_*^{\leq \mathbb{N}}$, a result of Brouwer's ensures that it is homeomorphic to $2^{\mathbb{N}}$ (see, e.g., [3, Theorem 7.4]).

§2. Meet embeddings. The *meet* of sequences $s, t \in \mathbb{N}^{< \mathbb{N}}$ is the sequence $r = s \wedge t$ of maximal length for which $r \sqsubseteq s$ and $r \sqsubseteq t$. A \wedge -embedding is an injection $\pi: \mathbb{N}^{< \mathbb{N}} \rightarrow \mathbb{N}^{< \mathbb{N}}$ such that $\pi(s \wedge t) = \pi(s) \wedge \pi(t)$ for all $s, t \in \mathbb{N}^{< \mathbb{N}}$.

PROPOSITION 2.1. *Suppose that $\pi: \mathbb{N}^{< \mathbb{N}} \rightarrow \mathbb{N}^{< \mathbb{N}}$. Then π is a \wedge -embedding if and only if the following conditions hold:*

- (1) $\forall i \in \mathbb{N} \forall t \in \mathbb{N}^{< \mathbb{N}} \pi(t) \sqsubset \pi(t \smallfrown (i))$.
- (2) $\forall i, j \in \mathbb{N} \forall t \in \mathbb{N}^{< \mathbb{N}} (i \neq j \implies \pi(t \smallfrown (i))(|\pi(t)|) \neq \pi(t \smallfrown (j))(|\pi(t)|))$.

PROOF. Suppose first that π is a \wedge -embedding. To see that condition (1) holds, observe that if $i \in \mathbb{N}$ and $t \in \mathbb{N}^{< \mathbb{N}}$, then $\pi(t) = \pi(t) \wedge \pi(t \smallfrown (i))$, so $\pi(t) \sqsubset \pi(t \smallfrown (i))$, thus $\pi(t) \sqsubset \pi(t \smallfrown (i))$. And to see that condition (2) holds, observe that if $i, j \in \mathbb{N}$ are distinct and $t \in \mathbb{N}^{< \mathbb{N}}$, then $\pi(t) = \pi(t \smallfrown (i)) \wedge \pi(t \smallfrown (j))$, so $\pi(t \smallfrown (i))(|\pi(t)|) \neq \pi(t \smallfrown (j))(|\pi(t)|)$.

Suppose now that π satisfies conditions (1) and (2). To see that π is a \wedge -embedding, suppose that $s, t \in \mathbb{N}^{< \mathbb{N}}$ are distinct, and define $r = s \wedge t$. By reversing the roles of s and t if necessary, we can assume that $|s| > |r|$, so $\pi(r) \sqsubset \pi(s)$, thus either $r = t$ or $(|t| > |r|$ and $\pi(s)(|\pi(r)|) \neq \pi(t)(|\pi(r)|))$. In both cases, it follows that $\pi(s) \neq \pi(t)$ and $\pi(r) = \pi(s) \wedge \pi(t)$. \dashv

REMARK 2.2. In particular, it follows that if $\pi: \mathbb{N}^{< \mathbb{N}} \rightarrow \mathbb{N}^{< \mathbb{N}}$ has the property that $\pi(t) \smallfrown (i) \sqsubseteq \pi(t \smallfrown (i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{< \mathbb{N}}$, then π is a \wedge -embedding.

The *composition* of a finite sequence $(\pi_i)_{i \leq n}$ of functions is given by $\circ_{i \leq n} \pi_i = \pi_0 \circ \dots \circ \pi_n$.

PROPOSITION 2.3. *Suppose that $(\pi_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ is a sequence of \wedge -embeddings with the property that $\pi_t(\mathbb{N}^{<\mathbb{N}}) \subseteq \mathcal{N}_t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$. Then the function $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ given by $\pi(t) = (\circ_{n \leq |t|} \pi_{t \upharpoonright n})(t)$ is also a \wedge -embedding.*

PROOF. Note that if $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, then $t \frown (i) \sqsubseteq \pi_{t \frown (i)}(t \frown (i))$, so Proposition 2.1 ensures that $(\circ_{n \leq |t|} \pi_{t \upharpoonright n})(t \frown (i)) \sqsubseteq \pi(t \frown (i))$, thus $\pi(t) \sqsubseteq (\circ_{n \leq |t|} \pi_{t \upharpoonright n})(t \frown (i)) \sqsubseteq \pi(t \frown (i))$. It also implies that if $i \neq j$, then $(\circ_{n \leq |t|} \pi_{t \upharpoonright n})(t \frown (i))(|\pi(t)|) \neq (\circ_{n \leq |t|} \pi_{t \upharpoonright n})(t \frown (j))(|\pi(t)|)$, so $\pi(t \frown (i))(|\pi(t)|) \neq \pi(t \frown (j))(|\pi(t)|)$. One last application of Proposition 2.1 therefore ensures that π is a \wedge -embedding. ⊣

We next consider the connection between \wedge -embeddings and closed continuous embeddings.

PROPOSITION 2.4. *Every \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ has a unique extension to a (necessarily injective) continuous map $\bar{\pi}: \mathbb{N}_*^{\leq \mathbb{N}} \rightarrow \mathbb{N}_*^{\leq \mathbb{N}}$, given by $\bar{\pi}(b) = \bigcup_{i \in \mathbb{N}} \pi(b \upharpoonright i)$ and $\bar{\pi}(t \frown (\infty)) = \pi(t) \frown (\infty)$ for all $b \in \mathbb{N}^{\mathbb{N}}$ and $t \in \mathbb{N}^{<\mathbb{N}}$.*

PROOF. Suppose that $\bar{\pi}: \mathbb{N}_*^{\leq \mathbb{N}} \rightarrow \mathbb{N}_*^{\leq \mathbb{N}}$ is a continuous extension of π . If $b \in \mathbb{N}^{\mathbb{N}}$, then $b \upharpoonright i \rightarrow b$, and since $(\pi(b \upharpoonright i))_{i \in \mathbb{N}}$ is strictly increasing by Proposition 2.1, it follows that $\bar{\pi}(b) = \bigcup_{i \in \mathbb{N}} \pi(b \upharpoonright i)$. If $t \in \mathbb{N}^{<\mathbb{N}}$, then $t \frown (i) \rightarrow t \frown (\infty)$, and since $\pi(t) = \pi(t) \frown (i) \wedge \pi(t) \frown (j)$ for all distinct $i, j \in \mathbb{N}$, it follows that $\bar{\pi}(t) \frown (\infty) = \pi(t) \frown (\infty)$.

To see that these constraints actually define a continuous function, note that if $t \in \mathbb{N}^{<\mathbb{N}}$, then either $\bar{\pi}^{-1}(\mathcal{N}_t) = \emptyset$ or there exists $s \in \mathbb{N}^{<\mathbb{N}}$ of minimal length with $t \sqsubseteq \pi(s)$, in which case $\bar{\pi}^{-1}(\mathcal{N}_t) = \mathcal{N}_s$.

To see that $\bar{\pi}$ is injective, it is enough to check that its restriction to $\mathbb{N}^{\mathbb{N}}$ is injective. Towards this end, suppose that $a, b \in \mathbb{N}^{\mathbb{N}}$ are distinct, fix $i \in \mathbb{N}$ least for which $a(i) \neq b(i)$, set $t = a \upharpoonright i = b \upharpoonright i$, and observe that $\pi(t \frown (a(i)))(|\pi(t)|) \neq \pi(t \frown (b(i)))(|\pi(t)|)$ by Proposition 2.1, thus $\bar{\pi}(a)$ and $\bar{\pi}(b)$ are distinct. ⊣

REMARK 2.5. It follows that the extension associated with the composition of two \wedge -embeddings is the composition of their extensions.

Given a function $\phi: X \rightarrow Y$ and sets $X' \subseteq X$ and $Y' \supseteq \phi(X')$, let $\phi \upharpoonright X' \rightarrow Y'$ denote the function $\psi: X' \rightarrow Y'$ given by $\phi(x) = \psi(x)$ for all $x \in X'$. Compactness ensures that if π is a \wedge -embedding, then $\bar{\pi}$ and $\bar{\pi} \upharpoonright \mathbb{N}_*^{\mathbb{N}}$ are closed continuous embeddings. The following observations show that so too are $\bar{\pi} \upharpoonright \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ and $\bar{\pi} \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$.

