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On the IAA version of the Doi–Edwards model versus the K-BKZ rheological model for polymer fluids: A global existence result for shear flows with small initial data†

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This paper establishes the existence of smooth solutions for the Doi–Edwards rheological model of viscoelastic polymer fluids in shear flows. The problem turns out to be formally equivalent to a K-BKZ equation but with constitutive functions spanning beyond the usual mathematical framework. We prove, for small enough initial data, that the solution remains in the domain of hyperbolicity of the equation for all $t \geqslant 0$.

Key words: Doi–Edwards polymer model; K-BKZ viscoelastic fluid; shear flows; convolution operator; evolutionary integro-differential equation

1 Introduction

Modelling industrial flows of non-Newtonian and viscoelastic fluids using molecular models and theories is nowadays of utter importance, see e.g. [1, 2, 4, 5, 7, 10, 11, 14–17, 21, 26, 28, 29, 32, 34]. However, this activity success hinges on users ability to cope with significant mathematical difficulties. This paper presents existence results for the shear flow of a Doi–Edwards/K-BKZ fluid, and before getting to the matter, for sake of clarity, we quickly introspect the key physical considerations laying at the model foundation.

In the realm of unfilled polymeric liquids, one distinguishes dilute systems from concentrated and melt ones and notices a physico-mathematical divide: constitutive laws and related mathematical technicalities are to a good extent different. When polymer concentration increases beyond a critical value, molecules overlap and excluded volume interactions, hydrodynamic interactions and entanglement interactions all strongly affect the molecular motion making calculations significantly complicated. At the present, there

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are two theories – which are reasonably supported by experiments – that account for the molecular dynamics, called the density (concentration) fluctuation theory and the kinetical theory. Although they describe different aspects of the dynamics, they are nonetheless seen as related to each other; however, the interrelation is far from being conspicuous and remains a matter of debate. The kinetical theory of Bird, Curtiss, Armstrong and Hassager [6] on one hand, and that of Doi and Edwards [20] (DE for the short) based on de Gennes' reptation diffusion [18] on the other, are considered of paramount importance. Here, we shall talk about the later.

In the DE model, a polymer chain – which consists of articulated segments – is seen as confined within a tube made-up of the surrounding chains and is free to wriggle out of it under imposed strain in a snake-like diffusion called reptation. As DE put it, this is not to say the tube vanished, but the protruding part moves into a newly formed tube segment while the portion that has been vacated on this occasion ceases to exist. Put it otherwise, the chain uses the degrees of freedom of its extremities to gradually change its shape by reptating. Under the influence of an external field, both the macromolecule and the tube are deformed in a cooperative manner. The chain configuration is actually determined by the new tube-like surrounding it just moved in, and is a many body problem of great complexity. DE first assumed that each chain segment deforms independently, hypothesis known as the Independent Alignment Approximation (IAA). The chain segment motion is seen as a concatenation of three processes which chronologically are: (a) an instantaneous deformation, followed by (b) a quick retraction to its original length, and eventually (c) undergoes a slow reptation motion out of the tube it initially occupied. Processes (a) and (b) affect the chain orientation without stretching it, while a relaxation occurs at stage (c). A more realistic view (albeit still an idealization of the real diffusion) is to consider (d) that the deformation (i.e. the flow) causes an extension of the segments at a rate proportional to the macroscopic velocity gradient. In this case, one needs to account for a retraction–extension mechanism by which the chain keeps its curvilinear length constant. When this is done, one gets what is commonly termed by physicists "the full non-linear DE model", in the sense that a more complicated diffusion equation results in a more complex stress tensor expression, hence the non-linear model.

Let $s \in (0, L)$ be the curvilinear coordinate along the chain, and $u = (u_1, u_2, u_3)$ a unitary vector belonging to the unit sphere S_2 that gives, for any s, the orientation of the chain segment.

At the heartcore of any kinetical model one finds a configurational probability diffusion equation the solution of which – here for convenience denoted F – is needed to obtain the stress tensor (or put it differently, to get the corresponding constitutive equation, CE). The solution F is a probability density with respect to the variable u, and solves ([13, 20, 35, 36]):

$$\begin{aligned}
&\frac{\partial F}{\partial t} + (V \cdot \nabla_X)F = D\frac{\partial^2 F}{\partial s^2} - \frac{\partial}{\partial u} \cdot [\mathscr{M}(t,u)F] + \epsilon F\kappa : uu - \epsilon\frac{\partial}{\partial s}\left[F\kappa : \lambda(F)\right], \\
&\text{for } (X,t,u,s) \in \widetilde{\Omega} \times (0,+\infty) \times S_2 \times (0,L) \\
&F(s=0) = F(s=L) = \frac{1}{4\pi};\ F(t=0) = F_0(X,s,u) \text{ a given function,} \qquad (1.1)
\end{aligned}$$

where $(X,t,u,s) \in \widetilde{\Omega} \times (0,+\infty) \times S_2 \times (0,L)$, and

$$\lambda(F)(X,t,s) = \int_0^s \left[\int_{S_2} F(X,t,u,s') uu \mathrm{d}\sigma \right] \mathrm{d}s' \tag{1.2}$$

and

$$\mathscr{M}(t,u) = \kappa \cdot u - (\kappa : uu)u. \tag{1.3}$$

In the above, $D > 0$ the diffusion coefficient, $X = (x_1, x_2, x_3) \in \widetilde{\Omega} \subset \mathbb{R}^3$ is the macroscopic Eulerian variable, t the time. $F = F(X,t,u,s)$ is the configurational probability, $\kappa = \nabla_X V$ the velocity gradient and $\epsilon \geqslant 0$ a model parameter. Einstein's usual summation convention over repeated indices applies wherever needed; classical tensor calculus notations are also used, e.g. uu stands for $u \otimes u$, etc. Note the presence of velocity V in equation (1.1). If one sets $\epsilon = 0$, the simplified diffusion equation is proper to the IAA version of the DE model, where chain segment extension–retraction mechanism is neglected. If $\epsilon > 0$, then all mechanisms from (a) to (d) are taken into consideration and one deals with a non-linear diffusion equation. A more detailed discussion on the physical meaning of the four terms in the r.h.s. of (1.2) is given e.g. in [13, 20, 35, 36].

The stress tensor $\tilde{\Sigma}$ reads [13, 35, 36]:

$$\tilde{\Sigma} = \alpha \left(H + \epsilon \widetilde{H} \right) \tag{1.4}$$

$$H(X,t) = \frac{1}{L} \int_{S_2} \int_0^L (uu - \delta)\, F(X,t,u,s) \mathrm{d}s \mathrm{d}\sigma \tag{1.5}$$

$$\widetilde{H}(X,t) = \frac{1}{L} \int_{S_2} \int_0^L uuF(t,u,s,X) \ln\left[4\pi F(X,t,u,s)\right] \mathrm{d}s \mathrm{d}\sigma, \tag{1.6}$$

where $\alpha > 0$ a physical constant; observe the stress tensor $\tilde{\Sigma}$ is made up of two parts H and $\widetilde{H}$ given in (1.5) and (1.6), δ being the unit tensor.

The fluid flow is governed by the classical momentum balance equation:

$$\begin{aligned} &\frac{\partial V}{\partial t} + (V \cdot \nabla_X)\, V = -\nabla p + \nabla_X \cdot \tilde{\Sigma} + \tilde{f}, \text{ in } \widetilde{\Omega} \\ &\nabla_X \cdot V = 0, \text{ in } \widetilde{\Omega} \end{aligned} \tag{1.7}$$

with p denoting the pressure. For the system of equations (1.1)–(1.7), energy and entropy estimates can be obtained by usual means. However, estimates on higher-order derivatives are necessary in order to prove the existence of a solution to the aforementioned system.

In [13], we proved the existence and uniqueness of solutions to the diffusion equation (1.1) assuming $V(t)$ is given and independent of X, using the Schauder fixed point theorem and the Galerkin's approximation method (for the existence and uniqueness of solutions of the corresponding stationary equation see [12]).

Compared to this manuscript framework, previously published papers on other polymer molecular models – e.g. [3, 27, 31, 32, 39] or [43] – deal with a different type of system of equations in that one notices the presence of Newtonian fluid related term, i.e. the Laplacian $\Delta_X V$, in the corresponding momentum balance equations. There, this term originates from physical considerations regarding the choice of a suitable constitutive law for the stress tensor $\tilde{\Sigma}$. Specifically, if one deals with a dilute polymeric fluid, then it is customary to assume that the total Cauchy extra-stress tensor is given by the sum between the solvent contribution which generally behaves like an ideal Newtonian fluid – hence the $\Delta_X V$ term – and a purely non-Newtonian or viscoelastic contribution. This assumption is in excellent agreement with experimental observations on dilute polymer liquids. However, in this work case, the fluid under scrutiny is solvent-free. Consequently, there is no Newtonian contribution to the stress tensor and we must deal with the additional difficulty that springs off from the absence of the otherwise convenient $\Delta_X V$ term.

In view of this difficulty, we choose here to solve the important problem of existence of a solution to (1.1)–(1.7) in the case of a simple shear flow for which $V(X,t) = (v(x_2,t), 0, 0)$, $F = F(x_2, t, u, s)$, $p = p(x_2, t)$, $\tilde{f} = (f(x_2, t), 0, 0))$, and taking $\epsilon = 0$ which corresponds to the IAA version of the model. For simplicity, from here on we denote x_2 by x: e.g. $v(x_2, t)$ reads $v(x, t)$.

The governing equations for the shear flow are given below:

$$\frac{\partial v}{\partial t} = \frac{\partial \tilde{\Sigma}_{12}}{\partial x} + f \tag{1.8}$$

$$\frac{\partial p}{\partial x} = \frac{\partial \tilde{\Sigma}_{22}}{\partial x} \tag{1.9}$$

$$\frac{\partial \tilde{\Sigma}_{32}}{\partial x} = 0, \tag{1.10}$$

where

$$\tilde{\Sigma}_{j2} = \frac{1}{L}\int_0^L \int_{S_2} u_2 u_j F \mathrm{d}u \mathrm{d}s - \frac{4\pi}{3}\delta_{j2},\ j = 1, 2, 3 \tag{1.11}$$

$$\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial s^2} - \frac{\partial v}{\partial x}\frac{\partial}{\partial u}\cdot(\mathcal{H}_0(u)F). \tag{1.12}$$

In the above, $\mathcal{H}_0(u) = M_0 \cdot u - (M_0 : uu)u$, with $(M_0)_{ij} = \delta_{i1}\delta_{2j}$.

To the system of equations (1.8)–(1.12), we assign the following boundary and initial conditions:

$$\begin{cases} v = 0, \text{ for } x \in \partial\Omega \\ v = v_0, \text{ for } t = 0 \\ F = \dfrac{1}{4\pi}, \text{ for } s = 0 \text{ or } s = L \\ F = F_0, \text{ for } t = 0, \end{cases} \tag{1.13}$$

where $\Omega \subset \mathbb{R}$ is the range for x, while $v_0(x)$ and $F_0(x, u, s)$ are initial data.

From [19], one sees the equation (1.12) for F can be solved allowing for the obtainment of $\tilde{\Sigma}$ as a function of the velocity gradient $\frac{\partial v}{\partial x}$. How this may be done is detailed in Appendix 1 (see Section A.1); here below we summarize – for sake of clarity – the aforementioned calculations and give the algebraic expression of $\tilde{\Sigma}_{j2}$, $j = 1, 2, 3$.

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The solution F is obtained via the method of characteristics which allows for arbitrary initial data. One gets $F(x,t,u,s)=\sum_{k=1}^{+\infty}F_k(x,t,u)\sin\left(\frac{k\pi}{L}s\right)$, wherever

$$F_k(x,t,u)=\frac{e^{-k^2\pi^2Dt/L^2}}{\left\|\chi\left(\int_0^t\frac{\partial v}{\partial x}(x,\lambda)\mathrm{d}\lambda,u\right)\right\|^3}F_{0k}\left(x,\frac{\chi\left(\int_0^t\frac{\partial v}{\partial x}(x,\lambda)\mathrm{d}\lambda,u\right)}{\left\|\chi\left(\int_0^t\frac{\partial v}{\partial x}(x,\lambda)\mathrm{d}\lambda,u\right)\right\|}\right)$$
$$+\frac{\pi}{4}\frac{D}{L^2}a_kk^2\int_0^t\frac{e^{-k^2\pi^2D(t-\tau)/L^2}}{\left\|\chi\left(\int_\tau^t\frac{\partial v}{\partial x}(x,\lambda)\mathrm{d}\lambda,u\right)\right\|^3}\mathrm{d}\tau \tag{1.14}$$

with

$$\chi(y,u)=(u_1-yu_2,u_2,u_3),\ \forall y\in\mathbb{R},\ \forall u=(u_1,u_2,u_3)\in S_2.$$

$F_{0k}(x,u)$ is such that

$$F_0(x,u,s)=\sum_{k=1}^{+\infty}F_{0k}(x,u)\sin\left(\frac{k\pi}{L}s\right) \tag{1.15}$$

$$1=\sum_{k=1}^{+\infty}a_k\sin\left(\frac{k\pi}{L}s\right),\ a_k\in\mathbb{R}. \tag{1.16}$$

Function F allows for calculating $\tilde{\Sigma}_{j2},\ j=1,2,3$ in (1.11) as a function of the gradient $\frac{\partial v}{\partial x}(x,t)$. Observe that (1.8) now becomes the main equation, of unknown v. Assuming it has a solution v – actually the goal of the present paper is to prove the existence of such a solution – one gets the pressure field p from (1.9).

Next, one sees that for any $k\in\mathbb{N}^*$ function $F_{0k}(x,u_1,u_2,u_3)$ – as a function of u_3 – depends on u_3^2, and so does F_k for any k as well. Consequently, $\tilde{\Sigma}_{32}=0$ and (1.10) always holds true.

In this work, we shall assume that

$$F_0=\frac{1}{4\pi} \tag{1.17}$$

which amounts to assuming $F_{0k}=\frac{a_k}{4\pi},\ \forall k\in\mathbb{N}^*$. Use (1.17) into (1.14), and since $a_k=\frac{2}{L}\int_0^L\sin(\frac{k\pi}{L}s)\mathrm{d}s=\frac{4}{k\pi}$ for any odd k and $a_k=0$ for any even k, one formally gets

$$F(x,t,u,s)=\frac{\tilde{a}(t,s)}{\left\|\chi\left(\int_0^t\frac{\partial v}{\partial x}(x,\lambda)\mathrm{d}\lambda,u\right)\right\|^3}-\int_0^t\frac{\frac{\partial\tilde{a}}{\partial t}(t-\tau,s)}{\left\|\chi\left(\int_\tau^t\frac{\partial v}{\partial x}(x,\lambda)\mathrm{d}\lambda,u\right)\right\|^3}\mathrm{d}\tau, \tag{1.18}$$

where

$$\tilde{a}(t,s)=\frac{1}{\pi^2}\sum_{p=1}^{+\infty}\frac{1}{2p+1}e^{-D(2p+1)^2\pi^2t/L^2}\sin\left(\frac{(2p+1)\pi}{L}s\right),\ \forall t\geqslant 0,\ s\in(0,L).$$

The assumption (1.17) is critical to ensure a K-BKZ type of equation in v is obtained (see equation (1.23)); altering it would lead to a different kind of problem hence a different mathematical setting.

Posing $v(x,\tau) = 0$ for $\tau < 0$ leads to

$$\int_{-\infty}^{0} \frac{\frac{\partial \tilde{a}}{\partial t}(t-s,s)}{\left\| \chi\left(\int_\tau^t \frac{\partial v}{\partial x}(x,\lambda)\mathrm{d}\lambda, u\right)\right\|^3}\mathrm{d}\tau = -\frac{\tilde{a}(s,t)}{\left\| \chi\left(\int_0^t \frac{\partial v}{\partial x}(x,\lambda)\mathrm{d}\lambda, u\right)\right\|^3} \tag{1.19}$$

and with help of (1.18),

$$F(x,t,u,s) = -\int_{-\infty}^{t} \frac{\frac{\partial \tilde{a}}{\partial t}(t-s,s)}{\left\| \chi\left(\int_\tau^t \frac{\partial v}{\partial x}(x,\lambda)\mathrm{d}\lambda, u\right)\right\|^3}\mathrm{d}\tau \tag{1.20}$$

which is equation (3.10) of [19] (reference in which the result is obtained via a Green's function technique), unanimously used when the IAA version theory is used. As an aside, we mention that in the rheology literature one commonly states that the "configurational probability density must vanish in the sufficiently distant past" i.e. $F_0 = 0$ for $t = -\infty$. But this gives $F_0 = \frac{1}{4\pi}$ for $t = 0$, hence (1.17). However, taking $F_0 \neq \frac{1}{4\pi}$ leads to a different mathematical problem, unrelated to the fluid under scrutiny here.

Moreover, a straightforward calculation shows that

$$\frac{1}{L}\int_0^L \tilde{a}(t,s)\mathrm{d}s = \frac{2}{\pi^3} a_{\mathrm{DE}}(t),$$

where the Doi–Edwards kernel (a.k.a. the Doi–Edwards relaxation function) a_{DE} is given by

$$a_{\mathrm{DE}}(t) = \sum_{p=1}^{+\infty} \frac{1}{(2p+1)^2} e^{-D(2p+1)^2\pi^2 t/L^2}, \ \forall t \geqslant 0.$$

Letting now $g_{\mathrm{DE}} : \mathbb{R} \to \mathbb{R}$,

$$g_{\mathrm{DE}}(y) = -\frac{2\alpha}{\pi^3}\int_{S_2} \frac{u_1 u_2}{\left[(u_1 - u_2 y)^2 + u_2^2 + u_3^2\right]^{3/2}}\mathrm{d}u, \ \forall y \in \mathbb{R}, \tag{1.21}$$

we obtain

$$\tilde{\Sigma}_{12} = -g_{\mathrm{DE}}\left(\int_0^t \frac{\partial v}{\partial x}(x,\tau)\mathrm{d}\tau\right) a_{\mathrm{DE}}(t) + \int_0^t g_{\mathrm{DE}}\left(\int_\tau^t \frac{\partial v}{\partial x}(x,r)\mathrm{d}r\right) a'_{\mathrm{DE}}(t-\tau)\mathrm{d}\tau. \tag{1.22}$$

Before proceeding further, we make at this stage a final remark: had one chosen $F_0 \neq \frac{1}{4\pi}$, then equation (1.23) would have contained an additional term of the form $\frac{\partial \varphi}{\partial x}(t, \int_0^t \frac{\partial v}{\partial x}(x,r)\mathrm{d}r)$, where φ is a "small" function whenever $(F_0 - \frac{1}{4\pi})$ is "small", resulting in a different mathematical problem.

From the above considerations, one infers the shear flow problem under scrutiny is tantamount to solving for v the below integro-differential equation:

$$\frac{\partial v}{\partial t} = -\frac{\partial}{\partial x} g_{DE}\left(\int_0^t \frac{\partial v}{\partial x}(x,\tau)\mathrm{d}\tau\right) a_{DE}(t) + \frac{\partial}{\partial x}\int_0^t g_{DE}\left(\int_\tau^t \frac{\partial v}{\partial x}(x,r)\mathrm{d}r\right) a'_{DE}(t-\tau)\mathrm{d}\tau + f,\ t>0. \tag{1.23}$$

Now equation (1.23) – here obtained on molecular dynamics grounds – has been focused on within the area of viscoelastic fluids as it appears when one studies shear flows for the K-BKZ fluids. There is no contingency here as in their 1978 original paper [19], DE have shown the simplified IAA version of their non-linear model actually enters the class of K-BKZ integral models, which are based on continuum mechanics concepts (for more on see [6, 33, 39]). Consequently, when undertaking the study of certain particular flows of the IAA-version of DE fluids, one may capitalize on previously obtained results for K-BKZ liquids.

In this paper, we study equation (1.23) with more general functions g and a replacing g_{DE} and a_{DE}, respectively. We prove a global in time solution existence result for small enough data. Equation (1.23) – as well as variants of it – was studied by various authors, see Renardy, Hrusa and Nohel [38], Engler [22], Brandon and Hrusa [8] and references cited therein.

The existence of local in time solutions [38] and of global solutions [8, 22] are known under more restrictive conditions compared to those stated in this paper. One of the assumptions in [22] and [8] is $g'(y) < -\gamma$, for any $y \in \mathbb{R}$, with $\gamma > 0$, which is not verified by the function $g = g_{DE}$. This is a consequence of the fact that $g_{DE}(0) = 0$, $g'_{DE}(0) < 0$ and $\lim_{y\to+\infty} g_{DE}(y) = 0$. Here, we make use of the less restrictive assumption $g'(y) < 0$, for any $y \in [-\theta, \theta]$, with $\theta > 0$ – assumption verified by g_{DE} – and show that the argument of g' is confined to $[-\theta, \theta]$. The requirement $g' < 0$ in a neighbourhood of 0 is a necessary hyperbolicity condition for the solution local existence. For the work presented in this paper, this condition being valid only locally makes it necessary to control, w.r.t. time t, the argument $\int_0^t \frac{\partial v}{\partial x}(x,\tau)\mathrm{d}\tau$ of g'. Observe that at a first sight, this argument may become large with increasing t; we obtain estimates for this term using the maximal function concept. Notice that in [22], the problem is studied in n-dimensional framework but in a weak functional setting which is unsuitable for our problem because the condition $g'(y) < -\gamma$, $\forall y \in \mathbb{R}$, does not hold in our case.

