FINITE GROUPS THE CENTRALIZERS OF WHOSE INVOLUTIONS HAVE NORMAL 2-COMPLEMENTS

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1. Introduction. In this paper we shall classify all finite groups in which the centralizer of every involution has a normal 2-complement. For brevity, we call such a group an I-group. To state our classification theorem precisely, we need a preliminary definition.

As is well-known, the automorphism group $G = \mathrm{P\Gamma L}(2, q)$ of $H = \mathrm{PSL}(2, q)$, $q = p^n$, is of the form $G = LF$, where $L = \mathrm{PGL}(2, q)$, $L \lhd G$, F is cyclic of order n, $L \cap F = 1$, and the elements of F are induced from semilinear transformations of the natural vector space on which $\mathrm{GL}(2, q)$ acts; cf. (**3**, Lemma 2.1) or (**7**, Lemma 3.3). It follows at once (**4**, Lemma 2.1; **8**, Lemma 3.1) that the groups H and L are each I-groups. Moreover, when q is an odd square, there is another subgroup of G in addition to L that contains H as a subgroup of index 2 and which is an I-group. Indeed, in this case $|F|$ is even and $|L:H| = 2$. Consequently, there are exactly two subgroups of G other than L containing H as a subgroup of index 2, one of which is contained in HF and the other which is not. One checks directly that the latter group, which we shall denote by $\mathrm{PGL}^*(2, q)$, is an I-group. As we shall show in Lemma 2.3 below, the group $\mathrm{PGL}^*(2, q)$ has semidihedral Sylow 2-subgroups. We remark that the group $\mathrm{PGL}^*(2, 9)$ is the group M_9 in the notation of Zassenhaus (**14**), of order 720, which is the projective group in one variable over the near-field with nine elements.

We shall follow the terminology of (**5**). In particular, we recall that in any group G, $O(G)$ and $S(G)$ denote the largest normal subgroup of G of odd order and the largest solvable normal subgroup of G, respectively. Moreover, the normalizer in G of a non-trivial p-subgroup of G, p a prime, is called a p-local subgroup of G.

Our main result is the following.

THEOREM A. *A non-solvable* I-*group G has one of the following structures:*

(i) *$G/O(G)$ contains a normal subgroup of odd index isomorphic to* $\mathrm{PSL}(2, q)$, $\mathrm{PGL}(2, q)$, $\mathrm{PGL}^*(2, q)$, *q odd, $q > 3$, or to A_7;*

(ii) *$G/O(G)$ is isomorphic to* $\mathrm{PSL}(2, 2^n)$ *or* $\mathrm{Sz}(2^n)$, *$n \geqq 3$, or to* $\mathrm{PSL}(3, 4)$*;*

(iii) *$S(G) = O_{2',2}(G) \supset O(G)$, and $G/S(G)$ is isomorphic to* $\mathrm{PSL}(2, 2^n)$, *$n \geqq 2$, or to* $\mathrm{Sz}(2^n)$, *$n \geqq 3$.*

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It is not difficult to show that any group of the form (i) or (ii) is an I-group and, moreover, that there exist I-groups of the form (iii). Furthermore, using the ideas of Higman (**9**, Lemma 2.4), Thompson has shown that in an I-group of the form (iii), $O_{2',2}(G)/O(G)$ is necessarily abelian.

As an immediate corollary to Theorem A, we have the following theorem.

THEOREM B. *A simple* I*-group of composite order is isomorphic to* PSL(2, q), $q > 3$, Sz(2^n), $n \geqq 3$, A_7, *or* PSL(3, 4).

In the course of the proof of Theorem A we shall also derive some properties of solvable I-groups.

The over-all proof of Theorem A is very similar to that of the classification of finite groups with abelian Sylow 2-subgroups in which the centralizer of every involution is solvable (**4**). One first argues by induction that either Theorem A holds or else there exists a simple I-group G with the following properties:

(1) $\mathrm{SCN}_3(S)$ is non-empty, S an S_2-subgroup of G;

(2) Every non-solvable proper subgroup of G satisfies the conclusion of Theorem A;

(3) S normalizes, but does not centralize, a subgroup of G of odd order.

We can also show that G has an additional property, the statement of which involves the concepts of both a *strongly embedded* subgroup of G (that is, a proper subgroup of G of even order containing the centralizer in G of each of its involutions and the normalizer in G of each of its S_2-subgroups) and a *uniqueness subgroup* for some odd prime p (that is, a p-local subgroup of G which is the unique maximal element of a certain well-specified collection of p-local subgroups of G). For definitions and properties of such subgroups, see (**5**, §§ 8.6, 9.2, and 9.3).

We can also prove that G must satisfy the following property:

(4) If M is a strongly embedded subgroup of G containing S and if M is also a uniqueness subgroup for some odd prime p, then S centralizes $O_p(M)$.

The reduction to the case of a group G satisfying conditions (1)–(4) requires, in particular, application of three major classification theorems:

(a) *Groups in which* $\mathrm{SCN}_3(S)$ *is empty and the centralizer of every involution is solvable* (Janko and Thompson (**10**)).

(b) *Groups in which the centralizer of every involution is* 2*-closed and, in particular, is nilpotent* (Suzuki (**11**; **12**; **13**)).

(c) I*-groups in which two elements of S conjugate in G are conjugate in the normalizer of S in G* (Glauberman (**3**)).

Once this reduction is achieved, our main concern is to show that an I-group G satisfying conditions (1), (2), and (3) must possess a subgroup M which violates condition (4). The proof of this result is patterned very closely after that of the analogous result established in (**4**), but is considerably simpler. This simplification is due, on the one hand, to Glauberman's ZJ-theorem

(**2**; **5**) which makes the construction of M in the so-called π_4-case extremely easy and, on the other hand, to a very strong transitivity theorem that holds for the prime 2 in the present situation; cf. Theorem 4.2 below. As a consequence, the prime 3 no longer plays the exceptional role that it did in (**4**). In addition, the concept of weak p-constraint, which was introduced in (**4**), is not required inasmuch as the hypotheses of those theorems of (**7**) which are needed for the argument can now be directly verified.

2. Preliminary lemmas. We shall need a variety of properties of groups which satisfy the conclusion of Theorem A. For convenience we call such a group an I_1-*group*, an I_2-*group*, or an I_3-*group* according as it is of the form (i), (ii), or (iii) of the theorem, respectively. For completeness, we call a solvable I-group an I_0-group.

LEMMA 2.1. *Let H be an I_0-group in which $O(H) = 1$. Let X be an $\mathrm{S}_{2'}$-subgroup of H and let $Y = O_2(H)$. Then*

(i) *If $X \neq 1$, then XY is a Frobenius group with kernel Y;*
(ii) *The Sylow subgroups of X are cyclic;*
(iii) *$XY \lhd H$, and H/XY is a cyclic 2-group;*
(iv) *If $XY \subset H$, then H contains an involution not in Y;*
(v) *If $3 \in \pi(X)$, then* $\mathrm{cl}(Y) \leqq 2$.

Proof. We first claim that no element x of $X^\#$ centralizes any non-trivial 2-element u of H. Assume that this is false. $C_H(u)$ has a normal 2-complement. However u, being a 2-element, centralizes some element y of $Z(Y)^\#$. Then $[y, x] \in O(C_H(u)) \cap Y = 1$, whence x centralizes y. But then $C_H(y)$ contains Y and also has a normal 2-complement, whence $[Y, x] \subseteq O(C_H(y)) \cap Y = 1$. Thus x centralizes Y. However, since H is a solvable group in which $O(H) = 1$, we have that $C_H(Y) \subseteq Y$ by a basic result of Hall and Higman (**5**, Theorem 6.3.2), and therefore $x \in Y$, a contradiction.

In particular, if $X \neq 1$, our argument shows that XY is a Frobenius group with kernel Y and complement X. But then as $|X|$ is odd, (**5**, Theorem 10.3.1) implies that the Sylow subgroups of X are all cyclic. Thus (i) and (ii) hold.

In proving (iii) and (iv) we may clearly assume that $XY \subset H$. Set $\bar{H} = H/Y$ and let $\bar{X}$ be the image of X in $\bar{H}$. Then $\bar{X} \subset \bar{H}$ and $\bar{X}$ is an $\mathrm{S}_{2'}$-subgroup of $\bar{H}$. In particular, $O(\bar{H}) \subseteq \bar{X}$. Moreover, no element $\bar{x}$ of $\bar{X}^\#$ can centralize a non-trivial 2-element of $\bar{H}$; otherwise, a representative x of $\bar{x}$ in X would necessarily centralize a non-trivial 2-element of H, contrary to what we have shown above. Since $\bar{H}$ has even order, it follows that $O(\bar{H})$ is inverted by an involution of $\bar{H}$, and therefore is abelian. On the other hand, the Sylow subgroups of $\bar{X}$ and $O(\bar{H})$ are cyclic by (ii), whence $O(\bar{H})$ is, in fact, cyclic. However, $C_{\bar{H}}(O(\bar{H})) \subseteq O(\bar{H})$ by another application of (**5**, Theorem 6.3.2); whence, $\bar{H}/O(\bar{H})$ is isomorphic to a subgroup of Aut $O(\bar{H})$. Since the automorphism group of a cyclic group is abelian, we conclude that

$\bar{H}/O(\bar{H})$ is abelian. Since $\bar{X}$ contains $O(\bar{H})$, this yields $\bar{X} \lhd \bar{H}$; whence, $XY \lhd H$.

On the other hand, since H is solvable, any other $S_{2'}$-subgroup of H is conjugate to X, whence $H = YN_H(X)$ by the Frattini argument. Let R be an S_2-subgroup of $N_H(X)$. Then $R \neq 1$ as $XY \subset H$. Moreover, $Y \cap R$ centralizes X; whence, $Y \cap R = 1$ by (i). In particular, every involution of R lies in $H - R$, proving (iv). Furthermore, by the first paragraph of the proof, no element of $R^{\#}$ centralizes an element of $X^{\#}$. Since the Sylow subgroups of X are cyclic, this forces R to be cyclic; thus (iii) also holds.

Finally, if $3 \in \pi(X)$, then X contains an element of order 3 which acts regularly on Y. It is well known that this forces Y to have class at most 2; cf. (**5**, Exercise 5.21).