PROPOSITION 2.6. *Suppose that $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ is a \wedge -embedding. Then $\bar{\pi} \upharpoonright \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ is closed.*

PROOF. It is sufficient to show that every sequence $(b_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{N}^{\mathbb{N}}$ for which $(\bar{\pi}(b_n))_{n \in \mathbb{N}}$ converges to an element of $\mathbb{N}^{\mathbb{N}}$ is itself convergent to an element of $\mathbb{N}^{\mathbb{N}}$. As $(\bar{\pi}(b_n) \upharpoonright i)_{n \in \mathbb{N}}$ is eventually constant for all $i \in \mathbb{N}$, a simple induction shows that $(b_n \upharpoonright i)_{n \in \mathbb{N}}$ is also eventually constant for all $i \in \mathbb{N}$, so $(b_n)_{n \in \mathbb{N}}$ converges to an element of $\mathbb{N}^{\mathbb{N}}$. ⊣

PROPOSITION 2.7. *Suppose that $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ is a \wedge -embedding. Then $\bar{\pi} \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ is closed.*

PROOF. It is sufficient to show that every sequence $(s_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{N}^{<\mathbb{N}}$ such that $(\pi(s_n))_{n \in \mathbb{N}}$ converges to $t \smallfrown (\infty)$ for some $t \in \mathbb{N}^{<\mathbb{N}}$ has a subsequence converging to an element of $\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$. By passing to a subsequence, we can assume that $\pi(s_m) \wedge \pi(s_n) = t$ for all distinct $m, n \in \mathbb{N}$. Let s be the \sqsubseteq -minimal element of $\mathbb{N}^{<\mathbb{N}}$ for which $t \sqsubseteq \pi(s)$. Then $s_m \wedge s_n = s$ for all distinct $m, n \in \mathbb{N}$, thus $s_n \rightarrow s \smallfrown (\infty)$. \dashv

A set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is \sqsubseteq -dense if $\forall s \in \mathbb{N}^{<\mathbb{N}} \exists t \in T \ s \sqsubseteq t$. More generally, a set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is \sqsubseteq -dense below $r \in \mathbb{N}^{<\mathbb{N}}$ if $\forall s \in \mathbb{N}^{<\mathbb{N}} \exists t \in T \ r \smallfrown s \sqsubseteq t$.

PROPOSITION 2.8. *Suppose that $T \subseteq \mathbb{N}^{<\mathbb{N}}$. Then there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ such that $\pi(\mathbb{N}^{<\mathbb{N}}) \subseteq T$ or $\pi(\mathbb{N}^{<\mathbb{N}}) \subseteq \sim T$.*

PROOF. Fix $S \in \{T, \sim T\}$ which is \sqsubseteq -dense below some $s \in \mathbb{N}^{<\mathbb{N}}$, and recursively construct a function $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathcal{N}_s \cap S$ with the property that $\pi(t) \smallfrown (i) \sqsubseteq \pi(t \smallfrown (i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. \dashv

PROPOSITION 2.9. *Suppose that $C \subseteq \mathbb{N}^{\mathbb{N}}$ is a nonmeager set with the Baire property. Then there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ with the property that $\bar{\pi}(\mathbb{N}^{\mathbb{N}}) \subseteq C$.*

PROOF. Fix $s \in \mathbb{N}^{<\mathbb{N}}$ for which C is comeager in $\mathcal{N}_s \cap \mathbb{N}^{\mathbb{N}}$, as well as dense open sets $U_n \subseteq \mathcal{N}_s \cap \mathbb{N}^{\mathbb{N}}$ with the property that $\bigcap_{n \in \mathbb{N}} U_n \subseteq C$. Set $T_n = \{t \in \mathbb{N}^{<\mathbb{N}} \mid \mathcal{N}_t \cap \mathbb{N}^{\mathbb{N}} \subseteq U_n\}$ for all $n \in \mathbb{N}$, and recursively construct a function $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathcal{N}_s \cap \mathbb{N}^{<\mathbb{N}}$ such that $\pi(\mathbb{N}^{\mathbb{N}}) \subseteq T_n$ for all $n \in \mathbb{N}$ and $\pi(t) \smallfrown (i) \sqsubseteq \pi(t \smallfrown (i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. \dashv

§3. Baire measurable functions on $\mathbb{N}^{\mathbb{N}}$. Here we provide a basis for the class of Baire measurable functions from $\mathbb{N}^{\mathbb{N}}$ to separable metric spaces.

PROPOSITION 3.1. *Suppose that X is a second countable topological space and $\phi: \mathbb{N}^{\mathbb{N}} \rightarrow X$ is Baire measurable. Then there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ for which $\phi \circ \bar{\pi}$ is continuous.*

PROOF. Fix a comeager set $C \subseteq \mathbb{N}^{\mathbb{N}}$ on which ϕ is continuous, and appeal to Proposition 2.9 to obtain a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ with the property that $\bar{\pi}(\mathbb{N}^{\mathbb{N}}) \subseteq C$. \dashv

PROPOSITION 3.2. *Suppose that X is a metric space and $\phi: \mathbb{N}^{\mathbb{N}} \rightarrow X$ is continuous. Then there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ with the property that $\text{diam} \phi(\mathcal{N}_{\pi(t)}) \rightarrow 0$.*

PROOF. Fix a sequence $(\varepsilon_i)_{i \in \mathbb{N}^{<\mathbb{N}}}$ of positive real numbers converging to zero, note that the continuity of ϕ ensures that for all $t \in \mathbb{N}^{<\mathbb{N}}$ the set $T_t = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \text{diam} \phi(\mathcal{N}_s) < \varepsilon_t\}$ is \sqsubseteq -dense, and recursively construct a function $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ such that $\pi(t) \in T_t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$ and $\pi(t) \smallfrown (i) \sqsubseteq \pi(t \smallfrown (i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. \dashv

Given a countable set I and a topological space X , we say that a sequence $(X_i)_{i \in I}$ of subsets of X converges to a point $x \in X$, or $X_i \rightarrow x$, if for every open neighborhood U of x , all but finitely many $i \in I$ have the property that $X_i \subseteq U$. We say that $(X_i)_{i \in I}$ is discrete if for all $x \in X$ there is an open neighborhood U of x such that all but finitely many $i \in I$ have the property that $U \cap X_i = \emptyset$.

PROPOSITION 3.3. *Suppose that X is a metric space and $\phi: \mathbb{N}^{\mathbb{N}} \rightarrow X$ has the property that $\text{diam} \phi(\mathcal{N}_{t \smallfrown (i)}) \rightarrow 0$ for all $t \in \mathbb{N}^{<\mathbb{N}}$. Then there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ such that $(\phi(\mathcal{N}_{\pi(t \smallfrown (i))}))_{i \in \mathbb{N}}$ is convergent or discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$.*

PROOF. For each $t \in \mathbb{N}^{<\mathbb{N}}$, the fact that $\text{diam} \phi(\mathcal{N}_{t \smallfrown (i)}) \rightarrow 0$ ensures that there is an injection $\iota_t: \mathbb{N} \rightarrow \mathbb{N}$ for which $(\phi(\mathcal{N}_{t \smallfrown (\iota_t(i))}))_{i \in \mathbb{N}}$ is convergent or discrete. Define $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ by choosing $\pi(\emptyset) \in \mathbb{N}^{<\mathbb{N}}$ arbitrarily and setting $\pi(t \smallfrown (i)) = \pi(t) \smallfrown (\iota_{\pi(t)}(i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. \dashv

We say that a function $\phi: X \rightarrow Y$ is *nowhere constant* if there is no nonempty open set $U \subseteq X$ on which ϕ is constant.

PROPOSITION 3.4. *Suppose that X is a metric space and $\phi: \mathbb{N}^{\mathbb{N}} \rightarrow X$ is continuous and nowhere constant. Then there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ such that*

$$\forall i \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}} \overline{\phi(\mathcal{N}_{\pi(t \smallfrown (i))})} \cap \overline{\bigcup_{j \in \mathbb{N} \setminus \{i\}} \phi(\mathcal{N}_{\pi(t \smallfrown (j))})} = \emptyset.$$

PROOF. Clearly each $\phi(\mathcal{N}_i)$ is infinite.

LEMMA 3.5. *For all $t \in \mathbb{N}^{<\mathbb{N}}$, there is a function $\iota_t: \mathbb{N} \rightarrow \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ such that $(\iota_t(i)(0))_{i \in \mathbb{N}}$ is injective and the closures of $\phi(\mathcal{N}_{t \smallfrown (\iota_t(i))})$ and $\bigcup_{j \in \mathbb{N} \setminus \{i\}} \phi(\mathcal{N}_{t \smallfrown (\iota_t(j))})$ are disjoint for all $i \in \mathbb{N}$.* \dashv

PROOF. As each $\phi(\mathcal{N}_{t \smallfrown (i)})$ is infinite, there are extensions $b_i \in \mathbb{N}^{\mathbb{N}}$ of $t \smallfrown (i)$ such that $\phi(b_i) \notin \{\phi(b_j) \mid j < i\}$ for all $i \in \mathbb{N}$. Fix a subsequence $(a_i)_{i \in \mathbb{N}}$ of $(b_i)_{i \in \mathbb{N}}$ for which $\{\phi(a_i) \mid i \in \mathbb{N}\}$ is discrete. For each $i \in \mathbb{N}$, fix $\varepsilon_i > 0$ such that $\phi(a_j) \notin \mathcal{B}(\phi(a_i), \varepsilon_i)$ for all $j \in \mathbb{N} \setminus \{i\}$, as well as $\iota_t(i) \in \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ with $t \smallfrown \iota_t(i) \sqsubseteq a_i$ and $\phi(\mathcal{N}_{t \smallfrown (\iota_t(i))}) \subseteq \mathcal{B}(\phi(a_i), \varepsilon_i/3)$.

