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New relaxation theorems with applications to strong materials

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Recently, Sychev showed that conditions both necessary and sufficient for lower semicontinuity of integral functionals with p-coercive extended-valued integrands are the $W^{1,p}$-quasi-convexity and the validity of a so-called matching condition (M). Condition (M) is so general that we conjecture whether it always holds in the case of continuous integrands. In this paper we develop the relaxation theory under the validity of condition (M). It turns out that a better relaxation theory is available in this case. This motivates our research since it is an important old open problem to develop the relaxation theory in the case of extended-value integrands. Then we discuss applications of the general relaxation theory to some concrete cases, in particular to the theory of strong materials.

Keywords: lower semicontinuity; relaxation; $W^{1;p}$-quasi-convexity; extended-valued integrand; strong materials

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1. Preliminaries

Let $m, N \geqslant 1$ be two integers and let $\mathbb{M}$ be the space of all real $m \times N$ matrices. In what follows, we denote the Lebesgue measure of a Borel subset A of $\mathbb{R}^N$ by $|A|$. Let $p > 1$ be a real number and let $E\colon W^{1,p}(\Omega;\mathbb{R}^m) \to [0,\infty]$ be defined by

$$E(u) := \int_\Omega L(\nabla u(x))\,\mathrm{d}x,$$

where Ω is a bounded open set such that $|\partial\Omega| = 0$ and $L\colon \mathbb{M} \to [0,\infty]$ is Borel measurable and p-coercive, i.e. $L(\cdot) \geqslant c|\cdot|^p$ for some $c > 0$. Let $\bar{E}\colon W^{1,p}(\Omega;\mathbb{R}^m) \to [0,\infty]$ be the lower semi-continuous envelope of E with respect to the strong convergence of $L^p(\Omega;\mathbb{R}^m)$, i.e.

$$\bar{E}(u) := \inf\Big\{ \varliminf_{n\to\infty} E(u_n) \colon u_n \to u \text{ in } L^p(\Omega;\mathbb{R}^m)\Big\}.$$

Due to the p-coercivity of L, strong convergence in L^p could be replaced by weak convergence in $W^{1,p}$, as this frequently happens in the literature. However, in this paper we shall use strong convergence in L^p. It is well known that $\bar{E}$ is lower semi-continuous with respect to the strong convergence in L^p (weak convergence in $W^{1,p}$).

Recently, Sychev suggested conditions both necessary and sufficient for lower semicontinuity of the functional E with respect to the strong convergence in L^p (see condition (M) and theorem 1.1).

(M) For every $\xi \in \mathbb{M}$ and every $\{u_n\}_n \subset W^{1,p}(Y;\mathbb{R}^m)$ such that $u_n \to l_\xi$ in $L^p(Y;\mathbb{R}^m)$ and $\varlimsup_{n\to\infty} I(u_n) < \infty$, there exist a subsequence $\{u_n\}_n$ (not relabelled) and a sequence $\{\phi_n\} \subset l_\xi + W_0^{1,p}(Y;\mathbb{R}^m)$ such that

$$\varliminf_{n\to\infty} \{I(u_n) - I(\phi_n)\} \geqslant 0, \quad \text{where } I(u) := \int_Y L(\nabla u(y))\,\mathrm{d}y.$$

Hereafter, l_ξ will denote the linear function with gradient equal to $\xi \in \mathbb{M}$ and $Y :=]-\frac{1}{2}, \frac{1}{2}[^N$.

THEOREM 1.1 (Sychev [23]). *Assume that $L\colon \mathbb{M} \to [0,\infty]$ is continuous. Then, the functional E is L^p-lower semi-continuous if and only if L is $W^{1,p}$-quasi-convex and the matching condition (M) holds.*

Recall that L is called $W^{1,p}$-quasi-convex at ξ if

$$\int_\Omega L(\nabla\phi(x))\,\mathrm{d}x \geqslant |\Omega| L(\xi) \quad \text{for all } \phi \in l_\xi + W_0^{1,p}(\Omega;\mathbb{R}^m).$$

The definition of $W^{1,p}$-quasi-convexity does not depend on Ω (see [7]), and it was shown in [7] that this condition is necessary for lower semicontinuity with respect to the weak convergence in $W^{1,p}$.

Theorem 1.1 also holds for Borel measurable integrands L, as this follows from theorem 2.1. Note that condition (M) is rather general and it is unknown whether it always holds in the case of continuous integrands (in the case of Borel measurable integrands the condition may fail; see § 2).

In the general case considered in this paper, the theory of integral representations of relaxed functionals $\bar{E}$ is not well developed, since it requires dealing with $W^{1,p}$-quasi-convexifications $\mathcal{Q}_p L$, where

$$\mathcal{Q}_p L(\xi) := \inf\Big\{ \int_Y L(\nabla\phi(y))\,\mathrm{d}y \colon \phi \in l_\xi + W_0^{1,p}(Y;\mathbb{R}^m)\Big\},$$

and there is not much information about the behaviour of such integrands (see, for example, [13–16]). A better theory is available when $\mathcal{Q}_p L = \mathcal{Q}_\infty L$ and L is a finite-valued function [1,11,21,22] (see [15] for a list of appropriate references). The study of extended-valued integrands $L\colon \mathbb{M} \to [0,\infty]$ is dictated by the theory of elasticity as suggested by Ball (see, for example, [4–6]) in order to account for the physical behaviour of materials when their volume is compressed to zero, i.e. the determinant of the deformation gradient tends to (or is equal to) 0. This is a major problem in elasticity and a challenge in the calculus of variations, as standard cut-offs cannot preserve the determinant constraint. The aim of this paper is to contribute to the development of the relaxation theory in the case of extended-valued integrands (see, for example, [2, 10, 20–22] for related works).

We shall use the following two standard lemmas about $W^{1,p}$-quasi-convexifications.

Lemma 1.2. *For every $\xi \in \mathbb{M}$ and every bounded open set $U \subset \mathbb{R}^N$ with $|\partial U| = 0$, we have*

$$\mathcal{Q}_p L(\xi) = \inf\left\{ ⨍_U L(\nabla\phi(y))\,\mathrm{d}y \colon \phi \in l_\xi + W_0^{1,p}(U;\mathbb{R}^m) \right\}. \tag{1.1}$$

Proof of lemma 1.2. Let $\xi \in \mathbb{M}$. Let $U \subset \mathbb{R}^N$ be a bounded open set with $|\partial U| = 0$, and denote the right-hand side of (1.1) by $\mathcal{Q}_p L(\xi, U)$. By Vitali's covering theorem there exists a finite or countable family $\{a_i + \alpha_i U\}_{i\in I}$ of disjoint subsets of Y, where $a_i \in \mathbb{R}^N$ and $0 < \alpha_i < 1$, such that $|Y \setminus \bigcup_{i\in I}(a_i + \alpha_i U)| = 0$ (and so $|U|\sum_{i\in I}\alpha_i^N = |Y| = 1$). Fix any $\phi \in l_\xi + W_0^{1,p}(Y;\mathbb{R}^m)$. For each $i \in I$, let $\hat{\phi}_i \in l_\xi + W_0^{1,p}(U;\mathbb{R}^m)$ be defined by

$$\hat{\phi}_i(x) := l_\xi(x) + \frac{1}{\alpha_i}(\phi(a_i + \alpha_i x) - l_\xi(a_i + \alpha_i x)).$$

Then

$$\int_U L(\nabla\hat{\phi}_i(x))\,\mathrm{d}x \geqslant |U|\mathcal{Q}_p L(\xi, U)$$

for all $i \in I$. But,

$$\begin{aligned}\int_Y L(\nabla\phi(y))\,\mathrm{d}y &= \sum_{i\in I}\int_{a_i+\alpha_i U} L(\nabla\phi(y))\,\mathrm{d}y \\ &= \sum_{i\in I}\alpha_i^N \int_U L(\nabla\phi(a_i + \alpha_i x))\,\mathrm{d}x \\ &= \sum_{i\in I}\alpha_i^N \int_U L(\nabla\hat{\phi}_i(x))\,\mathrm{d}x.\end{aligned}$$

Hence, recalling that $\sum_{i\in I}\alpha_i^N = 1/|U|$, it follows that

$$\begin{aligned}\int_Y L(\nabla\phi(y))\,\mathrm{d}y &\geqslant \sum_{i\in I}\alpha_i^N(|U|\mathcal{Q}_p L(\xi, U)) \\ &= \mathcal{Q}_p L(\xi, U).\end{aligned}$$