Next, among the restrictive hypotheses invoked by the authors of [8] for function a is that $a'' \in L^1(0, +\infty)$, which $a = a_{DE}$ does not verify. Comparatively, here we shall place significantly less restrictions on a and accordingly will construct a class of totally monotone functions, an element of which is $a = a_{DE}$. Moreover, we prove and make use of a new inversion formula for the operator $u \mapsto a * u$, a technique different from the (classical) one used in [8].

The manuscript is organized as following:

In Section 2, we introduce the problem and enunciate the main result.

Section 3 is devoted to the proof of several necessary results such as a Gårding type inequality and an inversion formula for the operator $u \mapsto a * u$.

In Section 4, we introduce an approximated problem and obtain useful estimates for its solution. The proof of the main result is achieved in Section 5.

In the ending Section 6, we construct a class of totally monotone functions that is compatible with the hypothesis made about a.

2 Presentation of the problem, of the main result, and the proof strategy

Let from now on $\Omega \subset \mathbb{R}$ be a bounded, open interval. Let the functions $f : \Omega \times [0, +\infty) \to \mathbb{R}$, $g : I \subset \mathbb{R} \to \mathbb{R}$, with $I \ni 0$ an open interval, $v_0 : \Omega \to \mathbb{R}$, $a : [0, +\infty) \to \mathbb{R}$.

The aim is to search for a solution $v : \Omega \times [0, +\infty) \to \mathbb{R}$ to the below given initial boundary value problem:

$$v_t(x,t) = -a(t)\frac{\partial}{\partial x} g\left(\int_0^t v_x(x,s)\mathrm{d}s\right) + \frac{\partial}{\partial x}\int_0^t g\left(\int_s^t v_x(x,\tau)\mathrm{d}\tau\right) a'(t-s)\mathrm{d}s + f(x,t) \quad (2.1)$$

$$v(x, t=0) = v_0(x),\ \forall x \in \Omega, \text{ and } v(x,t) = 0,\ \forall t < 0 \quad (2.2)$$

$$v = 0,\ \forall x \in \partial\Omega, \forall t \geqslant 0. \quad (2.3)$$

In the above, $v_x \equiv \frac{\partial v}{\partial x}$ and a' stands for the derivative of a. Throughout this paper, any function defined for $t \geqslant 0$ is understood as being set equal to 0 for $t < 0$, i.e. it has domain $\mathbb{R}$. Moreover, for a function $\varphi \in W^{k,1}(0, +\infty)$ we denote by $\varphi^{(k)}$ the distributional derivative of φ on $\mathbb{R}_+^*$, derivative which is understood to be extended to $\mathbb{R}$ by 0. Define

$$\bar{v}^t(x,s) := \int_{t-s}^t v(x,\tau)\mathrm{d}\tau,\ 0 \leqslant s, t;\ x \in \Omega.$$

Equation (2.1) now takes on a simpler form:

$$v_t(x,t) = \int_0^{+\infty} a'(s)\frac{\partial}{\partial x} g\left(\bar{v}_x^t(x,s)\right)\mathrm{d}s + f(x,t). \quad (2.4)$$

Drawing inspiration from [8], (2.4) can be re-written as

$$v_t(x,t) + g'(0)\int_0^t a(t-s)v_{xx}(x,s)\mathrm{d}s = f(x,t) + \mathcal{G}(x,t), \quad (2.5)$$

where

$$\begin{aligned}\mathcal{G}(x,t) &= \int_0^{+\infty} a'(s)\left[g'\left(\bar{v}_x^t(x,s)\right) - g'(0)\right]\bar{v}_{xx}^t(x,s)\mathrm{d}s \\ &= \int_0^t v_{xx}(x,s)\int_{t-s}^{+\infty} a'(\tau)\left[g'\left(\bar{v}_x^t(x,\tau)\right) - g'(0)\right]\mathrm{d}\tau\mathrm{d}s. \quad (2.6)\end{aligned}$$

Convolution with respect to t is denoted as usually by $*$; therefore (2.5) can be re-written in a more close form as

$$v_t + g'(0)a * v_{xx} = f + \mathcal{G}. \quad (2.7)$$

We now proceed to presenting several constitutive assumptions. The function g is taken such that

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(g_1) there exist $\theta \in [0,1]$ and $K > 0$, such that $g \in \mathscr{C}^3([-\theta,\theta],\mathbb{R})$ and $\left|g^{(3)}(y) - g^{(3)}(0)\right| \leqslant K|y|$, $\forall y \in [-\theta,\theta]$

(g_2) $g(0) = g''(0) = 0$

(g_3) $g'(0) < 0$.

The function f is such that

(f_1) $f, f_x, f_t \in \mathscr{C}_b^0\left([0,+\infty); L^2(\Omega)\right) \cap L^2\left([0,+\infty); L^2(\Omega)\right)$,

(f_2) $f_{tt} \in L^2\left([0,+\infty); L^2(\Omega)\right)$, $\int_0^t f(x,s)\mathrm{d}s \in \mathscr{C}_b^0\left([0,+\infty); H^1(\Omega)\right)$,

where $\mathscr{C}_b^0([0,+\infty); X)$ is the set of all functions $w : [0,+\infty) \to X$ which are bounded and continuous, and X is a Banach space.

Next, let v_0 be such that

$(v_0)_1$ $v_0 \in H^2(\Omega)$.

We assume that f and v_0 are compatible with the already stated initial-boundary conditions:

$$v_0(x) = f(x, t=0) = 0, \ \forall x \in \partial\Omega. \tag{2.8}$$

Let the measures associated to f and v_0 be defined as

$$F(f) := \sup_{t\geqslant 0} \int_\Omega \left[f^2 + f_x^2 + f_t^2 + \left(\int_0^t f(x,s)\mathrm{d}s\right)^2 + \left(\int_0^t f_x(x,s)\mathrm{d}s\right)^2\right]\mathrm{d}x \tag{2.9}$$

$$+ \int_0^{+\infty} \int_\Omega \left(f^2 + f_x^2 + f_t^2 + f_{tt}^2\right)(x,t)\mathrm{d}x\mathrm{d}t \tag{2.10}$$

$$V_0(v_0) = \|v_0\|^2_{H^2(\Omega)} = \int_\Omega \left[v_0^2 + (v_0')^2 + (v_0'')^2\right](x)\mathrm{d}x. \tag{2.11}$$

For any function $\varphi \in L^1((0,+\infty))$, we denote by $\mathcal{F}\varphi$ (or alternatively by $\hat{\varphi}$) and $\mathcal{L}\varphi$ the corresponding Fourier and Laplace transforms, i.e.

$$\mathcal{F}\varphi(\omega) := \int_0^{+\infty} \varphi(t)e^{-i\omega t}\mathrm{d}t, \ \forall \omega \in \mathbb{R}$$

$$\mathcal{L}\varphi(z) := \int_0^{+\infty} \varphi(t)e^{-zt}\mathrm{d}t, \ \forall z \in \mathbb{C}, \mathrm{Re}z \geqslant 0.$$

Let us now assume the function a satisfies the below given hypotheses (a_1)–(a_5):

(a_1) $a \in W^{1,1}(0,+\infty)$, $a'(t) \leqslant 0$ a.e. $t \geqslant 0$.

There exists a sequence of functions $(a_n)_{n\in\mathbb{N}}$, $a_n \in \mathscr{C}^2\left([0,+\infty) \cap W^{2,\infty}([0,+\infty)\right)$ s.t.

(a_2) $a_n'(t) \leqslant 0$, $\forall t \geqslant 0$, $(a_n)_{n\in\mathbb{N}}$ bounded in $W^{1,1}(0,+\infty)$ and $a_n \xrightarrow[n\to+\infty]{\mathscr{D}'(0,+\infty)} a$,

(a_3) $\sup_{n\in\mathbb{N}}\left[\int_0^1 t\,|a_n''(t)|\,\mathrm{d}t + \int_1^{+\infty} \sqrt{t}\,|a_n''(t)|\,\mathrm{d}t + \int_1^{+\infty} t^2\,|a_n'(t)|\,\mathrm{d}t\right] < +\infty$,

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(a_4) there exist constants $M_1 > 0$ and $n_0 \in \mathbb{N}$ s.t. $\mathrm{Re}(\mathcal{F}a_n(\omega)) \geqslant \frac{M_1}{1+\omega^2}$, $\forall n \in \mathbb{N}$, $n \geqslant n_0$, $\forall \omega \in \mathbb{R}$; observe that this is a strong positivity condition, common for this type of problems (see [8]).

(a_5) there exist constants $M_2 > 0$, $n_0 \in \mathbb{N}$, $p \in \mathbb{N}^*$, $p \geqslant 2$, s.t. $\frac{[\mathcal{F}(a_n')]^p}{\mathcal{F}a_n} \in \mathcal{F}\left(B_{L^1(\mathbb{R})}(0, M_2)\right)$, $\forall n \in \mathbb{N}$, $n \geqslant n_0$, where $B_{L^1(\mathbb{R})}(0, M_2)$ denotes the ball in $L^1(\mathbb{R})$ centered at 0 and of radius M_2; this assumption will be used to obtain a representation for the solution u of $a_n * u = b$ (see Theorem 3.1).

Remark 2.1 *In Section 6, we shall construct a class of functions compliant with assumptions* (a_1) *to* (a_5). *This class contains the Doi–Edwards relaxation kernel* $a_{DE} : [0, +\infty) \to \mathbb{R}$,

$$a_{DE}(t) = \sum_{k \geqslant 1} \frac{1}{(2k+1)^2} e^{-(2k+1)^2 \pi^2 D t / L^2}. \tag{2.12}$$

Also, since $g_{DE} \in \mathscr{C}^\infty(\mathbb{R})$ *is an odd function and* $g_{DE}'(0) = -\frac{6\alpha}{\pi^3} \int_{S_2} u_1^2 u_2^2 \mathrm{d}u < 0$, *then* g_{DE} *also verifies* (g_1)–(g_3) *and this paper results equally apply to the function* g_{DE}.

The main result of this paper is stated below:

Theorem 2.1 (*Main result*) *Assume that the hypotheses on the data given in* (g_1)–(g_3), (f_1)–(f_2), $(v_0)_1$, (a_1)–(a_5) *and* (2.8) *hold true. Then, there exists a* $\delta > 0$ *such that, if the additional smallness assumption* $F(f) + V_0(v_0) \leqslant \delta$ *is verified, then there exists at least a solution*

$$v \in \left\{ \bigcap_{m=0}^{2} W^{m,\infty}\left((0, +\infty); H^{2-m}(\Omega)\right) \right\} \cap \left\{ \bigcap_{m=0}^{2} W^{m,2}\left((0, +\infty); H^{2-m}(\Omega)\right) \right\}$$

with

$$\int_0^t v(x, s) \mathrm{d}s \in L^\infty\left((0, +\infty); H^3(\Omega)\right)$$

to the problem (2.4), (2.2), (2.3).

Next, we take on to introducing – and explaining – the proof stages for the aforementioned Theorem 2.1. In short, first we obtain a regularized problem (P_n) obtained from (2.5) with a being replaced by a sequence a_n satisfying hypotheses (a_1) to (a_5). Doing this allows to obtain a local in time existence and uniqueness result capitalizing on Renardy's result in [38]. Next goal is to obtain estimates independent of n granting the global existence of the solution for the approximated problem (P_n) and in the end, letting $n \to +\infty$, obtaining our result. How to get these estimates is explained below.

Let $u(x, t) = \int_0^t v(x, \tau) \mathrm{d}\tau$. For any $t > 0$, let $\mathcal{E}(t)$ stand for the sum of squared $L_t^\infty L_x^2$ norms of all derivatives in x and t of u up to third order and of all squared $L_t^2 L_x^2$ norms of all derivatives in x and t of v up to second order (see (4.4)). We prove that if $\mathcal{E}(t)$ is "small" for t close to 0 (a consequence of the assumption made on data v_0 and f), then $\mathcal{E}(t)$ stays "small" for any t. We do this by obtaining an inequality of the type

$$\mathcal{E}(t) \leqslant \frac{1}{2}\mathcal{E}(t) + \text{"small enough" quantities depending uniquely on } V_0 \text{ and } F. \tag{2.13}$$

To achieve that, we calculate three energy estimates (in a way similar in nature with that of Brandon and Hrusa [8]: we derivate (2.7) i-times (with $i \in \{0,1,2\}$) w.r.t. time t , then multiply the result by $\frac{d^i v}{dt^i}$ and integrate on $Q_t := \Omega \times (0,t)$. To calculate the second order derivative, one uses a finite difference operator $\triangle_h w(t) = w(t+h) - w(t)$, see (3.2). We sum up the resulting three equations and get an equality in which the most important term originates from the convolution part in the *l.h.s.* of (2.7). This term reads

$$-g'(0)\left[Q\left(v_x,t,a\right)+Q\left(v_{xt},t,a\right)+Q\left(v_{xtt},t,a\right)\right], \tag{2.14}$$

where $Q(w,t,a) = \int_0^t \int_\Omega w(x,s)\left(a * w\right)(x,s)\mathrm{d}x\mathrm{d}s$ (see (3.1)). We lower bound (2.14) using a Gårding type inequality: see Lemma 3.1.

The terms denoted by $\mathcal{G}$ in (2.7) can be controlled w.r.t. well-chosen norms by carrying out an integration by parts w.r.t. time t, switching the time derivatives onto a and using the fact that $ta'' \in L^1(0,1)$ (see assumption (a_3)). This allows to upper bound the $L^\infty_t L^2_x$ norms of v, v_x, v_t, v_{xt}, v_{tt}, and the $L^2_t L^2_x$ norms of v, v_x, v_t, v_{xt}. The results are gathered into $\mathcal{E}_1$, see (4.6). We point out that the aforementioned energy estimates do not provide norm estimates for v_{xx}. To cope with this difficulty, we use (2.7) which allows to express v_{xx} as a function of v_t, f and $\mathcal{G}$ with the help of an inversion Theorem for the operator $w \mapsto a * w$ and using the previously obtained estimates. We cannot use the resolvent kernel technique like in Brandon and Hrusa [9] because in this paper case $r' \notin L^1(\mathbb{R})$ (as $a'' \notin L^1(\mathbb{R}_+)$). Because of that we prove a point-wise inversion Theorem for the convolution of a assuming the rather pretty weak constraints (a_1)–(a_5) on a: see Theorem 3.1.

3 Preliminaries

For any $T > 0$, $w \in \mathscr{C}^0\left([0,T];L^2(\Omega)\right)$, $b \in L^1(0,+\infty)$ and $t \in [0,T]$, we define

$$\begin{aligned} Q(w,t,b) &:= \int_0^t \int_\Omega w(x,s) \int_0^s b(s-\tau)w(x,\tau)\mathrm{d}\tau\mathrm{d}x\mathrm{d}s \\ &= \int_0^t \int_\Omega w(x,s)(b*w)(x,s)\mathrm{d}x\mathrm{d}s, \end{aligned} \tag{3.1}$$

where w is considered as extended by 0 on $(T,+\infty)$. For any $T > 0$ and $h \in (0,T)$, we define the finite difference operator $\varDelta_h$

$$\left(\varDelta_h w\right)(x,t) = w(x,t+h) - w(x,t) \tag{3.2}$$

as a linear operator from $\mathscr{C}^0\left([0,T-h];L^2(\Omega)\right)$ onto $\mathscr{C}^0\left([0,T];L^2(\Omega)\right)$.

Moreover, if $X(J)$ denotes a space of functions defined on $J \subset \mathbb{R}$ and $I \subset J$, then $X_I(J)$ stands for the subspace of functions $X(J)$ the supports of which are included in I (i.e. that vanish on $J - I$).

Recall that $b \in L^1\left(\mathbb{R}_+\right)$ is of positive type if, for any $t \geqslant 0$ and any $\varphi \in L^2\left(\mathbb{R}_+\right)$, it satisfies $\int_0^t \varphi(s) \int_0^s b(s-\tau)\varphi(\tau)\mathrm{d}\tau\mathrm{d}s \geqslant 0$. Next, b is said to be of strong positive type if there exists $\epsilon > 0$ s.t. the function $b(t) - \epsilon e^{-t}$ is of positive type. Moreover, $Q_t := \Omega \times (0,t)$.

The following Lemma is a Gårding type inequality with boundary terms. It is proved in [9] using preliminary results due to Staffans [40] (see also [24] and [42]). Here, we shorten the original proof of [9] and remove the extraneous assumptions $b \in W^{3,1}(0,+\infty)$, $b'' \geqslant 0$.

Lemma 3.1 *Assume $b \in L^1_{\mathbb{R}_+}(\mathbb{R})$ is such that $Re(\mathcal{F}b(\omega)) \geqslant \frac{M_1}{1+\omega^2}$, for any $\omega \in \mathbb{R}$, where $M_1 > 0$. Then, for any $T > 0$, $w \in \mathscr{C}^1\left([0,T], L^2(\Omega)\right)$ and $t \in [0,T)$, we have*

$$\int_\Omega w^2(x,t)\mathrm{d}x + \int_0^t \int_\Omega w^2(x,s)\mathrm{d}x\mathrm{d}s \leqslant C\left[\frac{1}{M_1}Q(w,t,b) + \frac{1}{M_1}Q(w_t,t,b) + \int_\Omega w^2(x,0)\mathrm{d}x\right] \tag{3.3}$$

with $C > 0$ independent of T, t, w and b.

Moreover, if $w \in \mathscr{C}^0\left([0,T], L^2(\Omega)\right)$, then, for any $t \in [0,T]$,

$$\int_\Omega w^2(x,t)\mathrm{d}x + \int_0^t \int_\Omega w^2(x,s)\mathrm{d}x\mathrm{d}s \leqslant C\left[\frac{1}{M_1}Q(w,t,b) + \frac{1}{M_1}\liminf_{h\to 0_+}\frac{1}{h^2}Q(\triangle_h w,t,b) + \int_\Omega w^2(x,0)\mathrm{d}x\right]. \tag{3.4}$$

Proof Assuming that inequality (3.3) holds true, we undertake to proving (3.4). Let $w \in \mathscr{C}^0\left([0,T], L^2(\Omega)\right)$ and $t \in [0,T)$ be fixed. For $0 < h < (T-t)/2$, define the function $w_h \in \mathscr{C}^1\left(\left[0,(t+T)/2\right], L^2(\Omega)\right)$ by

$$w_h(s) := \frac{1}{h}\int_s^{s+h} w(\sigma)\mathrm{d}\sigma,\ s \in \left[0,(t+T)/2\right). \tag{3.5}$$

Applying (3.3) to w_h and passing to the limit $\liminf_{h\to 0_+}$ gives (3.4).

We now prove (3.3). Let $w \in \mathscr{C}^1\left([0,t], L^2(\Omega)\right)$, $t \in [0,T)$ be fixed, and let $\tilde{w} \in L^2_{[0,t)}\left(\mathbb{R}, L^2(\Omega)\right)$ be defined by $\tilde{w} = w$ a.e. in $[0,t]$ and $\tilde{w} = 0$ outside. Denote by $D\tilde{w}$ the distributional derivative of $\tilde{w}$ and by $\tilde{w}'$ its regular part, i.e.

$$D\tilde{w} = \tilde{w}' + w(0)\delta_0 - w(t)\delta_t. \tag{3.6}$$

Due to the Parseval identity, we have

$$Q(w,t,b) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_\Omega \mathrm{Re}\left(\hat{b}(\tau)\right)\left|\widehat{\tilde{w}}(x,\tau)\right|^2 \mathrm{d}x\mathrm{d}\tau \tag{3.7}$$

and a similar equation with w' instead of w as well. For $\lambda > 0$ (to be later determined) define $I(w)$ by

$$I(w) := Q\left(\tilde{w}',t,b\right) + \lambda Q\left(\tilde{w},t,b\right) + \frac{3M_1}{2}\int_\Omega w^2(x,0)\mathrm{d}x \tag{3.8}$$

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By (3.6) and (3.7) and the strong positivity of b,

$$I(w) \geqslant \frac{M_1}{2\pi} \int_{\mathbb{R}} \int_{\Omega} \left(\left| i\tau\widehat{w}(\tau) + w(t)e^{-i\tau t} - w(0) \right|^2 + \lambda \left| \widehat{w}(\tau) \right|^2 + 3|w(0)|^2 \right) \mathrm{d}x \frac{\mathrm{d}\tau}{1+\tau^2}. \quad (3.9)$$

Since for any $(a,b,c) \in \mathbb{C}^3$, we have $|a+b+c|^2 \geqslant \frac{|a|^2+|b|^2}{2} - 2|a||b| - 3|c|^2$, inequality (3.9) implies

$$I(w) \geqslant \frac{M_1}{2\pi} \int_{\mathbb{R}} \int_{\Omega} \left(\frac{|\tau|^2 + 2\lambda}{2} \left| \widehat{w}(\tau) \right|^2 + |w(t)|^2 \beta \sqrt{|\tau|} - 2|w(t)||\tau| \left| \widehat{w}(\tau) \right| \right) \mathrm{d}x \frac{\mathrm{d}\tau}{1+\tau^2} \quad (3.10)$$

with

$$\beta = \frac{1}{2} \left(\int_{\mathbb{R}} \frac{\mathrm{d}\tau}{1+\tau^2} \right) \Big/ \left(\int_{\mathbb{R}} \frac{\sqrt{|\tau|}}{1+\tau^2} \mathrm{d}\tau \right). \quad (3.11)$$

But

$$\begin{aligned} 2|w(t)||\tau| \left| \widehat{w}(\tau) \right| &\leqslant \frac{\beta}{2} \sqrt{|\tau|} |w(t)|^2 + \frac{2}{\beta} |\tau|^{3/2} \left| \widehat{w}(\tau) \right|^2 \\ &\leqslant \frac{\beta}{2} \sqrt{|\tau|} |w(t)|^2 + \left(\frac{|\tau|^2}{4} + L \right) \left| \widehat{w}(\tau) \right|^2 \end{aligned} \quad (3.12)$$

with $L > 0$ independent of t, w, b. Choose $\lambda = L + 1/4$. By (3.10) and (3.12), we get

$$I(w) \geqslant \frac{M_1}{2\pi} \int_{\mathbb{R}} \int_{\Omega} \left(\frac{|\tau|^2 + 1}{4} \left| \widehat{w}(\tau) \right|^2 + \frac{\beta\sqrt{|\tau|}}{2} |w(t)|^2 \right) \mathrm{d}x \frac{\mathrm{d}\tau}{1+\tau^2} \quad (3.13)$$

which is (3.3).