Suppose, towards a contradiction, that there exists $i \in \mathbb{N}$ for which some $x \in X$ is in the closures of $\phi(\mathcal{N}_{t \smallfrown (\iota_t(i))})$ and $\bigcup_{j \in \mathbb{N} \setminus \{i\}} \phi(\mathcal{N}_{t \smallfrown (\iota_t(j))})$. Then there exist $j \in \mathbb{N} \setminus \{i\}$ and $y \in \phi(\mathcal{N}_{t \smallfrown (\iota_t(j))})$ with the property that $d(x, y) \leq \varepsilon_i/3$, in which case

$$\begin{aligned} d(\phi(a_i), \phi(a_j)) &\leq d(\phi(a_i), x) + d(x, y) + d(y, \phi(a_j)) \\ &< \varepsilon_i/3 + \varepsilon_i/3 + \varepsilon_j/3 \\ &\leq \max\{\varepsilon_i, \varepsilon_j\}, \end{aligned}$$

so $\phi(a_i) \in \mathcal{B}(\phi(a_j), \varepsilon_j)$ or $\phi(a_j) \in \mathcal{B}(\phi(a_i), \varepsilon_i)$, a contradiction. \dashv

Define $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ by choosing $\pi(\emptyset) \in \mathbb{N}^{<\mathbb{N}}$ arbitrarily and setting $\pi(t \smallfrown (i)) = \pi(t) \smallfrown \iota_{\pi(t)}(i)$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$.

We now obtain our main result stabilizing the topological behavior of Baire measurable functions from $\mathbb{N}^{\mathbb{N}}$ to separable metric spaces.

THEOREM 3.6. *Suppose that X is a separable metric space and $\phi: \mathbb{N}^{\mathbb{N}} \rightarrow X$ is Baire measurable. Then there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ such that $\phi \circ \bar{\pi}$ is constant or extends to a closed continuous embedding on $\mathbb{N}^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}_*$.*

PROOF. By Remark 2.5, we are free to replace ϕ by its composition with the extension of any \wedge -embedding. For example, by Proposition 3.1, we can assume that ϕ is continuous.

If there exists $s \in \mathbb{N}^{<\mathbb{N}}$ for which $\phi \upharpoonright \mathcal{N}_s$ is constant, then define $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ by $\pi(t) = s \frown t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$, so $\phi \circ \bar{\pi}$ is constant. Otherwise, Propositions 2.8, 3.2, 3.3, and 3.4 yield a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ such that $\text{diam} \phi(\mathcal{N}_{\pi(t)}) \rightarrow 0$, $(\phi(\mathcal{N}_{\pi(t \frown (i))}))_{i \in \mathbb{N}}$ is convergent for all $t \in \mathbb{N}^{<\mathbb{N}}$ or discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$, and

$$\forall i \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}} \overline{\phi(\mathcal{N}_{\pi(t \frown (i))})} \cap \overline{\bigcup_{j \in \mathbb{N} \setminus \{i\}} \phi(\mathcal{N}_{\pi(t \frown (j))})} = \emptyset.$$

As $\bar{\pi}(\mathcal{N}_t) \subseteq \mathcal{N}_{\pi(t)}$ for all $t \in \mathbb{N}^{<\mathbb{N}}$, it follows that

$$\forall i \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}} \overline{(\phi \circ \bar{\pi})(\mathcal{N}_{t \frown (i)})} \cap \overline{\bigcup_{j \in \mathbb{N} \setminus \{i\}} (\phi \circ \bar{\pi})(\mathcal{N}_{t \frown (j)})} = \emptyset.$$

So by replacing ϕ with $\phi \circ \bar{\pi}$, we can assume that $\text{diam} \phi(\mathcal{N}_t) \rightarrow 0$, $(\phi(\mathcal{N}_{t \frown (i)}))_{i \in \mathbb{N}}$ is convergent for all $t \in \mathbb{N}^{<\mathbb{N}}$ or discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$, and

$$\forall i \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}} \overline{\phi(\mathcal{N}_{t \frown (i)})} \cap \overline{\bigcup_{j \in \mathbb{N} \setminus \{i\}} \phi(\mathcal{N}_{t \frown (j)})} = \emptyset. \tag{\dagger}$$

To see that ϕ is injective, note that if $a, b \in \mathbb{N}^{\mathbb{N}}$ are distinct, then there is a least $i \in \mathbb{N}$ for which $a(i) \neq b(i)$. Setting $t = a \upharpoonright i = b \upharpoonright i$, it follows from (\dagger) that $\phi(\mathcal{N}_{t \frown (a(i))})$ and $\phi(\mathcal{N}_{t \frown (b(i))})$ are disjoint, thus $\phi(a)$ and $\phi(b)$ are distinct.

We next check that if $(\phi(\mathcal{N}_{t \frown (i)}))_{i \in \mathbb{N}}$ is discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$, then ϕ is a closed continuous embedding. It is sufficient to show that every sequence $(b_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{N}^{\mathbb{N}}$ for which $(\phi(b_n))_{n \in \mathbb{N}}$ converges to some $x \in X$ is itself convergent. But a straightforward recursive argument yields $b \in \mathbb{N}^{\mathbb{N}}$ such that x is in the closure of $\phi(\mathcal{N}_{b \upharpoonright i})$ for all $i \in \mathbb{N}$, so (\dagger) ensures that x is not in the closure of $\bigcup_{j \in \mathbb{N} \setminus \{b(i)\}} \phi(\mathcal{N}_{b \upharpoonright i \frown (j)})$ for all $i \in \mathbb{N}$, thus $(b_n \upharpoonright i)_{n \in \mathbb{N}}$ is eventually constant with value $b \upharpoonright i$ for all $i \in \mathbb{N}$, hence $b_n \rightarrow b$.

It remains to check that if $(\phi(\mathcal{N}_{t \frown (i)}))_{i \in \mathbb{N}}$ is convergent for all $t \in \mathbb{N}^{<\mathbb{N}}$, then the extension of ϕ to $\mathbb{N}^{\mathbb{N}}$ given by $\bar{\phi}(t \frown (\infty)) = \lim_{i \rightarrow \infty} \phi(\mathcal{N}_{t \frown (i)})$ for all $t \in \mathbb{N}^{<\mathbb{N}}$ is a closed continuous embedding. To see that $\bar{\phi}$ is injective, note that if $c, d \in \mathbb{N}^{\mathbb{N}}$ are distinct, then there is a least $i \in \mathbb{N}$ with $c(i) \neq d(i)$. By reversing the roles of c and d if necessary, we can assume that $c(i) \neq \infty$. Set $t = c \upharpoonright i = d \upharpoonright i$, and appeal to (\dagger) to see that $\bar{\phi}(c)$ is in the closure of $\phi(\mathcal{N}_{t \frown (c(i))})$ but $\bar{\phi}(d)$ is not, so $\bar{\phi}(c) \neq \bar{\phi}(d)$. To see that $\bar{\phi}$ is continuous, suppose that $c \in \mathbb{N}^{\mathbb{N}}$ and U is an open neighborhood of $\bar{\phi}(c)$, and fix an open neighborhood V of $\bar{\phi}(c)$ whose closure is contained in U . If $c \in \mathbb{N}^{\mathbb{N}}$, then there exists $i \in \mathbb{N}$ for which $\phi(\mathcal{N}_{c \upharpoonright i}) \subseteq V$, thus $\mathcal{N}_{c \upharpoonright i}$ is an open neighborhood of c whose image under $\bar{\phi}$ is contained in U . Otherwise, there exists $t \in \mathbb{N}^{<\mathbb{N}}$ for which $c = t \frown (\infty)$, as well as $i \in \mathbb{N}$ for which $\phi(\mathcal{N}_t \setminus \bigcup_{j < i} \mathcal{N}_{t \frown (j)}) \subseteq V$. Then $\mathcal{N}_t \setminus \bigcup_{j < i} \mathcal{N}_{t \frown (j)}$ is an open neighborhood of c whose image under $\bar{\phi}$ is contained in U . \dashv

For each topological space X , let c_X denote the unique function from X to the trivial topological space $\{\infty\}$. Given topological spaces $X \subseteq Y$, define $\iota_{X,Y}: X \rightarrow Y$ by $\iota_{X,Y}(x) = x$ for all $x \in X$.