Thus, $\mathcal{Q}_p L(\xi) \geqslant \mathcal{Q}_p L(\xi, U)$. By the same reasoning we prove the converse inequality, and the proof is complete. □

Lemma 1.3. *Given $\xi \in \mathbb{M}$ and a bounded open set $U \subset \mathbb{R}^N$ with $|\partial U| = 0$, there exists a sequence $\{\phi_n\}_n \subset l_\xi + W_0^{1,p}(U;\mathbb{R}^m)$ such that $\phi_n \to l_\xi$ in $L^p(U;\mathbb{R}^m)$ and*

$$\lim_{n\to\infty} ⨍_U L(\nabla\phi_n(x))\,\mathrm{d}x = \mathcal{Q}_p L(\xi).$$

Proof of lemma 1.3. Given $\xi \in \mathbb{M}$, there exists $\{\phi_n\}_n \subset l_\xi + W_0^{1,p}(Y;\mathbb{R}^m)$ such that

$$\lim_{n\to\infty} \int_Y L(\nabla\phi_n(y))\,\mathrm{d}y = \mathcal{Q}_p L(\xi). \tag{1.2}$$

Fix any $n \geqslant 1$ and $k \geqslant 1$. By Vitali's covering theorem there exists a finite or countable family $\{a_i + \alpha_i Y\}_{i\in I}$ of disjoint subsets of U, where $a_i \in \mathbb{R}^N$ and $0 < \alpha_i < 1/k$, such that $|U \setminus \bigcup_{i\in I}(a_i + \alpha_i Y)| = 0$ (and so $\sum_{i\in I} \alpha_i^N = |U|$). Define $\phi_{n,k} \in l_\xi + W_0^{1,p}(U;\mathbb{R}^m)$ by

$$\phi_{n,k}(x) := l_\xi(x) + \alpha_i\left[\phi_n\left(\frac{x-a_i}{\alpha_i}\right) - l_\xi\left(\frac{x-a_i}{\alpha_i}\right)\right] \text{ if } x \in a_i + \alpha_i Y.$$

Clearly,

$$\|\phi_{n,k} - l_\xi\|^p_{L^p(U;\mathbb{R}^m)} \leqslant \frac{|U|}{k^p}\|\phi_n - l_\xi\|^p_{L^p(Y;\mathbb{R}^m)}.$$

Hence, $\lim_{k\to\infty} \|\phi_{n,k} - l_\xi\|_{L^p(U;\mathbb{R}^m)} = 0$ for all $k \geqslant 1$, and consequently

$$\lim_{n\to\infty}\lim_{k\to\infty} \|\phi_{n,k} - l_\xi\|_{L^p(U;\mathbb{R}^m)} = 0. \tag{1.3}$$

On the other hand, we have

$$\int_U L(\nabla\phi_{n,k}(x))\,\mathrm{d}x = \sum_{i\in I}\alpha_i^N \int_Y L(\nabla\phi_n(y))\,\mathrm{d}y = |U|\int_Y L(\nabla\phi_n(y))\,\mathrm{d}y$$

for all $n \geqslant 1$ and all $k \geqslant 1$. Using (1.2) we deduce that

$$\lim_{n\to\infty}\lim_{k\to\infty} \fint_U L(\nabla\phi_{n,k}(x))\,\mathrm{d}x = \mathcal{Q}_p L(\xi), \tag{1.4}$$

and the result follows from (1.3) and (1.4) by diagonalization. □

In this paper we study the relaxation theory when condition (M) holds, i.e. when $\mathcal{Q}_p L = \mathcal{T}_p L$, where

$$\mathcal{T}_p L(\xi) := \inf\left\{ \varliminf_{n\to\infty} \int_Y L(\nabla\phi_n(y))\,\mathrm{d}y \colon W^{1,p}(Y;\mathbb{R}^m) \ni \phi_n \to l_\xi \text{ in } L^p(Y;\mathbb{R}^m)\right\}.$$

Remark 1.4. From lemma 1.3 we see that

$$\mathcal{Q}_p L(\xi) = \inf\left\{ \varliminf_{n\to\infty} \fint_U L(\nabla\phi_n(x))\,\mathrm{d}x \colon \\ l_\xi + W_0^{1,p}(U;\mathbb{R}^m) \ni \phi_n \to l_\xi \text{ in } L^p(Y;\mathbb{R}^m)\right\}$$

for all bounded open sets $U \subset \mathbb{R}^N$ with $|\partial U| = 0$. In particular, we always have $\mathcal{T}_p L \leqslant \mathcal{Q}_p L$. So, condition (M) holds if and only if $\mathcal{T}_p L = \mathcal{Q}_p L$.

It turns out that a better relaxation theory is available in this case.

First, $W^{1,p}$-quasi-convexifications $\mathcal{Q}_p L$ of the original integrands are lower semi-continuous functions, and therefore the integral functionals with $\mathcal{Q}_p L$ integrands are well defined. This follows from a simple lemma.

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LEMMA 1.5. *The function $\mathcal{T}_pL$ is lower semi-continuous.*

Proof of lemma 1.5. Let $I\colon W^{1,p}(Y;\mathbb{R}^m)\to[0,\infty]$ be defined by

$$I(\phi):=\int_Y L(\nabla\phi(y))\,\mathrm{d}y.$$

Then $\mathcal{T}_pL(\xi)=\bar{I}(l_\xi)$ for all $\xi\in\mathbb{M}$, where $\bar{I}$ denotes the lower semi-continuous envelope of I with respect to the strong convergence in $L^p(Y;\mathbb{R}^m)$. As $\bar{I}$ is L^p-lower semi-continuous, we have, in particular, $\varliminf_{n\to\infty}\bar{I}(l_{\xi_n})\geqslant\bar{I}(l_\xi)$ for all $\xi\in\mathbb{M}$ and all $\{\xi_n\}_n\subset\mathbb{M}$ such that $l_{\xi_n}\to l_\xi$ in $L^p(Y;\mathbb{R}^m)$. Hence, $\varliminf_{n\to\infty}\mathcal{T}_pL(\xi_n)\geqslant\mathcal{T}_pL(\xi)$ for all $\xi\in\mathbb{M}$ and all $\{\xi_n\}_n\subset\mathbb{M}$ such that $\xi_n\to\xi$ in $\mathbb{M}$. □

Moreover, for $\mathcal{T}_pL$ we have the following.

LEMMA 1.6. *For every $\xi\in\mathbb{M}$ and every bounded open set $U\subset\mathbb{R}^N$ with $|\partial U|=0$, we have*

$$\mathcal{T}_pL(\xi)=\inf\left\{\varliminf_{n\to\infty}⨍_U L(\nabla\phi_n(x))\,\mathrm{d}x\colon \phi_n\to l_\xi \text{ in } L^p(U;\mathbb{R}^m)\right\}. \tag{1.5}$$

Proof of lemma 1.6. Let $\xi\in\mathbb{M}$. Let $U\subset\mathbb{R}^N$ be a bounded open set with $|\partial U|=0$ and denote the right-hand side of (1.5) by $\mathcal{T}_pL(\xi,U)$. By Vitali's covering theorem there exists a finite or countable family $\{a_i+\alpha_iU\}_{i\in I}$ of disjoint subsets of Y, where $a_i\in\mathbb{R}^N$ and $0<\alpha_i<1$, such that $|Y\setminus\bigcup_{i\in I}(a_i+\alpha_iU)|=0$ (and so $|U|\sum_{i\in I}\alpha_i^N=|Y|=1$). Fix any sequence $\{\phi_n\}_n\subset W^{1,p}(Y;\mathbb{R}^m)$ such that $\phi_n\to l_\xi$ in $L^p(Y;\mathbb{R}^m)$ and fix any $n\geqslant 1$. For each $i\in I$, let $\hat{\phi}_{n,i}\in W^{1,p}(U;\mathbb{R}^m)$ be defined by

$$\hat{\phi}_{n,i}(x):=l_\xi(x)+\frac{1}{\alpha_i}(\phi_n(a_i+\alpha_ix)-l_\xi(a_i+\alpha_ix)).$$