□

We now prove that, under suitable assumptions application $w \mapsto b * w$ is invertible, and obtain an inversion formula. We use truncated Neumann series and a special assumption (see (b_3) below) in order to control the remainder term.

For $b \in L^1(\mathbb{R})$, let the k-times convolution de denoted as $b^{*k} := \underbrace{b * b * \cdots * b}_{k \text{ times}}$. For $1 \leqslant q \leqslant +\infty$ and $t_0 \in (0, +\infty]$, the mapping $\mathcal{R}_{t_0,q}$ is defined by

$$\mathcal{R}_{t_0,q} : \begin{cases} L^q_{[0,t_0)}(-\infty, t_0) \longrightarrow W^{1,q}_{[0,t_0)}(-\infty, t_0). \\ w \qquad\qquad\qquad \mapsto b * w \end{cases}$$

Here, $b * w(t) := \int_0^t b(t-s)w(s)\mathrm{d}s$, for any $t < t_0$. We always write $\mathcal{R}$ in place of $\mathcal{R}_{+\infty,2}$.

Next, function b is assumed to comply with

(b_1) $b \in W^{1,1}(0, +\infty)$, $b(0_+) \neq 0$,

(b_2) there exists $M > 0$, $\beta > 0$ s.t.

$$|\mathcal{L}b(z)| \geqslant \frac{M}{1+|z|^\beta}, \forall z \in \mathbb{C}, \mathrm{Re}(z) \geqslant 0, \quad (3.14)$$

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(b_3) there exists $p \in \mathbb{N}^*$, $p \geqslant 2$ s.t.

$$\mathcal{F}^{-1}\left[\frac{\left(\mathcal{F}b'\right)^p}{\mathcal{F}b}\right] \in L^1(\mathbb{R}). \tag{3.15}$$

Our goal is to prove the following inversion Theorem:

Theorem 3.1 (*Inversion theorem*) *Let the assumptions* (b_1)–(b_3) *hold true. Then,*

(*i*) *for any* $1 \leqslant q \leqslant +\infty$ *and* $t_0 \in (0, +\infty]$, *the mapping* $\mathcal{R}_{t_0,q}$ *is a Banach isomorphism;*

(*ii*) *functions* B_1, B_2 *that depend only on* b *and are being given by*

$$B_1 = \sum_{k=1}^{p-1}(-1)^k \frac{(b')^{*k}}{b^{k+1}(0_+)} \tag{3.16}$$

$$B_2 = \frac{(-1)^p}{b^p(0_+)}\mathcal{F}^{-1}\left[\frac{\left(\mathcal{F}b'\right)^p}{\mathcal{F}b}\right], \tag{3.17}$$

belong to $L^1_{\mathbb{R}_+}(\mathbb{R})$*;*

(*iii*) *for any* $l \in W^{1,q}_{[0,t_0)}(-\infty, t_0)$, *one has*

$$\mathcal{R}^{-1}_{t_0,q}(l) = \frac{l'}{b(0_+)} + B_1 * l' + B_2 * l. \tag{3.18}$$

For the proof, we first need to introduce and prove two preliminary Lemmas.

Lemma 3.2 *Assume that* $b \in W^{1,1}\left(\mathbb{R}^*_+\right)$, $b(0_+) \neq 0$. *Let* $1 \leqslant q \leqslant +\infty$, $t_0 \in (0, +\infty)$. *Then,* $\mathcal{R}_{t_0,q}$ *is a continuous injection.*

Proof We begin by showing $\mathcal{R}_{t_0,q}$ is well defined and continuous. Since $b \in W^{1,1}\left(\mathbb{R}^*_+\right)$, it is clear that for any $w \in L^q_{[0,t_0)}(-\infty, t_0)$, the function $b * w$ belongs to $W^{1,q}_{[0,t_0)}(-\infty, t_0)$. Moreover, $(b * w)' = \left[b(0_+)w + b' * w\right]$. Hence,

$$\|\mathcal{R}_{t_0,q}(w)\|_{W^{1,q}(0,t_0)} \leqslant \left[|b(0_+)| + \|b\|_{W^{1,1}\left(\mathbb{R}^*_+\right)}\right] \|w\|_{L^q(0,t_0)} \tag{3.19}$$

which proves $\mathcal{R}_{t_0,q}$ is indeed continuous.

Next, assume $w \in L^q_{[0,t_0)}(-\infty, t_0)$ satisfies $\mathcal{R}_{t_0,q}(w) = 0$. Derivating the later leads to

$$w(s) + \int_0^s \frac{b'(s-\tau)}{b(0_+)} w(\tau)\mathrm{d}\tau = 0, \text{ a.e. } s < t_0. \tag{3.20}$$

Multiply (3.20) by $e^{-\theta s}$, $\theta > 0$, and set $w_1(s) = e^{-\theta s}w(s)$, $b_1(s) = \frac{b'(s)}{b(0_+)}e^{-\theta s}$. Equality (3.20) can now be re-written as

$$w_1(s) + \int_0^s b_1(s-\tau)w_1(\tau)\mathrm{d}\tau = 0, \text{ a.e. } s < t_0. \tag{3.21}$$

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It implies that

$$\|w_1\|_{L^q(0,t_0)} \leqslant \|b_1\|_{L^1(\mathbb{R}_+^*)} \|w_1\|_{L^q(0,t_0)}. \tag{3.22}$$

Notice that $\|b_1\|_{L^1} = \int_0^{+\infty} e^{-\theta s} \frac{|b'(s)|}{|b(0_+)|} ds \longrightarrow_{\theta\to+\infty} 0$. Pick up a $\theta > 0$ large enough s.t. $\|b_1\|_{L^1(\mathbb{R}_+^*)} < 1$. From (3.22), we get $\|w_1\|_{L^q(0,t_0)} = 0$. Finally, $w = 0$ and $\mathcal{R}_{t_0,q}$ is an injection mapping.

□

Lemma 3.3 *The Theorem 3.1 holds true for $t_0 = +\infty$ and $q = 2$.*

Proof The proof consists of four steps.

Step 1 First, we prove the existence, as a consequence of assumptions (b_1)–(b_2) on b, of a constant $M_3 > 0$ s.t.

$$|\mathcal{F}(b)(\omega)| \geqslant \frac{M_3}{1+|\omega|}, \quad \forall \omega \in \mathbb{R}. \tag{3.23}$$

Indeed, one sees that (b_2) implies

$$|\mathcal{F}(b)(\omega)| \geqslant \frac{M}{1+|\omega|^\beta}, \quad \forall \omega \in \mathbb{R}.$$

Then, it suffices to prove the existence of $m_1, m_2 > 0$ s.t. $|\mathcal{F}(b)(\omega)| \geqslant \frac{m_1}{|\omega|}$, $\forall \omega \in \mathbb{R}$ with $|\omega| \geqslant m_2$. This follows from the fact that $|\mathcal{F}(b)(\omega)| = \frac{1}{i\omega}\left[\mathcal{F}(b')(\omega) + b(0_+)\right]$, that $\mathcal{F}(b')(\omega) \to 0$ for $|\omega| \to \infty$, and $b(0_+) \neq 0$.

Step 2 Now, we prove $\mathcal{R}$ is a Banach isomorphism. Due to Lemma 3.2, one only needs to prove $\mathcal{R}$ is surjective. To begin with, one establishes that, for any $w \in L^2_{\mathbb{R}_+}(\mathbb{R})$, one has (with $M_3 > 0$ the constant in (3.23))

$$\|w\|_{L^2(\mathbb{R})} \leqslant \frac{1}{\sqrt{\pi} M_3} \|\mathcal{R}(w)\|_{H^1(\mathbb{R})}. \tag{3.24}$$

Actually using Parseval's identity and (3.23), one gets

$$\sqrt{2\pi}\|w\|_{L^2(\mathbb{R})} = \|\mathcal{F}w\|_{L^2(\mathbb{R})} = \left\| \frac{\mathcal{F}\mathcal{R}(w)}{\mathcal{F}b} \right\|_{L^2(\mathbb{R})} \leqslant \frac{1}{M_3} \|(1+|\omega|)\mathcal{F}\mathcal{R}(w)\|_{L^2(\mathbb{R})}. \tag{3.25}$$

Since $(1+|\omega|) \leqslant \sqrt{2(1+\omega^2)}$, inequality (3.25) implies inequality (3.24). Next, inequalities (3.19) and (3.24) prove that $\mathcal{R}(L^2_{\mathbb{R}_+}(\mathbb{R}))$ is closed. Therefore, in order to prove that $\mathcal{R}$ is surjective it is sufficient to show that the dense subset $(\mathscr{C}_c^\infty)_{(0,+\infty)}(\mathbb{R})$ of $H^1_{\mathbb{R}_+}(\mathbb{R})$ is included in $\mathcal{R}(L^2_{\mathbb{R}_+}(\mathbb{R}))$.

Let $r \in (\mathscr{C}_c^\infty)_{(0,+\infty)}(\mathbb{R})$. We search for $w \in L^2_{\mathbb{R}_+}(\mathbb{R})$ s.t. $b * w = r$. Since we are unable to identify the support of w by Fourier transform, we use Laplace transform instead. Consider the function

$$z \in \{z \in \mathbb{C} / \operatorname{Re}(z) \geqslant 0\} \mapsto \frac{\mathcal{L}r(z)}{\mathcal{L}b(z)} \in \mathbb{C}$$

which is well defined based on (b_2) and the fact that $r \in (\mathscr{C}_c^\infty)_{(0,+\infty)}(\mathbb{R})$. This function is clearly continuous on $\operatorname{Re}(z) \geqslant 0$ and analytic on $\operatorname{Re}(z) > 0$. As for any $z \in \mathbb{C}$ and $\gamma \in \mathbb{N}$,

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$\mathcal{L}r^{(\gamma)}(z) = z^\gamma \mathcal{L}r(z)$, and as $r^{(\gamma)} \in L^1(\mathbb{R})$, we deduce that there exists $m_3 \geqslant 0$ s.t.

$$|\mathcal{L}r(z)| \leqslant \frac{m_3}{1+|z|^{\beta+2}}, \ \forall z \in \mathbb{C},\ \mathrm{Re}(z) \geqslant 0.$$

Now, it easily follows the existence of $m_4 \geqslant 0$ s.t.

$$\left|\frac{\mathcal{L}r(z)}{\mathcal{L}b(z)}\right| \leqslant \frac{m_4}{1+|z|^2}, \ \forall z \in \mathbb{C},\ \mathrm{Re}(z) \geqslant 0. \tag{3.26}$$

Next, with the help of Bromwich–Mellin formula, for any $t \in \mathbb{R}$ and for fixed $x > 0$, define w as

$$w(t) := \frac{1}{2\pi i} \int_{\mathbb{R}} e^{t(x+iy)} \frac{\mathcal{L}r}{\mathcal{L}b}(x+iy)\mathrm{d}y. \tag{3.27}$$

Owing to Cauchy's formula and invoking (3.26), w thus defined is independent of $x > 0$. Also, for fixed $t < 0$, letting $x \to +\infty$ in (3.27) leads to $w(t) = 0$. This is $w(t) = 0$ for any $t < 0$. Next, for any fixed $t \in \mathbb{R}$, using Lebesgue's Theorem we calculate the limit for $x \to 0$ of (3.27) and obtain $w = \mathcal{F}^{-1}(\frac{\mathcal{F}r}{\mathcal{F}b})$. By Parseval's identity and by (3.26), w is clearly an element of $L^2_{\mathbb{R}_+}(\mathbb{R})$ and satisfies $\mathcal{R}(w) = r$. Therefore, $\mathcal{R}$ is surjective.

Step 3 The task now is proving the representation formula. Let $w \in L^2_{\mathbb{R}_+}(\mathbb{R})$ and set $l = \mathcal{R}(w)$. Derivation of the later gives

$$w + \frac{b'}{b(0_+)} * w = \frac{l'}{b(0_+)}. \tag{3.28}$$

Convolute (3.28) with the operator $\sum_{k=0}^{p-1}(-1)^k(\frac{b'}{b(0_+)})^{*k}*$ (by convention $(\frac{b'}{b(0_+)})^{*0} = \delta_0$). We obtain

$$w = \frac{l'}{b(0_+)} + \left(B_1 * l'\right) + \frac{(-1)^p}{b^p(0_+)}\left[(b')^{*p} * w\right]. \tag{3.29}$$

Since $l = b * w$, we get $\mathcal{F}l = \mathcal{F}b\mathcal{F}w$. Hence,

$$\mathcal{F}\left[(b')^{*p} * w\right] = \left(\mathcal{F}b'\right)^p \frac{\mathcal{F}l}{\mathcal{F}b}. \tag{3.30}$$

By hypothesis (b_3), $\frac{(\mathcal{F}b')^p}{\mathcal{F}b} \in L^\infty(\mathbb{R})$, which proves that inequality (3.30) holds in $L^2(\mathbb{R})$ since $\mathcal{F}l \in L^2(\mathbb{R})$. This fact allows to state that $\frac{(-b')^{*p}}{b^p(0_+)} * w = B_2 * l$ with B_2 given by (3.17). Now, for any $w \in L^2_{\mathbb{R}_+}(\mathbb{R})$ and $l = \mathcal{R}(w)$, (3.29) gives the representation formula

$$w = \frac{l'}{b(0_+)} + B_1 * l' + B_2 * l. \tag{3.31}$$

Step 4 Let us now show that the support of B_1 and that of B_2 are included in $\mathbb{R}_+$.

Since the support of b' is in $\mathbb{R}_+$, B_1 also has its support in $\mathbb{R}_+$ due to formula (3.16). Let $\rho \in \mathscr{D}_{\mathbb{R}_+}(\mathbb{R})$ and set $w = \mathcal{R}^{-1}(\rho)$ (see **Step 2.**). Equation (3.31) now ensures that, a.e. $t < 0$,

$$0 = w(t) = \frac{\rho'(t)}{b(0_+)} + \left(B_1 * \rho'\right)(t) + \left(B_2 * \rho\right)(t). \tag{3.32}$$

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Since $\rho'(s) = 0$ a.e. $s < 0$ and since B_1 has support in $\mathbb{R}_+$, we get

$$(B_2 * \rho)(t) = 0, \text{ a.e. } t < 0. \tag{3.33}$$

Take $\rho \geqslant 0$, $\rho \neq 0$, and set $\rho_n(t) = n\rho(nt)$, $n \in \mathbb{N}^*$, $t \in \mathbb{R}$. We know that

$$B_2 * \rho_n \xrightarrow[n \to +\infty]{L^1(\mathbb{R})} \|\rho\|_{L^1} B_2. \tag{3.34}$$

Taking $\rho = \rho_n$ in (3.33) and using (3.34), we obtain $B_2 = 0$ a.e. $t < 0$. Hence, B_2 has support in $\mathbb{R}_+$.

□

We are now in a position allowing to prove the previously stated Inversion Theorem 3.1.

Proof of the inversion Theorem 3.1 Let $q \in [1, +\infty)$ and $t_0 \in \mathbb{R}_+^* \cup \{+\infty\}$. Define the mapping $\mathcal{S}_{t_0,q}$ by

$$\mathcal{S}_{t_0,q} = \begin{cases} W^{1,q}_{[0,t_0)}(-\infty, t_0) \longrightarrow L^q_{[0,t_0)}(-\infty, t_0) \\ l \qquad\qquad\qquad \mapsto \dfrac{l'}{b(0_+)} + B_1 * l' + B_2 * l \end{cases}$$

with $B_1, B_2 \in L^1_{\mathbb{R}_+}(\mathbb{R})$ given by (3.16)–(3.17). Clearly, $\mathcal{S}_{t_0,q}$ is well defined and continuous.

We begin by studying the case $t_0 = +\infty$.

Notice that $\mathcal{S}_{+\infty,q} \circ \mathcal{R}_{+\infty,q}$ restricted to $D = L^q_{\mathbb{R}_+}(\mathbb{R}) \cap L^2_{\mathbb{R}_+}(\mathbb{R})$ is the identity (see Lemma 3.3). Since D is dense in $L^q_{\mathbb{R}_+}(\mathbb{R})$, and $\mathcal{S}_{+\infty,q}$ and $\mathcal{R}_{+\infty,q}$ are continuous, we find that $\mathcal{S}_{+\infty,q} \circ \mathcal{R}_{+\infty,q}$ is the identity on $L^q_{\mathbb{R}_+}(\mathbb{R})$. Similarly, $\mathcal{R}_{+\infty,q} \circ \mathcal{S}_{+\infty,q}$ is the identity on $W^{1,q}_{\mathbb{R}_+}(\mathbb{R})$. This proves the Theorem for $t_0 = +\infty$.

Assume now that $t_0 > 0$ and $q \in [1, +\infty]$. We know from Lemma 3.2 that $\mathcal{R}_{t_0,q}$ is continuous and injective. We now prove that $\mathcal{R}_{t_0,q}$ is surjective and that $\mathcal{S}_{t_0,q}$ is its inverse. Let $l \in W^{1,q}_{[0,t_0)}(-\infty, t_0)$ and extend l into $L \in W^{1,q}_{[0,2t_0)}(\mathbb{R})$ by reflexion

$$L(t) = \begin{cases} l(t) & \text{for } t < t_0 \\ l(2t_0 - t) & \text{for } t > t_0 \end{cases}.$$

Let $W = (\mathcal{S}_{+\infty,q})(L)$ and define $w \in L^q_{[0,t_0)}(-\infty, t_0)$ as the restriction of W to $(-\infty, t_0)$. Then, $b * w = b * W = l$ on $(-\infty, t_0)$, and

$$w = W = \frac{L'}{b(0_+)} + B_1 * L' + B_2 * L, \text{ on } (-\infty, t_0). \tag{3.35}$$

This is $w = \mathcal{S}_{t_0,q}(l)$. This proves the Theorem.

□

We now have the following result:

Proposition 3.1 *There exists $n_0 \in \mathbb{N}$ s.t. for any $n \in \mathbb{N}$, $n \geqslant n_0$, the function a_n introduced*

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in the string of assumptions (a_2)–(a_5) *satisfies the hypotheses* (b_1)–(b_3) *of Theorem 3.1. Moreover, functions* B_1, B_2 *of equations* (3.16)–(3.17)*, having* b *substituted by* a_n*, belong to a ball of radius independent of* n *that is included into* $L^1(\mathbb{R})$.

Proof One has

$$a_n(0_+) = \frac{1}{\pi} \lim_{k\to+\infty} \int_{-k}^{k} \mathcal{F}b(\omega)\mathrm{d}\omega = \frac{1}{\pi} \lim_{k\to+\infty} \int_{-k}^{k} \mathrm{Re}\,[\mathcal{F}b(\omega)]\,\mathrm{d}\omega.$$

Invoking the hypothesis (a_4) leads to

$$a_n(0_+) \geqslant M_1 \tag{3.36}$$

meaning a_n satisfies (b_1). From (a_5), we see that a_n satisfies (b_3).

Moreover, one has $(\mathcal{F}e^{-t})\omega = \frac{1}{1+\omega^2}$. This fact, together with Theorem 2.4 on page 494 of [25] imply that the function $t \in [0, +\infty) \to a_n(t) - M_4 e^{-t}$ is of positive type. From the same Theorem, one also gets $\mathrm{Re}(\mathcal{L}(a_n - M_1 e^{-t})(z)) \geqslant 0$, for any $z \in \mathbb{C}$ s.t. $\mathrm{Re}(z) \geqslant 0$. The later in turn implies $\mathrm{Re}(\mathcal{L}a_n(z)) \geqslant M_1 \frac{1+z_1}{(1+z_1)^2+z_2^2}$, for any $z = z_1 + iz_2$, $z_1 \geqslant 0$. Then, a_n satisfies (b_2) with $\beta = 2$.