PROPOSITION 3.7. *Suppose that X is a separable metric space, $\phi: \mathbb{N}^{\mathbb{N}} \rightarrow X$ is Baire measurable, $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ is a \wedge -embedding, and $\phi \circ \bar{\pi}$ is constant or extends to a closed continuous embedding on $\mathbb{N}^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$. Then there exist $\phi_0 \in \{c_{\mathbb{N}^{\mathbb{N}}}\} \cup \{\iota_{\mathbb{N}^{\mathbb{N}}, Z} \mid Z \in \{\mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}_*\}\}$ and $\psi: \overline{\phi_0(\mathbb{N}^{\mathbb{N}})} \rightarrow \overline{\phi(\mathbb{N}^{\mathbb{N}})}$ with the property that $(\bar{\pi} \upharpoonright \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}, \psi)$ is a closed continuous embedding of ϕ_0 into ϕ .*

PROOF. If $\phi \circ \bar{\pi}$ is constant, then set $\phi_0 = c_{\mathbb{N}^{\mathbb{N}}}$ and let ψ be the unique function from $c_{\mathbb{N}^{\mathbb{N}}}(\mathbb{N}^{\mathbb{N}})$ to $(\phi \circ \bar{\pi})(\mathbb{N}^{\mathbb{N}})$. If $\phi \circ \bar{\pi}$ extends to a closed continuous embedding ψ on $Z \in \{\mathbb{N}^{\mathbb{N}}, \mathbb{N}_*^{\mathbb{N}}\}$, then set $\phi_0 = \iota_{\mathbb{N}^{\mathbb{N}}, Z}$. ⊣

§4. Baire-class-one functions that are not σ -continuous with closed witnesses.

Here we strengthen [4, Theorem 3.1] by providing a basis for the class of non- σ -continuous-with-closed-witnesses Baire-class-one functions from analytic metric spaces to separable metric spaces.

PROPOSITION 4.1. *Suppose that X is a metric space and $\phi: \mathbb{N}_*^{\mathbb{N}} \rightarrow X$ has the property that $\phi \upharpoonright \mathbb{N}^{\mathbb{N}}$ is continuous. Then there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ such that either $(\phi \circ \bar{\pi})(\mathbb{N}^{\mathbb{N}}) \cap (\phi \circ \bar{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}) = \emptyset$ or $\phi \circ \bar{\pi}$ is continuous at every point of $\mathbb{N}^{\mathbb{N}}$.*

PROOF. We can assume that there is no $s \in \mathbb{N}^{<\mathbb{N}}$ with the property that $\inf \{d(\phi(s \smallfrown b), \phi(s \smallfrown t \smallfrown (\infty))) \mid b \in \mathbb{N}^{\mathbb{N}} \text{ and } t \in \mathbb{N}^{<\mathbb{N}}\} > 0$, since otherwise the \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ given by $\pi(t) = s \smallfrown t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$ has the property that $(\phi \circ \bar{\pi})(\mathbb{N}^{\mathbb{N}}) \cap (\phi \circ \bar{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}) = \emptyset$.

LEMMA 4.2. *Suppose that $\varepsilon > 0$ and $s \in \mathbb{N}^{<\mathbb{N}}$. Then there exists $t \in \mathbb{N}^{<\mathbb{N}}$ with $d(\phi(s \smallfrown t \smallfrown b), \phi(s \smallfrown t \smallfrown (\infty))) < \varepsilon$ for all $b \in \mathbb{N}^{\mathbb{N}}$.* ⊣

PROOF. Fix $\delta < \varepsilon$ and $u \in \mathbb{N}^{<\mathbb{N}}$ with $\text{diam} \phi(\mathcal{N}_{s \smallfrown u} \cap \mathbb{N}^{\mathbb{N}}) < \delta$, and $b \in \mathbb{N}^{\mathbb{N}}$ and $v \in \mathbb{N}^{<\mathbb{N}}$ with $d(\phi(s \smallfrown u \smallfrown b), \phi(s \smallfrown u \smallfrown v \smallfrown (\infty))) < \varepsilon - \delta$, and set $t = u \smallfrown v$. ⊣

Fix a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive real numbers converging to zero, and recursively construct a function $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ with the property that $d(\phi(\pi(t) \smallfrown b), \phi(\pi(t) \smallfrown (\infty))) < \varepsilon_{|t|}$ for all $b \in \mathbb{N}^{\mathbb{N}}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, and $\pi(t) \smallfrown (i) \sqsubseteq \pi(t \smallfrown (i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$.

We say that a metric space is ε -discrete if all distinct points have distance at least ε from one another.

PROPOSITION 4.3. *Suppose that X is a metric space, $\phi: \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \rightarrow X$, $\varepsilon > 0$, and $t \in \mathbb{N}^{<\mathbb{N}}$. Then there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathcal{N}_t \cap \mathbb{N}^{<\mathbb{N}}$ with the property that $\phi \circ \bar{\pi}$ is an injection into an ε -discrete set or $(\phi \circ \bar{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$ is contained in the ε -ball around a point of $\phi(\mathcal{N}_t)$.*

PROOF. If for no finite set $F \subseteq \phi(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$ and extension u of t is it the case that $\phi(\mathcal{N}_u) \subseteq \mathcal{B}(F, \varepsilon)$, then fix an enumeration $(t_n)_{n \in \mathbb{N}}$ of $\mathbb{N}^{<\mathbb{N}}$ with the property that $t_m \sqsubseteq t_n \implies m \leq n$ for all $m, n \in \mathbb{N}$, and recursively construct $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathcal{N}_t \cap \mathbb{N}^{<\mathbb{N}}$ such that $\phi(\pi(t_n) \smallfrown (\infty)) \notin \mathcal{B}(\{\phi(\pi(t_m) \smallfrown (\infty)) \mid m < n\}, \varepsilon)$ and $\pi(t'_n) \smallfrown (n) \sqsubseteq \pi(t_n)$ for all $n > 0$, where t'_n is the maximal proper initial segment of t_n .

Otherwise, there exists $x \in \phi(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$ with the property that the set $S = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \phi(s \smallfrown (\infty)) \in \mathcal{B}(x, \varepsilon)\}$ is \sqsubseteq -dense below some extension u of t , in which case we can recursively construct a function $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathcal{N}_u \cap S$ with the property that $\pi(v) \smallfrown (i) \sqsubseteq \pi(v \smallfrown (i))$ for all $i \in \mathbb{N}$ and $v \in \mathbb{N}^{<\mathbb{N}}$. ⊣

PROPOSITION 4.4. *Suppose that X is a metric space and $\phi: \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \rightarrow X$. Then there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ such that $\phi \circ \bar{\pi}$ is an injection into an ε -discrete set for some $\varepsilon > 0$ or $\text{diam}(\phi \circ \bar{\pi})(\mathcal{N}_t) \rightarrow 0$.*

PROOF. Suppose that for no $\varepsilon > 0$ is there a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ such that $\phi \circ \bar{\pi}$ is an injection into an ε -discrete set, fix a sequence $(\varepsilon_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ of positive real numbers converging to zero, and recursively apply Proposition 4.3 to the functions $\phi_t = \phi \circ (\circ_{n < |t|} \bar{\pi}_t \upharpoonright n)$ to obtain \wedge -embeddings $\pi_t: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathcal{N}_t \cap \mathbb{N}^{<\mathbb{N}}$ such that $(\phi \circ (\circ_{n \leq |t|} \bar{\pi}_t \upharpoonright n))(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$ is contained in an ε_t -ball for all $t \in \mathbb{N}^{<\mathbb{N}}$. Let π be the \wedge -embedding obtained from applying Proposition 2.3 to $(\pi_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$, and observe that $\text{diam}(\phi \circ \bar{\pi})(\mathcal{N}_t) \rightarrow 0$. -†

Define $p: \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ by setting $p(t \frown (\infty)) = t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$. Let $\mathbb{N}_*^{<\mathbb{N}} = \mathbb{N}^{<\mathbb{N}} \cup \{\infty\}$ denote the one-point compactification of $\mathbb{N}^{<\mathbb{N}}$.

THEOREM 4.5. Suppose that X is an analytic metric space, Y is a separable metric space, and $\phi: X \rightarrow Y$ is a Baire-class-one function that is not σ -continuous with closed witnesses. Then there exists $\phi_0 \in \{c_{\mathbb{N}^{\mathbb{N}}}\} \cup \{I_{\mathbb{N}^{\mathbb{N}}, Z} \mid Z \in \{\mathbb{N}^{\mathbb{N}}, \mathbb{N}_*^{\mathbb{N}}\}\}$ for which there is a closed continuous embedding of $\phi_0 \cup p$ into ϕ .

PROOF. By the Jayne-Rogers theorem (see, e.g., [2, Theorem 1]), we can assume that ϕ is not G_δ -measurable. Hurewicz’s dichotomy theorem for F_σ sets then yields a closed continuous embedding $\psi: \mathbb{N}_*^{\mathbb{N}} \rightarrow X$ with $\overline{(\phi \circ \psi)(\mathbb{N}^{\mathbb{N}})} \cap (\phi \circ \psi)(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}) = \emptyset$ (see, e.g., [1, Theorem 4.2]). As $(\psi, \text{id}_{\overline{(\phi \circ \psi)(\mathbb{N}_*^{\mathbb{N}})}})$ is a closed continuous embedding of $\phi \circ \psi$ into ϕ , by replacing the latter with the former, we can assume that $X = \mathbb{N}_*^{\mathbb{N}}$ and $\overline{(\phi \circ \psi)(\mathbb{N}^{\mathbb{N}})} \cap (\phi \circ \psi)(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}) = \emptyset$.