As $a_i+\alpha_iU\subset Y$ we have

$$\|\hat{\phi}_{n,i}-l_\xi\|^p_{L^p(U;\mathbb{R}^m)}\leqslant\alpha_i^{N-p}\|\phi_n-l_\xi\|^p_{L^p(Y;\mathbb{R}^m)},$$

and so $\hat{\phi}_{n,i}\to l_\xi$ in $L^p(U;\mathbb{R}^m)$. Then

$$\varliminf_{n\to\infty}\int_U L(\nabla\hat{\phi}_{n,i}(x))\,\mathrm{d}x\geqslant|U|\mathcal{T}_pL(\xi,U)$$

for all $i\in I$. But,

$$\begin{aligned}\varliminf_{n\to\infty}\int_Y L(\nabla\phi_n(y))\,\mathrm{d}y&=\varliminf_{n\to\infty}\left(\sum_{i\in I}\int_{a_i+\alpha_iU}L(\nabla\phi_n(y))\,\mathrm{d}y\right)\\&=\varliminf_{n\to\infty}\left(\sum_{i\in I}\alpha_i^N\int_U L(\nabla\phi_n(a_i+\alpha_ix))\,\mathrm{d}x\right)\\&=\varliminf_{n\to\infty}\left(\sum_{i\in I}\alpha_i^N\int_U L(\nabla\hat{\phi}_{n,i}(x))\,\mathrm{d}x\right)\\&\geqslant\sum_{i\in I}\alpha_i^N\varliminf_{n\to\infty}\int_U L(\nabla\hat{\phi}_{n,i}(x))\,\mathrm{d}x.\end{aligned}$$

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Hence, recalling that $\sum_{i\in I}\alpha_i^N = 1/|U|$, it follows that

$$\varliminf_{n\to\infty}\int_Y L(\nabla\phi_n(y))\,\mathrm{d}y \geqslant \sum_{i\in I}\alpha_i^N(|U|\mathcal{T}_pL(\xi,U)) = \mathcal{T}_pL(\xi,U).$$

Thus, $\mathcal{T}_pL(\xi)\geqslant\mathcal{T}_pL(\xi,U)$. By the same reasoning, we prove the converse inequality, and the proof is complete. □

Second, the values of the lower semi-continuous envelope of the original functional are given by the values of the integral functional with a $\mathcal{Q}_pL$ integrand at all continuous piecewise affine functions (see corollary 3.4).

Third, for all Sobolev functions in $W^{1,p}$ we have

$$\bar{E}(u)\geqslant\int_\Omega \mathcal{Q}_pL(\nabla u(x))\,\mathrm{d}x.$$

Finally, by using the approximation property ($\mathrm{C}_{[H]}$) with $H=\mathcal{Q}_pL$, we obtain a complete integral representation for $\bar{E}$ (see theorem 3.5).

In § 2, we prove extensions of the lower semicontinuity (theorem 1.1) to the case of Borel measurable integrands. The theory for relaxation under validity of condition (M) is presented in § 3. In § 4 we suggest applications of the general relaxation theory to the case when $W^{1,p}$-quasi-convexifications have convex growth. We discuss concrete cases of strong materials when this is so. The results of this paper were previously announced in the note [17].

2. General lower semicontinuity theorems

The main result of this section is the following.

THEOREM 2.1. *The relaxed functional $\bar{E}$ is equal to E if and only if, for every $\xi\in\mathbb{M}$, E is L^p-lower semi-continuous at l_ξ, i.e.*

$$\varliminf_{n\to\infty} E(\phi_n)\geqslant E(l_\xi)\quad \text{for all } \{\phi_n\}_n\subset W^{1,p}(\Omega;\mathbb{R}^m) \text{ with } \phi_n\to l_\xi \text{ in } L^p(\Omega;\mathbb{R}^m).$$

Proof of theorem 2.1. We only need to prove that if E is L^p-lower semi-continuous at l_ξ for every $\xi\in\mathbb{M}$, then E is L^p-lower semi-continuous, i.e. for every $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ and every $\{u_n\}_n\subset W^{1,p}(\Omega;\mathbb{R}^m)$ with $u_n\to u$ in $L^p(\Omega;\mathbb{R}^m)$, we have

$$\varliminf_{n\to\infty} E(u_n)\geqslant E(u). \tag{2.1}$$

STEP 1 (localization and blow-up). We can assume that $\lim_{n\to\infty}E(u_n)$ exists and is finite. Recalling that L is p-coercive, there is no loss of generality in assuming that $\{\nabla u_n\}_n$ generates a Young measure $(\nu_x)_{x\in\Omega}$ and $L(\nabla u_n(\cdot))\,\mathrm{d}x \overset{*}{\rightharpoonup}\mu$, where μ is a (positive) Radon measure. The measure μ can be represented as $\mu=\mu_a+\mu_s$, where $\mu_a=f(\cdot)\,\mathrm{d}x$ with $f\in L^1(\Omega)$ and μ_s is a singular measure, i.e. its support is contained in $\bigcup_{j=1}^\infty K_j\subset\Omega$, where K_j is a compact set with $|K_j|=0$ for each $j\geqslant 1$.

For almost every (a.e.) $x_0\in\Omega$ we have that x_0 is a Lebesgue point of the function $f(\cdot)$, and x_0 is a Lebesgue point of $\nu(\cdot)$ in the ρ-metric, where ρ is a metric

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responsible for weak-$*$ convergence of probability measures (see [18] for the latter fact).

For each $k \geqslant 1$ and each $n \geqslant 1$, consider the function $u_n^k\colon Y \to \mathbb{R}^m$ given by

$$u_n^k(y) = \frac{1}{k} u_n(x_0 + ky).$$

For a fixed $k \geqslant 1$ we have that $\{\nabla u_n^k\}_n$ generates a Young measure $(\nu_{(x_0+ky)})_{y \in Y/k}$ and $L(\nabla u_n^k(\cdot))\,\mathrm{d}y \overset{*}{\rightharpoonup} f(x_0 + ky)\,\mathrm{d}y + \mu_s^k$ in Y as $n \to \infty$. We can isolate a sequence $\{k_n\}_n$, with $k_n \to \infty$ as $n \to \infty$, in such a way that $\{\nabla u_n^{k_n}\}_n$ generates a homogeneous Young measure ν_{x_0}. We also have that, for a.e. $x_0 \in \Omega$,

$$L(\nabla u_n^{k_n}(\cdot))\,\mathrm{d}y \overset{*}{\rightharpoonup} f(x_0)\,\mathrm{d}y. \tag{2.2}$$

Then, the sequence $\{\nabla u_n^{k_n}\}_n$ is bounded in $W^{1,p}(Y;\mathbb{R}^m)$ and, as a consequence,

$$u_n^{k_n} \to l_{\nabla u(x_0)} \quad \text{in } L^p(Y;\mathbb{R}^m). \tag{2.3}$$

Step 2 (using L^p-lower semicontinuity of E at each l_ξ). Using lemma 1.6, we see that the L^p-lower semicontinuity of E at each l_ξ is equivalent to $\mathcal{T}_p L(\xi) = L(\xi)$ at each $\xi \in \mathbb{M}$. Hence, taking (2.3) into account, we can assert that

$$\varliminf_{n\to\infty} \int_Y L(\nabla u_n^{k_n}(y))\,\mathrm{d}y \geqslant L(\nabla u(x_0)). \tag{2.4}$$

From (2.2) and (2.4) it follows that

$$f(x_0) \geqslant L(\nabla u(x_0)).$$

Therefore, $f(x) \geqslant L(\nabla u(x))$ for almost all (a.a.) $x \in \Omega$, and (2.1) follows. □

Theorem 2.1 can be reformulated as follows.

Theorem 2.2. *$\bar{E} = E$ if and only if $\mathcal{T}_p L = L$.*

Proof of theorem 2.2. Theorem 2.1 asserts that $\bar{E} = E$ if and only if, for every $\xi \in \mathbb{M}$, E is L^p-lower semi-continuous at l_ξ. But, by lemma 1.6, this is also equivalent to $\mathcal{T}_p L = L$. □

Theorem 2.1 can also be reformulated as follows.

Corollary 2.3. *$\bar{E} = E$ if and only if $\mathcal{Q}_p L = L$ and condition (M) holds.*

Proof of corollary 2.3. It suffices to remark that condition (M) is equivalent to $\mathcal{T}_p L = \mathcal{Q}_p L$ (see remark 1.4). □

Instead of using two conditions in theorem 1.1, we can use only one condition.