LEMMA 2.2. *Let H be an* I_0*-group and let S be an* S_2*-subgroup of H.*

(i) *If H has only one class of involutions or if $S \subseteq [H, H]$, then $H = O(H)N_H(S)$.*

(ii) *If $H = O(H)N_H(S)$, then two elements of S conjugate in H are conjugate in $N_H(S)$.*

Proof. The assumptions of (i) clearly carry over to $H/O(H)$. In either case, Lemma 2.1 implies that $SO(H)/O(H) \lhd H/O(H)$; whence, $H = O(H)N_H(S)$ by the Frattini argument. Thus (i) holds. Moreover, if $u, v \in S$ and $v = u^h$, $h \in H$, then under the assumption of (ii), we have that $h = na$, where $n \in N_H(S)$ and $a \in O(H)$. Setting $w = u^n$, we obtain $v = w^a$ and $w^a \in S$, whence $w^{-1}w^a \in O(H) \cap S = 1$. Thus, $w = v$, and therefore $v = u^n$, proving (ii).

LEMMA 2.3. *The following conditions hold in* $\mathrm{PGL}^*(2, q)$, $q = p^m$, *p an odd prime:*

(i) S_2*-subgroups are semidihedral;*

(ii) *a 4-subgroup normalizes no non-trivial* p*-subgroup.*

Proof. By definition of $\mathrm{PGL}^*(2, q)$, q is odd and $q = r^2$ for some integer r. Set $G = \Gamma L(2, q)$ and $L = \mathrm{GL}(2, q)$. Then, as with $\mathrm{PSL}(2, q)$, we have that $G = LF$, where F is cyclic of even order, $L \cap F = 1$, and the elements of F are induced from automorphisms of $\mathrm{GF}(q)$. Let K be the normal subgroup of G consisting of all scalar matrices and set $\bar{G} = G/K$, so that $\bar{G} = \mathrm{P\Gamma L}(2, q)$. We use bars for images of elements and subgroups of G in $\bar{G}$. Thus, $\bar{L} = \mathrm{PGL}(2, q)$ and $\bar{G} = \bar{L}\bar{F}$ with $\bar{F}$ cyclic of even order and disjoint from $\bar{L}$. Let $\bar{H}$ be the subgroup of $\bar{L}$ equal to $\mathrm{PSL}(2, q)$.

Let 2^m be the highest power of 2 dividing $q - 1$ and let α be a primitive 2^mth root of unity in $\mathrm{GF}(q)$. Since $r^2 \equiv 1 \pmod 8$, we have that $m \geqq 3$. Let

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so that $|a| = 2^m$, $|b| = 2$, and $a, b \in L$. Since

$$b^{-1}ab = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} = a^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \equiv a^{-1} \pmod{K},$$

we have that $\bar{b}^{-1}\bar{a}\bar{b} = \bar{a}^{-1}$. Thus, $\bar{R} = \langle \bar{a}, \bar{b} \rangle$ is dihedral of order 2^{m+1} and so is an S_2-subgroup of $\bar{L}$. Moreover,

$$a^2 = \begin{pmatrix} \alpha^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \equiv \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \pmod{K},$$

and therefore $\bar{a}^2 \in \bar{H}$. Since $\bar{b} \in \bar{H}$ also, we have that $\bar{R} \cap \bar{H} = \langle \bar{a}^2, \bar{b} \rangle$ is an S_2-subgroup of $\bar{H}$.

Finally, let c be the unique element of order 2 in F. Then c centralizes b and

$$c^{-1}ac = \begin{pmatrix} \alpha^r & 0 \\ 0 & 1 \end{pmatrix} = a^r.$$

Thus

$$\bar{c}^{-1}\bar{b}\bar{c} = \bar{b} \quad \text{and} \quad \bar{c}^{-1}\bar{a}\bar{c} = \bar{a}^r.$$

In particular, $\bar{c}$ normalizes both $\bar{R}$ and $\bar{R} \cap \bar{H}$. Now by definition, $PGL^*(2, q)$ contains $\bar{H}$ as a normal subgroup of index 2 and is not equal to either $\bar{L}$ or $\langle \bar{H}, \bar{c} \rangle$. Thus, in fact, $PGL^*(2, q) = \langle \bar{H}, \bar{a}\bar{c} \rangle$ and therefore $\bar{S} = \langle \bar{R} \cap \bar{H}, \bar{a}\bar{c} \rangle = \langle \bar{a}^2, \bar{b}, \bar{a}\bar{c} \rangle$ is an S_2-subgroup of $PGL^*(2, q)$ and is of order 2^{m+1}.

We proceed to determine the structure of $\bar{S}$. Consider first the case $r \equiv 1 \pmod 4$, in which case $r = 1 + 2^{m-1}\lambda$ for some odd integer λ. In particular, $\frac{1}{2}(r + 1)$ is odd. Setting $\bar{x} = \bar{a}\bar{c}$, we have that $\bar{x}^2 = (\bar{a}\bar{c})^2 = \bar{a}(\bar{c}^{-1}\bar{a}\bar{c}) = \bar{a}^{r+1}$. Consequently, $\langle \bar{x}^2 \rangle = \langle \bar{a}^2 \rangle$, and therefore $|\bar{x}| = 2^m$. Furthermore, $\bar{b}^{-1}\bar{x}\bar{b} = \bar{a}^{-1}\bar{c} = \bar{a}^{-2}\bar{a}\bar{c} = \bar{a}^{-2}\bar{x}$. However, it is immediate from the form of r that $(r + 1)(2^{m-2} - 1) \equiv -2 \pmod{2^m}$; whence, $(\bar{x}^2)^{2^{m-2}-1} = (\bar{a}^{r+1})^{2^{m-1}-1} = \bar{a}^{-2}$. This yields $\bar{b}^{-1}\bar{x}\bar{b} = \bar{a}^{-2}\bar{x} = \bar{x}^{-1+2^{m-1}}$, and therefore $\bar{S} = \langle \bar{b}, \bar{x} \rangle$ is semidihedral when $r \equiv 1 \pmod 4$.

Now assume that $r \equiv -1 \pmod 4$. In this case we set $\bar{x} = \bar{b}\bar{a}\bar{c}$ and compute that $\bar{x}^2 = \bar{a}^{r-1}$; whence, $\langle \bar{x}^2 \rangle = \langle \bar{a}^2 \rangle$ and $|\bar{x}| = 2^m$. Furthermore, $\bar{b}^{-1}\bar{x}\bar{b} = \bar{b}\bar{a}^{-1}\bar{c} = \bar{a}^2(\bar{b}\bar{a}\bar{c}) = \bar{a}^2\bar{x}$. This time we have that $r = -1 + 2^{m-1}\lambda$, λ odd, and thus $(r - 1)(2^{m-2} - 1) \equiv 2 \pmod{2^m}$, whence $(\bar{x}^2)^{2^{m-2}-1} = \bar{a}^2$. Thus, $\bar{b}^{-1}\bar{x}\bar{b} = \bar{a}^2\bar{x} = \bar{x}^{-1+2^{m-1}}$, and therefore $\bar{S} = \langle \bar{b}, \bar{x} \rangle$ is semidihedral in this case as well. Thus (i) holds.

One easily verifies that every involution of $\bar{S}$ lies in $\bar{H}$, the subgroup of $PGL^*(2, q)$ which is equal to $PSL(2, q)$. Hence, any 4-subgroup $\bar{T}$ of $PGL^*(2, q)$ lies in $PSL(2, q)$. Since $q = p^m$, p odd, $\bar{T}$ normalizes no non-trivial p-subgroup of $\bar{H}$ (**8**, Lemma 3.1 (vii)), and (ii) follows.

LEMMA 2.4. *Let H be an I_1-group. Then*

(i) *The S_2-subgroups of H are dihedral or semidihedral;*

(ii) *H contains a 4-subgroup T which is normalized, but not centralized, by a 3-element of H and such that T^H is non-solvable.*

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Proof. Without loss we may assume that $O(H) = 1$, whence H contains a normal subgroup of odd index isomorphic to PSL(2, q), PGL(2, q), PGL*(2, q), q odd, $q > 3$, or to A_7. But now (i) follows from (**8**, Lemmas 3.1 (iii) and 3.2 (i)) and Lemma 2.3 (i) above. Moreover, H contains a normal subgroup K isomorphic to either PSL(2, q) or to A_7. However, if T is any 4-subgroup of K, $N_K(T)$ contains an element of order 3 which does not centralize T (**5**, Theorems 7.7.1 and 7.7.3) as K is simple. For the same reason T^K is non-solvable; therefore, both parts of the lemma hold.

LEMMA 2.5. *Let H be an I_i-group, $2 \leqq i \leqq 3$, and let S be an S_2-subgroup of H. Then*

(i) *The centralizer of every involution of $H/O(H)$ is a 2-group;*

(ii) *$N_H(S) \supset C_H(S)$ and S^H is non-solvable.*

Proof. In proving (i) we may assume that $O(H) = 1$. Suppose that $R = O_2(H) \neq 1$. Since H/R is simple, $C_H(R) \subseteq R$. But then by Lemma 2.1 (i), any element of odd order in $H^\#$ acts regularly on R; whence, the centralizer of any involution in R is a 2-group. Thus, (i) will hold if it holds in H/R. However, the centralizer of every involution of PSL(2, 2^n), Sz(2^n), and PSL(3, 4) is known to be a 2-group; see (**11**; **12**).

It clearly suffices to prove (ii) in the case that H is isomorphic to PSL(2, 2^n), $n \geqq 2$, Sz(2^n), $n \geqq 3$, or PSL(3, 4). Since H is simple, obviously S^H is non-solvable. Moreover, $N_H(S)/S$ is cyclic of order $2^n - 1$, $2^n - 1$, or 3, respectively, as can be directly verified.

LEMMA 2.6. *Let H be a simple normal I_i-subgroup of the* I*-group G, $1 \leqq i \leqq 3$, such that $C_G(H) = 1$. Then G is an I_i-group.*

Proof. Since $C_G(H) = 1$, we can identify G with a subgroup of Aut H. Without loss we can assume that $H \subset G$. If $H = A_7$, then G must be isomorphic to the symmetric group S_7. However, a transposition of S_7 has a non-solvable centralizer, contrary to the fact that G is an I-group. Thus $H \neq A_7$. In the remaining cases (**4**, Lemma 2.2 (i); **8**, Lemma 3.3 (i); **12**, Theorem 11), show that $G \subseteq LF$, where $L =$ PGL(2, q) if $H =$ PSL(2, q) and $L = H$ if $H =$ Sz(q), and where F is cyclic, $L \cap F = 1$, and the elements of F are induced from automorphisms of the underlying field GF(q). If $H =$ PSL(3, 4), then, as is well known, we reach the same conclusion with $L =$ PGL(3, 4).