By Proposition 3.1, there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ for which $(\phi \circ \bar{\pi}) \upharpoonright \mathbb{N}^{\mathbb{N}}$ is continuous. By composing π with the \wedge -embedding given by Proposition 4.1, we can assume that $\overline{(\phi \circ \bar{\pi})(\mathbb{N}^{\mathbb{N}})} \cap \overline{(\phi \circ \bar{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})} = \emptyset$ or $\phi \circ \bar{\pi}$ is continuous at every point of $\mathbb{N}^{\mathbb{N}}$. As ϕ is Baire class one, the former possibility would imply that the preimages of $\overline{(\phi \circ \bar{\pi})(\mathbb{N}^{\mathbb{N}})}$ and $\overline{(\phi \circ \bar{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})}$ under $\phi \circ \bar{\pi}$ are disjoint dense G_δ subsets of $\mathbb{N}_*^{\mathbb{N}}$, so the latter holds. By Proposition 4.4, we can assume that either there exists $\varepsilon > 0$ for which $(\phi \circ \bar{\pi}) \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ is an injection into an ε -discrete set, or $\text{diam}(\phi \circ \bar{\pi})(\mathcal{N}_t \cap (\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})) \rightarrow 0$. As the former possibility contradicts the facts that $(\phi \circ \bar{\pi})(\mathbb{N}^{\mathbb{N}}) \cap (\phi \circ \bar{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}) = \emptyset$ and $(\phi \circ \bar{\pi})(\mathbb{N}^{\mathbb{N}}) \subseteq \overline{(\phi \circ \bar{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})}$, it follows that the latter holds. By applying Proposition 4.3 with any $\varepsilon > 0$ and $t \in \mathbb{N}^{<\mathbb{N}}$, but replacing the given metric on X by one with respect to which all pairs of distinct points have distance at least ε from one another, we can assume that $(\phi \circ \bar{\pi}) \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ is either constant or injective.

LEMMA 4.6. Suppose that $(s_n)_{n \in \mathbb{N}}$ is an injective sequence of elements of $\mathbb{N}^{<\mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ is a sequence of elements of $\mathbb{N}^{\mathbb{N}}$ such that $s_n \sqsubseteq b_n$ for all $n \in \mathbb{N}$. Then $d_X((\phi \circ \bar{\pi})(b_n), (\phi \circ \bar{\pi})(s_n \frown (\infty))) \rightarrow 0$.

PROOF. Simply note that $(\phi \circ \bar{\pi})(b_n) \in \overline{(\phi \circ \bar{\pi})(\mathcal{N}_{s_n} \cap (\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}))}$ for all $n \in \mathbb{N}$ and $\text{diam}(\phi \circ \bar{\pi})(\mathcal{N}_{s_n} \cap (\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})) \rightarrow 0$. -†

Along with the facts that $(\phi \circ \bar{\pi})(\mathbb{N}^{\mathbb{N}}) \cap (\phi \circ \bar{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}) = \emptyset$ and $(\phi \circ \bar{\pi})(\mathbb{N}^{\mathbb{N}}) \subseteq \overline{(\phi \circ \bar{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})}$, Lemma 4.6 ensures that $(\phi \circ \bar{\pi}) \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ is not constant, and is therefore injective. Along with the fact that $(\phi \circ \bar{\pi})(\mathbb{N}^{\mathbb{N}}) \cap (\phi \circ \bar{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}) = \emptyset$, Lemma 4.6 ensures that $(\phi \circ \bar{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$ is discrete.

By Theorem 3.6, we can assume that $(\phi \circ \bar{\pi}) \upharpoonright \mathbb{N}^{\mathbb{N}}$ is constant or extends to a closed continuous embedding on $\mathbb{N}^{\mathbb{N}}$ or $\mathbb{N}_*^{\mathbb{N}}$.

We will now complete the proof by showing that there exist $\phi_0 \in \{c_{\mathbb{N}^{\mathbb{N}}}\} \cup \{t_{\mathbb{N}^{\mathbb{N}}, Z} \mid Z \in \{\mathbb{N}^{\mathbb{N}}, \mathbb{N}_*^{\mathbb{N}}\}\}$ and $\psi: \overline{\phi_0(\mathbb{N}_*^{\mathbb{N}})} \cup \mathbb{N}^{<\mathbb{N}} \rightarrow \overline{\phi(X)}$ for which $(\bar{\pi} \upharpoonright \mathbb{N}_*^{\mathbb{N}} \rightarrow \mathbb{N}_*^{\mathbb{N}}, \psi)$ is a closed continuous embedding of $\phi_0 \cup p$ into ϕ .

If $(\phi \circ \bar{\pi}) \upharpoonright \mathbb{N}^{\mathbb{N}}$ is constant with value $y \in Y$, then set $\phi_0 = c_{\mathbb{N}^{\mathbb{N}}}$, and note that the extension ψ of $\phi \circ \bar{\pi} \circ p^{-1}$ to $\mathbb{N}_*^{<\mathbb{N}}$ given by $\psi(\infty) = y$ is injective. As Lemma 4.6 ensures that $(\phi \circ \bar{\pi})(s_n \frown (\infty)) \rightarrow y$ for every injective sequence $(s_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{N}^{<\mathbb{N}}$, it follows that ψ is continuous, so the compactness of $\mathbb{N}_*^{<\mathbb{N}}$ ensures that ψ is a closed continuous embedding.

If $(\phi \circ \bar{\pi}) \upharpoonright \mathbb{N}^{\mathbb{N}}$ is a closed continuous embedding, then set $\phi_0 = t_{\mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}}$, and note that the extension ψ of $\phi \circ \bar{\pi} \circ p^{-1}$ to $\mathbb{N}^{<\mathbb{N}}$ given by $\psi \upharpoonright \mathbb{N}^{\mathbb{N}} = (\phi \circ \bar{\pi}) \upharpoonright \mathbb{N}^{\mathbb{N}}$ is a continuous injection. To see that it is closed, it is enough to show that every injective sequence $(a_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{N}^{<\mathbb{N}}$ for which $(\psi(a_n))_{n \in \mathbb{N}}$ converges to some point $y \in Y$ has a subsequence converging to a point of $\mathbb{N}^{\mathbb{N}}$. As $\mathbb{N}_*^{<\mathbb{N}}$ is compact, by passing to a subsequence, we can assume that $(a_n)_{n \in \mathbb{N}}$ converges to a point of $\mathbb{N}_*^{<\mathbb{N}}$. As every point of $\mathbb{N}^{<\mathbb{N}}$ is isolated, it therefore converges to a point of $\mathbb{N}_*^{\mathbb{N}}$. And if there exists $t \in \mathbb{N}^{<\mathbb{N}}$ for which $a_n \rightarrow t \frown (\infty)$, then there are extensions $b_n \in \mathbb{N}^{\mathbb{N}}$ of a_n for all $n \in \mathbb{N}$, in which case $b_n \rightarrow t \frown (\infty)$ and $\psi(b_n) \rightarrow y$ by Lemma 4.6. Fix $n \in \mathbb{N}$ sufficiently large that $(\phi \circ \bar{\pi})(b_m) \neq y$ for all $m \geq n$, and observe that $\{b_m \mid m \geq n\}$ is a closed subset of $\mathbb{N}^{\mathbb{N}}$ whose image under $\phi \circ \bar{\pi}$ is not closed, contradicting the fact that $(\phi \circ \bar{\pi}) \upharpoonright \mathbb{N}^{\mathbb{N}}$ is closed.

If $(\phi \circ \bar{\pi}) \upharpoonright \mathbb{N}^{\mathbb{N}}$ extends to a closed continuous embedding ψ' on $\mathbb{N}_*^{\mathbb{N}}$, then set $\phi_0 = t_{\mathbb{N}^{\mathbb{N}}, \mathbb{N}_*^{\mathbb{N}}}$, and note that the extension ψ of $\phi \circ \bar{\pi} \circ p^{-1}$ to $\mathbb{N}_*^{<\mathbb{N}}$ given by $\psi \upharpoonright \mathbb{N}_*^{\mathbb{N}} = \psi' \upharpoonright \mathbb{N}_*^{\mathbb{N}}$ is injective. To see that it is continuous, suppose that $(t_n)_{n \in \mathbb{N}}$ is an injective sequence of elements of $\mathbb{N}^{<\mathbb{N}}$ converging to $t \frown (\infty)$ for some $t \in \mathbb{N}^{<\mathbb{N}}$, fix $b_n \in \mathcal{N}_{t_n} \cap \mathbb{N}^{\mathbb{N}}$ for all $n \in \mathbb{N}$, and observe that the continuity of ψ' ensures that $\psi(b_n) \rightarrow \psi(t \frown (\infty))$, thus Lemma 4.6 implies that $\psi(t_n) \rightarrow \psi(t \frown (\infty))$. As $\mathbb{N}_*^{<\mathbb{N}}$ is compact, it follows that ψ is a closed continuous embedding.

§5. Functions on $\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$. Here we provide a basis for the class of all functions from $\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ to analytic metric spaces.