The last statement of the Proposition is obvious.

□

4 Approximated problems and estimates

4.1 Approximated and local problems. Preliminary notations and estimates

Remark that a is not smooth enough to ensure a straightforward local in time existence result for a solution v to our problem. As a consequence, we study the following approximated problem which we denote by P_n.

Problem P_n Find $v_n : \Omega \times [0, +\infty) \to \mathbb{R}$ s.t.

$(P_n)_1$ $(v_n)_t = \int_0^{+\infty} a_n'(s) \frac{\partial}{\partial x} g\left(\left(\overline{v}_n^t\right)_x\right)(x, s)\mathrm{d}s + f(x, t)$

$(P_n)_2$ $v_n = 0$ on $\partial\Omega$, $v_n(t) = 0$, $\forall t < 0$

$(P_n)_3$ $v_n(x, 0) = v_0(x)$ for $x \in \Omega$.

Given the assumptions on g, we conclude there exist $\gamma > 0$ and $\theta \in [0, 1]$ s.t.

$$g'(y) < -\gamma,\ \forall y \in [-\theta, \theta]. \tag{4.1}$$

Clearly, we can take the same θ as in assumption (g_1). Moreover, there exists $K > 0$ s.t.

$$|g'(y) - g'(0)| \leqslant K y^2,\ \forall y \in [-\theta, \theta]. \tag{4.2}$$

In the above, one may consider the same K as in (g_1).

Let us denote, for almost every $x \in \Omega$,

$$u_n(x, t) = \int_0^t v_n(x, s)\, ds.$$

The proof of the next Proposition is very similar to that of Theorem 3.10 in [38] and is omitted.

Proposition 4.1 *Assume that the hypotheses* (g_1)–(g_3), (f_1)–(f_4), (v_0) *and* (a_1)–(a_5) *on the data hold true. Then, the initial value problem* $(P_n)_1$, $(P_n)_2$, $(P_n)_3$ *has a unique solution* v_n *defined on a maximal time interval* $[0, T_n)$, $T_n > 0$, *and s.t.* $v_n \in \mathscr{C}^0([0,T_n);H^2(\Omega))$, $(v_n)_t \in \mathscr{C}^0([0,T_n);H^1(\Omega)), (v_n)_{tt} \in \mathscr{C}^0\left([0,T_n);L^2(\Omega)\right)$ *and* $u_n \in \mathscr{C}^0\left([0,T_n);H^3(\Omega)\right)$. *Moreover, if*

$$\sup_{t\in[0,T_n)} \left\{ \|v_n(\cdot,t)\|^2_{H^2(\Omega)} + \|(v_n)_t(\cdot,t)\|^2_{H^1(\Omega)} + \|(v_n)_{tt}(\cdot,t)\|^2_{L^2(\Omega)} + \|u_n(\cdot,t)\|^2_{H^3(\Omega)} \right\} < \infty \tag{4.3}$$

and

$$\sup_{\substack{x\in\Omega \\ 0\leqslant t\leqslant T_n}} |(u_n)_x(x,t)| \leqslant \frac{\theta}{2}$$

with θ *as in* (g_4), *then* $T_n = +\infty$.

Notice that our functional framework is different from that of [8]. As a consequence, here it is necessary to obtain new estimates on $\|u_n\|_{H^3(\Omega)}$.

In this Section, we obtain the necessary estimates to proving $T_n = +\infty$. These estimates will be proved to be independent of n, fact which allows to pass to the limit as $n \to +\infty$. To simplify notations, we drop the subscript n of a_n, v_n and T_n.

Drawing inspiration from [8], we introduce the following expressions:

$$\begin{aligned}
\mathcal{E}(t) &= \sup_{s\in[0,t)} \left[\int_\Omega \left(v^2 + v_x^2 + v_t^2 + v_{xx}^2 + v_{xt}^2 + v_{tt}^2 + u^2 + u_x^2 + u_{xx}^2 + u_{xxx}^2\right)(x,s)\mathrm{d}x \right] \\
&\quad + \int_0^t \int_\Omega \left(v^2 + v_x^2 + v_t^2 + v_{xx}^2 + v_{xt}^2 + v_{tt}^2\right)(x,s)\mathrm{d}x\mathrm{d}s \\
&= \sup_{s\in[0,t)} \left(\|v\|^2_{H^2(\Omega)} + \|v_t\|^2_{H^1(\Omega)} + \|v_{tt}\|^2_{L^2(\Omega)} + \|u\|^2_{H^3(\Omega)} \right)(s) + \|v\|^2_{H^2(\Omega\times[0,t))}
\end{aligned} \tag{4.4}$$

and

$$\nu(t) = \sup_{\substack{x\in\Omega \\ s\in[0,t)}} \left[\sqrt{\left(v^2 + v_x^2 + v_t^2\right)(x,s)} \right] + \sqrt{\int_0^t \sup_{x\in\Omega} (v_x(x,s))^2 \,\mathrm{d}s}. \tag{4.5}$$

For simplicity, let us denote

$$\begin{aligned}
\mathcal{E}_1(t) &= \sup_{s\in[0,t)} \left[\int_\Omega \left(v^2 + v_x^2 + v_t^2 + v_{xt}^2 + v_{tt}^2\right)(x,s)\mathrm{d}x \right] \\
&\quad + \int_0^t \int_\Omega \left(v^2 + v_x^2 + v_t^2 + v_{xt}^2\right)(x,s)\mathrm{d}x\mathrm{d}s.
\end{aligned} \tag{4.6}$$

In fact, $\mathcal{E}_1(t)$ collects the terms of $\mathcal{E}(t)$ which will be estimated in a first step with the help of energy estimates.

Remark that, due to Sobolev inequalities, there exists a constant $C_\Omega > 0$ s.t.

$$\nu(t) \leqslant C_\Omega \sqrt{\mathcal{E}(t)}, \ \forall t \in [0, T) \tag{4.7}$$

and

$$\sup_{x\in\Omega} |u_x(x,t)| \leqslant C_\Omega \sqrt{\mathcal{E}(t)}, \ \forall t \in [0,T). \tag{4.8}$$

Next, from (2.6) we get

$$\begin{aligned}\mathcal{G}_t(x,t) &= v_{xx}(x,t)\int_0^{+\infty} a'(s)\left[g'\left(\overline{v}_x^t(x,s)\right) - g'(0)\right] \mathrm{d}s \\ &\quad - \int_0^t v_{xx}(x,s)a'(t-s)\left[g'\left(\overline{v}_x^t(x,t-s)\right) - g'(0)\right] \mathrm{d}s \\ &\quad + \int_0^t v_{xx}(x,s)\int_{t-s}^{+\infty} a'(\tau)g''\left(\overline{v}_x^t(x,\tau)\right)[v_x(x,t) - v_x(x,t-\tau)]\, \mathrm{d}\tau \mathrm{d}s.\end{aligned} \tag{4.9}$$

All subsequent estimates will be obtained under the following smallness hypothesis on $\mathcal{E}(t)$:

$$\mathcal{E}(t) \leqslant \frac{\theta^2}{4C_\Omega^2}, \ \forall t \in [0,T) \tag{4.10}$$

which implies

$$\sup_{\substack{x\in\Omega \\ 0\leqslant t\leqslant T}} |u_x(x,t)| \leqslant \frac{\theta}{2} \tag{4.11}$$

Then,

$$\sup_{\substack{x\in\Omega \\ 0\leqslant s\leqslant t\leqslant T}} \left|\overline{v}_x^t(x,s)\right| \leqslant \theta, \text{ a.e. } x \in \Omega. \tag{4.12}$$

We shall frequently employ the following inequalities:

$$|xy| \leqslant \mu x^2 + \frac{1}{4\mu} y^2, \ x, y \in \mathbb{R}, \mu > 0 \tag{4.13}$$

and

$$\|F_1 * F_2\|_{L^p(0,T)} \leqslant \|F_1\|_{L^1(0,+\infty)} \|F_2\|_{L^p(0,T)}, \tag{4.14}$$

for any $T > 0$, $F_1 \in L^1(0,+\infty)$, and $F_2 \in L^p(0,T)$, with $p \geqslant 1$; functions F_1 and F_2 are extended to $\mathbb{R}$ by 0.

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For future reference, we prove the following Lemmas:

Lemma 4.1 *Let the mappings φ and $s \mapsto s\varphi(s)$ be elements of $L^1(\mathbb{R}_+)$. Then, the function $s \mapsto \int_s^{+\infty} \varphi(\tau)\mathrm{d}\tau$ belongs to $L^1(\mathbb{R}_+)$ and we have the estimate*

$$\int_0^{+\infty} \left| \int_s^{+\infty} \varphi(\tau)\mathrm{d}\tau \right| \mathrm{d}s \leqslant \int_0^{+\infty} |s\varphi(s)| \, \mathrm{d}s.$$

Proof The proof is a direct consequence of Fubini's Theorem. □

Lemma 4.2 *Let $\varphi \in L^1(\mathbb{R}_+)$. Then,*

(i) for any $w_1, w_2 \in L^2(Q_t)$ we have

$$\left| \int_0^t \int_\Omega w_1(x,s)(w_2 * \varphi)(x,s)\mathrm{d}s \right| \leqslant \|\varphi\|_{L^1(\mathbb{R}_+)} \|w_1\|_{L^2(Q_t)} \|w_2\|_{L^2(Q_t)}; \tag{4.15}$$

(ii) for any $w_3 \in L^2(\Omega)$, $w_4 \in L^\infty\left(0, T; L^2(\Omega)\right)$ we have

$$\left| \int_\Omega w_3(x)(\varphi * w_4)(x,t)\mathrm{d}x \right| \leqslant \|\varphi\|_{L^1(0,T)} \|w_3\|_{L^2(\Omega)} \sup_{0 \leqslant \tau \leqslant t} \|w_4(\tau)\|_{L^2(\Omega)}, \text{ a.e. } t \in [0,T). \tag{4.16}$$

Proof Part (i): observe that

$$\begin{aligned} \left| \int_0^t \int_\Omega w_1(x,s)(w_2 * \varphi)(x,s)\mathrm{d}s \right| &\leqslant \int_\Omega \|w_1(x,\cdot)\|_{L^2(0,t)} \|(w_2 * \varphi)(x,\cdot)\|_{L^2(0,t)} \, \mathrm{d}x \\ &\leqslant \|\varphi\|_{L^1(\mathbb{R}_+)} \int_\Omega \|w_1(x,\cdot)\|_{L^2(0,t)} \|w_2(x,\cdot)\|_{L^2(0,t)} \, \mathrm{d}x \end{aligned} \tag{4.17}$$

which gives the result.

Part (4.2): one has

$$\left| \int_\Omega w_3(x)(\varphi * w_4)(x,t)\mathrm{d}x \right| \leqslant \|w_3\|_{L^2(\Omega)} \int_0^t \|w_4(x, t-\tau)\|_{L^2(\Omega)} |\varphi(\tau)| \mathrm{d}\tau \tag{4.18}$$

and the result follows. □

Let $r_0 : \mathbb{R}_+ \to \mathbb{R}_+$, $r_0(s) := \min\left\{s, \sqrt{s}\right\}$. We have the following estimates:

Lemma 4.3 *Let $t \in [0,T)$, assume (4.10) is satisfied. Then,*

(i) $\left|g^{(j)}\left(\overline{v}_x^t(x,s)\right) - g^{(j)}(0)\right| \leqslant K \min\left\{v(t)r_0(s), \theta\right\}$ *a.e.* $x \in \Omega$, $s \in [0,t]$, $j = 0,1,2,3$

(ii) $|\mathcal{G}(x,t)| \leqslant K v(t)[|v_{xx}(x,\cdot)| * \psi](t)$, *a.e.* $x \in \Omega$

(iii) $|\mathcal{G}_t(x,t)| \leqslant K v(t)\overline{a}\,|v_{xx}(x,t)| + K v(t) \left[|v_{xx}(x,\cdot)| * \psi\right](t)$, *a.e.* $x \in \Omega$,

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where

$$\overline{a} = \int_0^{+\infty} |a'(s)|\, r_0(s)\mathrm{d}s. \tag{4.19}$$

$$\psi(t) = |a'(t)|\, r_0(t) + 2\int_t^{+\infty} |a'(\tau)|\, r_0(\tau)\mathrm{d}\tau \tag{4.20}$$

Remark 4.1 *Lemma 4.1 and the assumptions made about function a grant the fact that ψ in* (4.20) *is s.t.* $\psi \in L^1(\mathbb{R}_+)$.

Proof

(i) On one hand, as a consequence of (g_1) and (4.12), we have

$$\left| g^{(j)}\left(\overline{v}_x^t(x,s)\right) - g^{(j)}(0)\right| \leqslant K\left|\overline{v}_x^t(x,s)\right|,\ j = 0,1,2,3. \tag{4.21}$$

On the other hand,

$$\left|\overline{v}_x^t(x,s)\right| \leqslant \int_{t-s}^{t} |v_x(x,\lambda)|\,\mathrm{d}\lambda \leqslant s \sup_{t-s\leqslant\lambda\leqslant t} |v_x(x,\lambda)| \leqslant sv(t) \tag{4.22}$$

and

$$\left|\overline{v}_x^t(x,s)\right| \leqslant \sqrt{s}\left[\int_{t-s}^{t} |v_x(x,\lambda)|^2\,\mathrm{d}\lambda\right]^{1/2} \leqslant \sqrt{s}v(t) \tag{4.23}$$

which gives the result.

(ii) From (2.6) and (i) above, one gets

$$\begin{aligned}|\mathcal{G}(x,t)| &\leqslant K v(t)\int_0^t |v_{xx}(x,s)| \int_{t-s}^{+\infty} a'(\tau)\min\{\tau,\sqrt{\tau}\}\mathrm{d}\tau\mathrm{d}s\\ &\leqslant K v(t)\int_0^t |v_{xx}(x,s)|\,\psi(t-s)\mathrm{d}s\end{aligned} \tag{4.24}$$

from which the result follows.

(iii) We use (4.9), (g_1), (i), the fact that $g''(0) = 0$ and $0 \leqslant \theta \leqslant 1$ to obtain

$$\begin{aligned}|\mathcal{G}_t(x,t)| \leqslant{}& K\,|v_{xx}(x,t)|\, v(t)\int_0^{+\infty} |a'(s)| r_0(s)\mathrm{d}s\\ &+ K v(t)\int_0^t |v_{xx}(x,s)|\,|a'(t-s)|\, r_0(t-s)\mathrm{d}s\\ &+ 2K\theta v(t)\int_0^t |v_{xx}(x,s)| \int_{t-s}^{+\infty} |a'(\tau)| r_0(\tau)\mathrm{d}\tau\mathrm{d}s\end{aligned} \tag{4.25}$$

which gives the result.

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4.2 Energy estimates

□

The next Lemmas give energy estimates for the terms in $\mathcal{E}_1(t)$ (see (4.6)), as in [8].

In what follows, the notation $C > 0$ stands for a generic constant that is independent of n.

Lemma 4.4 *Assume the inequality* (4.10) *holds true. Then,*

$$\int_\Omega v^2(x,t)\mathrm{d}x - 2g'(0)Q\,(v_x,t,a) \leqslant V_0 + 2\sqrt{F}\sqrt{\mathcal{E}(t)} + 2K\,\|\psi\|_{L^1(\mathbb{R}_+)}\nu(t)\mathcal{E}(t). \tag{4.26}$$

Proof For a fixed $t \in (0, T_0)$, we multiply (2.5) by $v(x,t)$ and integrate on Ω and on $(0,t)$. We get

$$\begin{aligned}&\frac{1}{2}\int_\Omega v^2(x,t)\mathrm{d}x - \frac{1}{2}\int_\Omega v_0^2\mathrm{d}x - g'(0)Q(v_x,t,a)\\ &\quad= \int_0^t\int_\Omega f(x,s)v(x,s)\mathrm{d}x\mathrm{d}s + \int_0^t\int_\Omega \mathcal{G}(x,s)v(x,s)\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{4.27}$$

Observe that $\int_0^t\int_\Omega fv\mathrm{d}x\mathrm{d}s \leqslant \|f\|_{L^2(Q_t)}\|v\|_{L^2(Q_t)} \leqslant \sqrt{F}\sqrt{\mathcal{E}}$.

Now, using Lemma 4.3 we get

$$\left|\int_0^t\int_\Omega \mathcal{G}(x,s)v(x,s)\mathrm{d}x\mathrm{d}s\right| \leqslant K\nu(t)\int_0^t\int_\Omega |v(x,s)|\,\left(|v_{xx}| * |\psi|\right)(x,s)\mathrm{d}x\mathrm{d}s.$$

Using part (i) of Lemma 4.2 with $w_1 = v$, $w_2 = v_{xx}$ and $\varphi = |\psi|$, one gets

$$\left|\int_0^t\int_\Omega \mathcal{G}(x,s)v(x,s)\mathrm{d}x\mathrm{d}s\right| \leqslant K\nu(t)\|\psi\|_{L^1(\mathbb{R}_+)}\mathcal{E}(t),$$

thus ending the proof. □

Lemma 4.5 *Let $\overline{a}$ and ψ be given by* (4.19) *and* (4.20)*, respectively. Under the assumption that* (4.10) *is fulfilled , one has the following inequality:*

$$\begin{aligned}\int_\Omega v_t^2(x,t)\mathrm{d}x - 2g'(0)Q\,(v_{xt},t,a) &\leqslant F + 2\|a\|_{L^1(\mathbb{R}_+)}\sqrt{V_0}\sqrt{\mathcal{E}(t)}\\ &+ 2\sqrt{F}\sqrt{\mathcal{E}(t)} + 2K\,\left(\|\psi\|_{L^1(\mathbb{R}_+)} + \overline{a}\right)\nu(t)\mathcal{E}(t).\end{aligned} \tag{4.28}$$

Proof First, we derivate (2.5) w.r.t. t and obtain

$$v_{tt}(x,t) + g'(0)a(0)v_{xx}(x,t) + g'(0)\int_0^t a'(t-s)v_{xx}(x,s)\mathrm{d}s = f_t + \mathcal{G}_t. \tag{4.29}$$

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Second, multiplying the above by v_t and integrating on Ω and on $[0,t]$ leads to

$$\frac{1}{2}\int_\Omega v_t^2(x,t)\mathrm{d}x - \frac{1}{2}\int_\Omega v_t^2(x,0)\mathrm{d}x - g'(0)a(0)\int_0^t\int_\Omega v_x v_{xt}\mathrm{d}x\mathrm{d}s$$
$$- g'(0)\int_0^t\int_\Omega\int_0^s a'(s-\tau)v_x(\tau)\mathrm{d}\tau v_{xt}(s)\mathrm{d}x\mathrm{d}s = \int_0^t\int_\Omega f_t v_t\mathrm{d}x\mathrm{d}s$$
$$+ \int_0^t\int_\Omega \mathcal{G}_t v_t\mathrm{d}x\mathrm{d}s. \tag{4.30}$$

Observe now that

$$\int_0^s a'(s-\tau)v_x(\tau)\mathrm{d}\tau = -a(0)v_x(s) + a(s)v_x(0) + \int_0^s a(s-\tau)v_{xt}(\tau)\mathrm{d}\tau. \tag{4.31}$$

One now gets

$$\frac{1}{2}\int_\Omega v_t^2(x,t)\mathrm{d}x - g'(0)Q\left(v_{xt},t,a\right) = \frac{1}{2}\int_\Omega v_t^2(x,0)\mathrm{d}x - g'(0)\int_0^t\int_\Omega a(s)v_0''(x)v_t(x,s)\mathrm{d}x\mathrm{d}s$$
$$+ \int_0^t\int_\Omega \left(f_t v_t\right)(x,s)\mathrm{d}x\mathrm{d}s + \int_0^t\int_\Omega \left(\mathcal{G}_t v_t\right)(x,s)\mathrm{d}x\mathrm{d}s. \tag{4.32}$$

Notice that

$$v_t(x,0) = f(x,0) \tag{4.33}$$

which gives $\int_\Omega v_t^2(x,0)\mathrm{d}x \leqslant F$. We also have

$$\left|\int_0^t\int_\Omega a(s)v_0''(x)v_t(x,s)\mathrm{d}x\mathrm{d}s\right| \leqslant \|v_0''\|_{L^2(\Omega)}\|a\|_{L^1(\mathbb{R}_+)}\sup_{0\leqslant s\leqslant t}\|v_t(\cdot,s)\|_{L^2(\Omega)}$$
$$\leqslant \|a\|_{L^1(\mathbb{R}_+)}\sqrt{V_0}\sqrt{\mathcal{E}(t)} \tag{4.34}$$

and

$$\int_0^t\int_\Omega \left(f_t v_t\right)(x,s)\mathrm{d}x\mathrm{d}s \leqslant \sqrt{F}\sqrt{\mathcal{E}(t)}. \tag{4.35}$$

Finally, invoking part (iii) of Lemma 4.3 and part (i) of Lemma 4.2, we deduce that

$$\int_0^t\int_\Omega \left(\mathcal{G}_t v_t\right)(x,s)\mathrm{d}x\mathrm{d}s \leqslant K\overline{a}v(t)\mathcal{E}(t) + Kv(t)\|\psi\|_{L^1(\mathbb{R}_+)}\mathcal{E}(t) \tag{4.36}$$

and with the obtainment of this last estimates the proof ends.