Definition 2.4. A Borel measurable function $L\colon \mathbb{M} \to [0,\infty]$ is said to be lim-$W^{1,p}$-quasi-convex if

$$\varliminf_{n\to\infty} \int_Y L(\nabla \phi_n(y))\,\mathrm{d}y \geqslant L(\xi)$$

for all $\xi \in \mathbb{M}$ and all $\{\phi_n\}_n \subset W^{1,p}(Y;\mathbb{R}^m)$ with $\phi_n \to l_\xi$ in $L^p(Y;\mathbb{R}^m)$.

Remark 2.5. L is lim-$W^{1,p}$-quasi-convex if and only if $L = \mathcal{T}_p L$, which is also equivalent to $\mathcal{Q}_p L = L$ and the validity of condition (M). So, if L is lim-$W^{1,p}$-quasi-convex, then L is lower semi-continuous and $W^{1,p}$-quasi-convex.

The interest of definition 2.4 comes from the following result, which is yet another formulation of theorem 2.1.

Theorem 2.6. *The functional E is L^p-lower semi-continuous if and only if L is* lim-$W^{1,p}$*-quasi-convex.*

An open question is the following.

Question 2.7. Do continuity and $W^{1,p}$-quasi-convexity imply lim-$W^{1,p}$-quasi-convexity?

This question is equivalent to clarifying whether continuous and $W^{1,p}$-quasi-convex integrands always satisfy condition (M).

Remark 2.8. In the case of Borel measurability and even lower semicontinuity there exist $W^{1,p}$-quasi-convex integrands that are not lim-$W^{1,p}$-quasi-convex. Consider for example, in the scalar case, $L\colon \mathbb{R}^2 \to [0,\infty]$ given by

$$L(\xi) = \begin{cases} 1 & \text{if } \xi = (0,0), \\ 0 & \text{if } \xi \in \{(-1,0),(1,0)\}, \\ \infty & \text{otherwise.} \end{cases}$$

The function L is not lim-$W^{1,p}$-quasi-convex but L is lower semi-continuous and $W^{1,p}$-quasi-convex (but not convex) because all nonlinear functions from $Y \subset \mathbb{R}^2$ to $\mathbb{R}$ with a linear boundary datum give infinite energy, since it is impossible for the gradient to stay in a line due to Cellina's result (see [8, 9]).

3. General relaxation theorems

Denote the space of all continuous piecewise affine functions from Ω to $\mathbb{R}^m$ by $\mathrm{Aff}(\Omega;\mathbb{R}^m)$.

Theorem 3.1. *The following two inequalities always hold:*

$$\bar{E}(u) \geqslant \int_\Omega \mathcal{T}_p L(\nabla u(x))\,\mathrm{d}x \quad \textit{for all } u \in W^{1,p}(\Omega;\mathbb{R}^m); \tag{3.1}$$

$$\bar{E}(u) \leqslant \int_\Omega \mathcal{Q}_p L(\nabla u(x))\,\mathrm{d}x \quad \textit{for all } u \in \mathrm{Aff}(\Omega;\mathbb{R}^m). \tag{3.2}$$

Proof of theorem 3.1. We begin by proving (3.1). Consider $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and $\{u_n\}_n \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that

$$\|u_n - u\|_{L^p(\Omega;\mathbb{R}^m)} \to 0, \tag{3.3}$$

and prove that

$$\varliminf_{n\to\infty} E(u_n) \geqslant \int_\Omega \mathcal{T}_p L(\nabla u(x))\,\mathrm{d}x. \tag{3.4}$$

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Step 1 (localization). Without loss of generality we can assume that

$$\infty > \varliminf_{n\to\infty} E(u_n) = \lim_{n\to\infty} E(u_n), \quad \text{and so } \sup_n \int_\Omega L(\nabla u_n(x))\,\mathrm{d}x < \infty. \tag{3.5}$$

For each $n \geqslant 1$, we define the (positive) Radon measure λ_n on Ω by

$$\lambda_n := L(\nabla u_n(\cdot))\,\mathrm{d}x.$$

From (3.5) we see that $\sup_n \lambda_n(\Omega) < \infty$, and so there exists a (positive) Radon measure λ on Ω such that (up to a subsequence) $\lambda_n \overset{*}{\rightharpoonup} \lambda$, i.e.

$$\lim_{n\to\infty} \int_\Omega \phi\,\mathrm{d}\lambda_n = \int_\Omega \phi\,\mathrm{d}\lambda \quad \text{for all } \phi \in C_{\mathrm{c}}(\Omega),$$

or, equivalently, the following two equivalent conditions hold:

(a)

$$\varliminf_{n\to\infty} \lambda_n(U) \geqslant \lambda(U) \quad \text{for all open sets } U \subset \Omega,$$
$$\varlimsup_{n\to\infty} \lambda_n(K) \leqslant \lambda(K) \quad \text{for all compact sets } K \subset \Omega;$$

(b) $\lim_{n\to\infty} \lambda_n(B) = \lambda(B)$ for all bounded Borel sets $B \subset \Omega$ with $\lambda(\partial B) = 0$.

By Lebesgue's decomposition theorem, we have $\lambda = \lambda_a + \lambda_s$, where λ_a and λ_s are (positive) Radon measures such that $\lambda_a \ll \mathrm{d}x$ and $\lambda_s \perp \mathrm{d}x$, and from the Radon–Nikodým theorem we deduce that there exists $f \in L^1(\Omega; [0,\infty[)$ given by

$$f(x) = \lim_{r\to 0} \frac{\lambda_a(x + rY)}{r^N} = \lim_{r\to 0} \frac{\lambda(x + rY)}{r^N} \quad \text{for a.a. } x \in \Omega, \tag{3.6}$$

such that

$$\lambda_a(A) = \int_A f\,\mathrm{d}x \quad \text{for all measurable sets } A \subset \Omega.$$

To prove (3.4) it suffices to show that

$$f(x) \geqslant \mathcal{T}_p(\nabla u(x)) \quad \text{for a.a. } x \in \Omega. \tag{3.7}$$

Indeed, from (a) we see that

$$\varliminf_{n\to\infty} E(u_n) = \varliminf_{n\to\infty} \lambda_n(\Omega) \geqslant \lambda(\Omega) = \lambda_a(\Omega) + \lambda_s(\Omega) \geqslant \lambda_a(\Omega) = \int_\Omega f(x)\,\mathrm{d}x.$$

But, by (3.7), we have

$$\int_\Omega f(x)\,\mathrm{d}x \geqslant \int_\Omega \mathcal{T}_p(\nabla u(x))\,\mathrm{d}x,$$

and (3.4) follows.

Step 2 (blow up). Fix $x_0 \in \Omega \setminus N$, where $N \subset \Omega$ is a suitable set such that $|N| = 0$, and prove that

$$f(x_0) \geqslant \mathcal{T}_p L(\nabla u(x_0)). \tag{3.8}$$

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As $\lambda(\Omega) < \infty$ we have $\lambda(x_0 + rY) = 0$ for all $r \in]0, 1] \setminus D$, where D is a countable set. From (b) and (3.6) we deduce that

$$f(x_0) = \lim_{r\to 0} \lim_{n\to\infty} \frac{\lambda_n(x_0 + rY)}{r^N} = \lim_{r\to 0} \lim_{n\to\infty} \fint_{x_0+rY} L(\nabla u_n(x))\,\mathrm{d}x. \tag{3.9}$$

As $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ it follows that u is almost everywhere L^p-differentiable (see [25, theorem 3.4.2, p. 129]), i.e. for a.e. $x_0 \in \Omega$,

$$\lim_{r\to 0} \frac{1}{r^{N+p}} \|u(x_0 + \cdot) - u(x_0) - \nabla u(x_0)y\|^p_{L^p(rY;\mathbb{R}^m)} = 0. \tag{3.10}$$

From (3.3) we see that (up to a subsequence), for a.e. $x_0 \in \Omega$,

$$|u_n(x_0) - u(x_0)|^p \to 0. \tag{3.11}$$

Without loss of generality we can assume that $x_0 \in \Omega \setminus N$ is such that (3.9)–(3.11) hold. Fix $r_0 > 0$ such that $x_0 + rY \subset \Omega$ for all $r \in]0, r_0]$. For each $n \geqslant 1$ and each $r \in]0, r_0]$, let $u^r_n \in W^{1,p}(Y; \mathbb{R}^m)$ be given by

$$u^r_n(y) := \frac{1}{r}(u_n(x_0 + ry) - u_n(x_0)).$$

Then (3.9) can be rewritten as

$$f(x_0) = \lim_{r\to 0} \lim_{n\to\infty} \int_Y L(\nabla u^r_n(x))\,\mathrm{d}x. \tag{3.12}$$