PROPOSITION 5.1. *Suppose that X is a topological space, $\phi: \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \rightarrow X$ is injective, and $x \in X$. Then there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ such that $x \notin (\phi \circ \bar{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$.*

PROOF. Fix $s \in \mathbb{N}^{<\mathbb{N}}$ such that $x \notin \phi(\mathcal{N}_s)$, and define $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ by $\pi(t) = s \frown t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$. ⊥

PROPOSITION 5.2. *Suppose that X is a metric space and $\phi: \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \rightarrow X$. Then there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ with the property that $((\phi \circ \bar{\pi})(t \frown (i, \infty)))_{i \in \mathbb{N}}$ is convergent or $\{(\phi \circ \bar{\pi})(t \frown (i, \infty)) \mid i \in \mathbb{N}\}$ is closed and discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$.*

PROOF. For each $t \in \mathbb{N}^{<\mathbb{N}}$, there is an injection $\iota_t: \mathbb{N} \rightarrow \mathbb{N}$ for which $(\phi(t \frown (\iota_t(i), \infty)))_{i \in \mathbb{N}}$ is convergent or $\{\phi(t \frown (\iota_t(i), \infty)) \mid i \in \mathbb{N}\}$ is closed and discrete. Define $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ by choosing $\pi(\emptyset) \in \mathbb{N}^{<\mathbb{N}}$ arbitrarily and setting $\pi(t \frown (i)) =$

$\pi(t) \frown (t_{\pi(t)}(i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, and note that $(\phi \circ \bar{\pi})(t \frown (i, \infty)) = \phi(\pi(t \frown (i)) \frown (\infty)) = \phi(\pi(t) \frown (t_{\pi(t)}(i), \infty))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{\mathbb{N}}$. \dashv

PROPOSITION 5.3. *Suppose that X is a metric space, $\phi: \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \rightarrow X$, $F \subseteq X$ is finite, and $t \in \mathbb{N}^{<\mathbb{N}}$. Then there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathcal{N}_t \cap \mathbb{N}^{<\mathbb{N}}$ such that either $((\phi \circ \bar{\pi})(u \frown (\infty)))_{u \in \mathbb{N}^{<\mathbb{N}}}$ converges to an element of F or the closure of $(\phi \circ \bar{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$ is disjoint from F .*

PROOF. If the set $S_\varepsilon = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \phi(s \frown (\infty)) \in \mathcal{B}(F, \varepsilon)\}$ is \sqsubseteq -dense below t for all $\varepsilon > 0$, then there exist an extension u of t and $x \in F$ such that the set $S_{\varepsilon,x} = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \phi(s \frown (\infty)) \in \mathcal{B}(x, \varepsilon)\}$ is \sqsubseteq -dense below u for all $\varepsilon > 0$. Fix a sequence $(\varepsilon_v)_{v \in \mathbb{N}^{<\mathbb{N}}}$ of positive real numbers converging to zero, and recursively construct a function $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathcal{N}_u \cap \mathbb{N}^{<\mathbb{N}}$ such that $\pi(v) \in S_{\varepsilon_v,x}$ for all $v \in \mathbb{N}^{<\mathbb{N}}$ and $\pi(v) \frown (i) \sqsubseteq \pi(v \frown (i))$ for all $i \in \mathbb{N}$ and $v \in \mathbb{N}^{<\mathbb{N}}$, and observe that $(\phi \circ \bar{\pi})(v \frown (\infty)) \rightarrow x$.

Otherwise, fix $\varepsilon > 0$ and an extension u of t with the property that $\mathcal{N}_u \cap S_\varepsilon = \emptyset$, define $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathcal{N}_u \cap \mathbb{N}^{<\mathbb{N}}$ by $\pi(v) = u \frown v$, and note that the closure of $(\phi \circ \bar{\pi})(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$ is disjoint from F . \dashv

For the rest of this section, it will be convenient to fix an enumeration $(t_n)_{n \in \mathbb{N}}$ of $\mathbb{N}^{<\mathbb{N}}$ such that $t_m \sqsubseteq t_n \implies m \leq n$ for all $m, n \in \mathbb{N}$.

PROPOSITION 5.4. *Suppose that X is a metric space and $\phi: \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \rightarrow X$. Then there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ with the property that $((\phi \circ \bar{\pi})(t \frown (\infty)))_{t \in \mathbb{N}^{<\mathbb{N}}}$ converges or for no natural numbers $m < n$ is $(\phi \circ \bar{\pi})(t_m \frown (\infty))$ or a limit point of $\{(\phi \circ \bar{\pi})(t_m \frown (i, \infty)) \mid i \in \mathbb{N}\}$ in the closure of $(\phi \circ \bar{\pi})(\mathcal{N}_{t_m})$.*

PROOF. Suppose that for no \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ is the sequence $((\phi \circ \bar{\pi})(t \frown (\infty)))_{t \in \mathbb{N}^{<\mathbb{N}}}$ convergent. By Proposition 5.2, we can assume that $(\phi(t \frown (i, \infty)))_{i \in \mathbb{N}}$ is convergent or $\{\phi(t \frown (i, \infty)) \mid i \in \mathbb{N}\}$ is closed and discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$. By recursively applying Lemma 5.3 to the functions $\phi_t = \phi \circ (\circ_{k < |t|} \bar{\pi}_{t|k})$, we obtain \wedge -embeddings $\pi_t: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathcal{N}_t \cap \mathbb{N}^{<\mathbb{N}}$ such that there do not exist natural numbers $m < n$ for which $(\phi \circ (\circ_{k \leq |t_m|} \bar{\pi}_{t_m|k}))(t_m \frown (\infty))$ or a limit point of $\{(\phi \circ (\circ_{k \leq |t_m|} \bar{\pi}_{t_m|k}))(t_m \frown (i, \infty)) \mid i \in \mathbb{N}\}$ in the closure of $(\phi \circ (\circ_{k \leq |t_m|} \bar{\pi}_{t_m|k}))(\mathcal{N}_{t_m})$. Let π be the \wedge -embedding obtained from applying Proposition 2.3 to $(\pi_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$, and observe that for no natural numbers $m < n$ is it the case that $(\phi \circ \bar{\pi})(t_m \frown (\infty))$ or a limit point of $\{(\phi \circ \bar{\pi})(t_m \frown (i, \infty)) \mid i \in \mathbb{N}\}$ in the closure of $(\phi \circ \bar{\pi})(\mathcal{N}_{t_m})$. \dashv

THEOREM 5.5. *Suppose that X is an analytic metric space and $\phi: \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \rightarrow X$. Then there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ such that $\phi \circ \bar{\pi}$ is constant, $\phi \circ \bar{\pi}$ extends to a closed continuous embedding on $\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ or $\mathbb{N}_*^{\mathbb{N}}$, or $\phi \circ \bar{\pi} \circ p^{-1}$ extends to a closed continuous embedding on $\mathbb{N}^{<\mathbb{N}}$, $\mathbb{N}_*^{<\mathbb{N}}$, $\mathbb{N}_*^{\leq \mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$, $\mathbb{N}^{\leq \mathbb{N}}$, or $\mathbb{N}_*^{\leq \mathbb{N}}$.*

PROOF. As before, we will repeatedly precompose ϕ with appropriate \wedge -embeddings, albeit this time so as to stabilize the behavior of the function $\psi = \phi \circ p^{-1}$, as opposed to that of the function ϕ itself. By applying Proposition 4.3 with any $\varepsilon > 0$ and $t \in \mathbb{N}^{<\mathbb{N}}$, but replacing the given metric on X by one with respect to which all pairs of distinct points have distance at least ε from one another, we can assume that ψ is either constant or injective. As ϕ is constant in the former case, we can assume that we are in the latter.

By Proposition 4.4, we can ensure that $\psi(\mathbb{N}^{<\mathbb{N}})$ is closed and discrete or $\text{diam } \psi(\mathcal{N}_i) \rightarrow 0$. As ψ is a closed continuous embedding in the former case, we can assume that we are in the latter.

Let $\bar{\psi}$ be the extension of ψ to the set $D = \mathbb{N}^{<\mathbb{N}} \cup \{b \in \mathbb{N}^{\mathbb{N}} \mid \lim_{i \rightarrow \infty} \psi(b \upharpoonright i) \text{ exists}\} \cup \{t \frown (\infty) \in \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \mid \lim_{i \rightarrow \infty} \psi(t \frown (i)) \text{ exists}\}$ given by $\bar{\psi}(b) = \lim_{i \rightarrow \infty} \psi(b \upharpoonright i)$ and $\bar{\psi}(t \frown (\infty)) = \lim_{i \rightarrow \infty} \psi(t \frown (i))$ for all $b \in D \cap \mathbb{N}^{\mathbb{N}}$ and $t \frown (\infty) \in D \cap (\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$. Note that D could potentially be any set of the form $\mathbb{N}^{<\mathbb{N}} \cup A$, where $A \subseteq \mathbb{N}_*^{\mathbb{N}}$ is analytic (simply consider the inclusion map from $\mathbb{N}^{<\mathbb{N}}$ to $\mathbb{N}^{\leq \mathbb{N}} \cup A$).