We first consider the case $H =$ PSL(2, 2^n) or Sz(2^n), $n \geqq 3$, whence $L = H$ and $G \cap F \neq 1$. We can choose F to normalize the S_2-subgroup S of H consisting of the appropriate lower triangular matrices. Then $G \cap F$ centralizes the subgroup $S_0 \neq 1$ of S whose elements have entries in the prime field GF(2). But then $(G \cap F)S$ is not a Frobenius group, and consequently $O(G \cap F)$ centralizes S by Lemma 2.1 (i). However, one checks directly that only the identity element of F acts trivially on S. Hence, $O(G \cap F) = 1$, and therefore $G \cap F$ is a 2-group. Since n is necessarily odd if $H =$ Sz(2^n)

(**12**, Theorem 7), this forces $H = \mathrm{PSL}(2, 2^n)$. However, if x is the involution of $G \cap F$, then $C_H(x) = \mathrm{PSL}(2, 2^m)$, where $n = 2m$ (**4**, Lemma 2.2 (ii)). Since $n \geqq 3$, $m \geqq 2$, and consequently $C_H(x)$ is non-solvable, a contradiction.

Next, suppose that $H = \mathrm{PSL}(3, 4)$. In this case, $|F| = 2$ and $C_H(F) = \mathrm{PSL}(3, 2)$, which is non-solvable. Thus, $G \cap F = 1$. Since $|L/H| = 3$ in this case, the only possibility is that $G = L$. However, the S_2-subgroup R of $\mathrm{GL}(3, 4)$ consisting of the appropriate lower triangular matrices is normalized by all diagonal matrices of $\mathrm{GL}(3, 4)$, which form an elementary abelian group of order 27. It follows at once that the image S of R in H is normalized by an elementary abelian 3-subgroup P of G of order 9 with $C_P(S) = 1$. But then $O_3(SP) = 1$, and we see that Lemma 2.1 (ii) is contradicted.

We conclude that $H = \mathrm{PSL}(2, q)$, $q = p^n$, where either p is an odd prime or $p = n = 2$. We may assume that $|G{:}H|$ is even, since otherwise G is clearly an I_1-group. If $x \in F$ has order k, then (**8**, Lemma 3.3 (i)) $k|n$ and $C_H(x)$ contains $\mathrm{PSL}(2, p^{n/k})$. But then, if x is an involution of $F \cap G$, we see that $C_G(x)$ does not have a normal 2-complement except in the case that $p = n = 2$. Since $\mathrm{PSL}(2, 4)$ and $\mathrm{PSL}(2, 5)$ are isomorphic, it is immediate in the latter case that G is isomorphic to $\mathrm{PGL}(2, 5)$, and therefore is an I_1-group. Thus, we may assume that $|G \cap F|$ is odd and that p is odd. In this case, $|L{:}H| = 2$. This forces $|G{:}H|_2 \leqq 2$, since otherwise $|F \cap G|$ would necessarily be even. Thus, $|G{:}H|_2 = 2$. Since LF/H is abelian, it follows that G possesses a normal subgroup K of odd index in G such that $|K{:}H| = 2$. Since $K \cap F = 1$ and F is cyclic, there are exactly two possibilities for K, namely, $K = L$ or $K = \mathrm{PGL}^*(2, q)$. In either case, G is an I_1-group and the lemma is proved.

LEMMA 2.7. *Let T be a 2-subgroup of the I_i-group H, $0 \leqq i \leqq 3$, and assume that T contains a non-cyclic abelian subgroup of order 8. Then we have that*

(i) *every T-invariant subgroup of H of odd order lies in $O(H)$;*

(ii) *for any odd prime p, any two maximal T-invariant* p-*subgroups of H are conjugate by an element of $O(C_H(T))$.*

Proof. First, (ii) is an immediate consequence of (i). Moreover, in proving (i) we can assume without loss that $O(H) = 1$. Since an I_1-group has dihedral or semidihedral S_2-subgroups by Lemma 2.4 (i), our conditions imply that H is not an I_1-group.

Suppose that K is a non-trivial T-invariant subgroup of H of odd order. Since T contains a 4-subgroup, $C_K(x) \neq 1$ for some involution x of T. But then H is not an I_2-group or an I_3-group by Lemma 2.5 (i). Thus, H is an I_0-group. By Lemma 2.1 (iii), $T_0 = T \cap O_2(H) \neq 1$. Clearly, K centralizes T_0; whence, $O_2(H)K$ is not a Frobenius group, contrary to Lemma 2.1 (i).

LEMMA 2.8. *Let T be a 4-subgroup of the I_i-group H, $0 \leqq i \leqq 3$. Then for for any odd prime p, any two maximal T-invariant* p-*subgroups of H are conjugate by an element of $N_H(T)$.*

Proof. Suppose that H is not an I_1-group. Then the argument of the second paragraph of the preceding lemma can be repeated without change to yield that $O(H)$ contains every T-invariant p-subgroup of H and the lemma follows at once. Thus, we may assume that H is an I_1-group. In this case, the lemma follows from (**8**, Lemma 3.6 (i)). We note that the lemma applies even if the S_2-subgroups of H are semidihedral since, as we have noted in the proof of Lemma 2.4 (ii), the image of T in $H/O(H)$ lies in the normal subgroup of $H/O(H)$ isomorphic to PSL$(2, q)$.

LEMMA 2.9. *Let H be an I_i-group having only one class of involutions, $0 \leqq i \leqq 3$. Then*
(i) $i \neq 3$; *and*
(ii) *if S is an S_2-subgroup of H, then either $\Omega_1(S) \subseteq [H, H]$ or H has a normal 2-complement.*

Proof. It will suffice to prove the lemma for $H/O(H)$; therefore, without loss, we can assume that $O(H) = 1$. Suppose that $i = 3$, in which case $O_2(H) \neq 1$, all involutions of H lie in $O_2(H)$, and $\bar{H} = H/O_2(H)$ is isomorphic to PSL$(2, 2^n)$, $n \geqq 2$, or Sz(2^n), $n \geqq 3$. In either case, $\bar{H}$ contains an element $\bar{x} \neq \bar{1}$ of odd order which is inverted by an involution $\bar{y}$ of $\bar{H}$. However, if H_0 is the inverse image of $\langle \bar{x}, \bar{y} \rangle$ in H, we see that H_0 is an I_0-group, that $O(H_0) = 1$, and that $O_2(H_0) = O_2(H)$. Since $O_2(H_0)$ is thus not an S_2-subgroup of H_0, it follows from Lemma 2.1 (iv) that H_0 possesses an involution not in $O_2(H_0) = O_2(H)$, a contradiction. Thus (i) holds.

Next, suppose that $i = 0$. Since $O(H) = 1$, either H is a 2-group or by Lemma 2.1 (iv), H is a Frobenius group. Clearly, $\Omega_1(S) \subseteq [H, H]$ in the latter case, and hence (ii) holds when $i = 0$. If $i = 1$, our conditions imply that H contains a normal subgroup K of odd index isomorphic to PSL$(2, q)$ or PGL*$(2, q)$, q odd, or to A_7. Moreover, if K is isomorphic to PGL*$(2, q)$, all involutions of S lie in the subgroup of K isomorphic to PSL$(2, q)$. Hence, in each case, $\Omega_1(S)$ is contained in a simple normal subgroup of H. Furthermore, if $i = 2$, the same conclusion holds since then H itself is simple. Therefore, (ii) also holds when $i = 1$ or 2.

LEMMA 2.10. *Let H be an I_i-group, $0 \leqq i \leqq 3$, that is not* p*-constrained, p an odd prime. Then $O(H)$ contains every S_p-subgroup of $O_{p',p}(H)$.*

Proof. By (**5**, Theorem 6.3.2), H is not p-solvable. In particular, $i > 0$. Since H has exactly one non-solvable composition factor, we must have that $O_{p'}(H) \subseteq S(H)$; otherwise, H would be p-solvable. Hence, $O_{p',p}(H) \subseteq S(H)$. However, by definition of an I_i-group, $1 \leqq i \leqq 3$, $S(H)$ has a normal 2-complement. The lemma follows.

LEMMA 2.11. *Let H be an I_i-group in which* SCN$_3(2)$ *is non-empty*, $0 \leqq i \leqq 3$. *Let S be an S_2-subgroup of H and let P be a normal* p*-subgroup of H, p an odd prime. If S does not centralize P, then both $C_H(P)$ and $O_{p',p}(H)$ are contained in $S(H)$.*

Proof. Assume the contrary. Since $O_{p'}(H)$ clearly centralizes P, we necessarily have that $C = C_H(P) \nsubseteq S(H)$. In particular, $i > 0$. Since SCN$_3(S)$

is non-empty, Lemma 2.4 (i) implies that $i \neq 1$. It will suffice to prove that $H = O(H)C$, for then C will contain every S_2-subgroup of H, in which case S centralizes P, contrary to hypothesis. Since $C \lhd H$ and C is non-solvable, this will indeed be the case if $i = 2$, inasmuch as $H/O(H)$ is then simple.

Finally, suppose that $i = 3$ and let $\bar{C}$ be the image of C in $\bar{H} = H/O(H)$. It will suffice to prove that $\bar{C} = \bar{H}$. Since $\bar{H}/O_2(\bar{H})$ is simple and $\bar{C}$ is a non-solvable normal subgroup of $\bar{H}$, we have that $\bar{H} = O_2(\bar{H})\bar{C}$. If $O_2(\bar{C}) = O_2(\bar{H})$, then $\bar{C} = \bar{H}$, as required; thus, we may assume that $O_2(\bar{C}) \subset O_2(\bar{H})$. However, $\bar{C}$ contains an element $\bar{v} \neq \bar{1}$ of odd order and $[\bar{v}, O_2(\bar{H})] \subseteq \bar{C} \cap O_2(\bar{H}) \subseteq O_2(\bar{C})$; whence, $\bar{v}$ centralizes $O_2(\bar{H})/O_2(\bar{C})$. Lemma 2.1 (i), applied to $\langle O_2(\bar{H}), \bar{v}\rangle$, now yields that $\bar{v}$ centralizes $O_2(\bar{H})$, contrary to Lemma 2.5 (i).

LEMMA 2.12. *Let H be an I_i group, $0 \leqq i \leqq 3$, let T be a 2-subgroup of H which contains a 4-subgroup, and let P be a non-cyclic T-invariant* p*-subgroup of H, p an odd prime. If $[T, P] = P$, then T does not centralize $P \cap O(H)$.*

Proof. Let $P_0 = P \cap O(H)$ and suppose, by way of contradiction, that T centralizes P_0. Then $[P_0, T, P] = 1$ and $[P, P_0, T] = 1$; therefore, $[T, P, P_0] = [P, P_0] = 1$ by the 3-subgroup lemma. Thus, $P_0 \subseteq Z(P)$. If P/P_0 were cyclic, it would follow that P is abelian, whence $P = C_P(T) \times [P, T]$ (**5**, Theorem 5.2.3). But then $C_P(T) = 1$, whence $P_0 = 1$ and P is cyclic, contrary to assumption. It follows that P/P_0 is non-cyclic. Since $[T, P/P_0] = P/P_0$, we see that our conditions carry over to $H/O(H)$. Hence, without loss we can assume, to begin with, that $O(H) = 1$.