□

Next, in order to obtain energy estimates for $\int_\Omega v_{tt}^2(x,t)\mathrm{d}x$, we shall use the difference operator $\left(\triangle_h w\right)(x,t) = w(x,t+h) - w(x,t)$, for $h>0$ small enough.

Lemma 4.6 *Under the assumption that* (4.10) *is fulfilled, one has*

$$\int_{\Omega} v_{tt}^2(x,t)\mathrm{d}x - 2g'(0)\lim_{h\to 0_+}\frac{1}{h^2}Q\left(\triangle_h v_{xt},t,a\right) \leqslant C\Big\{F+\sqrt{F}\sqrt{\mathcal{E}(t)} + \left[v(t)+v^3(t)\right]\mathcal{E}(t)+\sqrt{V_0}\mathcal{E}(t)\Big\}. \tag{4.37}$$

For the Proof, see the Appendix Section.

Since $v(t)$ and $\mathcal{E}(t)$ are non-increasing functions in t, we obtain as a consequence of Lemmas 4.4–4.6, 3.1 and Sobolev embeddings, that:

Lemma 4.7 *Under the assumption stated in* (4.10)*, one has*

$$\mathcal{E}_1(t) \leqslant C\left\{V_0+F+\left(\sqrt{V_0}+\sqrt{F}\right)\sqrt{\mathcal{E}(t)}+\left[v(t)+v^3(t)\right]\mathcal{E}(t)+\sqrt{V_0}\mathcal{E}(t)\right\}. \tag{4.38}$$

4.3 Non-energy estimates

In the following, we obtain estimates for the other constitutive terms of $\mathcal{E}(t)$.

Now, from (2.5) and using for a.e. $x\in\Omega$ the result of Theorem 3.1 with $b=a$ (see Proposition 3.1),
$l(t)=\frac{1}{g'(0)}\left[f(x,t)+\mathcal{G}(x,t)-v_t(x,t)\right]$, and $w(t)=v_{xx}(x,t)$, we deduce the equality

$$v_{xx}=\frac{1}{g'(0)}\left[\frac{1}{a(0)}\left(f_t+\mathcal{G}_t-v_{tt}\right)+A_1*\left(f_t+\mathcal{G}_t-v_{tt}\right)+A_2*\left(f+\mathcal{G}-v_t\right)\right], \tag{4.39}$$

where $A_1,A_2\in L^1_{[0,+\infty)}(\mathbb{R})$ are two functions that depend on a_n, with bounded L^1 norms which are independent of n, due to Proposition 3.1.

We have the following estimate:

Lemma 4.8 *Under the assumption stated in* (4.10)*, one has*

$$\int_{\Omega} v_{xx}^2(x,t)\mathrm{d}x+\int_0^t\int_{\Omega} v_{xx}^2(x,s)\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\Omega} v_{tt}^2(x,s)\mathrm{d}x\mathrm{d}s \leqslant C\left[F+\mathcal{E}_1(t)+v(t)\mathcal{E}(t)\right]. \tag{4.40}$$

Proof

Step 1 We multiply (4.39) by v_{xx} and integrate on Ω. It is clear that, for any $\eta>0$, we have

$$\left|\int_{\Omega}\left(f_t-v_{tt}\right)v_{xx}\mathrm{d}x\right| \leqslant \eta\int_{\Omega} v_{xx}^2\mathrm{d}x+\frac{1}{2\eta}\int_{\Omega}\left(f_t^2+v_{tt}^2\right)\mathrm{d}x. \tag{4.41}$$

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From part (iii) in Lemma 4.3, we obtain

$$\left|\int_\Omega \mathcal{G}_t v_{xx}\mathrm{d}x\right| \leqslant K\nu(t)\int_\Omega |v_{xx}(x,t)|\left(|v_{xx}| * |\psi|\right)(x,t)\mathrm{d}x + \bar{a}K\nu(t)\int_\Omega |v_{xx}(x,t)|^2\,\mathrm{d}x. \tag{4.42}$$

Further, with the help of part (ii) in Lemma 4.2 we obtain

$$\left|\int_\Omega \mathcal{G}_t v_{xx}\mathrm{d}x\right| \leqslant K\nu(t)\,\|v_{xx}(\cdot,t)\|_{L^2(\Omega)}\,\|\psi\|_{L^1(\mathbb{R}_+)}\sup_{0\leqslant\tau\leqslant t}\|v_{xx}(\cdot,\tau)\|_{L^2(\Omega)} + \bar{a}K\nu(t)\,\|v_{xx}(\cdot,t)\|^2_{L^2(\Omega)} \leqslant K\nu(t)\left[\|\psi\|_{L^1(\mathbb{R}_+)}+\bar{a}\right]\mathcal{E}(t). \tag{4.43}$$

For any $\eta > 0$, one has

$$\left|\int_\Omega A_1 * (f_t - v_{tt})\,v_{xx}\mathrm{d}x\right| \leqslant \|A_1\|_{L^1(\mathbb{R}_+)}\,\|v_{xx}(\cdot,t)\|_{L^2(\Omega)}\sup_{0\leqslant\tau\leqslant t}\left[\|f_t(\cdot,\tau)\|_{L^2(\Omega)}+\|v_{tt}(\cdot,\tau)\|_{L^2(\Omega)}\right]$$
$$\leqslant \eta\,\|v_{xx}(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2\eta}\,\|A_1\|^2_{L^1(\mathbb{R}_+)}\sup_{0\leqslant\tau\leqslant t}[\|f_t(\cdot,\tau)\|^2_{L^2(\Omega)}+\|v_{tt}(\cdot,\tau)\|^2_{L^2(\Omega)}] \tag{4.44}$$

and also

$$\left|\int_\Omega A_2 * (f - v_t)\,v_{xx}\mathrm{d}x\right|$$
$$\leqslant \eta\,\|v_{xx}(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2\eta}\,\|A_2\|^2_{L^1(\mathbb{R}_+)}\sup_{0\leqslant\tau\leqslant t}\left[\|f(\cdot,\tau)\|^2_{L^2(\Omega)}+\|v_t(\cdot,\tau)\|^2_{L^2(\Omega)}\right]. \tag{4.45}$$

We now have,

$$\left|\int_\Omega (A_1 * \mathcal{G}_t)(x,t)v_{xx}(x,t)\mathrm{d}x\right| \leqslant \bar{a}K\nu(t)\int_\Omega (|A_1| * |v_{xx}|)(x,t)|v_{xx}(x,t)|\mathrm{d}x + K\nu(t)\int_\Omega (|A_1| * |\psi| * |v_{xx}(x,t)|)(x,t)|v_{xx}(x,t)|\mathrm{d}x. \tag{4.46}$$

Then,

$$\left|\int_\Omega (A_1 * \mathcal{G}_t)(x,t)v_{xx}(x,t)\mathrm{d}x\right|$$
$$\leqslant K\nu(t)\left[\bar{a}\,\|A_1\|_{L^1(\mathbb{R}_+)} + \|\,|A_1| * |\psi|\,\|_{L^1(\mathbb{R}_+)}\right]\|v_{xx}(\cdot,t)\|_{L^2(\Omega)}\sup_{0\leqslant\tau\leqslant t}\|v_{xx}(\cdot,\tau)\|_{L^2(\Omega)}. \tag{4.47}$$

This gives

$$\left|\int_\Omega (A_1 * \mathcal{G}_t)(x,t)v_{xx}(x,t)\mathrm{d}x\right| \leqslant C\nu(t)\mathcal{E}(t). \tag{4.48}$$

Likewise,

$$\left|\int_\Omega (A_2 * \mathcal{G})(x,t)v_{xx}(x,t)\mathrm{d}x\right| \leqslant C\nu(t)\mathcal{E}(t). \tag{4.49}$$

Now, from the above estimates (4.41)–(4.45), (4.48) and (4.49), with $\eta > 0$ small enough leads to

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$$\sup_{0\leqslant s\leqslant t}\int_\Omega v_{xx}^2(x,s)\mathrm{d}x \leqslant C\left[F+\mathcal{E}_1(t)+\nu(t)\mathcal{E}(t)\right]. \tag{4.50}$$

Step 2 We multiply (4.39) by v_{xx} and integrate on $(0,t)$ and on Ω. Proceeding as in **Step 1.**, using part (i) in Lemma 4.2, one gets for any $\eta>0$ that

$$\int_{Q_t}\left[f_t+\mathcal{G}_t+A_1*f_t+A_2*(f-v_t)+A_1*\mathcal{G}_t+A_2*\mathcal{G}\right]v_{xx}\mathrm{d}x\mathrm{d}s$$
$$\leqslant \eta\int_{Q_t}v_{xx}^2\mathrm{d}x\mathrm{d}s+\frac{C}{\eta}\left[F+\mathcal{E}_1(t)\right]+C\nu(t)\mathcal{E}(t). \tag{4.51}$$

We are left to focus on terms that contain v_{tt}. Invoking density arguments,

$$\int_{Q_t}\left(v_{tt}v_{xx}\right)(x,s)\mathrm{d}x\mathrm{d}s=\int_\Omega\left(v_{xx}v_t\right)(x,t)\mathrm{d}x-\int_\Omega v_0''(x)v_t(x,0)\mathrm{d}x+\int_{Q_t}v_{xt}^2\mathrm{d}x\mathrm{d}s \tag{4.52}$$

which gives, using (4.33),

$$\left|\int_{Q_t}\left(v_{tt}v_{xx}\right)(x,s)\mathrm{d}x\mathrm{d}s\right|\leqslant \|v_{xx}(\cdot,t)\|_{L^2(\Omega)}\|v_t(\cdot,t)\|_{L^2(\Omega)}$$
$$+\|v_0''\|_{L^2(\Omega)}\|f(\cdot,0)\|_{L^2(\Omega)}+\int_{Q_t}v_{xt}^2(x,s)\mathrm{d}x\mathrm{d}s. \tag{4.53}$$

Finally, we have

$$\int_{Q_t}\left(A_1*v_{tt}\right)(x,s)v_{xx}(x,s)\mathrm{d}x\mathrm{d}s=\int_{Q_t}\left(A_1*v_t\right)_t v_{xx}(x,s)\mathrm{d}x\mathrm{d}s$$
$$-\int_{Q_t}A_1(s)v_t(x,0)v_{xx}(x,s)\mathrm{d}x\mathrm{d}s. \tag{4.54}$$

Again, calling in the density arguments leads to

$$\int_{Q_t}\left(A_1*v_t\right)_t(x,s)v_{xx}(x,s)\mathrm{d}x\mathrm{d}s=\int_\Omega\left(A_1*v_t\right)(x,t)v_{xx}(x,t)\mathrm{d}x$$
$$+\int_{Q_t}\left(A_1*v_{xt}\right)v_{xt}\mathrm{d}x\mathrm{d}s. \tag{4.55}$$

From equalities (4.54) and (4.55), one easily gets

$$\left|\int_{Q_t}\left(A_1*v_{tt}\right)v_{xx}(x,s)\mathrm{d}x\mathrm{d}s\right|\leqslant\|A_1\|_{L^1(\mathbb{R}_+)}$$
$$\left[\int_{Q_t}v_{xt}^2\mathrm{d}x\mathrm{d}s+\|v_{xx}(\cdot,t)\|_{L^2(\Omega)}\sup_{0\leqslant\tau\leqslant t}\|v_t(\cdot,t)\|_{L^2(\Omega)}+\|f(\cdot,0)\|_{L^2(\Omega)}\sup_{0\leqslant s\leqslant t}\|v_{xx}(\cdot,s)\|_{L^2(\Omega)}\right]. \tag{4.56}$$

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Now, adding inequalities (4.51), (4.53), (4.56) and upon using (4.50) it allows us to get

$$\int_{Q_t} v_{xx}^2(x,t)\mathrm{d}x \leqslant C\left[F + \mathcal{E}_1(t) + v(t)\mathcal{E}(t)\right]. \tag{4.57}$$

Step 3 We now multiply (4.29) by v_{tt} and integrate on Q_t. We have the listed below results:

$$\left|\int_{Q_t} v_{xx}v_{tt}\mathrm{d}x\mathrm{d}s\right| \leqslant \eta \int_{Q_t} v_{tt}^2\mathrm{d}x\mathrm{d}s + \frac{1}{4\eta}\int_{Q_t} v_{xx}^2\mathrm{d}x\mathrm{d}s \tag{4.58}$$

$$\begin{aligned}\int_{Q_t}\left(a' * v_{xx}\right) v_{tt}\mathrm{d}x\mathrm{d}s &\leqslant \|a'\|_{L^1(\mathbb{R}_+)}\,\|v_{xx}\|_{L^2(Q_t)}\,\|v_{tt}\|_{L^2(Q_t)}\\ &\leqslant \eta\,\|v_{tt}\|^2_{L^2(Q_t)} + \frac{1}{4\eta}\|a'\|^2_{L^1(\mathbb{R}_+)}\,\|v_{xx}\|^2_{L^2(Q_t)}\end{aligned} \tag{4.59}$$

$$\int_{Q_t} f_t v_{tt}\mathrm{d}x\mathrm{d}s \leqslant \eta\,\|v_{tt}\|^2_{L^2(Q_t)} + \frac{1}{4\eta}\,\|f_t\|^2_{L^2(Q_t)} \tag{4.60}$$

$$\begin{aligned}\int_{Q_t}\mathcal{G}_t v_{tt}\mathrm{d}x\mathrm{d}s &\leqslant \overline{a}kv(t)\int_{Q_t}|v_{xx}|\,|v_{tt}|\,\mathrm{d}x\mathrm{d}s + kv(t)\int_{Q_t}\left(|v_{xx}| * |\psi|\right)|v_{tt}|\,\mathrm{d}x\mathrm{d}s\\ &\leqslant kv(t)\left(\overline{a} + \|\psi\|_{L^1(\mathbb{R}_+)}\right)\mathcal{E}(t).\end{aligned} \tag{4.61}$$

We then obtain, taking η small enough and using (4.57), that

$$\int_{Q_t} v_{xx}^2(x,t)\mathrm{d}x\mathrm{d}s \leqslant C\left[F + \mathcal{E}_1(t) + v(t)\mathcal{E}(t)\right]. \tag{4.62}$$

Now from estimates (4.50), (4.57) and (4.62), we obtain the result of Lemma 4.8.

□

Now, we take on to obtaining estimates for u defined as $u(x,t) = \int_0^t v(x,s)\mathrm{d}s$. The idea is to integrate (4.39) w.r.t. t; one gets

$$u_{xx} = \frac{1}{g'(0)}\left\{\frac{f + \mathcal{G} - v_t}{a(0)} + \int_0^t \left[A_1 * (f_t + \mathcal{G}_t - v_{tt})\right](x,s)\mathrm{d}s + \int_0^t \left[A_2 * (f + \mathcal{G} - v_t)\right](x,s)\mathrm{d}s\right\}. \tag{4.63}$$

We shall use in the following the below Lemma:

Lemma 4.9 *Suppose that $A \in L^1(0,T)$, $\varphi \in W^{1,1}(0,T)$. Then, for any $t \in (0,T)$, we have*

$$\int_0^t (A * \varphi')(s)\mathrm{d}s = A * \left[\varphi - \varphi(0)H\right]. \tag{4.64}$$

Proof The proof is a direct consequence of Fubini's Theorem. □

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Recall from (4.33) that $(f + \mathcal{G} - v_t)(x,0) = 0$. Then, (4.63) can be re-written in the form

$$u_{xx} = \frac{1}{g'(0)}\left\{\frac{f+\mathcal{G}-v_t}{a(0)} + A_1 * (f+\mathcal{G}-v_t) + A_2 * \left[\int_0^t f(x,s)\mathrm{d}s + \int_0^t \mathcal{G}(x,s)\mathrm{d}s - v + v_0\right]\right\}. \tag{4.65}$$

We deduce from the above equation that

$$u_{xxx} = \frac{1}{g'(0)}\left\{\frac{f_x+\mathcal{G}_x-v_{xt}}{a(0)} + A_1 * (f_x+\mathcal{G}_x-v_{xt}) + A_2 * \left[\int_0^t f_x(x,s)\mathrm{d}s + \int_0^t \mathcal{G}_x(x,s)\mathrm{d}s - v_x + v_0'\right]\right\}. \tag{4.66}$$

We can now prove the following:

Lemma 4.10 *Consider the assumption formulated in* (4.10) *holds true. Then,*

$$\sup_{0\leqslant s\leqslant t} \|u_{xx}(\cdot,s)\|^2_{L^2(\Omega)} \leqslant C\left\{V_0 + F + v^2(t)\mathcal{E}(t) + \mathcal{E}^3(t) + \mathcal{E}_1(t)\right\} \tag{4.67}$$

and

$$\sup_{0\leqslant s\leqslant t} \|u_{xxx}(\cdot,s)\|^2_{L^2(\Omega)} \leqslant C\left\{V_0 + F + v^2(t)\mathcal{E}(t) + v^2(t)\mathcal{E}^2(t) + \mathcal{E}^3(t) + \mathcal{E}_1(t)\right\}, \tag{4.68}$$

where $C > 0$ is a constant which is independent of n.

Proof The proof is performed in two steps.

Step 1 Here, we obtain the necessary estimates for $\mathcal{G}(t)$, $\int_0^t \mathcal{G}(s)\mathrm{d}s$, $\mathcal{G}_x(t)$ and for $\int_0^t \mathcal{G}_x(s)\mathrm{d}s$. Using (2.6) and part (i) of Lemma 4.3, we have

$$|\mathcal{G}(t)| \leqslant K v(t) \int_0^{+\infty} |a'(s)|\, r_0(s)\, |u_{xx}(x,t) - u_{xx}(x,t-s)|\, \mathrm{d}s \tag{4.69}$$

and this gives

$$\|\mathcal{G}(\cdot,t)\|_{L^2(\Omega)} \leqslant 2K v(t) \int_0^{+\infty} |a'(s)|\, r_0(s)\mathrm{d}s \left(\sup_{0\leqslant s\leqslant t} \|u_{xx}(\cdot,s)\|_{L^2(\Omega)}\right) \leqslant C v(t)\sqrt{\mathcal{E}(t)}. \tag{4.70}$$

On the other hand, using (2.6) and (4.2), we have that

$$\left|\int_0^t \mathcal{G}(x,s)\mathrm{d}s\right| \leqslant K \int_0^{+\infty} |a'(\tau)| \int_0^t |\overline{v}^s_x(x,\tau)|^2\, |u_{xx}(x,s) - u_{xx}(x,s-\tau)|\, \mathrm{d}s\mathrm{d}\tau \tag{4.71}$$

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which implies, taking the $L^2(\Omega)$-norm, that

$$\left\|\int_0^t \mathcal{G}(\cdot,s)\mathrm{d}s\right\|_{L^2(\Omega)} \leqslant 2K\left(\sup_{0\leqslant\tau\leqslant t}\|u_{xx}(\cdot,\tau)\|_{L^2(\Omega)}\right) \int_0^{+\infty}|a'(\tau)|\int_0^t\|\overline{v}_x^s(\cdot,\tau)\|^2_{L^\infty(\Omega)}\,\mathrm{d}s\mathrm{d}\tau. \tag{4.72}$$

Now, we have by Sobolev inclusions

$$\|\overline{v}_x^s(\cdot,\tau)\|_{L^\infty(\Omega)} \leqslant C\int_{s-\tau}^s\|v(\cdot,\lambda)\|_{H^2(\Omega)}\,\mathrm{d}\lambda \leqslant 2C\tau\mathcal{M}\left(\|\tilde{v}\|_{H^2(\Omega)}\right)(s), \tag{4.73}$$

where $\tilde{v}(x,s)$ is the function defined on $\Omega\times\mathbb{R}$ by

$$\tilde{v}(x,s)=\begin{cases} v(x,s) & \text{for } s\in[0,t) \\ 0 & \text{for } s\in\mathbb{R}-[0,t)\end{cases} \tag{4.74}$$

and

$$\mathcal{M}\left(\|\tilde{v}\|_{H^2(\Omega)}\right)(s)=\sup_{\rho>0}\frac{1}{2\rho}\int_{s-\rho}^{s+\rho}\|\tilde{v}(\cdot,\tau)\|_{H^2(\Omega)}\,\mathrm{d}\tau \tag{4.75}$$

is the maximal function of $s\mapsto\|\tilde{v}(\cdot,s)\|_{H^2(\Omega)}$ (see [41]).