By Proposition 5.2, we can assume that $\{\psi(t \frown (i)) \mid i \in \mathbb{N}\}$ has a limit point $\implies t \frown (\infty) \in \text{dom}(\bar{\psi})$ for all $t \in \mathbb{N}^{<\mathbb{N}}$.

As each point of $\mathbb{N}^{<\mathbb{N}}$ is isolated, $\text{diam } \psi(\mathcal{N}_{b \upharpoonright i}) \rightarrow 0$ for all $b \in \mathbb{N}^{\mathbb{N}}$, and $\text{diam } \psi(\mathcal{N}_{t \frown (i)}) \rightarrow 0$ for all $t \in \mathbb{N}^{<\mathbb{N}}$, it follows that $\bar{\psi}$ is continuous. To see that $\bar{\psi}$ is closed, it is sufficient show that every injective sequence $(c_n)_{n \in \mathbb{N}}$ of points in the domain of $\bar{\psi}$ for which $(\bar{\psi}(c_n))_{n \in \mathbb{N}}$ is convergent has a subsequence converging to a point in the domain of $\bar{\psi}$. By passing to a subsequence, we can assume that the sequence converges to a point of $\mathbb{N}_*^{\leq \mathbb{N}}$. As each point of $\mathbb{N}^{<\mathbb{N}}$ is isolated, the sequence converges to a point of $\mathbb{N}_*^{\mathbb{N}}$, so the facts that $\text{diam } \psi(\mathcal{N}_{b \upharpoonright i}) \rightarrow 0$ for all $b \in \mathbb{N}^{\mathbb{N}}$, $\text{diam } \psi(\mathcal{N}_{t \frown (i)}) \rightarrow 0$ for all $t \in \mathbb{N}^{<\mathbb{N}}$, and $\{\psi(t \frown (i)) \mid i \in \mathbb{N}\}$ has a limit point $\implies t \frown (\infty) \in \text{dom}(\bar{\psi})$ for all $t \in \mathbb{N}^{<\mathbb{N}}$ ensure that it converges to a point of the domain of $\bar{\psi}$.

By Proposition 2.8, we can assume that one of the following holds:

- (1) $\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \subseteq \text{dom}(\bar{\psi})$ and $\forall t \in \mathbb{N}^{<\mathbb{N}} \bar{\psi}(t) = \bar{\psi}(t \frown (\infty))$.
- (2) $\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \subseteq \text{dom}(\bar{\psi})$ and $\forall t \in \mathbb{N}^{\leq \mathbb{N}} \bar{\psi}(t) \neq \bar{\psi}(t \frown (\infty))$.
- (3) $(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}) \cap \text{dom}(\bar{\psi}) = \emptyset$.

As the domain of $\bar{\psi}$ is analytic, so too is its intersection with $\mathbb{N}^{\mathbb{N}}$. It follows that the latter intersection has the Baire property, so Proposition 2.9 allows us to assume that one of the following holds:

- (a) The domain of $\bar{\psi}$ is disjoint from $\mathbb{N}^{\mathbb{N}}$.
- (b) The domain of $\bar{\psi}$ contains $\mathbb{N}^{\mathbb{N}}$.

In the special case that condition (b) holds, Theorem 3.6 allows us to assume that $\bar{\psi} \upharpoonright \mathbb{N}^{\mathbb{N}}$ is either constant or injective.

Proposition 5.4 allows us to assume that $(\psi(t))_{t \in \mathbb{N}^{<\mathbb{N}}}$ converges to some $x \in X$ or for no natural numbers $m < n$ is $\psi(t_m)$ or $\bar{\psi}(t_m \frown (\infty))$ in the closure of $\psi(\mathcal{N}_{t_n})$. In the former case, Proposition 5.1 allows us to assume that $\psi(\mathbb{N}^{<\mathbb{N}})$ is discrete, so the extension of ψ to $\mathbb{N}_*^{\leq \mathbb{N}}$ sending ∞ to x is a closed continuous embedding, thus we can assume that we are in the latter.

LEMMA 5.6. *Suppose that $c, d \in \text{dom}(\bar{\psi})$ are distinct but $\bar{\psi}(c) = \bar{\psi}(d)$. Then there exists $t \in \mathbb{N}^{<\mathbb{N}}$ such that $\{c, d\} = \{t, t \frown (\infty)\}$.*

PROOF. To see that $\bar{\psi} \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ is injective, observe that if $m < n$, both $t_m \frown (\infty)$ and $t_n \frown (\infty)$ are in the domain of $\bar{\psi}$, and moreover $\bar{\psi}(t_m \frown (\infty)) = \bar{\psi}(t_n \frown (\infty))$, then $\bar{\psi}(t_m \frown (\infty))$ is in the closure of $\psi(\mathcal{N}_{t_n})$, a contradiction.

To see that $\bar{\psi} \upharpoonright \mathbb{N}^{\mathbb{N}}$ is injective when $\mathbb{N}^{\mathbb{N}}$ is contained in the domain of $\bar{\psi}$, note that otherwise it is constant, and let x be this constant value. Then for each $t \in \mathbb{N}^{<\mathbb{N}}$, there is a sequence $(u_i)_{i \in \mathbb{N}}$ of elements of $\mathbb{N}^{<\mathbb{N}}$ such that $\psi(t \frown (i) \frown (u_i)) \rightarrow x$, so

the fact that $\text{diam } \psi(\mathcal{N}_{t \smallfrown (i)}) \rightarrow 0$ ensures that $\overline{\psi}(t \smallfrown (\infty)) = x$, contradicting the fact that $\overline{\psi} \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ is injective.

To see that $\overline{\psi}(\mathbb{N}^{\mathbb{N}}) \cap \psi(\mathbb{N}^{<\mathbb{N}}) = \emptyset$, note that if $b \in \text{dom}(\overline{\psi}) \cap \mathbb{N}^{\mathbb{N}}$, $t \in \mathbb{N}^{<\mathbb{N}}$, and $\overline{\psi}(b) = \psi(t)$, then there exist $m < n$ with $t_m = t$ and $t_n \sqsubset b$, so $\psi(t_m)$ is in the closure of $\psi(\mathcal{N}_{t_m})$, a contradiction.

To see that $\overline{\psi}(\mathbb{N}^{\mathbb{N}}) \cap \overline{\psi}(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}) = \emptyset$, note that if $b \in \text{dom}(\overline{\psi}) \cap \mathbb{N}^{\mathbb{N}}$, $t \in \mathbb{N}^{<\mathbb{N}}$, $t \smallfrown (\infty) \in \text{dom}(\overline{\psi})$, and $\overline{\psi}(b) = \overline{\psi}(t \smallfrown (\infty))$, then there exist $m < n$ with $t_m = t$ and $t_n \sqsubset b$, in which case $\overline{\psi}(t_m \smallfrown (\infty))$ is in the closure of $\psi(\mathcal{N}_{t_m})$, a contradiction.

Observe finally that if $s, t \in \mathbb{N}^{<\mathbb{N}}$ are distinct, $t \smallfrown (\infty) \in \text{dom}(\overline{\psi})$, and $\psi(s) = \overline{\psi}(t \smallfrown (\infty))$, then there exist $m \neq n$ such that $t_m = s$ and $t_n = t$. Then $\psi(t_m)$ is in the closure of $\psi(\mathcal{N}_{t_m})$ and $\overline{\psi}(t_n \smallfrown (\infty))$ is in $\psi(\mathcal{N}_{t_m})$, a contradiction. ⊥

If (1a) or (1b) holds, then $\overline{\psi} \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ or $\overline{\psi} \upharpoonright \mathbb{N}_*^{\mathbb{N}}$ is an extension of ϕ to a closed continuous embedding. If (2a), (2b), (3a), or (3b) holds, then $\overline{\psi}$ is an extension of ψ to a closed continuous embedding on $\mathbb{N}_*^{\leq \mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$, $\mathbb{N}_*^{\leq \mathbb{N}}$, $\mathbb{N}^{<\mathbb{N}}$, or $\mathbb{N}^{\leq \mathbb{N}}$.