On the other hand, we have

$$\begin{aligned}\|u^r_n - l_{\nabla u(x_0)}\|^p_{L^p(Y;\mathbb{R}^m)} &= \int_Y |u^r_n(y) - l_{\nabla u(x_0)}(y)|^p\,\mathrm{d}y \\ &= \frac{1}{r^{N+p}} \|u_n(x_0 + \cdot) - u_n(x_0) - l_{\nabla u(x_0)}\|^p_{L^p(rY;\mathbb{R}^m)},\end{aligned}$$

and, consequently,

$$\begin{aligned}\|u^r_n - l_{\nabla u(x_0)}\|^p_{L^p(Y;\mathbb{R}^m)} \leqslant{}& \frac{c}{r^{N+p}} \|u_n - u\|^p_{L^p(\Omega;\mathbb{R}^m)} + \frac{c}{r^{N+p}} |u_n(x_0) - u(x_0)|^p \\ &+ \frac{c}{r^{N+p}} \|u(x_0 + \cdot) - u(x_0) - l_{\nabla u(x_0)}\|^p_{L^p(rY;\mathbb{R}^m)}\end{aligned}$$

with $c > 0$, which depends only on p. Using (3.3), (3.11) and (3.10) we deduce that

$$\lim_{r\to 0} \lim_{n\to\infty} \|u^r_n - l_{\nabla u(x_0)}\|_{L^p(Y;\mathbb{R}^m)} = 0. \tag{3.13}$$

According to (3.12) and (3.13), by diagonalization there exists a mapping $n \to r_n$ decreasing to 0 such that

$$\phi_n \to l_{\nabla u(x_0)} \text{ in } L^p(Y; \mathbb{R}^m), \tag{3.14}$$

$$f(x_0) = \lim_{n\to\infty} \int_Y L(\nabla \phi_n(y))\,\mathrm{d}y, \tag{3.15}$$

where $\phi_n := u^{r_n}_n$.

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Step 3 (end of the proof of (3.1)). According to (3.14), by the definition of $\mathcal{T}_pL$ we see that

$$\varliminf_{n\to\infty}\int_Y L(\nabla\phi_n(y))\,\mathrm{d}y \geqslant \mathcal{T}_pL(\nabla u(x_0))$$

and (3.8) follows by using (3.15).

We now prove (3.2). Given $u\in \mathrm{Aff}(\Omega;\mathbb{R}^m)$ there exists a finite family $\{U_i\}_{i\in I}$ of open disjoint subsets of Ω such that $|\Omega\setminus\bigcup_{i\in I}U_i|=0$ and, for each $i\in I$, $|\partial U_i|=0$ and $\nabla u(x)=\xi_i$ in U_i with $\xi_i\in\mathbb{M}$. Thus,

$$\int_\Omega \mathcal{Q}_pL(\nabla u(x))\,\mathrm{d}x=\sum_{i\in I}|U_i|\mathcal{Q}_pL(\xi_i). \tag{3.16}$$

Using lemma 1.3, for each $i\in I$, we can assert that there exists $\{\phi_n^i\}_n\subset l_{\xi_i}+W_0^{1,p}(U_i;\mathbb{R}^m)$ such that

$$\lim_{n\to\infty}\|\phi_n^i-l_{\xi_i}\|_{L^p(U_i;\mathbb{R}^m)}=0, \tag{3.17}$$

$$\lim_{n\to\infty}⨍_{U_i}L(\nabla\phi_n^i(x))\,\mathrm{d}x=\mathcal{Q}_pL(\xi_i). \tag{3.18}$$

Define $\{u_n\}_n\subset W^{1,p}(\Omega;\mathbb{R}^m)$ by

$$u_n(x):=u(x)+\phi_n^i(x)-l_{\xi_i}(x)\quad\text{if } x\in U_i.$$

Using (3.17) it is easy to see that $\|u_n-u\|_{L^p(\Omega;\mathbb{R}^m)}\to 0$, and combining (3.18) with (3.16) we deduce that

$$\lim_{n\to\infty}L(\nabla u_n(x))\,\mathrm{d}x=\int_\Omega \mathcal{Q}_pL(\nabla u(x))\,\mathrm{d}x,$$

and (3.2) follows. □

Remark 3.2. Analysing the previous proof, it is easily seen that we have in fact proved the following lemma.

Lemma 3.3. *Let $H\colon\mathbb{M}\to[0,\infty]$ be a Borel measurable function. For every $u\in$ $\mathrm{Aff}(\Omega;\mathbb{R}^m)$ there exists $\{u_n\}_n\subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that*

$$\lim_{n\to\infty}\|u_n-u\|_{L^p(\Omega;\mathbb{R}^m)}=0,$$
$$\lim_{n\to\infty}\int_\Omega H(\nabla u_n(x))\,\mathrm{d}x=\int_\Omega \mathcal{Q}_pH(\nabla u(x))\,\mathrm{d}x.$$

As a consequence of theorem 3.1 we have the following.

Corollary 3.4. *If $\mathcal{T}_pL=\mathcal{Q}_pL$, or, equivalently, condition (M) is satisfied, then*

$$\bar{E}(u)\geqslant\int_\Omega \mathcal{Q}_pL(\nabla u(x))\,\mathrm{d}x\quad\text{for all } u\in W^{1,p}(\Omega;\mathbb{R}^m);$$
$$\bar{E}(u)\leqslant\int_\Omega \mathcal{Q}_pL(\nabla u(x))\,\mathrm{d}x\quad\text{for all } u\in \mathrm{Aff}(\Omega;\mathbb{R}^m).$$

In particular, we have

$$\bar{E}(u) = \int_\Omega \mathcal{Q}_p L(\nabla u(x))\,\mathrm{d}x \tag{3.19}$$

for all $u \in \mathrm{Aff}(\Omega;\mathbb{R}^m)$.

For each Borel measurable function $H\colon \mathbb{M} \to [0,\infty]$ we consider the following condition:

($\mathrm{C}_{[H]}$) for every $u \in W^{1,p}(\Omega;\mathbb{R}^m) \setminus \mathrm{Aff}(\Omega;\mathbb{R}^m)$ there exists $\{u_k\}_k \subset \mathrm{Aff}(\Omega;\mathbb{R}^m)$ such that

$$\lim_{k\to\infty} \|u_k - u\|_{L^p(\Omega;\mathbb{R}^m)} = 0,$$
$$\overline{\lim_{k\to\infty}} \int_\Omega H(\nabla u_k(x))\,\mathrm{d}x \leqslant \int_\Omega H(\nabla u(x))\,\mathrm{d}x.$$

THEOREM 3.5. *If* $\mathcal{T}_p L = \mathcal{Q}_p L$, *or, equivalently, condition (M) is satisfied, and if* ($\mathrm{C}_{[\mathcal{Q}_p L]}$) *holds, then (3.19) holds for all* $u \in W^{1,p}(\Omega;\mathbb{R}^m)$.

In particular, $\mathcal{Q}_p L$ *is* $W^{1,p}$*-quasi-convex and condition (M) holds for* $\mathcal{Q}_p L$.