Since P is non-cyclic, $i > 0$ by Lemma 2.1 (ii). If $i > 1$, then $C_H(x)$ is a 2-group for any involution x of T by Lemma 2.5 (i). Since T contains a 4-subgroup, $C_P(x) \neq 1$ for some involution x, a contradiction. Hence $i = 1$. In this case, H contains a normal subgroup L of odd index isomorphic to $\mathrm{PSL}(2, q)$, $\mathrm{PGL}(2, q)$, $\mathrm{PGL}^*(2, q)$, q odd, or to A_7. Then $T \subset L$, and thus T centralizes $P/P \cap L$. Since $P = [T, P]$, it follows that $P \subset L$. One checks that A_7 does not contain subgroups T, P satisfying the given conditions; therefore, H is not isomorphic to A_7. Since the S_p-subgroups of $\mathrm{PSL}(2, q)$ are cyclic if $q \neq p^m$ (**8**, Lemma 3.1 (v)), we must have that $q = p^m$. However, by (**8**, Lemma 3.1 (vii)) and Lemma 2.3 (ii), T normalizes no non-trivial p-subgroup of L. This proves the lemma.

LEMMA 2.13. *Let H be an I_i-group, $0 \leqq i \leqq 3$, and let A be an abelian* p*-subgroup of H, p an odd prime, with the following properties:*

(a) $A \in \mathrm{SCN}_3(p)$;

(b) $A \cap S(H) = 1$.

Then every A-invariant p′*-subgroup of H lies in $S(H)$.*

Proof. Let P be an A-invariant S_p-subgroup of $S(H)$. If $P \neq 1$, then $P \cap Z(PA) = P_0 \neq 1$ and A centralizes P_0. However, A is an S_p-subgroup of $C_H(A)$ by (a), whence $P_0 \subseteq A$, and therefore $A \cap S(H) \neq 1$, contrary to (b). Thus, $P = 1$ and $S(H)$ is a p′-group. It follows at once that our conditions carry over to $H/S(H)$. Hence, without loss we can assume, to begin with, that $S(H) = 1$.

By (**4**, Lemma 2.1 (v); **12**, Theorem 9), the groups $\mathrm{PSL}(2, 2^n)$ and $\mathrm{Sz}(2^n)$ do not contain subgroups of type (p, p, p) for any odd p. The same is true of A_7 and $\mathrm{PSL}(3, 4)$, as is easily verified. Hence, we can identify H with a subgroup of $G = \mathrm{P\Gamma L}(2, q)$, q odd. Moreover, we must have that $q = p^m$, since otherwise the S_p-subgroups of H would not contain a subgroup of type (p, p, p) (**8**, Lemma 3.3 (v)). As usual, $G = LF$, where $L = \mathrm{PGL}(2, q)$ and F is cyclic of order m. By (**8**, Lemma 3.3 (iii) and (iv)), A is an S_p-subgroup of L and F can be chosen to normalize A. By (**8**, Lemmas 3.1 (vii) and 3.3 (i)), A normalizes no non-trivial p′-subgroup of L and centralizes no element of $F^\#$. We conclude at once that A normalizes no non-trivial p′-subgroup of H.

LEMMA 2.14. *If H is an I_i-group, $0 \leqq i \leqq 3$, then H is* p-*stable for any odd prime p.*

Proof. Suppose the contrary. Then H possesses a section L/K isomorphic to $\mathrm{SL}(2, p)$, for some odd prime p (**5**, Theorem 3.8.3). Now the subgroups of $\mathrm{PSL}(2, q)$, $\mathrm{Sz}(q)$, A_7, and $\mathrm{PSL}(3, 4)$ are known and as a result, it is not difficult to verify that every subgroup of H is not only an I-group, but, in fact, an I_j-group for some j, $0 \leqq j \leqq 3$. (Actually, for the applications of this lemma, it would suffice to add this assertion to our hypothesis.) Hence, without loss we may assume that $L = H$. Since $H/O(H)$ must also involve $\mathrm{SL}(2, p)$, we can also assume that $O(H) = 1$. Since $\mathrm{SL}(2, p)$ is not an I-group, we must have that $K \neq 1$, which implies that $i = 0$ or 3. Moreover, $K \subseteq O_2(H)$ if $i = 3$ and the same conclusion follows from Lemma 2.1 if $i = 0$. Since $\mathrm{SL}(2, p)$ contains a subgroup isomorphic to $\mathrm{SL}(2, 3)$, it follows that H contains a subgroup H_1 with $H_1 \supset K$ and H_1/K isomorphic to $\mathrm{SL}(2, 3)$. But then H_1 is an I_0-group, and thus is a Frobenius group by Lemma 2.1. Therefore, an element x of order 3 in H_1 acts regularly on $O_2(H_1)$, and hence also on every homomorphic image of $O_2(H_1)$. However, x fixes the central involution of the quaternion group $O_2(H_1)/K$.

LEMMA 2.15. *Let H be an I_i-group, $1 \leqq i \leqq 3$, and let C be a non-solvable normal subgroup of H. Then C possesses a 2-group T with the following properties:*

(i) *$Z(T)$ is non-cyclic;*

(ii) *either T is an S_2-subgroup of H or T is a 4-group, and an S_2-subgroup of H is dihedral or semidihedral;*

(iii) *T^C in non-solvable;*

(iv) *$[u, T] = T$ for some element u of odd order in $N_C(T)$.*

Proof. If C is an I_1-group, then by Lemma 2.4 (ii), C contains a 4-subgroup T such that T^C is non-solvable and $[u, T] = T$ for some 3-element u of $N_C(T)$. If C is an I_2-group or an I_3-group, then by Lemma 2.5 (ii) an S_2-subgroup T of C is normalized, but not centralized, by an element u of C and T^C is non-solvable. Lemma 2.1 (i) then implies that $[u, T] = T$ and that $Z(T)$ is non-cyclic. Hence, to establish the lemma, it remains only to verify (ii). Since $CO(H)/O(H) \lhd H/O(H)$, it suffices to consider the case that $O(H) = 1$.

If $i = 1$, then an S_2-subgroup of H is dihedral or semidihedral by Lemma 2.4 (i), in which case C is clearly also an I_1-group. But then T is a 4-group and (ii) holds. If $i = 2$ or 3, we shall argue that $C = H$, whence T will be an S_2-subgroup of H and again (ii) will hold. If $i = 2$, this is indeed the case, for then H is simple as $O(H) = 1$. On the other hand, if $i = 3$, the argument of the final paragraph of the proof of Lemma 2.11 yields the desired conclusion $C = H$, inasmuch as C is a non-solvable normal subgroup of H and $O(H) = 1$.

3. Initial reductions. For the balance of the paper, G will denote a minimal counterexample to Theorem A.

LEMMA 3.1. *Every proper subgroup of G is an I_i-group, $i = 0, 1, 2,$ or 3.*

Proof. This follows at once from the minimality of G and the fact that the property of being an I-group is obviously inherited by subgroups.

THEOREM 3.2. *The centralizer of some involution of G is not a 2-group.*

Proof. Assume the contrary, in which case the results of Suzuki's papers (**11**; **12**) are applicable. Since G is non-solvable, $S(G) = O_2(G)$ (**11**, Theorem 4). Moreover, the centralizer of every involution in $\bar{G} = G/O_2(G)$ is a 2-group. Hence, by Suzuki's main results, $\bar{G}$ is isomorphic to one of the following groups: $PSL(2, p)$, p a Fermat or Mersenne prime with $p > 5$, $PSL(2, 9)$, $PGL^*(2, 9)$, $PSL(3, 4)$, $PSL(2, 2^n)$, $n \geqq 2$, or $Sz(2^n)$, $n \geqq 3$. Since G is a counterexample to Theorem A, we conclude that $O_2(G) \neq 1$ and that $\bar{G}$ is isomorphic to $PSL(2, 9)$, $PGL^*(2, 9)$, $PSL(3, 4)$, or $PSL(2, p)$, p a Fermat or Mersenne prime with $p > 5$. However, in each of the first three cases, an S_3-subgroup P of G is non-cyclic, as is easily verified. However, $H = PO_2(G)$ is an I_0-group and $O(H) = 1$ as $O(H)$ centralizes $O_2(G)$, contrary to Lemma 2.1 (ii).

Thus, $\bar{G}$ is isomorphic to $PSL(2, p)$, p a Fermat or Mersenne prime with $p > 5$. In particular, an S_2-subgroup of $\bar{G}$ is dihedral of order at least 8. Hence, if $\bar{T}$ is an arbitrary 4-subgroup of $\bar{G}$, then $\bar{N} = N_{\bar{G}}(\bar{T})$ is isomorphic to the symmetric group S_4 (**8**, Lemma 3.1 (iv)). Hence, if T and N denote the inverse images of $\bar{T}$ and $\bar{N}$, respectively, in G, then N/T is dihedral of order 6 and an element of order 3 in N acts regularly on T. In particular, $cl(T) \leqq 2$ by Lemma 2.1 (v). Since $O_2(G) \subset T$ and $C_G(O_2(G)) \subseteq O_2(G)$, we have, in fact, that $cl(T) = 2$.

Now let $\bar{T}_1$ be a 4-subgroup $\bar{N}$ distinct from $\bar{T}$ and let T_1 be the inverse image of $\bar{T}_1$ in G. Since $\bar{T}$ was arbitrary, we also have that $cl(T_1) = 2$. However, T_1 has index 2 in an S_2-subgroup of N and is distinct from T. It therefore follows (**9**, Lemma 2.4 (i)) that $cl(T_1) > cl(T) = 2$, a contradiction.

THEOREM 3.3. *G is simple.*

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Proof. It is immediate from the definition that $G/O(G)$ is an I-group. Hence, if $O(G) \neq 1$, $G/O(G)$ satisfies the conclusion of Theorem A by the minimality of G. But then so also does G, contrary to our choice of G. Thus $O(G) = 1$.

Suppose that $R = O_2(G) \neq 1$. By Theorem 3.2, there exists an involution x of G which centralizes a non-trivial element y of G of odd order. Then $\langle R, x\rangle \lhd K = \langle R, x, y\rangle$ and K is not a Frobenius group. Hence, by Lemma 2.1 (ii), $O(K) \neq 1$. However, $O(K) \subseteq C = C_G(R)$ and C has a normal 2-complement; whence, $O(C) \neq 1$. Since $O(C)$ char $C \lhd G$, it follows that $O(C) \subseteq O(G) = 1$, a contradiction. Thus, $O_2(G) = 1$, and consequently $S(G) = 1$.