Now, the maximal inequality (see Theorem 1, page 5 in [41]) in this case leads to

$$\int_{\mathbb{R}}\mathcal{M}\left(\|\tilde{v}(\cdot,s)\|^2_{H^2(\Omega)}\right)(s)\mathrm{d}s \leqslant 2\sqrt{10}\int_{\mathbb{R}}\|\tilde{v}(\cdot,s)\|^2_{H^2(\Omega)}(x,s)\mathrm{d}s = 2\sqrt{10}\int_0^t\|v(\cdot,s)\|^2_{H^2(\Omega)}(x,s)\mathrm{d}s. \tag{4.76}$$

Then, from (4.73) and (4.76) by Sobolev inclusions we have that

$$\int_0^t\|\overline{v}_x^s(\cdot,\tau)\|^2_{L^\infty(\Omega)}\,\mathrm{d}\tau \leqslant C\tau^2\int_0^t\|v(\cdot,s)\|^2_{H^2(\Omega)}\mathrm{d}s. \tag{4.77}$$

Next, with the help of (4.72) we deduce

$$\left\|\int_0^t\mathcal{G}(\cdot,s)\mathrm{d}s\right\|_{L^2(\Omega)} \leqslant CK\sup_{0\leqslant\tau\leqslant t}\|u_{xx}(\cdot,\tau)\|_{L^2(\Omega)}\int_0^t\|v(\cdot,s)\|^2_{H^2(\Omega)}\,\mathrm{d}s\int_0^{+\infty}|a'(\tau)|\,\tau^2\mathrm{d}\tau \tag{4.78}$$

that is

$$\left\|\int_0^t\mathcal{G}(\cdot,s)\mathrm{d}s\right\|_{L^2(\Omega)} \leqslant C\mathcal{E}^{3/2}(t). \tag{4.79}$$

Next, let $\mathcal{G}_x(x,t)=I_1+I_2$, where

$$I_1=\int_0^{+\infty}a'(s)g''\left(\overline{v}_x^t(s)\right)\left|\overline{v}_{xx}^t(s)\right|^2\mathrm{d}s \tag{4.80}$$

$$I_2=\int_0^{+\infty}a'(s)\left[g'\left(\overline{v}_x^t(s)\right)-g'(0)\right]\overline{v}_{xxx}^t(s)\mathrm{d}s \tag{4.81}$$

and also $\int_0^t \mathcal{G}_x(x,s)\mathrm{d}s = I_3 + I_4$, where

$$I_3 = \int_0^t \int_0^{+\infty} a'(\tau) g''\left(\overline{v}_x^s(\tau)\right) \left|\overline{v}_{xx}^s(\tau)\right|^2 \mathrm{d}\tau \mathrm{d}s \tag{4.82}$$

$$I_4 = \int_0^t \int_0^{+\infty} a'(\tau)\left[g'\left(\overline{v}_x^s(\tau)\right) - g'(0)\right] \overline{v}_{xxx}^s(\tau)\mathrm{d}\tau\mathrm{d}s. \tag{4.83}$$

Since $\overline{v}^t(s) = u(t) - u(t-s)$, using again part (i) in Lemma 4.3 we obtain

$$\begin{aligned}\|I_1\|_{L^2(\Omega)} &\leqslant 2K\,v(t)\int_0^{+\infty} |a'(s)|\, r_0(s)\left[\left\|u_{xx}^2(\cdot,t)\right\|_{L^2(\Omega)} + \left\|u_{xx}^2(\cdot,t-s)\right\|_{L^2(\Omega)}\right]\mathrm{d}s \\ &\leqslant 4K\,v(t)\sup_{0\leqslant s\leqslant t}\|u_{xx}(\cdot,s)\|_{L^4(\Omega)}^2 \int_0^{+\infty} |a'(s)|\, r_0(s)\mathrm{d}s. \end{aligned} \tag{4.84}$$

This gives further down by Sobolev inclusion,

$$\|I_1\|_{L^2(\Omega)} \leqslant 4K\left(\int_0^{+\infty} |a'(s)|\, r_0(s)\mathrm{d}s\right) v(t)\mathcal{E}(t). \tag{4.85}$$

Next, as in (4.70), one easily obtains that

$$\|I_2\|_{L^2(\Omega)} \leqslant 2K\left(\int_0^{+\infty} |a'(s)|\, r_0(s)\mathrm{d}s\right) v(t)\sqrt{\mathcal{E}(t)}. \tag{4.86}$$

Moreover,

$$\|I_3\|_{L^2(\Omega)} \leqslant \\ K\int_0^t\int_0^{+\infty} |a'(\tau)|\,\|\overline{v}_x^s(\cdot,\tau)\|_{L^\infty(\Omega)}\,\|u_{xx}(\cdot,s) - u_{xx}(\cdot,s-\tau)\|_{L^\infty(\Omega)}\,\|\overline{v}_{xx}^s(\cdot,\tau)\|_{L^2(\Omega)}\,\mathrm{d}\tau\mathrm{d}s. \tag{4.87}$$

As in the proof of (4.67), we have the following estimates:

$$\|\overline{v}_x^s(\tau)\|_{L^\infty(\Omega)} \leqslant 2\tau\mathcal{M}\left(\|\tilde{v}_x\|_{L^\infty(\Omega)}\right)(s)$$

$$\|\overline{v}_{xx}^s(\tau)\|_{L^2(\Omega)} \leqslant 2\tau\mathcal{M}\left(\|\tilde{v}_{xx}\|_{L^2(\Omega)}\right)(s)$$

which give

$$\begin{aligned}\|I_3\|_{L^2(\Omega)} \leqslant 8K \sup_{0\leqslant s\leqslant t}\|u_{xx}(\cdot,s)\|_{L^\infty(\Omega)}\int_0^{+\infty} |a'(\tau)|\,\tau^2\mathrm{d}\tau \\ \sqrt{\int_0^t \mathcal{M}\left(\|\tilde{v}_x\|_{L^\infty(\Omega)}\right)^2(s)\mathrm{d}s}\sqrt{\int_0^t \mathcal{M}\left(\|\tilde{v}_{xx}\|_{L^2(\Omega)}\right)^2(s)\mathrm{d}s.}\end{aligned} \tag{4.88}$$

Using again the maximal inequality from [41] and the Sobolev embeddings leads to

$$\|I_3\|_{L^2(\Omega)} \leqslant C \sup_{0\leqslant s\leqslant t} \|u(\cdot,s)\|_{H^3(\Omega)} \int_0^t \|v(\cdot,s)\|^2_{H^2(\Omega)}\, \mathrm{d}s \tag{4.89}$$

that is

$$\|I_3\|_{L^2(\Omega)} \leqslant C\mathcal{E}^{3/2}(t). \tag{4.90}$$

Finally, for I_4 we proceed as for obtaining (4.79) and get

$$\|I_4\|_{L^2(\Omega)} \leqslant C\mathcal{E}^{3/2}(t). \tag{4.91}$$

The above estimates lead to the below ones:

$$\|\mathcal{G}_x(\cdot,t)\|_{L^2(\Omega)} \leqslant C\nu(t)\left(\mathcal{E}(t) + \sqrt{\mathcal{E}(t)}\right) \tag{4.92}$$

$$\left\|\int_0^t \mathcal{G}_x(\cdot,s)\mathrm{d}s\right\|_{L^2(\Omega)} \leqslant C\mathcal{E}^{3/2}(t). \tag{4.93}$$

Step 2 From (4.65), we obtain

$$\begin{aligned}
\|u_{xx}(\cdot,t)\|_{L^2(\Omega)} \leqslant \frac{1}{|g'(0)|}\Bigg\{ &\frac{1}{a(0)}\left[\|f(\cdot,t)\|_{L^2(\Omega)} + \|\mathcal{G}(\cdot,t)\|_{L^2(\Omega)} + \|v_t(\cdot,t)\|_{L^2(\Omega)}\right] \\
&+ \|A_1\|_{L^1(\mathbb{R}_+)} \sup_{0\leqslant s\leqslant t}\left[\|f(\cdot,s)\|_{L^2(\Omega)} + \|\mathcal{G}(\cdot,s)\|_{L^2(\Omega)} + \|v_t(\cdot,s)\|_{L^2(\Omega)}\right] \\
&+ \|A_2\|_{L^1(\mathbb{R}_+)} \sup_{0\leqslant s\leqslant t}\left[\left\|\int_0^s f(\cdot,\tau)\mathrm{d}\tau\right\|_{L^2(\Omega)} + \left\|\int_0^s \mathcal{G}(\cdot,\tau)\mathrm{d}\tau\right\|_{L^2(\Omega)} + \|v(\cdot,s)\|_{L^2(\Omega)} + \|v_0\|_{L^2(\Omega)}\right]\Bigg\}.
\end{aligned} \tag{4.94}$$

Using now (4.70) and (4.79) and the fact that $\nu(t)$ and $\mathcal{E}(t)$ are increasing functions, we obtain (4.67). Next, (4.68) is obtained in a similar manner: one produces an equality like that of (4.94) satisfied by $\|u_{xxx}(\cdot,t)\|_{L^2(\Omega)}$ with f_x, $\mathcal{G}_x$, v_{tx}, v_x, v_0' in place of f, $\mathcal{G}$, v_t, v, v_0. Using (4.92) and (4.93), we get (4.68).

□

4.4 Smallness estimates

The next Proposition proves the uniform boundedness of $\mathcal{E}(t)$.

Proposition 4.2 *There exist two numbers $\overline{\mathcal{E}} > 0$ and $\delta > 0$ independent of n such that, whenever v_0 and f verify $F(f) + V_0(v_0) \leqslant \delta$, one has*

$$\mathcal{E}(t) \leqslant \frac{\overline{\mathcal{E}}}{2}, \ \forall t \in [0, T). \tag{4.95}$$

Proof Remark first that, capitalizing on (4.29) and (4.9), one has $v_t(x,0) = f(x,0)$, $v_{xt}(x,0) = f_x(x,0)$, $v_{tt}(x,0) = -g'(0)a(0)v_0''(x) + f_t(x,0)$. From the definition of $\mathcal{E}(t)$, we deduce

$$\mathcal{E}(0) \leqslant \left[1 + 2a^2(0)\,|g'(0)|^2\right] \|v_0\|^2_{H^2(\Omega)} + \int_\Omega \left[f^2(x,0) + f_x^2(x,0) + 2f_t^2(x,0)\right] \mathrm{d}x. \tag{4.96}$$

Therefore,

$$\mathcal{E}(0) \leqslant 2\left[1 + a^2(0)\,|g'(0)|^2\right](F + V_0). \tag{4.97}$$

We now use the fact that the seminorm $w \in H^2(\Omega) \mapsto \|w_{xx}\|_{L^2(\Omega)}$ is a norm on $H^2(\Omega) \cap H^1_0(\Omega)$, equivalent to the usual norm in $H^2(\Omega)$. We shall as well make use of the inequality $(\sqrt{V_0} + \sqrt{F})\sqrt{\mathcal{E}(t)} \leqslant \eta\mathcal{E}(t) + \frac{1}{2\eta}(V_0 + F)$, with $\eta > 0$ small enough.

From Lemmas 4.7, 4.8 and 4.10, we deduce

$$\mathcal{E}(t) \leqslant C\left\{V_0 + F + \left[v(t) + v^3(t)\right]\mathcal{E}(t) + \sqrt{V_0}\mathcal{E}(t) + \mathcal{E}^3(t) + v^2(t)\mathcal{E}^2(t)\right\} \tag{4.98}$$

provided (4.10) holds true.

Recall also the inequality (4.7):

$$v(t) \leqslant c_\Omega\sqrt{\mathcal{E}(t)},\ \forall t \in [0,T). \tag{4.99}$$

Then, we deduce from (4.98) that

$$\mathcal{E}(t) \leqslant c_1\left[V_0 + F + \mathcal{E}^3(t)\right] \tag{4.100}$$

with $c_1 > 0$ a constant independent of n.

Now, observe that we can choose $\overline{\mathcal{E}} > 0$ and $\delta > 0$ such that

$$\begin{cases} c_1\overline{\mathcal{E}}^2 \leqslant \dfrac{1}{2} \\ \overline{\mathcal{E}} < \dfrac{\theta^2}{4C_\Omega} \\ c_1\delta \leqslant \dfrac{\overline{\mathcal{E}}}{4} \\ 2\left[1 + a^2(0)\,|g'(0)|^2\right]\delta \leqslant \dfrac{\overline{\mathcal{E}}}{2} \end{cases}. \tag{4.101}$$

Let us now prove that, for any $t \in [0,T)$, (4.95) holds true. Indeed, if the contrary were true, then invoking the continuity w.r.t. time there exists $t_2 \in (0,T)$ s.t. $\mathcal{E}(t) \leqslant \overline{\mathcal{E}}$, for any $t \in (0,t_2)$, but inequality (4.95) is false on an interval (t_1,t_2) with $0 < t_1 < t_2$. From the second inequality in (4.101), we deduce that (4.100) is satisfied on $[0,t_2]$. Using once more (4.101), one gets $\mathcal{E}(t) \leqslant \frac{\mathcal{E}(t)}{2} + \frac{\overline{\mathcal{E}}}{4}$ which triggers $\mathcal{E}(t) \leqslant \frac{\overline{\mathcal{E}}}{2}$ on $[0,t_2]$, hence a contradiction. This later fact ends the proof.

□

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5 Proof of the main result

Remark that from Proposition 4.2, we actually deduce that for v_n – solution of $(P_n)_1$, $(P_n)_2$, $(P_n)_3$ – we have the following upper bounds:

$$\sup_{t\in[0,T_n)}\left[\|u_n(\cdot,t)\|^2_{H^3(\Omega)}+\|(u_n)_t(\cdot,t)\|^2_{H^2(\Omega)}+\|(u_n)_{tt}(\cdot,t)\|^2_{H^1(\Omega)}+\|(u_n)_{ttt}(\cdot,t)\|^2_{L^2(\Omega)}\right]$$
$$+\int_0^{T_n}\left\{\|v_n(\cdot,t)\|^2_{H^2(\Omega)}+\|(v_n)_t(\cdot,t)\|^2_{H^1(\Omega)}+\|(v_n)_{tt}(\cdot,t)\|^2_{L^2(\Omega)}\right\}\mathrm{d}t\leqslant\frac{\overline{\mathcal{E}}}{2}\tag{5.1}$$

and

$$\sup_{\substack{x\in\Omega\\0<s<t<T_n}}\left|\int_{t-s}^{t}(v_n)_x(x,\tau)\mathrm{d}\tau\right|\leqslant\theta.\tag{5.2}$$

We then deduce from Proposition 4.1 that $T_n=+\infty$, so (5.1) and (5.2) are valid upon replacing T_n by $+\infty$. It follows that there exist two limits

$$u\in\bigcap_{m=0}^{3}W^{m,\infty}\left((0,+\infty);H^{3-m}(\Omega)\right)$$

and

$$v\in\left\{\bigcap_{m=0}^{2}W^{m,\infty}\left((0,+\infty);H^{2-m}(\Omega)\right)\right\}\cap\left\{\bigcap_{m=0}^{2}W^{m,2}\left((0,+\infty);H^{2-m}(\Omega)\right)\right\}$$

with $u(x,t)=\int_0^t v(x,s)\,ds$ s.t. (up to a subsequence of n) we have

$$\frac{d^m u_n}{dt^m}\rightharpoonup\frac{d^m u}{dt^m}\quad\text{weakly}*\ \text{ in }\ L^\infty\left((0,+\infty);H^{3-m}(\Omega)\right),\quad m=0,1,2,3$$

and

$$\frac{d^m v_n}{dt^m}\rightharpoonup\frac{d^m v}{dt^m}\quad\text{weakly in }\ L^2\left((0,+\infty);H^{2-m}(\Omega)\right),\quad m=0,1,2.$$

By the trace theorem, we have $v=0$ for $x\in\partial\Omega$, $t\geqslant0$, and $v(x,0)=v_0(x)$, for $x\in\Omega$. Now, remark that the equation $(P_n)_1$ can be written in the form

$$(v_n)_t(x,t)=-\frac{\partial}{\partial x}\int_0^t a_n(t-s)g'\left((u_n)_x(x,t)-(u_n)_x(x,s)\right)(v_n)_x(x,s)\mathrm{d}s+f(x,t).\tag{5.3}$$

We now pass to the limit in (5.3) above, for any fixed $t\geqslant0$. By the trace theorem, it is clear that $(v_n)_t(\cdot,t)\xrightarrow[n\to+\infty]{L^2(\Omega)}v_t(\cdot,t)$ weakly. Next, we take on to proving that

$$\int_0^t a_n(t-s)g'\left((u_n)_x(x,t)-(u_n)_x(x,s)\right)(v_n)_x(x,s)\mathrm{d}s$$

weakly converges in $L^2(\Omega)$ towards

$$\int_0^t a(t-s)g'\left(u_x(x,t)-u_x(x,s)\right)v_x(x,s)\mathrm{d}s.$$

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Let $\phi \in L^2(\Omega)$ be fixed; we have to prove that

$$E_n \xrightarrow[n\to+\infty]{} E, \tag{5.4}$$

where

$$E_n = \int_{Q_t} \phi(x)a_n(t-s)g'\left((u_n)_x(x,t)-(u_n)_x(x,s)\right)(v_n)_x(x,s)\mathrm{d}x\mathrm{d}s \tag{5.5}$$

$$E = \int_{Q_t} \phi(x)a(t-s)g'\left(u_x(x,t)-u_x(x,s)\right)v_x(x,s)\mathrm{d}x\mathrm{d}s. \tag{5.6}$$

By Sobolev compact inclusion, we have that $(u_n)_x \xrightarrow[n\to+\infty]{C(\overline{Q_t})} u_x$ strongly and $(u_n)_x(\cdot,t) \xrightarrow[n\to+\infty]{C(\overline{\Omega})} u_x(\cdot,t)$ also strongly. From (5.2), with $T_n = +\infty$ we deduce

$$\sup_{\substack{x\in\Omega\\ 0<s<t}} \left|\int_{t-s}^{t} v_x(x,\tau)\mathrm{d}\tau\right| \leqslant \theta. \tag{5.7}$$

Making use of (4.2) leads to the strong convergence

$$g'\left((u_n)_x(x,t)-(u_n)_x(x,s)\right) \xrightarrow[n\to+\infty]{C(\overline{Q_t})} g'\left(u_x(x,t)-u_x(x,s)\right). \tag{5.8}$$

Since $(v_n)_x \xrightarrow[n\to+\infty]{L^2(Q_t)} v_x$ strongly and $a_n \xrightarrow[n\to+\infty]{L^2(0,t)} a$ strongly (consequence of assumption (a_2)), one easily gets (5.4) which ends the proof of Theorem 2.1.

6 A class of totally monotone functions compliant with hypotheses (a_1) to (a_5)

The goal here is to introduce a large class of functions a compliant with assumptions (a_1)–(a_5). The following Lemma gives sufficiently weak enough conditions so that (a_5) holds.

Lemma 6.1 *Assume that $b \in W^{1,1}(0,+\infty)$ satisfies the following conditions:*

(i) $tb' \in L^1(0,+\infty)$
(ii) there exists $M_4 > 0$ and $\alpha_1 > 0$ s.t. $|\mathcal{F}b(\omega)| \geqslant \frac{M_4}{1+|\omega|^{\alpha_1}}$, $\forall \omega \in \mathbb{R}$
(iii) there exists $M_5 > 0$ and $\alpha_2 > 0$ s.t. $|\mathcal{F}b'(\omega)| \leqslant \frac{M_5}{1+|\omega|^{\alpha_2}}$, $\forall \omega \in \mathbb{R}$
(iv) there exists $\alpha_3 \in \mathbb{R}$ s.t. the function $\mathbb{R} \ni t \mapsto tb(t) \in \mathbb{R}$ is an element of $H^{\alpha_3}(\mathbb{R})$.

Then, there exists $M_6 > 0$ depending only on M_4,M_5, α_1, α_2 and α_3, and $p \in \mathbb{N}^*$ depending only on α_1 and α_2 and α_3, s.t.

$$\frac{\left(\mathcal{F}b'\right)^p}{\mathcal{F}b} \in \mathcal{F}\left(B_{L^1(\mathbb{R})}(0,M_7)\right), \tag{6.1}$$

where

$$M_7 = M_6\left[1+\|tb'\|_{L^1(\mathbb{R})}+\|tb\|_{H^{\alpha_3}(\mathbb{R})}\right]. \tag{6.2}$$

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Proof Since $H^1(\mathbb{R}) \subset \mathcal{F}L^1(\mathbb{R})$ and $\|\mathcal{F}^{-1}w\|_{L^1(\mathbb{R})} \leqslant C\|w\|_{H^1(\mathbb{R})}$, $\forall w \in H^1(\mathbb{R})$ (see [30]), it suffices to consider the H^1 norm of $E \equiv \frac{[\mathcal{F}b']^p}{\mathcal{F}b}$. From hypotheses (ii) and (iii), it is clear that, for p large enough depending on α_1 and α_2, we have

$$\|E\|_{L^2(\mathbb{R})} \leqslant M_6, \tag{6.3}$$

where M_6 depends on M_4, M_5 and α_2. We also have $E' = E_1 - E_2$, with

$$E_1 := p\frac{[\mathcal{F}b']^{p-1}[\mathcal{F}b']'}{\mathcal{F}b} \tag{6.4}$$

$$E_2 := \frac{[\mathcal{F}b']^p[\mathcal{F}b]'}{(\mathcal{F}b)^2}. \tag{6.5}$$

Since $|(\mathcal{F}b')'| = |\mathcal{F}(tb')| \in L^\infty(\mathbb{R}_+)$, from the above mentioned assumptions we get there exists p large enough depending on α_1 and α_2 s.t.

$$\|E_1\|_{L^2(\mathbb{R})} \leqslant M_6\|tb'\|_{L^1(\mathbb{R})}. \tag{6.6}$$

From assumption (iv) and the fact that $|(\mathcal{F}b)'| = |\mathcal{F}(tb)|$, we have that the function $\omega \longrightarrow (1+\omega^2)^{\alpha_3/2}(\mathcal{F}b)'(\omega) \in L^2(\mathbb{R})$, and, $\|(1+\omega^2)^{\alpha_3/2}(\mathcal{F}b)'(\omega)\|_{L^2(\mathbb{R})} = \|tb\|_{H^{\alpha_3}(\mathbb{R})}$.