Proof of theorem 3.5. As $\mathcal{T}_p L = \mathcal{Q}_p L$, by corollary 3.4 we have

$$\bar{E}(u) \geqslant \int_\Omega \mathcal{Q}_p L(\nabla u(x))\,\mathrm{d}x \quad \text{for all } u \in W^{1,p}(\Omega;\mathbb{R}^m);$$
$$\bar{E}(u) \leqslant \int_\Omega \mathcal{Q}_p L(\nabla u(x))\,\mathrm{d}x \quad \text{for all } u \in \mathrm{Aff}(\Omega;\mathbb{R}^m).$$

Then, it is sufficient to prove that

$$\bar{E}(u) \leqslant \int_\Omega \mathcal{Q}_p L(\nabla u(x))\,\mathrm{d}x \quad \text{for all } u \in W^{1,p}(\Omega;\mathbb{R}^m) \setminus \mathrm{Aff}(\Omega;\mathbb{R}^m).$$

Let $u \in W^{1,p}(\Omega;\mathbb{R}^m) \setminus \mathrm{Aff}(\Omega;\mathbb{R}^m)$. By ($\mathrm{C}_{[\mathcal{Q}_p L]}$) there exists $\{u_k\}_k \subset \mathrm{Aff}(\Omega;\mathbb{R}^m)$ such that

$$\left.\begin{aligned} \lim_{k\to\infty} \|u_k - u\|_{L^p(\Omega;\mathbb{R}^m)} &= 0, \\ \overline{\lim_{k\to\infty}} \int_\Omega \mathcal{Q}_p L(\nabla u_k(x))\,\mathrm{d}x &\leqslant \int_\Omega \mathcal{Q}_p L(\nabla u(x))\,\mathrm{d}x. \end{aligned}\right\} \tag{3.20}$$

From lemma 3.3 we deduce that for every $k \geqslant 1$ there exists $\{u_{n,k}\}_n \subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that

$$\left.\begin{aligned} \lim_{n\to\infty} \|u_{n,k} - u_k\|_{L^p(\Omega;\mathbb{R}^m)} &= 0, \\ \lim_{n\to\infty} \int_\Omega L(\nabla u_{n,k}(x))\,\mathrm{d}x &= \int_\Omega \mathcal{Q}_p L(\nabla u_k(x))\,\mathrm{d}x. \end{aligned}\right\} \tag{3.21}$$

Combining (3.21) with (3.20), we conclude that

$$\left.\begin{aligned} \overline{\lim_{k\to\infty}} \lim_{n\to\infty} \|u_{n,k} - u\|_{L^p(\Omega;\mathbb{R}^m)} &= 0, \\ \overline{\lim_{k\to\infty}} \lim_{n\to\infty} \int_\Omega L(\nabla u_{n,k}(x))\,\mathrm{d}x &\leqslant \int_\Omega \mathcal{Q}_p L(\nabla u(x))\,\mathrm{d}x. \end{aligned}\right\} \tag{3.22}$$

and the result follows by diagonalization.

Theorem 1.1 implies that $\mathcal{Q}_p L$ is $W^{1,p}$-quasi-convex and condition (M) holds for $\mathcal{Q}_p L$. □

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4. Applications

In this section we everywhere assume that Ω is a star-shaped domain, i.e. there exists $x_0 \in \Omega$ such that $\overline{-x_0 + \Omega} \subset t(-x_0 + \Omega)$ for all $t > 1$. (This assumption is needed for applying lemma 4.3.)

We begin with the following theorem.

THEOREM 4.1. *If $\mathcal{T}_p L = \mathcal{Q}_p L$, or, equivalently, condition (M) is satisfied, and if $\mathcal{Q}_p L$ has continuous convex growth, i.e. there exists a continuous convex function $J\colon \mathbb{M} \to [0, \infty]$ such that*

$$\alpha J(\cdot) \leqslant \mathcal{Q}_p L(\cdot) \leqslant \beta(1 + J(\cdot)) \quad \textit{for some } \alpha, \beta > 0, \tag{4.1}$$

then (3.19) holds for all $u \in W^{1,p}(\Omega; \mathbb{R}^m)$.

Proof of theorem 4.1. Since J is continuous, its domain is necessarily open. Hence, the domain of $\mathcal{Q}_p L$ is also open because it is equal to that of J by (4.1). Thus, using lemma 4.2 with $H = L$, we can assert that $\mathcal{Q}_p L$ is continuous.

LEMMA 4.2 (Fonseca [13]). *Let $H\colon \mathbb{M} \to [0, \infty]$ be a Borel measurable function. Then, $\mathcal{Q}_p H$ is continuous on the interior of its domain.*

So, we can apply lemma 4.3 with $H = \mathcal{Q}_p L$, and we see that $(\mathrm{C}_{[\mathcal{Q}_p L]})$ holds.

LEMMA 4.3 (Anza Hafsa and Mandallena [3, § 3.3]). *If $H\colon \mathbb{M} \to [0, \infty]$ is continuous and has convex growth, then for every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ with $\int_\Omega H(\nabla u(x))\,\mathrm{d}x < \infty$ there exists $\{u_n\}_n \subset \mathrm{Aff}(\Omega; \mathbb{R}^m)$ such that*

$$u_n \to u \quad \textit{in } W^{1,p}(\Omega; \mathbb{R}^m),$$
$$\lim_{n\to\infty} \int_\Omega H(\nabla u_n(x)) = \int_\Omega H(\nabla u(x))\,\mathrm{d}x.$$

Hence, theorem 4.1 follows from theorem 3.5. □

Corollaries 4.4, 4.5 and 4.6 are consequences of theorem 4.1. These results, which are concerned with the case of extended-valued integrands, generalize known relaxation theorems for finite integrands, and are motivated by an open question of Ball in [5, 6] to prove that $W^{1,p}$-quasi-convexification gives the relaxation in the case of extended-valued integrands. These corollaries are contributions in this direction (see, for example, [2, 10, 20–22]).

COROLLARY 4.4. *Assume that $p > N$, and $L\colon \mathbb{M} \to [0, \infty[$ is such that*

$$\alpha G(|\cdot|) \leqslant L(\cdot) \leqslant \beta(1 + G(|\cdot|)) \quad \textit{for some } \alpha, \beta > 0, \tag{4.2}$$

where $G\colon [0, \infty[\to [0, \infty[$ is a non-decreasing function with the following property:

$$G(t + s) \leqslant \theta(|s|)G(t) + \hat{\theta}(|s|) \quad \textit{for all } t, s \in [0, \infty[, \tag{4.3}$$

where $\theta, \hat{\theta}\colon [0, \infty[\to [0, \infty[$ are non-decreasing functions. If $\mathcal{Q}_p G(|\cdot|)$ has a convex growth, then (3.19) holds for all $u \in W^{1,p}(\Omega; \mathbb{R}^m)$.

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Proof of corollary 4.4. The proof follows by applying theorem 4.1.

We first prove that $\mathcal{T}_pL = \mathcal{Q}_pL$ or, equivalently, that condition (M) is satisfied. For this, consider $\xi \in \mathbb{M}$ and $\{\phi_n\}_n \subset W^{1,p}(Y;\mathbb{R}^m)$ such that

$$\|\phi_n - l_\xi\|_{L^p(Y;\mathbb{R}^m)} \to 0, \tag{4.4}$$

and prove that there exists $\{\psi_n\}_n \subset l_\xi + W_0^{1,p}(Y;\mathbb{R}^m)$ such that

$$\varliminf_{n\to\infty} \int_Y L(\nabla\phi_n(y))\,\mathrm{d}y \geqslant \varlimsup_{n\to\infty} \int_Y L(\nabla\psi_n(y))\,\mathrm{d}y. \tag{4.5}$$

Without loss of generality we can assume that

$$\varliminf_{n\to\infty} \int_Y L(\nabla\phi_n(y))\,\mathrm{d}y = \lim_{n\to\infty} \int_Y L(\nabla\phi_n(y))\,\mathrm{d}y < \infty$$

and so

$$\sup_n \int_Y L(\nabla\phi_n(y))\,\mathrm{d}y < \infty. \tag{4.6}$$

Consequently, there exists a (positive) Radon measure λ on Y such that, up to a subsequence,

$$\lambda_n := L(\nabla\phi_n(\cdot))\,\mathrm{d}y \overset{*}{\rightharpoonup} \lambda \quad \text{in the sense of measure.}$$

Moreover, as L is p-coercive, from (4.6) we have

$$\sup_n \|\nabla\phi_n\|_{L^p(Y;\mathbb{R}^m)} < \infty. \tag{4.7}$$

As $p > N$, from (4.4) and (4.7) it follows that, up to a subsequence,

$$\|\phi_n - l_\xi\|_{L^\infty(Y;\mathbb{R}^m)} \to 0. \tag{4.8}$$

As $\lambda(Y) < \infty$ we have $\lambda(\partial(rY)) = 0$ for all $r \in]0,1[\setminus D$, where D is a countable set. Fix any $r \in]0,1[\setminus D$ and any $\varepsilon \in]0,r[$. Let $\psi \in C_c^\infty(Y;[0,1])$ such that $\psi = 1$ on $Q :=]\frac{1}{2}(\varepsilon - r), \frac{1}{2}(r-\varepsilon)[^N$ and $\psi = 0$ on $Y \setminus r\bar{Y}$ with $\|\nabla\psi\|_{L^\infty(Y)} \leqslant 2/\varepsilon$. Define $\psi_n \in l_\xi + W_0^{1,p}(Y;\mathbb{R}^m)$ by

$$\psi_n := l_\xi + \psi(\phi_n - l_\xi).$$

Then

$$\nabla\psi_n = \begin{cases} \nabla\phi_n & \text{on } Q, \\ \xi(1-\psi) + \psi\nabla\phi_n + \nabla\psi\otimes(\phi_n - l_\xi) & \text{on } S_\varepsilon := rY\setminus\bar{Q}, \\ l_\xi & \text{on } Y\setminus r\bar{Y}. \end{cases}$$