Now let H be a minimal normal subgroup of G. Since H is characteristically simple and $S(G) = 1$, H is necessarily the direct product of isomorphic non-abelian simple groups. If H itself were not simple, some involution of H would have a non-solvable centralizer in H, contrary to the fact that H is an I-group. Thus, H is simple. Suppose that $H \subset G$, in which case H is an I_i-group, $1 \leqq i \leqq 3$. Since G is an I-group, $C_G(H)$ must have odd order. Since $C_G(H) \lhd G$, it follows that $C_G(H) \subseteq O(G) = 1$. But now Lemma 2.6 implies that G satisfies the conclusion of Theorem A, which is not the case. Thus $G = H$ is simple.

We next prove the following result.

THEOREM 3.4. $\mathrm{SCN}_3(2)$ *is non-empty in* G.

Proof. Assume the contrary. Since the centralizer of every involution of G is certainly solvable, the results of Janko and Thompson (**10**) are applicable and yield that G is isomorphic to either $\mathrm{PSL}(2, q)$, q odd, $q > 3$, A_7, M_{11}, $\mathrm{PSL}(3, 3)$, $\mathrm{PSU}(3, 3)$, or $\mathrm{PSU}(3, 4)$. However, one easily checks that in each of the last four groups the centralizer of an involution does not have a normal 2-complement. Hence, the latter four groups are not I-groups, and consequently G is isomorphic to $\mathrm{PSL}\ (2, q)$, q odd, $q > 3$, or to A_7, and therefore satisfies the conclusion of Theorem A, which is not the case.

Throughout the balance of the paper S will denote a fixed S_2-subgroup of G. We now prove the following result.

THEOREM 3.5. S *normalizes, but does not centralize, some subgroup of* G *of odd order.*

Proof. Assume the contrary. We shall argue that the centralizer of every involution of G is 2-closed. Observe, first of all, that a 2-local subgroup H of G is either an I_0-group or an I_3-group. Since $C_H(O_2(H)) \subseteq O_{2',2}(H)$ in either case, it follows that H is 2-constrained. Hence, by the Thompson transitivity theorem and its corollary (**5**, Theorems 8.5.4 and 8.5.6), if $A \in \mathrm{SCN}_3(S)$, then for any odd prime p, a maximal A-invariant p-subgroup P of G is normalized by an S_2-subgroup S^* of G containing A. By our assumption, S^* centralizes P, and hence so does A. This implies that A centralizes every subgroup of odd order that it normalizes.

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Now let x be an involution of G and set $C = C_G(x)$. Without loss we may assume that $S \cap C$ is an S_2-subgroup of C. Since $A \in \mathrm{SCN}_3(S)$, $C_A(x) = A \cap C$ is non-cyclic, and therefore contains a 4-subgroup T. Let t_i be the involutions of T and set $C_i = C_G(t_i)$, $1 \leqq i \leqq 3$. Since $A \subseteq C_i$, A centralizes $O(C_i)$ for each i, $1 \leqq i \leqq 3$. However, C_i has a normal 2-complement, and therefore A centralizes $O(C) \cap C_i = C_{O(C)}(t_i)$, $1 \leqq i \leqq 3$, whence A centralizes $O(C)$. Thus, $O(C) \subseteq O(C_0)$, where $C_0 = C_G(A)$. However, S normalizes C_0 as $A \lhd S$; whence, S centralizes $O(C_0)$ by our assumption on S. It follows that S centralizes $O(C)$. Since C has a normal 2-complement, we conclude that $C = O(C) \times (S \cap C)$, and hence that C is 2-closed.

We can therefore apply the main results of (**13**). Since the centralizer of some involution of G is not a 2-group by Theorem 3.2, Suzuki's classification theorem yields that G is isomorphic to $\mathrm{PSL}(3, 2^n)$ or to $\mathrm{PSU}(3, 2^n)$, $n > 1$. However, one can verify directly that neither of these groups is an I-group.

By the theorem, S normalizes, but does not centralize, a p-subgroup of G for some odd prime p. We shall denote the set of all such odd primes by σ.

The main results of Glauberman's paper (**3**) yield the following result.

THEOREM 3.6. *One of the following conditions holds:*

(i) *S is elementary abelian and all involutions of S are conjugate in G; or*

(ii) *there exist two elements of S conjugate in G that are not conjugate in $N_G(S)$.*

Proof. Assume the contrary. Then by Glauberman's results, G is isomorphic to a Suzuki group, which is not the case.

Finally, we prove the following theorem.†

THEOREM 3.7. *If G possesses a strongly embedded subgroup M containing S which is a uniqueness subgroup for some odd prime p, then S centralizes $O_p(M)$.*

Proof. We first argue that $\Omega_1(S)$ centralizes $O_p(M)$; thus, assume the contrary. Let x be an involution of $Z(S)$, set $C = C_M(x) = C_G(x)$ and $H = CO_p(M)$. Since M has only one class of involutions (**5**, Theorem 9.2.1 (ii)), x does not centralize $O_p(M)$, and hence $H \supset C$. Since $S \subseteq C$ and C has a normal 2-complement, S normalizes an S_p-subgroup Q of H. Moreover, by Lemma 2.7 (i), $Q \subseteq O(M)$. Let P be an S-invariant S_p-subgroup of $O(M)$.

Now the argument of (**5**, Theorem 9.3.1) shows that (**5**, Theorem 9.2.2 (ii)) will be contradicted provided we prove that $C_P(y)$ contains a subgroup of type (p, p, p) for any y in $Q^\#$. To see that this is indeed the case, let B be a critical subgroup of P and set $D = \Omega_1(B)$. Since $\Omega_1(S)$ does not centralize P, it does not centralize either B or D, each of which is characteristic in P. Moreover, $\mathrm{cl}(D) \leqq 2$ and D has exponent p. If $\mathrm{SCN}_3(D)$ is empty, then $|D/\phi(D)| \leqq p^2$. Since $\mathrm{SCN}_3(S)$ is non-empty, some involution of S is thus forced to centralize D. However, $M = O(M)K$, where $K = N_M(P)$ by the Frattini argument, and therefore K has only one class of involutions. Since K

†Added in proof. It follows directly from a recent classification theorem of Bender that G does not contain a strongly embedded subgroup.

normalizes D, it follows that $\Omega_1(S)$ centralizes D, a contradiction. Thus, $\mathrm{SCN}_3(D)$ is non-empty. But now the proof of (**4**, Lemma 4.2) shows that either the desired conclusion holds or else D possesses a K-invariant normal series $\mathscr{D}$: $D = D_0 \supset D_1 \supset \ldots \supset D_m = 1$ such that $|D_{i-1}/D_i| = p$, $1 \leqq i \leqq m - 1$. However, in the latter case, we see that $K' = [K, K]$ stabilizes the chain $\mathscr{D}$. Furthermore, since S has more than one involution and all involutions of S are conjugate in K, Lemma 2.2 (ii) implies that K does not have a normal 2-complement. It follows therefore from Lemma 2.9 (ii) that $\Omega_1(S) \subseteq K'$. We conclude at once that $\Omega_1(S)$ centralizes $O_p(M)$, as asserted.

Since $\mathrm{SCN}_3(S)$ is non-empty and M has only one class of involutions, M is either an I_0-group or an I_2-group by Lemmas 2.4 (i) and 2.9 (i). If M is an I_2-group, then $M/O(M)$ is simple, and consequently $M = O(M)C_M(O_p(M))$. Thus, S centralizes $O_p(M)$ in this case. If M is an I_0-group, then all involutions of S are conjugate in $N_G(S)$ by Lemma 2.2 (ii). We argue that any two elements x, y of $S^\#$ that are conjugate in G are conjugate in $N_G(S)$. Indeed, if $y = x^a$, $a \in G$, then $y_1 = x_1{}^a$, where x_1 and y_1 are involutions that are powers of x and y, respectively. But then $y_1 = x_1{}^b$ for some b in $N_G(S)$. Thus, it suffices to show that x and $y^{b^{-1}}$ are conjugate by an element of $N_G(S)$, and therefore without loss we may assume that $y_1 = x_1$, whence $a \in C_G(x_1) \subseteq M$. However, $M = O(M)N_G(S)$ by Lemma 2.2 (i), and therefore it will also suffice to consider the case that $a \in O(M)$. However, $[a, x] \in O(M) \cap S = 1$, and therefore a centralizes x; whence, $y = x$ and the assertion is proved. But now S is elementary abelian by the preceding theorem, and therefore $S = \Omega_1(S)$ centralizes $O_p(M)$ in this case as well.

4. A transitivity theorem and some consequences. We shall need a variation of the Thompson transitivity theorem. We first prove a simple lemma.

LEMMA 4.1. *Let* $A \in \mathrm{SCN}_3(S)$ *and let* T *be an elementary abelian subgroup of* S. *Then*

(i) *if* $m(T) = 3$ *and* $T \not\subseteq A$, *then* $N_S(T)$ *contains an elementary abelian subgroup* T_0 *with* $m(T_0) = 3$ *such that* $m(T_0 \cap A) > m(T \cap A)$;

(ii) $T \subseteq T_1$ *for some elementary abelian subgroup* T_1 *with* $m(T_1) \geqq 3$.

Proof. We first prove (i). Now $\Omega_1(Z(S))$ is contained in both A and $C_S(T)$. Hence, if $\Omega_1(Z(S)) \not\subseteq T$, it is immediate that $C_S(T)$ contains a subgroup T_0 with the required properties. Thus, we may assume that $\Omega_1(Z(S)) \subseteq T$. Likewise, we may assume that $m(N_A(T)) \leqq 2$, otherwise we can choose T_0 to be a subgroup of $N_A(T)$. Set $A_1 = \Omega_1(A)$. Since $A_1 \lhd S$, it follows that $m(N_{A_1}(T)) > m(T \cap A_1)$. Hence, the only case left to consider is that in which $T \cap A_1 = \Omega_1(Z(S))$ is of order 2 and $A_0 = N_{A_1}(T)$ is a 4-group. However, T normalizes A_0 as $A_1 \lhd S$, and consequently some involution x of $T - \Omega_1(Z(S))$ centralizes A_0. Taking $T_0 = \langle x, A_0 \rangle$, we see that T_0 has the required properties. Thus (i) holds.

As for (ii), we can take $T = T_1$ if $m(T) \geqq 3$ and can take $T_1 = A_1 = \Omega_1(A)$ if $T \subseteq A_1$. Clearly, we need only treat the case that $\Omega_1(Z(S)) \subseteq T$. Hence, we are reduced to proving (ii) in the case that T is a 4-group and $T \cap A_1 = \Omega_1(Z(S))$ has order 2. Taking x in $T - \Omega_1(Z(S))$, we have that $m(C_{A_1}(x)) \geqq 2$ as $m(A_1) \geqq 3$; whence, $T_1 = \langle T, C_{A_1}(x)\rangle = TC_{A_1}(T)$ is elementary abelian of order at least 8 and contains T.