Then, there exists p large enough depending on α_1, α_2 and α_3 s.t.

$$\|E_2\|_{L^2(\mathbb{R})} \leqslant M_6\|tb\|_{H^{\alpha_3}(\mathbb{R})} \tag{6.7}$$

with M_6 as before. From (6.3), (6.6) and (6.7), the claimed result follows.

□

Let μ be a positive, finite and non-zero Borel measure on $\mathbb{R}_+$, satisfying

(μ_1) the function $\mathbb{R}_+ \ni \rho \mapsto \frac{1}{\rho^2}$ is an element of $L^1_\mu(0,+\infty)$

(μ_2) there exists $\gamma \in (0,1)$ s.t. the function $\mathbb{R}_+ \ni \rho \mapsto \rho^\gamma$ is an element of $L^1_\mu(0,+\infty)$.

Remark that, as a consequence of these hypotheses, the function $\mathbb{R}_+ \ni \rho \mapsto \rho^\beta$ is an element of $L^1_\mu(0,+\infty)$ for any $\beta \in [-2,\gamma]$.

We now consider the following totally monotone function (see [37])

$$\tilde{a} : [0,+\infty) \to \mathbb{R},\ \tilde{a}(t) = \int_{\mathbb{R}_+} e^{-\rho t}\mathrm{d}\mu(\rho),\ \forall t \geqslant 0. \tag{6.8}$$

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This Section main result is contained in the below theorem:

Theorem 6.1 *Assume the hypotheses* (μ_1) *and* (μ_2) *hold true. Then, the function* $\tilde{a}$ *given by* (6.8) *satisfies the hypotheses* (a_1)–(a_5) *of Section 2 with*

$$\tilde{a}_n(t) = \int_{[0,n)} e^{-\rho t} \mathrm{d}\mu(\rho), \ \forall t \geqslant 0, \ \forall n \in \mathbb{N}^*.$$

Proof Since the measure μ is finite, it is clear that $\tilde{a}_n \in \mathscr{C}^\infty(\mathbb{R}_+)$, and for any $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$, $(\tilde{a}_n)^{(k)}(t) = \int_{[0,n)} (-1)^k \rho^k e^{-\rho t} \mathrm{d}\mu(\rho)$. This gives $\tilde{a}_n \in W^{p,\infty}(0, +\infty)$, for any $p \in \mathbb{N}$ and also $\tilde{a}_n' < 0$.

Let $k \in \mathbb{N}$ and $q \in \mathbb{R}_+$. Then,

$$\begin{aligned}\int_0^{+\infty} t^q \left(\tilde{a}_n\right)^{(k)}(t) \mathrm{d}t &= (-1)^k \int_0^{+\infty} t^q \int_{[0,n)} \rho^k e^{-\rho t} \mathrm{d}\mu(\rho) \mathrm{d}t \\ &= (-1)^k \int_{[0,n)} \rho^k \left(\int_0^{+\infty} t^q e^{-\rho t} \mathrm{d}t \right) \mathrm{d}\mu(\rho).\end{aligned}$$

Taking $\tau = \rho t$ in the integral w.r.t. t leads to

$$\int_0^{+\infty} t^q \left| (\tilde{a}_n)^{(k)}(t) \right| \mathrm{d}t = \int_0^{+\infty} \tau^q e^{-\tau} \mathrm{d}\tau \int_{[0,n)} \rho^{k-q-1} \mathrm{d}\mu(\rho). \tag{6.9}$$

Invoking hypotheses (μ_1) and (μ_2) gives

$$\int_{[0,+\infty)} \rho^{k-q-1} \mathrm{d}\mu(\rho) < \infty \tag{6.10}$$

provided that

$$0 \leqslant q + 1 - k \leqslant 2. \tag{6.11}$$

For $q = 0$ and $k = 0$ or $k = 1$, one sees that (6.11) is verified, therefore (a_1) and (a_2) are valid.

For $q = 2$ and $k = 1$ (6.11) is also verified, then $\int_0^{+\infty} t^2 |\tilde{a}_n'(t)| \mathrm{d}t$ is bounded. The same for $q = 1$ and $k = 2$, with this time $\int_0^{+\infty} t |\tilde{a}_n''(t)| \mathrm{d}t$ bounded. The later grants (a_3) is valid.

Next, by Fubini's theorem we obtain, for $\omega \in \mathbb{R}$,

$$\mathcal{F}\tilde{a}_n(\omega) = \int_0^{+\infty} \int_{[0,n)} e^{-\rho t} \mathrm{d}\mu(\rho) e^{-i\omega t} \mathrm{d}t = \int_{[0,n)} \frac{\mathrm{d}\mu(\rho)}{\rho + i\omega}$$

from which one gets

$$\mathrm{Re}\left[\mathcal{F}\tilde{a}_n(\omega)\right] = \int_{[0,n)} \frac{\rho}{\rho^2 + \omega^2} \mathrm{d}\mu(\rho).$$

Now, assumption (μ_1) gives $\mu\left(\{0\}\right) = 0$, so, there exists $\underline{\mu}$ and $\overline{\mu}$ s.t. $0 < \underline{\mu} < \overline{\mu}$ and $\mu([\underline{\mu}, \overline{\mu}]) > 0$. Take $n > \overline{\mu}$ to get

$$\mathrm{Re}\left[\mathcal{F}\tilde{a}_n(\omega)\right] \geqslant \frac{\underline{\mu}}{\overline{\mu}^2 + \omega^2} \mu\left([\underline{\mu}, \overline{\mu}]\right), \ \forall \omega \in \mathbb{R}$$

which proves (a_4).

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Now, we prove that the hypotheses of Lemma 6.1 are verified for $b = \tilde{a}_n$, with constants independent of n.

The last inequality also proves that (ii) of Lemma 6.1 is verified with M_3 independent of n and $\alpha_1 = 2$. Taking $q = k = 1$ (which satisfy (6.11)), we deduce that part (i) of Lemma 6.1 is also verified, and that $\|t\tilde{a}_n'\|_{L^1(0,+\infty)}$ is bounded.

Next, on one hand, we easily calculate

$$\mathcal{F}\tilde{a}_n'(\omega) = -\int_{[0,n)} \frac{\rho}{\rho + i\omega} \mathrm{d}\mu(\rho)$$

which gives

$$|\mathcal{F}\tilde{a}_n'(\omega)| \leqslant \int_{[0,n)} \frac{\rho}{\sqrt{\rho^2 + \omega^2}} \mathrm{d}\mu(\rho). \tag{6.12}$$

We deduce that

$$|\mathcal{F}\tilde{a}_n'(\omega)| \leqslant \int_{\mathbb{R}_+} \mathrm{d}\mu(\rho). \tag{6.13}$$

On the other hand now, we use the fact that

$$\rho^{2(1-\gamma)}|\omega|^{2\gamma} \leqslant \gamma|\omega|^2 + (1-\gamma)\rho^2 \leqslant |\omega|^2 + \rho^2$$

to get from (6.12), for $\omega \neq 0$,

$$|\mathcal{F}\tilde{a}_n'(\omega)| \leqslant \int_{[0,n)} \frac{\rho}{\rho^{1-\gamma}|\omega|^\gamma} \mathrm{d}\mu(\rho) = \frac{1}{|\omega|^\gamma} \int_{[0,n)} \rho^\gamma \mathrm{d}\mu(\rho).$$

Invoke (μ_2) to get, for $\omega \neq 0$,

$$|\mathcal{F}\tilde{a}_n'(\omega)| \leqslant \frac{1}{|\omega|^\gamma} \int_{\mathbb{R}_+} \rho^\gamma \mathrm{d}\mu(\rho). \tag{6.14}$$

Then, (6.13) and (6.14) give

$$|\mathcal{F}\tilde{a}_n'(\omega)| \leqslant \frac{2}{1 + |\omega|^\gamma} \int_{\mathbb{R}_+} (1 + \rho^\gamma)\, \mathrm{d}\mu(\rho).$$

Then, the assumption formulated in (iii) of Lemma 6.1 is verified with $\alpha_2 = \gamma$ and a constant M_4 independent of n.

Finally, the inequality (6.11) is verified with $q = 1$ and $k = 0$. From (6.9) and assumption (μ_2), we get

$$\|t\tilde{a}_n\|_{L^1(\mathbb{R}_+)} \leqslant \int_0^{+\infty} \tau e^{-\tau} \mathrm{d}\tau \int_{\mathbb{R}_+} \rho^{-2} \mathrm{d}\mu(\rho) < \infty.$$

The above entails $t\tilde{a}_n$ is bounded in $H^{-1}(\mathbb{R})$; consequently hypothesis (iv) of Lemma 6.1 is verified with $\beta = -1$. We then deduce that the conclusion of Lemma 6.1 is verified with a constant $M_6 > 0$ independent of n. Then, hypothesis (a_5) is verified. □

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Remark 6.1 *The relaxation function of the Doi–Edwards theory,*
$a_{DE}(t) = \sum_{k=1}^{+\infty} \frac{1}{(2k+1)^2} e^{-(2k+1)^2\pi^2 Dt/L^2}$, $t \geqslant 0$,
is actually a particular case of (6.8) *with the measure* $\mu_{DE} = \sum_{k=1}^{+\infty} \frac{1}{(2k+1)^2} \delta_{(2k+1)^2\pi^2 D/L^2}$, *where* $\delta_{(2k+1)^2\pi^2 D/L^2}$ *is Dirac's measure at* $(2k+1)^2\pi^2 D/L^2$.

It is easy to see that the assumptions (μ_1), (μ_2) *are verified for this measure, and this paper results can be applied for the* a_{DE} *function.*

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Appendix A

A.1 Appendix 1

For sake of clarity, we show here how equation (1.12) can be solved employing the method of characteristics in a more general case, that is with $\frac{\partial v}{\partial x}\mathcal{H}_0(u)$ being replaced by the tensor $\mathscr{M}(t,u)$ given in (1.3) with κ an arbitrary tensor. In the end, we shall make the necessary particular assumptions to explicitly obtain the solution of (1.12).

Consider the problem:

$$\frac{\partial F}{\partial t} = D\frac{\partial^2 F}{\partial s^2} - \frac{\partial}{\partial u}\cdot[\mathscr{M}(t,u)F]\,,\ (t,u,s) \in [0,+\infty) \times S_2 \times (0,L) \tag{A 1}$$

$$F(s=0) = F(s=L) = \frac{1}{4\pi} \tag{A 2}$$

$$F(t=0) = F_0. \tag{A 3}$$

Given that $\frac{\partial}{\partial u}\cdot\mathscr{M} = -3\kappa u\cdot u \equiv -3\kappa : uu$, one may re-write (A 1) as

$$\frac{\partial F}{\partial t} = D\frac{\partial^2 F}{\partial s^2} + (3\kappa u\cdot u)F - \mathscr{M}\cdot\frac{\partial F}{\partial u}$$

and further on, with $\overline{F} = F - \frac{1}{4\pi}$ so that $\overline{F}(s=0) = \overline{F}(s=L) = 0$, as

$$\frac{\partial \overline{F}}{\partial t} - D\frac{\partial^2 \overline{F}}{\partial s^2} - (3\kappa u\cdot u)\overline{F} + \mathscr{M}\cdot\frac{\partial \overline{F}}{\partial u} = \frac{3}{4\pi}\kappa u\cdot u. \tag{A 4}$$

We now search for solutions $\overline{F}$ in the form

$$\overline{F}(t,u,s) = \sum_{k=1}^{+\infty}\overline{F}_k(t,u)\sin\frac{k\pi}{L}s.$$

Introducing the above into (A 4) gives, for any $k \in \mathbb{N}^*$,

$$\frac{\partial \overline{F}_k}{\partial t} + \left(\frac{k^2\pi^2 D}{L^2} - 3\kappa u\cdot u\right)\overline{F}_k + \mathscr{M}\cdot\frac{\partial \overline{F}_k}{\partial u} = \left(\frac{3}{4\pi}\kappa u\cdot u\right)a_k$$

with a_k satisfying (1.16). One gets

$$F(t,u,s) = \sum_{k=1}^{+\infty}F_k(t,u)\sin\frac{k\pi}{L}s \tag{A 5}$$

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with F_k solutions to

$$\frac{\partial F_k}{\partial t} + \left(\frac{k^2\pi^2 D}{L^2} - 3\kappa u \cdot u\right) F_k + \mathscr{M} \cdot \frac{\partial F_k}{\partial u} = \frac{\pi}{4}\frac{D}{L^2}k^2 a_k \tag{A 6}$$

$$F_k(t=0) = F_{0k} \tag{A 7}$$

with F_{0k} being given by (1.15). As an aside, in the above x is to be considered only as a parameter, not entering the calculations.

To solve (A 6), (A 7) we use the method of characteristics. Specifically, for all $t, \tau \geqslant 0$, denote $M(t,\tau)$ the unique solution of the differential system

$$\frac{\partial M}{\partial t} = \kappa(t) M,$$

$$M(t=\tau,\tau) = Id_3,$$

where Id_3 is the Kronecker's delta (unity) tensor. For any $w \in S_2$, set $\xi(t,\tau,w) = \frac{M(t,\tau)w}{\|M(t,\tau)w\|} \in S_2$. One easily checks that

$$\frac{\partial \xi}{\partial t} = \kappa(t)\xi - [\kappa(t)\xi \cdot \xi]\,\xi,\ t,\tau \geqslant 0$$

and that

$$\xi(t=\tau,\tau,w) = w,\ \tau \geqslant 0.$$

Equation (A 6) is now solved along the characteristic curves $\xi(t,0,w)$. To do so, for any fixed $w \in S_2$, let $A_k(t,w) = F_k\,(t, \xi(t,0,w))$; from (A 6), one obtains

$$\frac{\partial A_k}{\partial t} = \left[3\kappa(t)\xi(t,0,w)\cdot\xi(t,0,w) - \frac{k^2\pi^2 D}{L^2}\right] A_k + \frac{\pi}{4}\frac{D}{L^2}k^2 a_k$$

with

$$A_k(t=0) = F_{0k}(w).$$

One gets

$$\begin{aligned} A_k(t) &= \exp\left[-\frac{k^2\pi^2 D}{L^2}t + 3\int_0^t \kappa(\tau)\xi(\tau,0,w)\cdot\xi(\tau,0,w)\mathrm{d}\tau\right] F_{0k}(w) \\ &+ \frac{\pi}{4}\frac{D}{L^2}k^2 a_k \int_0^t \exp\left[-\frac{k^2\pi^2 D}{L^2}(t-\tau) + 3\int_0^t \kappa(r)\xi(r,0,w)\cdot\xi(r,0,w)\mathrm{d}r\right]\mathrm{d}\tau. \end{aligned} \tag{A 8}$$

Letting $u = \xi(t,0,w)$ implies $\xi(\tau,0,w) = \xi(\tau,t,u)$ (because $w = \xi(0,t,u)$). Next, for any $t_1, t_2 \geqslant 0$,

$$\begin{aligned} \int_{t_1}^{t_2} \kappa(\tau)\xi(\tau,t,u)\cdot\xi(\tau,t,u)\mathrm{d}\tau &= \int_{t_1}^{t_2} \kappa(\tau)\frac{M(\tau,t)w}{\|M(\tau,t)w\|}\cdot\frac{M(\tau,t)w}{\|M(\tau,t)w\|}\mathrm{d}\tau \\ &= \int_{t_1}^{t_2} \frac{\frac{\partial}{\partial\tau}(M(\tau,t)w)}{\|M(\tau,t)w\|}\cdot\frac{M(\tau,t)w}{\|M(\tau,t)w\|}\mathrm{d}\tau = \log\left(\frac{\|M(t_2,t)u\|}{\|M(t_1,t)u\|}\right). \end{aligned}$$

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with the help of the above and from (A 8), we get

$$F_k(t,u) = \frac{e^{-k^2\pi^2 Dt/L^2}}{\|M(0,t)u\|^3} F_{0k}\left(\frac{M(0,t)u}{\|M(0,t)u\|}\right) + \frac{\pi D k^2}{4L^2} a_k \int_0^t \frac{e^{-k^2\pi^2 D(t-\tau)/L^2}}{\|M(\tau,t)u\|^3} \mathrm{d}\tau. \tag{A 9}$$

We conclude that the solution of (A 1)–(A 3) is given by (A 5), with F_k from (A 9) above. In the particular case where $\mathscr{M}(t,u) = \frac{\partial v}{\partial x}\mathcal{H}_0(u)$ as in (1.12), one has

$$M(t,\tau) = Id_3 + \left[\int_\tau^t \frac{\partial v}{\partial x}(x,r)\mathrm{d}r\right] M_0,$$

for all $t, \tau \geqslant 0$.

This allows to obtain (1.14) knowing that

$$M(\tau,t)u = \left(u_1 - \left[\int_\tau^t \frac{\partial v}{\partial x}(x,r)\mathrm{d}r\right] u_2, u_2, u_3\right).$$

A.2 Appendix 2

The task here is to prove Lemma 4.6, relabelled below as Lemma A.2.

Let the function $\xi = \xi(s,t,x)$ be defined a.e. as $\xi(s,t,x) := a'(s)\left[g'\left(\overline{v}_x^t(x,s)\right) - g'(0)\right]$, $s \in [0,+\infty)$, $t \in [0,T)$, $x \in \Omega$. Let $D_T := \{(s,t) : s \in [0,+\infty),\, t \in [0,T),\, s \neq t\}$.

In the following, $\partial_1\xi$, $\partial_2\xi$, $\partial_{22}\xi$ stand for $\frac{\partial\xi}{\partial s}$, $\frac{\partial\xi}{\partial t}$, and $\frac{\partial^2\xi}{\partial t^2}$, respectively.

The first step is proving the following:

Lemma A.1 *Invoking the above defined notations,*

(*i*) *one has:* $\xi \in \mathscr{C}^1\left(D_T; H^1(\Omega)\right)$, $\frac{\partial^2\xi}{\partial t^2} \in \mathscr{C}^0\left(D_T; L^2(\Omega)\right)$;

(*ii*) *assuming* (4.10) *holds true, one has the following estimates a.e.* $x \in \Omega$, $s \in [0,+\infty)$

$$|\xi(s,t,x)| \leqslant K v(t)\,|a'(s)|\, r_0(s) \tag{A 10}$$

$$\left|\frac{\partial\xi}{\partial t}(s,t,x)\right| \leqslant 2K\theta v(t)\,|a'(s)| \tag{A 11}$$

$$\left|\frac{\partial\xi}{\partial s}(s,t,x)\right| \leqslant K v(t)\left[|a''(s)|\, r_0(s) + \theta\,|a'(s)|\right] \tag{A 12}$$

$$\left|\frac{\partial^2\xi}{\partial t^2}(s,t,x)\right| \leqslant 4v^2(t)\left[K\theta + \left|g^{(3)}(0)\right|\right]|a'(s)| + K v(t)\,|a'(s)|\, r_0(s)\left[|v_{xt}(x,t)| + |v_{xt}(x,t-s)|\right]. \tag{A 13}$$

The above derivatives may be considered in the classical sense, as they are defined for $s \neq t$.

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Proof Observe that

$$\frac{\partial \xi}{\partial t} = a'(s)g''\left(\overline{v}_x^t(s)\right)[v_x(t) - v_x(t-s)]$$

$$\frac{\partial \xi}{\partial s} = a''(s)\left[g'\left(\overline{v}_x^t(s)\right) - g'(0)\right] + a'(s)g''\left(\overline{v}_x^t(s)\right)v_x(t-s)$$

$$\frac{\partial^2 \xi}{\partial t^2} = a'(s)g^{(3)}\left(\overline{v}_x^t(s)\right)[v_x(t) - v_x(t-s)]^2 + a'(s)g''\left(\overline{v}_x^t(s)\right)[v_{xt}(t) - v_{xt}(t-s)]\,.$$

Repeated use of part (i) of Lemma 4.3 triggers the result.

□

For sake of clarity and – last but not least – reader's convenience, we restate Lemma's 4.6 content and then achieve its proof.