Hence, $|\nabla\psi_n(y)| \leqslant |\xi| + |\nabla\phi_n(y)| + (2/\varepsilon)\|\phi_n - l_\xi\|_{L^\infty(Y;\mathbb{R}^m)}$ for a.a. $y \in S_\varepsilon$. Using (4.8) we can assert that there exists $n_\varepsilon \geqslant 1$ such that

$$|\nabla\psi_n(y)| \leqslant |\xi| + 1 + |\nabla\phi_n(y)|$$

for all $n \geqslant n_\varepsilon$ and a.a. $y \in S_\varepsilon$. Recalling that G, θ and $\hat{\theta}$ are non-decreasing functions and using (4.3), we deduce that

$$G(|\nabla\psi_n(y)|) \leqslant \theta(|\xi|+1)G(|\nabla\phi_n(y)|) + \hat{\theta}(|\xi|+1),$$

and so

$$L(\nabla\psi_n(y)) \leqslant \gamma L(\nabla\phi_n(y)) + \delta \tag{4.9}$$

for all $n \geqslant n_\varepsilon$ and a.a. $y \in S_\varepsilon$ with $\gamma := \beta\theta(|\xi|+1)/\alpha$ and $\delta := \beta(1+\hat{\theta}(|\xi|+1))$. Fix any $n \geqslant n_\varepsilon$. It is clear that

$$\begin{aligned}\int_Y L(\nabla\psi_n(y))\,\mathrm{d}y &= \int_Q L(\nabla\psi_n(y))\,\mathrm{d}y + \int_{S_\varepsilon} L(\nabla\psi_n(y))\,\mathrm{d}y + \int_{Y\setminus r\bar{Y}} L(\nabla\psi_n(y))\,\mathrm{d}y \\ &\leqslant \int_Y L(\nabla\phi_n(y))\,\mathrm{d}y + \int_{S_\varepsilon} L(\nabla\psi_n(y))\,\mathrm{d}y + L(\xi)(1-r^N).\end{aligned}$$

But, using (4.9), we see that

$$\int_{S_\varepsilon} L(\nabla\psi_n(y))\,\mathrm{d}y \leqslant \gamma\int_{S_\varepsilon} L(\nabla\phi_n(y))\,\mathrm{d}y + \delta|S_\varepsilon| = \gamma\lambda_n(S_\varepsilon) + \delta|S_\varepsilon|,$$

and hence

$$\int_Y L(\nabla\psi_n(y))\,\mathrm{d}y \leqslant \int_Y L(\nabla\phi_n(y))\,\mathrm{d}y + \gamma\lambda_n(S_\varepsilon) + \delta|S_\varepsilon| + L(\xi)(1-r^N). \tag{4.10}$$

Moreover, as $S_\varepsilon \subset \bar{S}_\varepsilon$ we have $\lambda_n(S_\varepsilon) \leqslant \lambda_n(\bar{S}_\varepsilon)$, and so $\overline{\lim}_{n\to\infty}\lambda_n(S_\varepsilon) \leqslant \lambda(\bar{S}_\varepsilon)$. So, passing to the limsup in (4.10) we obtain

$$\overline{\lim_{n\to\infty}}\int_Y L(\nabla\psi_n(y))\,\mathrm{d}y \leqslant \lim_{n\to\infty}\int_Y L(\nabla\phi_n(y))\,\mathrm{d}y + \gamma\lambda(\bar{S}_\varepsilon) + \delta|S_\varepsilon| + L(\xi)(1-r^N)$$

for all $r \in \,]0,1[\,\setminus D$ and all $\varepsilon \in \,]0,r[$, and (4.5) follows by letting $\varepsilon \to 0$ (on noting that $\lim_{\varepsilon\to 0}\lambda(\bar{S}_\varepsilon) = \lambda(\bigcap_\varepsilon \bar{S}_\varepsilon) = \lambda(\partial(rY)) = 0)$ and then $r \to 1$.

Finally, as L satisfies (4.2) we have

$$\alpha\mathcal{Q}_pG(|\cdot|) \leqslant \mathcal{Q}_pL(\cdot) \leqslant \beta(1+\mathcal{Q}_pG(|\cdot|)).$$

Moreover, since $G(|\cdot|)$ is finite, $\mathcal{Q}_pG(|\cdot|)$ is continuous by applying lemma 4.2 with $H = G(|\cdot|)$. Hence, $\mathcal{Q}_pL$ has continuous convex growth because $\mathcal{Q}_pG(|\cdot|)$ is assumed also to have convex growth, and the proof is complete. □

Corollary 4.5. *Assume that $p > N$ and $L\colon \mathbb{M} \to [0,\infty]$ is such that*

$$\sum_{i=1}^m F_i(\xi^i) \leqslant L(\xi) \leqslant c\biggl(\sum_{i=1}^m F_i(\xi^i) + 1\biggr) \quad \textit{for all } \xi \in \mathbb{M} \textit{ and some } c \geqslant 1, \tag{4.11}$$

where, for each $i \in \{1,\dots,m\}$, ξ^i is the ith row of the matrix ξ and $F_i\colon \mathbb{R}^N \to [0,\infty]$ is a continuous function such that $\lim_{|v|\to\infty} F_i(v)/|v|^p = \infty$. Then (3.19) holds for all $u \in W^{1,p}(\Omega;\mathbb{R}^m)$.

Proof of corollary 4.5. As is well known in the scalar case $G\colon \mathbb{R}^N \to [0,\infty]$, the $\mathcal{Q}_pG$-quasi-convexification is given by convexification G^{c}, where

$$G^{\mathrm{c}}(v) = \inf\biggl\{\sum_{i=1}^k c_iG(v_i)\colon c_i \geqslant 0,\ v_i \in \mathbb{R}^N,\ \sum c_i = 1,\ \sum c_iv_i = v, k \geqslant 1\biggr\}$$

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(see, for example, [20]). In particular $G^{c} \colon \mathbb{R}^N \to [0,\infty]$ is a continuous convex function. Therefore, we have

$$\sum_{i=1}^{m} F_i^{c}(\xi^i) \leqslant \mathcal{Q}_p L(\xi) \leqslant c\biggl(\sum_{i=1}^{m} F_i^{c}(\xi^i) + 1\biggr).$$

So $\mathcal{Q}_p L$ has convex growth and is continuous. Then (4.1) is valid and, due to theorem 4.1, the corollary is valid if condition (M) holds.

The remainder of the proof is devoted to establishing validity of condition (M) for L satisfying (4.11).

Let $\xi \in \mathbb{M}$ and assume $u_n \to l_\xi$ in $L^p(Y;\mathbb{R}^m)$, where $u_n \in W^{1,p}(Y;\mathbb{R}^m)$. We have to show that for a subsequence u_n (not relabelled) there exists $\{\phi_n\}_n \subset l_\xi + W_0^{1,p}(Y;\mathbb{R}^m)$ such that

$$\varliminf_{n\to\infty} \int_Y \{L(\nabla u_n(y)) - L(\nabla \phi_n(y))\}\,\mathrm{d}y \geqslant 0. \tag{4.12}$$

Without loss of generality we can assume that

$$\lim_{n\to\infty} \int_Y L(\nabla u_n(y))\,\mathrm{d}y =: I < \infty$$

and, as L is p-coercive, that $\{\nabla u_n\}_n$ generates a Young measure $(\nu_y)_{y\in Y}$. Moreover, we can assume that $L(\nabla u_n) \overset{*}{\rightharpoonup} \mu$, where μ is a (positive) Radon measure.

Given $\delta > 0$ we can always isolate $0 < \delta_2 < \delta_1 < \delta$ such that if $Y_\delta =]-\frac{1}{2}+\delta, \frac{1}{2}-\delta[^N$ and $Y_1 = Y_{\delta_1}$, $Y_2 = Y_{\delta_2}$, then

$$\mu(\bar{Y}_2 \setminus Y_1) \leqslant \delta. \tag{4.13}$$

We also have

$$I \geqslant \int_Y \int_{\mathbb{M}} \langle L(\cdot);\nu_y\rangle\,\mathrm{d}y$$

(see, for example, [18]). Therefore, for a.a. $y \in Y$ we have $\langle L(\cdot);\nu_y\rangle < \infty$. Fix such a $y \in Y$ and define $\nu = \nu_y$. Note that ξ is the centre of mass of the probability measure ν.