We now prove the following theorem.

THEOREM 4.2. *Let T be a subgroup of S in which $Z(T)$ is non-cyclic. Then for any odd prime p, we have that*

(i) *$O(C_G(T))$ acts transitively by conjugation on the set of maximal T-invariant* p*-subgroups of G;*

(ii) *S normalizes some maximal T-invariant* p*-subgroup of G;*

(iii) *if P is a maximal T-invariant* p*-subgroup of G, then $N_G(T) = O(C_G(T))(N_G(T) \cap N_G(P))$.*

Proof. We show first that (i) implies (iii). If $x \in N_G(T)$, we have that $P^x = P^y$ for some y in $O(C_G(T))$ by (i). Since $xy^{-1} \in N_G(P)$, (iii) follows. In particular, it follows, if (i) holds for T, that $N_S(T)$ normalizes some maximal T-invariant p-subgroup of G.

We next prove (i) and (ii) when $m(Z(T)) \geqq 3$. The proof of (i) follows a now standard argument, in the course of which one must consider subgroups H of G with H either the centralizer of an involution of $Z(T)$ or a p-local subgroup of G containing T; cf. (**5**, § 8.5). Since $m(Z(T)) \geqq 3$, it follows in the first case from the definition of an I-group and in the second from Lemma 2.7 (i) that any maximal T-invariant p-subgroup of H lies in $O(H)$. Hence, and two such maximal p-subgroups of H are conjugate by an element of $C_{O(H)}(T)$. Since $C_{O(H)}(T) \subseteq O(C_G(T))$, we obtain the transitivity theorem stated in (i).

Next, let T_1 be an elementary abelian subgroup of $Z(T)$ of order 8. Then by the first paragraph, $N_S(T_1)$ normalizes a maximal T_1-invariant p-subgroup P_1 of G. Thus, P_1 is T-invariant; whence, $|P_1| \leqq |P|$ by (i), where P is a maximal T-invariant p-subgroup of G. On the other hand, T_1 normalizes P, so also $|P| \leqq |P_1|$ by (i). Thus, P_1 is a maximal T-invariant p-subgroup of G. We see then that it suffices to prove (ii) for T_1, and therefore without loss we may assume that T itself has order 8.

Now let $A \in SCN_3(S)$. If $T \subseteq A$, then $A \subseteq N_S(T)$ and therefore A normalizes some maximal T-invariant p-subgroup P of G by the first paragraph. Since $S \subseteq N_G(\Omega_1(A))$, it follows for the same reason that S normalizes a conjugate of P by an element of $O(C_G(\Omega_1(A)))$. Thus, (ii) holds in this case. Suppose then that $T \nsubseteq A$. We can assume (ii) for all elementary abelian subgroups T_0 of S of order 8 such that $m(T_0 \cap A) > m(T \cap A)$. By Lemma 4.1 (i), $N_S(T)$ possesses such a subgroup T_0. Then T_0 normalizes some maximal T-invariant p-subgroup P of G. On the other hand, by (ii) applied to T_0, S normalizes some maximal T_0-invariant p-subgroup P_0 of G. By (i), $|P_0| \geqq |P|$. However, T normalizes P_0 as $T \subseteq S$; therefore $|P_0| \leqq |P|$

by (i). Thus, P_0 is a maximal T-invariant p-subgroup of G and (ii) is established when $m(Z(T)) \geqq 3$.

Next, suppose that $m(Z(T)) = 2$ and set $Z = \Omega_1(Z(T))$. By Lemma 4.1 (ii), $Z \subset T_0$ for some elementary abelian subgroup T_0 of S of order 8. Hence, if $z \in Z^{\#}$, T_0 normalizes some S_p-subgroup R of $C_G(z)$. However, if P is a maximal S-invariant p-subgroup of G, it follows from (i) and (ii), applied to T_0, that $R^x \subseteq P$ for some x in $C_G(T_0)$. Thus, P contains an S_p-subgroup of $C_G(z)$ for each z in $Z^{\#}$. We conclude at once from this that P is a maximal T-invariant p-subgroup of G, proving (ii) in this case as well.

We next prove that if $T \supset Z$, then (i) holds, and if $T = Z$, then any two maximal T-invariant p-subgroups of G are conjugate by an element of $N_G(T)$. Suppose the contrary and let Q be a maximal T-invariant p-subgroup of G which is not conjugate to P by an element of $O(C_G(T))$ if $T \supset Z$ or by an element of $N_G(T)$ if $T = Z$ and chosen so that $D = P \cap Q$ has maximal order. Then $Q \neq 1$, and therefore $C_Q(z) \neq 1$ for some z in $Z^{\#}$. However, P contains an S_p-subgroup of $C_G(z)$ by the preceding paragraph. Since P is T-invariant, P contains, in fact, a T-invariant S_p-subgroup of $C_G(z)$. Since any two such S_p-subgroups are conjugate by an element of $O(C_G(T))$, it follows from our choice of Q that $D \neq 1$. Since $D \subset P$ and $D \subset Q$, consideration of $K = N_G(D)$ leads to a contradiction in the usual way, since by Lemmas 2.7 (ii) and 2.8 any two maximal T-invariant p-subgroups of K are conjugate by an element of $O(C_K(T))$ if $T \supset Z$ and by an element of $N_K(T)$ if $T = Z$. This proves the assertion.

Finally, consider the case $T = Z$. Then by Lemma 4.1 (ii), $T \subset T_0$ for some elementary abelian subgroup T_0 of S of order 8. Set $N = N_G(T)$, let R be an S_2-subgroup of N containing T_0, and set $R_0 = R \cap O_{2',2}(N)$, so that R_0 is an S_2-subgroup of $O_{2',2}(N)$ and $N = O(N)N_0$, where $N_0 = N_N(R_0)$. By (ii), R_0 normalizes some maximal T-invariant p-subgroup of G, which we may again denote by P. Now $O(N)$ centralizes T, and therefore lies in $O(C_G(T))$. Hence, by the preceding paragraph, (i) will hold provided we can show that for any x in N_0, $P^x = P^y$ for some y in $O(C_G(T))$. To do this, it will suffice to show that $T_0 \subseteq R_0$, for then T_0 will normalize P^x, which is certainly a maximal T_0-invariant p-subgroup of G as $T \subset T_0$. But then $P^x = P^y$ for some y in $O(C_G(T_0))$ by (i). Since $O(C_G(T_0)) \subseteq O(C_G(T))$, the desired conclusion will follow.

Now $C_G(T)$ has a normal 2-complement and $N/C_G(T)$ is isomorphic to a subgroup of the symmetric group S_3; whence, N is solvable. If N has a normal 2-complement, then $R = R_0$, in which case obviously $T_0 \subseteq R_0$. In the contrary case, it easily follows from Lemma 2.1 that $T \subseteq Z(R_0)$ and that $C_G(T) = O(N)R_0$. Since T_0 centralizes T and lies in R, we again have that $T_0 \subseteq R_0$. Thus, (i) holds in this case as well, and the theorem is proved.

This theorem has an important consequence. We recall that a proper subgroup H of a simple group G is said to be *weakly embedded* in G provided H contains an S_2-subgroup S of G, two elements of S conjugate in G are conjugate in H, and for any involution x of S, $C_G(x) = O(C_G(x))(C_G(x) \cap H)$; cf. (**5**, § 17.2).

We prove the following theorem.

THEOREM 4.3. *Let P be a maximal S-invariant* p*-subgroup of G, p an odd prime, with $P \neq 1$ and set $H = N_G(P)$. Then*

(i) *H is weakly embedded in G;*

(ii) *one of the following holds:*

(a) *S is elementary abelian and all involutions of S are conjugate in H;*

(b) *$H/O(H)$ is isomorphic to* PSL(3, 4)*;*

(c) *H is an* I_3*-group.*

Proof. Since $P \neq 1$ and P is S-invariant, H is a proper subgroup of G containing S. Let x be an involution of S. By Lemma 4.1 (ii), $x \in T$, where T is a non-cyclic elementary abelian subgroup of S. Let R be an S_2-subgroup of $C = C_G(x)$ containing T. By Theorem 4.2, R normalizes some maximal T-invariant p-subgroup Q of G and $P = Q^y$ for some y in $O(C_G(T))$. Since $x \in T$, $y \in O(C)$, and therefore R^y is an S_2-subgroup of C. However, R^y normalizes P, and thus $R^y \subseteq H$, whence $C = O(C)(C \cap H)$.

Next, suppose that there exist two elements of S that are conjugate in G, but are not conjugate in H. Then by Alperin's fusion theorem (**1**; **5**), there exist elements u, v of S with $u, v \in T$, where $T = S \cap S_1$ is a tame intersection of the S_2-subgroups S, S_1 of G, such that u and v are conjugate in $N = N_G(T)$, but u and v are not conjugate in H. However, if $Z(T)$ is non-cyclic, $N = O(N)(N \cap H)$ by Theorem 4.2 (iii). Hence, if $v = u^n$, $n \in N$, we have that $n = ha$, where $h \in N \cap H$ and $a \in O(N)$. Setting $w = u^h$, we have that $v = w^a$ and $u, w \in T$, whence $w^{-1}w^a \in O(N) \cap T = 1$. Thus, a centralizes w, and so $v = u^h$, contrary to our choice of u, v. Hence, $Z(T)$ must be cyclic.

We have that $|\Omega_1(Z(T))| = 2$, and consequently $K = N_G(\Omega_1(Z(T)))$ has a normal 2-complement. Since $N \subseteq K$, N also has a normal 2-complement. However, by definition of a tame intersection, $S \cap N$ is an S_2-subgroup of N, whence $N = O(N)(S \cap N) = O(N)(H \cap N)$, as $S \subset H$. This leads to the same contradiction as in the preceding case. We conclude that H is weakly embedded in G.

To prove (ii), observe first that by (i), if $u \in S$ and $u^{-1}u^g \in S$ for some g in G, then $u^{-1}u^g = u^{-1}u^h$ for suitable h in H. It therefore follows from the focal subgroup theorem (**5**, Theorem 7.3.4) that $S \cap [G, G] = S \cap [H, H]$. However, $S \subseteq [G, G]$ as G is simple, and hence $S \subseteq [H, H]$. In particular, if H is an I_0-group, Lemma 2.2 now yields that two elements of S conjugate in G are conjugate in $N_H(S)$. But then (ii)(a) holds by Theorem 3.6.