Lemma A.2 *Under the assumption that* (4.10) *is fulfilled, one has*

$$\int_\Omega v_{tt}^2(x,t)\mathrm{d}x - 2g'(0)\lim_{h\to 0_+}\frac{1}{h^2}Q\left(\triangle_h v_{xt}, t, a\right) \leqslant C\Big\{F + \sqrt{F}\sqrt{\mathcal{E}(t)} + \left[v(t) + v^3(t)\right]\mathcal{E}(t) + \sqrt{V_0}\mathcal{E}(t)\Big\}. \tag{A 14}$$

Proof Derivate (2.4) w.r.t. t and apply $\triangle_h$ on the resulting equation. One gets

$$\triangle_h v_{tt} = \int_0^{+\infty} a'(s)\triangle_h\left(g\left(\overline{v}_x^t(s)\right)\right)_{xt}\mathrm{d}s + \triangle_h f_t. \tag{A 15}$$

Multiply the above by $\triangle_h v_t$, integrate on $\Omega \times [0,t]$ to obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega [\triangle_h v_t(x,t)]^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega [\triangle_h v_t(x,0)]^2\,\mathrm{d}x \\ = -\int_0^t\int_\Omega\int_0^{+\infty} a'(\tau)\triangle_h g\left(\overline{v}_x^s(x,\tau)\right)_s\triangle_h v_{xt}(x,s)\mathrm{d}\tau\mathrm{d}x\mathrm{d}s \\ + \int_0^t\int_\Omega \triangle_h f_t(x,s)\triangle_h v_t(x,s)\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{A 16}$$

Observing that

$$g\left(\overline{v}_x^s(x,\tau)\right)_s = g'\left(\overline{v}_x^s(x,\tau)\right)[v_x(x,s) - v_x(x,s-\tau)]$$

leads to

$$-\int_0^t\int_\Omega\int_0^{+\infty} a'(\tau)\triangle_h g\left(\overline{v}_x^s(x,\tau)\right)_s\triangle_h v_{xt}(x,s)\mathrm{d}\tau\mathrm{d}x\mathrm{d}s = I_1 + I_2 + I_3 + I_4, \tag{A 17}$$

where

$$I_1 = -\int_0^t\int_\Omega\int_0^{+\infty} a'(\tau)\triangle_h v_{xt}(x,s)\triangle_h g'\left(\overline{v}_x^s(x,\tau)\right)[v_x(s+h) - v_x(s+h-\tau)]\,\mathrm{d}\tau\mathrm{d}x\mathrm{d}s \tag{A 18}$$

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$$I_2 = -\int_0^t \int_\Omega \int_0^{+\infty} a'(\tau)\triangle_h v_{xt}(x,s)\left[g'\left(\overline{v}_x^s(x,\tau)\right) - g'(0)\right]\triangle_h v_x(x,s)\mathrm{d}\tau\mathrm{d}x\mathrm{d}s \tag{A 19}$$

$$I_3 = g'(0)\int_0^t \int_\Omega \int_0^{+\infty} a'(\tau)\triangle_h v_{xt}(x,s)\left[\triangle_h v_x(s-\tau) - \triangle_h v_x(s)\right]\mathrm{d}\tau\mathrm{d}x\mathrm{d}s \tag{A 20}$$

$$I_4 = \int_0^t \int_\Omega \int_0^{+\infty} a'(\tau)\triangle_h v_{xt}(x,s)\left[g'\left(\overline{v}_x^s(x,\tau)\right) - g'(0)\right]\triangle_h v_x(s-\tau)\mathrm{d}\tau\mathrm{d}x\mathrm{d}s. \tag{A 21}$$

Integrating by parts w.r.t. s leads to $I_1 = I_{11} + I_{12}$, where

$$\begin{aligned} I_{11} = &-\int_\Omega \int_0^{+\infty} a'(\tau)\triangle_h v_x(x,t)\triangle_h g'\left(\overline{v}_x^t(x,\tau)\right)\left[v_x(x,t+h) - v_x(x,t+h-\tau)\right]\mathrm{d}\tau\mathrm{d}x \\ &+ \int_0^t \int_\Omega \int_0^{+\infty} a'(\tau)\triangle_h v_x(x,s)\triangle_h\left[g''\left(\overline{v}_x^s(x,\tau)\right)\left(v_x(x,s) - v_x(x,s-\tau)\right)\right] \\ &\times \left[v_x(x,s+h) - v_x(x,s+h-\tau)\right]\mathrm{d}\tau\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_\Omega \int_0^{+\infty} a'(\tau)\triangle_h v_x(x,s)\triangle_h g'\left(\overline{v}_x^s(x,\tau)\right)\left[v_{xt}(x,s+h) - v_{xt}(x,s+h-\tau)\right]\mathrm{d}\tau\mathrm{d}x\mathrm{d}s \end{aligned} \tag{A 22}$$

and

$$\begin{aligned} I_{12} = &\int_\Omega \int_0^{+\infty} a'(\tau)\triangle_h v_x(0)\triangle_h g'\left(\overline{v}_x^0(x,\tau)\right)\left[v_x(x,h) - v_x(x,h-\tau)\right]\mathrm{d}\tau\mathrm{d}x \\ &- \int_0^t \int_\Omega a'(s+h)\triangle_h g'\left(\int_0^s v_x(x,\lambda)\mathrm{d}\lambda\right) v_0'(x)\triangle_h v_x(x,s)\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{A 23}$$

Observe that

$$\begin{aligned} &\int_\Omega \int_0^{+\infty} a'(\tau)\triangle_h v_x(0)\triangle_h g'\left(\overline{v}_x^0(x,\tau)\right)\left[v_x(x,h) - v_x(x,h-\tau)\right]\mathrm{d}\tau\mathrm{d}x \\ &= \int_\Omega \left[v_x(h) - v_x(0)\right]\int_0^h a'(\tau)\left[g'\left(\int_{h-\tau}^h v_x(\lambda)\mathrm{d}\lambda\right) - g'(0)\right]\left[v_x(h) - v_x(h-\tau)\right]\mathrm{d}\tau \\ &\quad - \int_\Omega a(h)\left[v_x(h) - v_x(0)\right]\left[g'\left(\int_0^h v_x(\lambda)\mathrm{d}\lambda\right) - g'(0)\right] v_x(h)\mathrm{d}x. \end{aligned} \tag{A 24}$$

By integrating the first term by parts w.r.t. τ, one gets

$$\begin{aligned} I_{12} = &\int_\Omega \left[v_x(h) - v_x(0)\right]^2 a(h)\left[g'\left(\int_0^h v_x(\lambda)\mathrm{d}\lambda\right) - g'(0)\right]\mathrm{d}x \\ &- \int_\Omega \left[v_x(h) - v_x(0)\right]\int_0^h a(\tau)g''\left(\int_{h-\tau}^h v_x(\lambda)\mathrm{d}\lambda\right) v_x(h-\tau)\left[v_x(h) - v_x(h-\tau)\right]\mathrm{d}\tau\mathrm{d}x \\ &- \int_\Omega \left[v_x(h) - v_x(0)\right]\int_0^h a(\tau)\left[g'\left(\int_{h-\tau}^h v_x(\lambda)\mathrm{d}\lambda\right) - g'(0)\right] v_{xt}(h-\tau)\mathrm{d}\tau\mathrm{d}x \end{aligned}$$

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$$- \int_{\Omega} a(h) \left[v_x(h) - v_x(0)\right] \left[g' \left(\int_0^h v_x(\lambda) d\lambda \right) - g'(0)\right] v_x(h) dx$$
$$- \int_0^t \int_{\Omega} a'(s+h) \triangle_h g' \left(\int_0^s v_x(\lambda) d\lambda \right) v_0'(x) \triangle_h v_x(s) dx ds. \quad \text{(A 25)}$$

Next, dividing the above by h^2, passing to the limit for $h \to 0_+$ and using the fact that v and its derivatives up to order 2 belong to $\mathscr{C}^2\left([0,T); L^2(\Omega)\right)$ leads to

$$\frac{1}{h^2} I_1 \xrightarrow[h \to 0_+]{} J_1 + J_{01}, \quad \text{(A 26)}$$

where

$$J_1 = - \int_{\Omega} \int_0^{+\infty} \partial_2 \xi(\tau, t, x) v_{xt}(x,t) \left[v_x(x,t) - v_x(x, t-\tau)\right] d\tau dx$$
$$+ \int_0^t \int_{\Omega} \int_0^{+\infty} \partial_{22} \xi(\tau, s, x) v_{xt}(x,s) \left[v_x(x,s) - v_x(x, s-\tau)\right] d\tau dx ds$$
$$+ \int_0^t \int_{\Omega} \int_0^{+\infty} \partial_2 \xi(\tau, s, x) v_{xt}(x,s) \left[v_{xt}(x,s) - v_{xt}(x, s-\tau)\right] d\tau dx ds \quad \text{(A 27)}$$

and

$$J_{01} = - \int_0^t \int_{\Omega} a'(s) v_{xt}(x,s) g'' \left(\overline{v}_x^s(x,s)\right) v_x(x,s) v_0'(x) dx ds. \quad \text{(A 28)}$$

The term I_2 can be re-written as

$$I_2 = -\frac{1}{2} \int_0^t \int_{\Omega} \int_0^{+\infty} \xi(\tau, s, x) \frac{\partial}{\partial s} \left|\triangle_h v_x\right|^2 (x,s) d\tau dx ds$$
$$= -\frac{1}{2} \int_{\Omega} \int_0^{+\infty} \xi(\tau, t, x) \left|\triangle_h v_x(x,t)\right|^2 d\tau dx$$
$$+ \frac{1}{2} \int_0^t \int_{\Omega} \int_0^{+\infty} \partial_2 \xi(\tau, s, x) \left|\triangle_h v_x(x,s)\right|^2 d\tau dx ds. \quad \text{(A 29)}$$

Dividing by h^2 and passing to the limit for $h \to 0_+$, one obtains

$$\frac{1}{h^2} I_2 \xrightarrow[h \to 0_+]{} J_2, \quad \text{(A 30)}$$

where

$$J_2 = -\frac{1}{2} \int_{\Omega} \int_0^{+\infty} \xi(\tau, t, x) \left|v_{xt}(x,t)\right|^2 d\tau dx$$
$$+ \frac{1}{2} \int_0^t \int_{\Omega} \int_0^{+\infty} \partial_2 \xi(\tau, s, x) \left|v_{xt}(x,s)\right|^2 d\tau dx ds. \quad \text{(A 31)}$$

Next, $I_3 = I_{31} + I_{32} + I_{33}$, where

$$I_{31} = g'(0) \int_0^t \int_{\Omega} \int_0^s a'(\tau) \triangle_h v_{xt}(x,s) \triangle_h v_x(x, s-\tau) d\tau dx ds \quad \text{(A 32)}$$

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$$I_{32} = g'(0) \int_0^t \int_\Omega \int_s^{s+h} a'(\tau) \triangle_h v_{xt}(x,s) v_x(x, s+h-\tau) \mathrm{d}\tau \mathrm{d}x \mathrm{d}s \tag{A 33}$$

$$I_{33} = g'(0) a(0) \int_0^t \int_\Omega \triangle_h v_{xt}(x,s) \triangle_h v_x(x,s) \mathrm{d}x \mathrm{d}s. \tag{A 34}$$

Upon integration by parts w.r.t. τ leads to

$$\begin{aligned} I_{31} = {} & g'(0) \int_0^t \int_\Omega a(s) \triangle_h v_{xt}(x,s) \triangle_h v_x(x,0) \mathrm{d}x \mathrm{d}s \\ & - g'(0) a(0) \int_0^t \int_\Omega \triangle_h v_{xt}(x,s) \triangle_h v_x(x,s) \mathrm{d}x \mathrm{d}s \\ & + g'(0) Q\left(\triangle_h v_{xt}, a, t\right). \end{aligned} \tag{A 35}$$

The above implies, upon simplification and integration by parts w.r.t. s, that

$$\begin{aligned} I_3 = {} & g'(0) Q\left(\triangle_h v_{xt}, a, t\right) + g'(0) \int_\Omega a(t) \triangle_h v_x(t) \triangle_h v_x(0) \mathrm{d}x \\ & - g'(0) a(0) \int_\Omega \left(\triangle_h v_x(0)\right)^2 \mathrm{d}x - g'(0) \int_0^t \int_\Omega a'(s) \triangle_h v_x(s) \triangle_h v_x(0) \mathrm{d}x \mathrm{d}s \\ & - g'(0) \int_0^t \int_\Omega \int_s^{s+h} a'(\tau) \triangle_h v_t(x,s) v_{xx}(x, s+h-\tau) \mathrm{d}\tau \mathrm{d}x \mathrm{d}s. \end{aligned} \tag{A 36}$$

Divide the above by h^2 and taking the lower limit for $h \to 0_+$ gives

$$\liminf_{h \to 0_+} \frac{1}{h^2} I_3 = g'(0) \liminf_{h \to 0_+} \frac{1}{h^2} Q\left(\triangle_h v_{xt}, a, t\right) + J_3, \tag{A 37}$$

where

$$\begin{aligned} J_3 = g'(0) \Big\{ & a(t) \int_\Omega v_{xt}(x,t) v_{xt}(x,0) \mathrm{d}x - a(0) \int_\Omega v_{xt}^2(x,0) \mathrm{d}x \\ & - \int_0^t \int_\Omega a'(s) v_{xt}(x,s) v_{xt}(x,0) \mathrm{d}x \mathrm{d}s - \int_0^t \int_\Omega a'(s) v_{tt}(x,s) v_0''(x) \mathrm{d}x \mathrm{d}s \Big\}. \end{aligned} \tag{A 38}$$

Next, we end up with the same result as in (A 37) with $\left(\liminf_{h \to 0_+}\right)$ being replaced by $\left(\limsup_{h \to 0_+}\right)$.

Now, we can write I_4 in the form

$$I_4 = \int_0^t \int_\Omega \left[\int_0^s \xi(\tau,s) \triangle_h v_x(s-\tau) \mathrm{d}\tau + \int_s^{s+h} \xi(\tau,s) v_x(s+h-\tau) \mathrm{d}\tau \right] \triangle_h v_{xt}(x,s) \mathrm{d}x \mathrm{d}s. \tag{A 39}$$

An integration by parts w.r.t. s gives

$$I_4 = I_{41} + I_{42} + I_{43} + I_{44}, \tag{A 40}$$

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where

$$I_{41} = -\int_0^t \int_\Omega \int_0^s \left[\partial_2 \xi(\tau,s)\triangle_h v_x(x,s-\tau) + \xi(\tau,s)\triangle_h v_{xt}(x,s-\tau)\right] \mathrm{d}\tau \triangle_h v_x(x,s)\mathrm{d}x\mathrm{d}s \quad \text{(A 41)}$$

$$I_{42} = -\int_0^t \int_\Omega \int_s^{s+h} \left[\partial_2 \xi(\tau,s) v_x(x,s+h-\tau) + \xi(\tau,s) v_{xt}(x,s+h-\tau)\right] \mathrm{d}\tau \triangle_h v_x(x,s)\mathrm{d}x\mathrm{d}s \quad \text{(A 42)}$$

$$I_{43} = -\int_0^t \int_\Omega \left[\xi(s+h,s) - \xi(s,s)\right] v_0'(x)\triangle_h v_x(x,s)\mathrm{d}x\mathrm{d}s \quad \text{(A 43)}$$

and

$$I_{44} = \left[\int_\Omega \int_0^{s+h} \xi(\tau,s)\triangle_h v_x(x,s-\tau)\triangle_h v_x(x,s)\mathrm{d}\tau\mathrm{d}x\right]_{s=0}^{s=t}. \quad \text{(A 44)}$$

We now deal with the second term in I_{41}; we have

$$-\int_0^s \xi(\tau,s)\triangle_h v_{xt}(x,s-\tau)\mathrm{d}\tau = \xi(s,s)\left[v_x(h) - v_x(0)\right] - \int_0^s \partial_1 \xi(\tau,s)\triangle_h v_x(x,s-\tau)\mathrm{d}\tau \quad \text{(A 45)}$$

fact that allows to get

$$\begin{aligned} I_{41} = &-\int_0^t \int_\Omega \int_0^s \left[\partial_1 \xi(\tau,s) + \partial_2 \xi(\tau,s)\right] \triangle_h v_x(x,s-\tau)\triangle_h v_x(x,s)\mathrm{d}\tau\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_\Omega \xi(s,s)\left[v_x(h) - v_x(0)\right]\triangle_h v_x(x,s)\mathrm{d}x\mathrm{d}s. \end{aligned} \quad \text{(A 46)}$$

Now, we obtain

$$\frac{1}{h^2} I_4 \xrightarrow[h\to 0_+]{} J_4 + J_{04}, \quad \text{(A 47)}$$

where

$$\begin{aligned} J_4 = &-\int_0^t \int_\Omega \int_0^s \left[\partial_1 \xi(\tau,s) + \partial_2 \xi(\tau,s)\right] v_{xt}(x,s-\tau) v_{xt}(x,s)\mathrm{d}\tau\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_\Omega \xi(\tau,t) v_{xt}(x,t-\tau) v_{xt}(x,t)\mathrm{d}x\mathrm{d}\tau \end{aligned} \quad \text{(A 48)}$$

and

$$J_{04} = -\int_0^t \int_\Omega \left[\partial_1 \xi(s,s) + \partial_2 \xi(s,s)\right] v_0'(x) v_{xt}(x,s)\mathrm{d}x\mathrm{d}s. \quad \text{(A 49)}$$

Now, from (A 16), (A 17), (A 26), (A 27), (A 30), (A 31), (A 37), (A 38), (A 47), (A 48), we deduce that

$$\frac{1}{2}\int_{\Omega} v_{tt}^2(x,t)\mathrm{d}x - \frac{1}{2}\int_{\Omega} v_{tt}^2(x,0)\mathrm{d}x = g'(0)\lim_{h\to 0_+}\frac{1}{h^2}Q\left(\triangle_h v_{xt}, a, t\right)$$
$$+\int_0^t\int_{\Omega} v_{tt}(x,s)f_{tt}(x,s)\mathrm{d}x\mathrm{d}s + J_1 + J_2 + J_3 + J_4 + J_{01} + J_{04} \tag{A 50}$$

with J_1–J_4 being given by (A 27), (A 31), (A 38) and (A 48), respectively. One now needs to appropriately bound the terms J_1–J_4, J_{01} and J_{04}. It may be easily seen, using Lemma A.1, that all terms J_1, J_2 and J_4 can be bounded by one of the following type of expressions:

$$cv^k(t)\int_{\Omega}|w_1(x,t)|\,|w_2(x,t)|\,\mathrm{d}x \tag{A 51}$$

or

$$cv^k(t)\int_0^t\int_{\Omega}|w_1(x,s)|\,|w_2(x,s)|\,\mathrm{d}x\mathrm{d}s \tag{A 52}$$

or

$$cv^k(t)\int_0^t\int_{\Omega}\varphi(\tau)\,|w_1(x,t-\tau)|\,|w_2(x,t)|\,\mathrm{d}x\mathrm{d}\tau \tag{A 53}$$

or

$$cv^k(t)\int_0^t\int_{\Omega}\int_0^s\varphi(\tau)\,|w_1(x,s-\tau)|\,|w_2(x,s)|\,\mathrm{d}\tau\mathrm{d}x\mathrm{d}s, \tag{A 54}$$

where $\varphi \geqslant 0$ is a given function in $L^1(\mathbb{R}_+)$ depending on a, $c > 0$ is a constant, w_1, w_2 stand for either v or one of its derivatives up to second order, and $k \in \{1,2,3\}$. This is a consequence of assumption (a_3).

Terms like (A 51) and (A 52) can easily be bounded by $cv^k(t)\mathcal{E}(t)$. Using Lemma 4.2, terms like (A 53) and (A 54) can also be easily bounded by $cv^k(t)\mathcal{E}(t)$. We then obtain that there exists a constant $c > 0$ s.t.

$$J_1 + J_2 + J_4 \leqslant c\left[v(t) + v^3(t)\right]\mathcal{E}(t). \tag{A 55}$$

The estimates for J_3, J_{01} and J_{04} are simpler to obtain since they contain initial data. Using (4.33), we get $v_{xt}(x,0) = f_x(x,0)$. It easily follows that

$$|J_3| \leqslant |g'(0)|\left(|a(t)| + \|a'\|_{L^1(\mathbb{R}_+)}\right)\left[\left(\sqrt{F}+\sqrt{V_0}\right)\sqrt{\mathcal{E}(t)} + a(0)F\right] \tag{A 56}$$

$$|J_{01}| \leqslant K\theta\|a'\|_{L^1(\mathbb{R}_+)}\|v_0\|_{H^2(\Omega)}\mathcal{E}(t) \tag{A 57}$$

and

$$|J_{04}| \leqslant 3K\left(\theta\|a'\|_{L^1(\mathbb{R}_+)} + \|a''r_0\|_{L^1(\mathbb{R}_+)}\right)\|v_0\|_{H^2(\Omega)}v(t)\sqrt{\mathcal{E}(t)} \tag{A 58}$$

From (A 50), (A 55), (A 56) and (A 58), the result stated in the Lemma now follows.

□