We have

$$\biggl\langle \sum_{i=1}^{m} F_i;\nu \biggr\rangle = \sum_{i=1}^{m} \langle F_i;\nu^i\rangle, \tag{4.14}$$

where ν^i is a probability measure with support in $\mathbb{R}^N$ and is the projection of ν on the space of $v^i \in \mathbb{R}^N$ variable, $i \in \{1,\dots,m\}$.

By the results of [19] in the scalar case $G \colon Y \to [0,\infty]$, any probability measure λ with finite action on G is a homogeneous G-gradient Young measure, which means there exists a sequence of finitely piecewise affine functions $w_n \colon Y \to \mathbb{R}$ such that $w_n \in l_\xi + W_0^{1,\infty}(Y)$, where ξ is the centre of mass of λ, and

$$\lim_{n\to\infty} \int_Y G(\nabla w_n(y))\,\mathrm{d}y = \langle L;\lambda\rangle.$$

Here, finitely continuous piecewise affine means that the gradient of w_n takes only a finite collection of values. Then, by (4.14), for each $i \in \{1,\dots,m\}$, we can find a

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function $w^i \in l_{\xi^i} + W_0^{1,\infty}(Y)$, where ξ^i is the ith row of ξ such that w^i is finitely piecewise affine, i.e. $\nabla w^i \in \{v_1^i, \dots, v_{k_i}^i\}$ and

$$F_i(v_j^i) < C < \infty \quad \text{for all } i \in \{1, \dots, m\} \text{ and all } j \in \{1, \dots, k_i\}. \tag{4.15}$$

Given $\varepsilon > 0$, consider a piecewise affine function $f \in l_\xi + W_0^{1,\infty}(Y_2; \mathbb{R}^m)$ such that $\|f - l_\xi\|_{W^{1,\infty}} \leqslant \varepsilon$ and $f^i > l_{\xi^i}$ in Y_2 for all $i \in \{1, \dots, m\}$. In $Y \setminus Y_2$ we assume $f = l_\xi$.

Furthermore, without loss of generality we can assume that, for every $n \geqslant 1$ and every $i \in \{1, \dots, m\}$,

$$u_n^i > l_{\xi^i} \quad \text{in } Y.$$

(Recall that u_n^i converge uniformly to l_{ξ^i} as $n \to \infty$ since $p > N$.)

Now, given $n \geqslant 1$, define a function $\psi_n\colon Y \to \mathbb{R}^m$ as follows:

$$\psi_n^i = \min\{u_n^i, f^i\} \quad \text{for } i = 1, \dots, m.$$

Obviously, $\psi_n \in l_\xi + W_0^{1,p}(Y_2; \mathbb{R}^m)$ and $\psi_n = u_n$ in Ω_n, where $\Omega_n \subset Y_2$ is such that $|Y_2 \setminus \Omega_n| \to 0$ as $n \to \infty$ and for sufficiently large $n \geqslant 1$ we also have $Y_1 \subset \Omega_n$.

For each $n \geqslant 1$ and each $i \in \{1, \dots, m\}$ we have $\psi_n^i = u_n^i$ in $\Omega_n^i \subset Y_2$, where $\Omega_n \subset \Omega_n^i$. In $Y \setminus \Omega_n^i$ we have $\psi_n^i = f^i$.

We assume $\phi_n^i = u_n^i$ in Ω_n^i. In each piece $\hat{\Omega}$ of $Y \setminus \Omega_n^i$, where ψ_n^i is affine, i.e. $\nabla \psi_n^i = \zeta^i$, we assume $\phi_n^i = l_{\zeta^i} + \hat{w}^i - l_{\xi^i}$. Here $\hat{w}^i\colon \hat{\Omega} \to \mathbb{R}$ is a rescaling of $w^i\colon Y \to \mathbb{R}$ as suggested in the proof of lemma 1.3. In particular, $\hat{w}^i$ has the same values of the gradient in $\hat{\Omega}$ as w^i in Y and $\hat{w}^i \in l_{\xi^i} + W_0^{1,\infty}(\hat{\Omega})$. Then $\phi_n^i \in l_{\zeta^i} + W_0^{1,\infty}(\hat{\Omega})$ and in case of sufficiently small $\varepsilon > 0$ we have by (4.15) that

$$F_i(\nabla \phi_n^i(y)) < C \quad \text{for a.e. } y \in \hat{\Omega}. \tag{4.16}$$

Therefore, $\phi_n \in l_\xi + W_0^{1,p}(Y; \mathbb{R}^m)$ and we have

$$\int_Y \{L(\nabla u_n(y)) - L(\nabla \phi_n(y))\} \, \mathrm{d}y \geqslant -\int_{Y \setminus Y_1} L(\nabla \phi_n(y)) \, \mathrm{d}y.$$

But, by (4.13) and (4.16) we have

$$\begin{aligned} \varlimsup_{n\to\infty} \int_{Y_2 \setminus Y_1} L(\nabla \phi_n(y)) \, \mathrm{d}y &\leqslant c\mu(\bar{Y}_2 \setminus Y_1) + (2c + cmC)|\bar{Y}_2 \setminus Y_1| \\ &\leqslant (3c + cmC)\delta, \\ \varlimsup_{n\to\infty} \int_{Y \setminus Y_2} L(\nabla \phi_n(y)) \, \mathrm{d}y &\leqslant (cmC + c)|Y \setminus Y_2| \\ &\leqslant (cmC + c)\delta. \end{aligned}$$

Hence,

$$\varliminf_{n\to\infty} \int_Y \{L(\nabla u_n(y)) - L(\nabla \phi_n(y))\} \, \mathrm{d}y \geqslant -(4c + 2cmC)\delta,$$

and (4.12) follows by letting $\delta \to 0$. □

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In corollaries 4.4 and 4.5, we considered the case of so-called strong materials, i.e. when L is p-coercive with $p > N$. This is an old conjecture by Sychev: in this case the relaxation theory has a better shape (see [20–22, 24]). In corollaries 4.4 and 4.5, this conjecture results in validity of condition (M). For other conjectures about strong materials; see [24].

It is also possible to derive a new result in the scalar case.

Corollary 4.6. *Let $L\colon \mathbb{R}^N \to [0,\infty]$ be continuous and p-coercive. Then*

$$\bar{E}(u) = \int_{\Omega} L^{\mathrm{c}}(\nabla u(x))\,\mathrm{d}x \quad \text{for all } u \in W^{1,p}(\Omega),$$

where L^{c} is the convexification of L.

In the case of L with p-growth, this result is a classical theorem of Ekeland and Temam (see [12]). Sychev proved this result for $p > N$ in the context of his theory for strong materials (see [20]). As we shall see, the result remains valid for p-coercive integrands without additional requirements on $p > 1$.

The result is a straightforward consequence of theorem 4.1, since $W^{1,p}$-quasiconvexification of L is just L^{c} in the scalar case (see, for example, [20]), and condition (M) always holds in the scalar case.

Lemma 4.7. *Let $L\colon \mathbb{R}^N \to [0,\infty]$ be continuous and p-coercive. Then condition (M) holds.*

Proof. Without loss of generality we can assume that the gradients of a subsequence $\{u_n\}_n$ generate a Young measure $\{\nu_x\}_{x\in Y}$. Then,

$$\varliminf_{n\to\infty} I(u_n) \geqslant \int_Y \langle L; \nu_x\rangle\,\mathrm{d}x.$$

There exists an $x_0 \in Y$ such that

$$\langle L; \nu_{x_0}\rangle |Y| \leqslant \int_Y \langle L; \nu_x\rangle\,\mathrm{d}x.$$

But ν_{x_0} is a homogeneous L-gradient Young measure, which means there exists a sequence $\{\phi_n\}_n \subset l_\xi + W_0^{1,\infty}(Y)$ with the property

$$\lim_{n\to\infty} \int_Y L(\nabla \phi_n(x))\,\mathrm{d}x = \langle L; \nu_{x_0}\rangle |Y|$$

(see [19]). Then the sequence $\{\phi_n\}_n$ is just the appropriate one for condition (M) to hold. □

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