Since $SCN_3(S)$ is non-empty, H is not an I_1-group. If H is an I_3-group, then (ii)(c) holds. Finally, consider the case that H is an I_2-group. If $H/O(H)$ is isomorphic to PSL$(2, 2^n)$ or to PSL(3, 4), then correspondingly, (ii)(a) or (ii)(b) holds. On the other hand, if $H/O(H)$ is isomorphic to Sz(2^n), then S

is non-abelian; however, two elements of S conjugate in G, and hence in H, are conjugate in $N_H(S)$, contrary to Theorem 3.6, and (ii) is proved.

Remark. It is entirely possible that there exists a character-theoretic argument analogous to that given by Glauberman in (**3**) which will show that G does not possess a weakly embedded subgroup H of the form (ii)(b) or (ii)(c) in which $O(H) \neq 1$. If so, the proof of Theorem A could then be completed by invoking the main theorem of (**4**) to handle the case that H is of the form (ii)(a). However, it should be pointed out that Glauberman's argument cannot be extended to include this minimal case. Since the main results of (**4**) depend upon the existence of strongly embedded subgroups, it thus appears that a proof of Theorem A cannot be given which completely avoids the use of strongly embedded subgroups. The argument we shall present, which in effect is a repetition in somewhat simplified form of that of (**4**), will treat all the possibilities in (ii) uniformly. On the other hand, the main theorem of (**4**) will not be explicitly quoted.

We need the following lemma.

LEMMA 4.4. *Let P be a maximal S-invariant* p*-subgroup of G for any p in σ and set $H = N_G(P)$. Then S possesses an elementary abelian normal subgroup S_1 of order at least* 8 *with the following properties:*

(i) $[y, S_1] = S_1$ *for some element y of $N_H(S_1)$ of odd order;*

(ii) *for some* 4*-subgroup T of S_1, $C_P(x)$ contains a subgroup of type (p, p, p) for each x in $T^{\#}$. In particular,* $\mathrm{SCN}_3(P)$ *is non-empty.*

Proof. H has one of the three forms listed in Theorem 4.3 (ii). If H is of type (a), we can take $S = S_1$. If H is of type (b), then $N_H(S)$ contains a 3-element which normalizes, but does not centralize, an elementary abelian normal subgroup S_1 of S of order 16. If H is of type (c), we can take $S_1 = \Omega_1(Z(S \cap O_{2',2}(H)))$. In each case, $N_H(S_1)$ contains an element y of odd order which does not centralize S_1. But then $[y, S_1] = S_1$ by Lemma 2.1 (i), proving (i).

Once we know that (i) holds, the proof of (**4**, Lemma 4.3) can be repeated verbatim to yield the first assertion of (ii). The second assertion of (ii) then follows from (**5**, Theorem 5.4.15).

This enables us to prove the following result.

THEOREM 4.5. *Suppose that G satisfies the uniqueness condition for some prime p in σ and let M be a uniqueness subgroup for p. Then M is strongly embedded in G.*

Proof. Let P be a maximal S-invariant p-subgroup of G and let $\tilde{P}$ be an S_p-subgroup of G containing P. Without loss we may assume that $\tilde{P} \subseteq M$. Since $\mathrm{SCN}_3(P)$ is non-empty by the preceding lemma, $H = N_G(P) \subseteq M$ by definition of a uniqueness subgroup. We shall argue that M contains $C = C_G(x)$ for any involution x of S. Since H is weakly embedded in G by Theorem 4.3 (i), $C = O(C)(C \cap H)$. Since $H \subseteq M$, it will thus suffice to show that $O(C) \subseteq M$.

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Let T and S_1 be as in Lemma 4.4 (ii). First suppose that $x \in T^\#$, in which case $P \cap C$ contains a subgroup D of type (p, p, p). Since $O_{p'}(C) = \langle C_{O_{p'}(C)}(u) | u \in D^\# \rangle$, it follows that $O_{p'}(C) \subseteq M$. Likewise, if R is a D-invariant S_p-subgroup of $O_{p',p}(O(C))$, then $N_{O(C)}(R) \subseteq M$. Since $O(C) = O_{p'}(O(C))N_{O(C)}(R)$ by the Frattini argument, we conclude that $O(C) \subseteq M$ in this case. Next, suppose that $x \in S_1$. Since S_1 is abelian, $T \subseteq C$, and hence $O(C) = \langle C_{O(C)}(t) | t \in T^\# \rangle$. Since $O(C_G(t)) \subseteq M$ for each t in $T^\#$, we conclude in this case as well that $O(C) \subseteq M$. Finally, let x be an arbitrary involution of S. Since $m(S_1) \geqq 3$ and $S_1 \lhd S$, $C_{S_1}(x)$ contains a 4-group T_1. However, $O(C_G(t_1)) \subseteq M$ for each t_1 in $T_1{}^\#$ as $T_1 \subset S_1$. Since $T_1 \subseteq C$, we conclude as in the preceding case that $O(C) \subseteq M$.

We next claim that $N = N_G(S) \subseteq M$. By Theorem 4.3 (ii) $Z(S)$ is non-cyclic, and hence $N = O(C_G(S))(N \cap H)$ by Theorem 4.2 (iii). Since $O(C_G(S)) \subseteq M$ by the preceding paragraph and since $H \subseteq M$, the desired conclusion $N \subseteq M$ follows.

Since M is a proper subgroup of G and G is simple, not every involution of G lies in M. We therefore conclude from the definition that M is strongly embedded in G.

Remark. In view of Theorems 3.7 and 4.5, to complete the proof of Theorem A, it will suffice to show that for some prime p in σ, G satisfies the uniqueness condition and if M is a corresponding uniqueness subgroup containing S, then S does not centralize $O_p(M)$. This we shall do in the next two sections.

5. The set of tame primes. In order to apply the results of (**7**), we shall now verify that G is σ-tame, as this term is defined in (**7**). We carry this out in a sequence of lemmas.

LEMMA 5.1. *If $p \in \sigma$, then every* p*-local subgroup of G is* p*-constrained.*

Proof. Assume the contrary and let $K = N_G(D)$ be a non-p-constrained p-local subgroup of G, where D is a non-trivial p-subgroup of G. Then, if E is an S_p-subgroup of $O_{p',p}(K)$, $C = C_K(E) \not\subseteq O_{p',p}(K)$. By Lemma 2.10, $E \subseteq O(K)$. Since $C_{S(K)}(E) \subseteq O_{p',p}(S(K)) \subseteq O_{p',p}(K)$ (**5**, Theorem 6.3.2), it follows at once that C is non-solvable. Moreover, $C \lhd N_K(E)$ and $N_K(E)$ contains an S_2-subgroup of K by the Frattini argument, as $E \subseteq O(K)$. We therefore conclude from Lemma 2.15 that C contains a 2-group T with the following properties: (a) $Z(T)$ is non-cyclic; (b) either T is an S_2-subgroup of K or T is a 4-group and an S_2-subgroup of K is dihedral or semidihedral; (c) T^C is non-solvable; and (d) $[u, T] = T$ for some element u of odd order in $N_C(T)$.

Let P be a maximal T-invariant p-subgroup of G containing E and set $Q = C_P(T)$. First suppose that $Q \subset P$. Then T does not centralize P, and thus T does not centralize $P_0 = C_P(Q)$ (**5**, Theorem 5.3.4). On the other hand, if $H = N_G(P)$, it follows from (d) and Theorem 4.2 (iii) that $[y, T] = T$ for some element y of odd order in H. Then y normalizes P_0, and consequently

$P_1 = [P_0, T]$ is non-cyclic. However, $D \subseteq E \subseteq Q$, whence $P_1 \subset K$. Lemma 2.12 now yields that T does not centralize $P_1 \cap O(K)$. However, $P_1 \cap O(K) \subseteq E$ as P_1 centralizes E and $C_{O(K)}(E) \subseteq E$, contrary to the fact that T centralizes E. Thus, $Q = P$, and therefore T centralizes P. In view of Theorem 4.2 (i), this implies that T centralizes every p-subgroup of G that it normalizes.

Next, suppose that T is not a 4-group. Let P_0 now denote a T-invariant subgroup of P containing E of maximal order such that T^{C_0} is non-solvable, where $C_0 = C_G(P_0)$. Set $N_0 = N_G(P_0)$, so that $C_0 \subseteq N_0$. Since $P_1 = N_P(P_0)$ is a T-invariant p-subgroup of N_0, $P_1 \subseteq O(N_0)$ by Lemma 2.7. Let R be a T-invariant S_p-subgroup of $O(N_0)$ containing P_1. Then T centralizes R. Since $O(N_0)C_0 = O(N_0)N_{O(N_0)C_0}(R)$ by the Frattini argument, we conclude at once that the normal closure of T in $C_{N_0}(R)$ is non-solvable. Since $P_1 \subseteq R$, it follows that T^{C_1} is non-solvable, where $C_1 = C_G(P_1)$. But then $P_1 = P_0$ by our maximal choice of P_0, and therefore $P = P_0$. Thus $N_0 = H$. However, H contains an S_2-subgroup of G by Theorem 4.2 (ii). Without loss we may assume that $T \subseteq S \subset H$. Since $C_0 = C_G(P)$ is non-solvable, Lemma 2.11 now yields that S centralizes P, contrary to the fact that $p \in \sigma$. Thus, T is a 4-group and an S_2-subgroup of K is dihedral or semidihedral.

We shall now contradict this last conclusion. Indeed, since T centralizes P and S does not, it follows from Theorem 4.3 (ii) that H is an I_3-group. But then Lemma 2.11 implies that $S_0 = C_S(P) \subseteq O_{2',2}(H)$. Since $H = O_{2',2}(H)N_H(S_0)$ by the Frattini argument, S_0 must contain a non-cyclic abelian subgroup of order 8. However, $S_0 \subset K$ as S_0 centralizes $D \subseteq P$, giving the desired contradiction.

LEMMA 5.2. *For any odd prime p, every proper subgroup of G is* p-*stable.*

Proof. Since every proper subgroup of G is an I_i-group, $0 \leqq i \leqq 3$, the lemma follows from Lemma 2.14.

LEMMA 5.3. *G is* p-*tame for all p in σ.*

Proof. We first argue that G is weakly p-tame. By Lemma 4.4, $SCN_3(p)$ is non-empty. Since G is p-constrained and is p-stable by the preceding two lemmas, we need only show that if K is a proper subgroup of G which contains an element A of $SCN_3(p)$ such that $A \cap S(K) = 1$, then every p'-subgroup of K normalized by A lies in $S(K)$. However, this is the case, by Lemma 2.13.

