Canad. Math. Bull. Vol. **17** (4), 1974

A NOTE ON MAL'CEVIAN VARIETIES

BY
AHMAD SHAFAAT

By a homomorphic relation over an algebra A we mean a subalgebra of $A \times A$.

A variety [1] $\mathscr{V}$ of algebras will be called Mal'cevian [2] if the identities of $\mathscr{V}$ include two identities of the form $f(x, y, y)=x, f(x, x, y)=y$. In [3] many examples and interesting properties of Mal'cevian varieties have been quoted or proved. In [4] it is noted that every reflexive homomorphic relation over an algebra of a Mal'cevian variety is a congruence. The purpose of this short note is to observe that the property of Mal'cevian varieties noted in [4] is in fact characteristic of such varieties.

THEOREM. *A variety $\mathscr{V}$ is Mal'cevian if and only if:* ($M'5$) *reflexive homomorphic relations over its algebras are congruences.*

Proof. By [4] every Mal'cevian variety satisfies ($M'5$). Assume therefore that $\mathscr{V}$ is a variety satisfying ($M'5$). Let F be the free algebra of $\mathscr{V}$ freely generated by two elements, a, b. Let R be the subalgebra of $F \times F$ generated by $\langle a, a\rangle$, $\langle a, b\rangle$, $\langle b, b\rangle$. It is immediately seen that R is reflexive. By ($M'5$) this implies that R is symmetric and in particular $\langle b, a\rangle \in R$. This means that there is a polynomial or word $f(x, y, z)$ such that $f(\langle a, a\rangle, \langle a, b\rangle, \langle b, b\rangle)=\langle b, a\rangle$, or $f(a, a, b)=b$, $f(a, b, b)=a$. Since a, b freely generate F we have $f(x, x, y)=y$, $f(x, y, y)=x$ holding identically in F and therefore in all algebras of $\mathscr{V}$. This proves that $\mathscr{V}$ is Mal'cevian and hence the theorem.

Note that we have not used the full strength of ($M'5$) in the above proof; we only used the symmetry of a reflexive homomorphic relation.

In addition to ($M'5$) there are many other known characterizations of Mal'cevian varieties. Three of them are listed in [3] as ($M1$), ($M2$), ($M3$). One may add [5] to this list the following: ($M4$). For every subdirect product R of algebras A, $B \in \mathscr{V}$ there exist onto homomorphisms $\varphi: A \to C$, $\psi: B \to C$ such that $R=\{\langle a, b\rangle; a\varphi=b\psi\}$. ($M5$) Every reflexive homomorphic relation over an algebra of $\mathscr{V}$ is symmetric.

REFERENCES

1. P. M. Cohn, *Universal Algebra*, Harper and Row (1965).
2. A. I. Mal'cev, *On the general theory of algebraic systems*, Mat. Sbornik N.S., **35** (1954), 3–20.
3. J. Lambek, *Goursat's theorem and the Zassenhaus lemma*, Canad. J. Math. **10** (1957), 45–56.
4. G. D. Findlay, *Reflexive homomorphic relations*, Canad. Math. Bull. **3(2)** 1960, 131–132.
5. Isidore Fleischer, *A note on subdirect products*, Acta Math. Acad. Sci. Hungar., **6** (1955), 463–465.

DEPT. OF MATH.,
LAVAL UNIVERSITY,
QUEBEC, P.Q.
CANADA

https://doi.org/10.4153/CMB-1974-112-3 Published online by Cambridge University Press