PROPOSITION 5.7. *Suppose that X is an analytic metric space, $\phi: \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \rightarrow X$, $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ is a \wedge -embedding, and $\phi \circ \pi$ is constant, $\phi \circ \pi$ extends to a closed continuous embedding on $\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ or $\mathbb{N}_*^{\mathbb{N}}$, or $\phi \circ \pi \circ p^{-1}$ extends to a closed continuous embedding on $\mathbb{N}^{<\mathbb{N}}$, $\mathbb{N}_*^{<\mathbb{N}}$, $\mathbb{N}_*^{\leq \mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$, $\mathbb{N}^{\leq \mathbb{N}}$, or $\mathbb{N}_*^{\leq \mathbb{N}}$. Then there exist $\phi_0 \in \{c_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}}\} \cup \{I_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}, Z} \mid Z \in \{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}, \mathbb{N}_*^{\mathbb{N}}\}\} \cup \{I_{\mathbb{N}^{<\mathbb{N}}, Z} \circ p \mid Z \in \{\mathbb{N}^{<\mathbb{N}}, \mathbb{N}_*^{<\mathbb{N}}, \mathbb{N}_*^{\leq \mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\leq \mathbb{N}}, \mathbb{N}_*^{\leq \mathbb{N}}\}\}$ and $\psi: \phi_0(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}) \rightarrow \phi(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$ with the property that $(\pi \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}, \psi)$ is a closed continuous embedding of ϕ_0 into ϕ .*

PROOF. If $\phi \circ \pi$ is constant, then set $\phi_0 = c_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}}$ and let ψ be the unique function from $c_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}}(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$ to $(\phi \circ \pi)(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$. If $\phi \circ \pi$ extends to a closed continuous embedding ψ on $Z \in \{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}, \mathbb{N}_*^{\mathbb{N}}\}$, then set $\phi_0 = I_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}, Z}$. And if $\phi \circ \pi \circ p^{-1}$ extends to a closed continuous embedding ψ on $Z \in \{\mathbb{N}^{<\mathbb{N}}, \mathbb{N}_*^{<\mathbb{N}}, \mathbb{N}_*^{\leq \mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\leq \mathbb{N}}, \mathbb{N}_*^{\leq \mathbb{N}}\}$, then set $\phi_0 = I_{\mathbb{N}^{\leq \mathbb{N}}, Z} \circ p$. ⊥

§6. Borel functions that are not Baire class one. Here we provide bases for the classes of non-Baire-class-one Borel functions and non- σ -continuous-with-closed-witnesses Borel functions between analytic metric spaces.

PROPOSITION 6.1. *Suppose that X is a metric space and $\phi: \mathbb{N}_*^{\mathbb{N}} \rightarrow X$ has the property that $\phi \upharpoonright \mathbb{N}^{\mathbb{N}}$ is continuous and $\phi(\mathbb{N}^{\mathbb{N}}) \not\subseteq \overline{\phi(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})}$. Then there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ with the property that $(\phi \circ \pi)(\mathbb{N}^{\mathbb{N}}) \cap \overline{(\phi \circ \pi)(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})} = \emptyset$.*

PROOF. Fix $b \in \mathbb{N}^{\mathbb{N}}$ for which $\phi(b)$ is not in the closure of $\phi(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$. Then there is an open neighborhood U of $\phi(b)$ disjoint from $\phi(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})$, as well as an open neighborhood V of $\phi(b)$ whose closure is contained in U , in which case the continuity of $\phi \upharpoonright \mathbb{N}^{\mathbb{N}}$ yields a proper initial segment s of b for which $\phi(\mathcal{N}_s \cap \mathbb{N}^{\mathbb{N}}) \subseteq V$. Then the \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ given by $\pi(t) = s \smallfrown t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$ is as desired. ⊥

Given $\phi_{\mathbb{N}^{\mathbb{N}}}: \mathbb{N}^{\mathbb{N}} \rightarrow X$ and $\phi_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}}: \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \rightarrow Y$, let $\phi_{\mathbb{N}^{\mathbb{N}}} \sqcup \phi_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}}$ denote the corresponding function from $\mathbb{N}_*^{\mathbb{N}}$ to the disjoint union $X \sqcup Y$.

THEOREM 6.2. *Suppose that X and Y are analytic metric spaces and $\phi: X \rightarrow Y$ is a Borel function that is not Baire class one. Then there exist $\phi_{\mathbb{N}^{\mathbb{N}}} \in \{c_{\mathbb{N}^{\mathbb{N}}}\} \cup \{I_{\mathbb{N}^{\mathbb{N}}, Z} \mid Z \in \{\mathbb{N}^{\mathbb{N}}, \mathbb{N}_*^{\mathbb{N}}\}\}$ and $\phi_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}} \in \{c_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}}\} \cup \{I_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}, Z} \mid Z \in \{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}, \mathbb{N}_*^{\mathbb{N}}\}\} \cup \{I_{\mathbb{N}^{<\mathbb{N}}, Z} \circ p \mid Z \in \{\mathbb{N}^{<\mathbb{N}}, \mathbb{N}_*^{<\mathbb{N}}, \mathbb{N}_*^{\leq \mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\leq \mathbb{N}}, \mathbb{N}_*^{\leq \mathbb{N}}\}\}$ for which there is a closed continuous embedding of $\phi_{\mathbb{N}^{\mathbb{N}}} \sqcup \phi_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}}$ into ϕ .*

PROOF. Hurewicz’s dichotomy theorem for F_σ sets yields a closed continuous embedding $\psi: \mathbb{N}_*^{\mathbb{N}} \rightarrow X$ with $(\phi \circ \psi)(\mathbb{N}^{\mathbb{N}}) \cap \overline{(\phi \circ \psi)(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})} = \emptyset$. As $(\psi, \text{id}_{\overline{(\phi \circ \psi)(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})}})$ is a closed continuous embedding of $\phi \circ \psi$ into ϕ , by replacing the latter with the former, we can assume that $X = \mathbb{N}_*^{\mathbb{N}}$ and $\phi(\mathbb{N}^{\mathbb{N}}) \cap \overline{\phi(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})} = \emptyset$.

By Proposition 3.1, there is a \wedge -embedding $\pi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ for which $(\phi \circ \pi) \upharpoonright \mathbb{N}^{\mathbb{N}}$ is continuous. By composing π with the \wedge -embedding given by Proposition 6.1, we can assume that $\overline{(\phi \circ \pi)(\mathbb{N}^{\mathbb{N}})} \cap \overline{(\phi \circ \pi)(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})} = \emptyset$. By composing π with the \wedge -embedding given by Theorem 3.6, we can assume that $(\phi \circ \pi) \upharpoonright \mathbb{N}^{\mathbb{N}}$ is constant or extends to a closed continuous embedding on $\mathbb{N}^{\mathbb{N}}$ or $\mathbb{N}_*^{\mathbb{N}}$. And by composing π with the \wedge -embedding given by Theorem 5.5, we can assume that $(\phi \circ \pi) \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ is constant, $(\phi \circ \pi) \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ extends to a closed continuous embedding on $\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ or $\mathbb{N}_*^{\mathbb{N}}$, or $\phi \circ \pi \circ p^{-1}$ extends to a closed continuous embedding on $\mathbb{N}^{<\mathbb{N}}, \mathbb{N}_*^{<\mathbb{N}}, \mathbb{N}_*^{\leq \mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\leq \mathbb{N}}$, or $\mathbb{N}_*^{\leq \mathbb{N}}$.

By Proposition 3.7, there exist $\phi_{\mathbb{N}^{\mathbb{N}}} \in \{c_{\mathbb{N}^{\mathbb{N}}}\} \cup \{I_{\mathbb{N}^{\mathbb{N}}, Z} \mid Z \in \{\mathbb{N}^{\mathbb{N}}, \mathbb{N}_*^{\mathbb{N}}\}\}$ and $\psi_{\mathbb{N}^{\mathbb{N}}}: \overline{\phi_{\mathbb{N}^{\mathbb{N}}}(\mathbb{N}^{\mathbb{N}})} \rightarrow \overline{\phi(\mathbb{N}^{\mathbb{N}})}$ for which $(\overline{\pi} \upharpoonright \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}, \psi_{\mathbb{N}^{\mathbb{N}}})$ is a closed continuous embedding of $\phi_{\mathbb{N}^{\mathbb{N}}}$ into $\phi \upharpoonright \mathbb{N}^{\mathbb{N}}$. By Proposition 5.7, there exist $\phi_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}} \in \{c_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}}\} \cup \{I_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}, Z} \mid Z \in \{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}, \mathbb{N}_*^{\mathbb{N}}\}\} \cup \{I_{\mathbb{N}^{<\mathbb{N}}, Z} \circ p \mid Z \in \{\mathbb{N}^{<\mathbb{N}}, \mathbb{N}_*^{<\mathbb{N}}, \mathbb{N}_*^{\leq \mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\leq \mathbb{N}}, \mathbb{N}_*^{\leq \mathbb{N}}\}\}$ and $\psi_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}}: \overline{\phi_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}}(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})} \rightarrow \overline{\phi(\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}})}$ for which $(\overline{\pi} \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}, \psi_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}})$ is a closed continuous embedding of $\phi_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}}$ into $\phi \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$. Then $(\overline{\pi} \upharpoonright \mathbb{N}_*^{\mathbb{N}} \rightarrow \mathbb{N}_*^{\mathbb{N}}, \psi_{\mathbb{N}^{\mathbb{N}}} \sqcup \psi_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}})$ is a closed continuous embedding of $\phi_{\mathbb{N}^{\mathbb{N}}} \sqcup \phi_{\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}}$ into ϕ .

Theorems 4.5 and 6.2 together provide the promised twenty-seven element basis under closed continuous embeddability for the class of non- σ -continuous-with-closed-witnesses Borel functions between analytic metric spaces.

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