Let $\tilde{P}$ be an S_p-subgroup of G and let Q be a non-trivial $\tilde{P}$-invariant q-subgroup of G, q a prime, $q \neq p$. To prove that G is p-tame, we must show, in addition to the preceding results, that $\tilde{P} \cap N_G(Q) \neq 1$. Assume the contrary. Then as in the proof of (**4**, Proposition 6), we reduce to the case that Q is a maximal $\tilde{P}$-invariant subgroup, that $N = N_G(Q)$ is an I_1-group of characteristic p^m, that $SCN_3(\tilde{P})$ consists of a unique element A, and that for any subgroup R of $\tilde{P}$ containing A, N contains a conjugate of an S_2-subgroup of $N_G(R)$. Since an S_2-subgroup of N is dihedral or semidihedral, it will therefore suffice to show that R can be taken so that $N_G(R)$ contains an S_2-subgroup of G.

Let P be a maximal S-invariant subgroup of G and set $H = N_G(P)$. Without loss we may assume that $\tilde{P} \cap H$ is an S_p-subgroup of H. By the maximality of P, P is an S_p-subgroup of $O(H)$, and hence is an S_p-subgroup of $O_{p',p}(H)$ by Lemma 2.11. Since S does not centralize P, $C_H(P) \subseteq S(H)$ by the same lemma, and therefore $C_H(P) \subseteq O_{p',p}(H)$. Thus, H is p-constrained. Moreover, H is p-stable by Lemma 2.14. But then $A \subseteq P$ (**7**, Lemma 3.4), and therefore we can take P as R.

We now prove the following theorem.

THEOREM 5.4. *G is σ-tame.*

Proof. By the preceding lemma, we need only show that $p \sim q$ for p, q in σ. Assume the contrary and, for definiteness, suppose that $p > q$. Let P be a maximal S-invariant p-subgroup of G and set $H = N_G(P)$. By Lemma 4.4 there exists an elementary abelian subgroup S_1 of S with $m(S_1) \geqq 3$ such that $[y, S_1] = S_1$ for some element y of $N_H(S_1)$ of odd order and for some 4-subgroup T of S_1, $C_P(t)$ contains a subgroup of type (p, p, p) for each t in $T^{\#}$. But now reasoning exactly as in the proof of (**4**, Proposition 7), but with S_1 in place of the group S, we reduce to the case that S_1 does not centralize some S_1-invariant S_q-subgroup Q_1 of $O(H)$. But since $[y, S_1] = S_1$, Q_1 must contain a subgroup of type (q, q, q). Since Q_1 normalizes P, we conclude that $p \sim q$.

6. Elimination of the tame primes. We recall that a prime p in $\pi(G)$ for which $SCN_3(p)$ is non-empty is said to lie in π_3 or π_4 according as an S_p-subgroup of G normalizes some or no non-trivial p′-subgroup of G. We first prove the following theorem.

THEOREM 6.1. $\sigma \subseteq \pi_3$.

Proof. Assume the contrary. Since $SCN_3(p)$ is non-empty for p in σ by Lemma 5.3, we must have that $p \in \pi_4$ for some p in σ. By Lemmas 5.1 and 5.2, every p-local subgroup of G is p-stable and p-constrained. Moreover, if $\tilde{P}$ is an S_p-subgroup of G, then $\tilde{P}$ normalizes no non-trivial p′-subgroups of G. The proof of (**5**, Theorem 8.6.3) now shows that G satisfies the uniqueness condition for p and that $M = N_G(Z(J(\tilde{P})))$ is a uniqueness subgroup for p.

Since $\tilde{P} \subseteq M$ and $p \in \pi_4$, we have that $O_{p'}(M) = 1$, whence $O_{p',p}(M) = O_p(M)$. But then $C_M(O_p(M)) \subseteq O_p(M)$ as M is p-constrained. On the other hand, M is strongly embedded in G by Theorem 4.5, and therefore contains an S_2-subgroup of G, which without loss we may assume to be S. Then S does not centralize $O_p(M)$, contrary to Theorem 3.7.

We now prove the following theorem.

THEOREM 6.2. *For some prime p in σ and some S_p-subgroup $\tilde{P}$ of G, $\tilde{P}$ possesses a subgroup P^* with the following properties:*

(i) $SCN_3(P^*)$ *is non-empty;*

(ii) *S normalizes, but does not centralize, P^*;*

(iii) *P^* centralizes every $\tilde{P}$-invariant p′-subgroup of G.*

Proof. The proof of the theorem is essentially identical to that of (**4**, Lemmas 7.2 and 7.3), but with some simplifications. We shall give an outline of the argument. For the sake of clarity we conform closely to the notation of (**4**). First of all, since G is σ-tame by Theorem 5.4, (**7**, Theorem A) can be applied and yields the existence of a proper subgroup L of G which contains an S_p-subgroup of G for each p in σ such that if K is an S_σ-subgroup of $O(L)$, then $K \lhd L$ and $p \in \pi(K)$ for each p in σ. We show that the theorem holds for some prime in $\pi(F(K))$. To begin with, we let p be an arbitrary prime in $\pi(F(K))$. Let P be a maximal S-invariant p-subgroup of G and set $M_1 = N_G(P)$. Let $\tilde{P}$ be an S_p-subgroup of G such that $\tilde{P} \cap M_1$ is an S_p-subgroup of M_1. Without loss we may assume that $\tilde{P}$ is an S_p-subgroup of L.

Let $A \in SCN_3(\tilde{P})$, set $V = V(\text{ccl}_G(A); \tilde{P})$ and $N_1 = N_G(V)$. As in the final paragraph of the proof of Lemma 5.3, $A \subseteq P$, and consequently $M_1 \subseteq N_1$ (**7**, Lemma 5.4). By Theorem 4.3 (ii) and Lemma 2.5 (ii), $N_{M_1}(S)$ contains a cyclic subgroup R of odd order which does not centralize S. Then $[R, S] = S$ by Lemma 2.1 (i). We now conclude, as in the proof of (**4**, Lemma 7.2), that SR normalizes a non-trivial normal subgroup Z^* of $\tilde{P}$ with Z^* centralizing $O_{p'}(K)$.

Setting $M^* = N_G(Z^*)$, it then follows that $O_{p'}(K)$ is a Hall subgroup of $O(M^*)$ and we are able to reduce to the case that PSR normalizes $O_{p'}(K)$. Suppose first that $O_{p'}(K) \neq 1$ and set $H^* = N_G(O_{p'}(K))$. By Lemma 2.7 (i), $P \subseteq O(K)$. Since P is a maximal S-invariant p-subgroup of G, it follows that P is an S_p-subgroup of $O(H^*)$. However, $L \subseteq H^*$ as $O_{p'}(K) \lhd L$. Hence, by the Frattini argument, $O_{p'}(K)N_{H^*}(P)$ has the same properties as L and, in addition, contains SR. Thus, without loss we may assume, to begin with, that $SR \subset L$ if $O_{p'}(K) \neq 1$. On the other hand, if $O_{p'}(K) = 1$, then $L \subseteq N_1$ (**7**, Lemmas 5.4 and 5.5). If K_1 is an SR-invariant S_σ-subgroup of $O(N_1)$, then $N_{N_1}(K_1)$ has the same properties as L and contains SR. Hence, we may assume that $SR \subset L$ in this case as well.

Since $P \subseteq O(L)$ by Lemma 2.7 (i), $P \subseteq K$. If S centralizes $F(K)$, then so does $[P, S]$. However, $C_K(F(K)) \subseteq F(K)$ by a basic property of the Fitting subgroup of a solvable group (**5**, Theorem 6.1.3). Thus, $[P, S] \subseteq F(K)$; whence $[P, S, S] = 1$. But then S centralizes P (**5**, Theorem 5.3.6), which is not the case. Thus, S does not centralize $O_r(K)$ for some prime r. We can therefore also assume, to begin with, that S does not centralize $O_p(K)$.

We set $P^* = [S, O_p(K)]$ and argue that P^* centralizes every $\tilde{P}$-invariant p′-subgroup of G. As in (**4**, Lemma 7.3), P^* centralizes every $\tilde{P}$-invariant q-subgroup of G for q in $\sigma - \{p\}$; moreover, if Q is a maximal $\tilde{P}$-invariant q-subgroup of G for $q \notin \sigma$, some conjugate S^* of S with the property $P^* = [S^*, P^*]$ normalizes both P^* and Q. If q is odd, then S^* centralizes Q as $q \notin \sigma$, whence $P^* = [S^*, P^*]$ centralizes Q. On the other hand, if $q = 2$, then $Q \subseteq S^*$ as S^* is an S_2-subgroup of G; whence, $[Q, P^*] \subseteq Q \cap P^* = 1$, thus P^* centralizes Q in this case as well. We conclude at once that P^* centralizes every $\tilde{P}$-invariant p′-subgroup of G.

Finally, since P^* is SR-invariant and S does not centralize P^*, we have that $\mathrm{SCN}_3(P^*)$ is non-empty. Thus, all parts of the theorem hold.

On the basis of Theorem 6.2 we can now easily complete the proof of Theorem A. Let p, $\tilde{P}$, and P^* be as in the theorem and let B be the normal closure of P^* in $\tilde{P}$. If K is any $\tilde{P}$-invariant p′-subgroup of G, then P^* centralizes K, and hence $P^* \subseteq O_p(K\tilde{P})$ (**5**, Theorem 6.3.2). But then $B \subseteq O_p(K\tilde{P})$, whence $[K, B] \subseteq K \cap O_p(K\tilde{P}) = 1$. Thus, B centralizes every $\tilde{P}$-invariant p′-subgroup of G and B is a non-trivial normal subgroup of $\tilde{P}$. Since every p-local subgroup of G is both p-constrained and p-stable, the hypotheses of the Maximal Subgroup Theorem (**5**, Theorem 8.6.3) are therefore satisfied and we conclude that G satisfies the uniqueness condition for p.

Let M be the corresponding uniqueness subgroup containing $\tilde{P}$. Since $\mathrm{SCN}_3(P^*)$ is non-empty and $P^* \subseteq \tilde{P}$, $N_G(P^*) \subseteq M$. Since S normalizes P^*, we have that $S \subset M$. Set $Q = [S, P^*]$ and $F = F(O(M))$. Then $[S, Q] = Q \neq 1$ and $O_p(M) = O_p(F)$. Furthermore, S must centralize $O_p(M)$ by Theorems 3.7 and 4.5, whence also $Q = [S, Q]$ centralizes $O_p(M)$. However, Q also centralizes $O_{p'}(F)$ since $O_{p'}(F)$ is $\tilde{P}$-invariant and $Q \subseteq P^*$. Thus, Q centralizes $F = O_{p'}(F) \times O_p(F)$. On the other hand, $Q \subseteq O(M)$ by Lemma 2.7 (i). Since $C_{O(M)}(F) \subseteq F$, it follows that $Q \subseteq F$, whence $Q \subseteq O_p(M)$, contrary to the fact that S centralizes $O_p(M)$, but does not centralize Q.

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