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On finite time blow-up for the mass-critical Hartree equations

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We consider the fractional Schrödinger equations with focusing Hartree-type nonlinearities. When the energy is negative, we show that the solution blows up in a finite time. For this purpose, based on Glassey's argument, we obtain a virial-type inequality.

1. Introduction

In this paper we consider the Cauchy problem of the focusing fractional nonlinear Schrödinger equations

$$\left.\begin{aligned} \mathrm{i}\partial_t u &= |\nabla|^\alpha u + F(u) \quad \text{in } \mathbb{R}^{1+n}\times\mathbb{R}, \\ u(x,0) &= \varphi(x) \quad \text{in } \mathbb{R}^n, \end{aligned}\right\} \tag{1.1}$$

where $|\nabla| = (-\Delta)^{1/2}$, $n \geqslant 2$, $\alpha \geqslant 1$ and $F(u)$ is a non-local nonlinear term of Hartree type given by

$$F(u)(x) = -\left(\frac{\psi(\cdot)}{|\cdot|^\gamma} * |u|^2\right)(x)u(x) \equiv -V_\gamma(|u|^2)(x)u(x).$$

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Here, $0 \leqslant \psi \in L^\infty(\mathbb{R}^n)$ and $0 < \gamma < n$. We say that (1.1) is focusing since $-V_\gamma(|u|^2)$ serves as an attractive self-reinforcing potential. We also use the simplified notation V_γ to denote $V_\gamma(|u|^2)$.

When ψ is homogeneous of degree zero (for example, $\psi \equiv 1$), (1.1) has scaling invariance. In fact, if u is a solution of (1.1), then u_λ, $\lambda > 0$, given by

$$u_\lambda(t,x) = \lambda^{-(\gamma-\alpha)/2+n/2} u(\lambda^\alpha t, \lambda x),$$

is also a solution. We denote the critical Sobolev exponent by $s_c = (\gamma - \alpha)/2$. Under the scaling $u \to u_\lambda$, the $\dot{H}^{s_c}$-norm of data is preserved. The solution u of (1.1) formally satisfies the mass and energy conservation laws

$$\left.\begin{aligned} m(u) &= \|u(t)\|_{L^2}^2, \\ E(u) &= K(u) + V(u), \end{aligned}\right\} \tag{1.2}$$

where

$$K(u) = \tfrac{1}{2}\langle u, |\nabla|^\alpha u\rangle, \qquad V(u) = -\tfrac{1}{4}\langle u, V_\gamma(|u|^2)u\rangle.$$

Here $\langle\cdot,\cdot\rangle$ is the complex inner product in L^2. In view of scaling invariance and the conservation laws, when each conserved quantity is invariant under scaling, we say that (1.1) is mass critical if $\gamma = \alpha$ and energy critical if $\gamma = 2\alpha$.

The aim of this paper is to show the finite time blow-up of solutions to the fractional or higher-order equations when (1.1) is mass critical. If the energy is negative (i.e. the magnitude of the potential energy $V(u)$ is larger than that of kinetic part $K(u)$), then self-attracting power dominates the overall dynamics and so it may result in a collapse in a finite time. For the usual Schrödinger equations ($\alpha = 2$), Glassey [6] introduced a convexity argument to show existence of finite time blow-up solutions. Indeed, if $\psi \equiv 1$, $2 \leqslant \gamma < \min(n, 4)$, $n \geqslant 3$ and $\varphi \in H^{\gamma/2}(\mathbb{R}^n)$ with $x\varphi \in L^2$, then

$$\|xu(t)\|_{L^2}^2 \leqslant 8t^2 E(\varphi) + 4t\langle\varphi, A\varphi\rangle + \|x\varphi\|_{L^2}^2,$$

where A is the dilation operator $(1/2\mathrm{i})(\nabla\cdot x + x\cdot\nabla)$. This implies that if $E(\varphi) < 0$, then the maximal time of existence satisfies $T^* < \infty$. For details, see [1, § 6.5] and § 3.

In the fractional or high-order equations, a variant of the second moment is the quantity

$$\mathcal{M}(u) := \langle u, x\cdot|\nabla|^{2-\alpha}xu\rangle.$$

This was first used by Fröhlich and Lenzmann [5] in their study of the semi-relativistic nonlinear Schrödinger equations ($\alpha = 1$). More precisely, they obtained

$$\mathcal{M}(u(t)) \leqslant 2t^2 E(\varphi) + 2t(\langle\varphi, A\varphi\rangle + C\|\varphi\|_{L^2}^4) + \mathcal{M}(\varphi)$$

for $\psi = \mathrm{e}^{-\mu|x|}$ ($\mu \geqslant 0$), $\gamma = 1$, $\varphi \in H^2_{\mathrm{rad}}(\mathbb{R}^3)$ with $|x|^2\varphi \in L^2$. Here, the function space X_{rad} denotes the subspace X of radial functions. The quartic term $\|\varphi\|_{L^2}^4$ appears due to the commutator $[|x|^2 V_\gamma, |\nabla|]$ and in $\mathbb{R}^3$ it is controlled by Newton's theorem.

Unlike the usual case ($\alpha = 2$), when $\alpha \neq 2$ the presence of $|\nabla|^{2-\alpha}$ gives rise to certain types of singular integrals that necessitate the use of commutators. So the main issue is how to estimate the commutator $[|x|^2 V_\gamma, |\nabla|^{2-\alpha}]$ since Newton's theorem is generally not available except for $\alpha = 1$ in $\mathbb{R}^3$. In order to obtain the desired estimate, we use the Stein–Weiss inequality (2.12) and combine this with a convolution estimate in lemma 2.3. To close our argument, we furthermore need an estimate for the moments $\|xu\|_{L^2}$ and $\||x|\nabla u\|_{L^2}$ for t contained in the existence time interval (see proposition 3.1). It is done under some regularity assumption[1], which we need to impose to get estimates for the commutators $[|\nabla|^\alpha, |x|^2]$ and $[V_\gamma, \nabla \cdot (x \cdot |\nabla|^{2-\alpha} x)\nabla]$.

The Hartree nonlinearity is essentially a cubic one, though it is convolved with the potential. Thus, by a fairly standard argument, one can show the local well-posedness of the Cauchy problem for suitably regular initial data. Indeed, we have local well-posedness for $s \geqslant \gamma/2$ so that, within the maximal existence time interval $[0, T^*)$, there is a unique solution $u \in C([0, T^*); H^s) \cap C^1([0, T^*); H^{s-\alpha})$ and $\lim_{t \nearrow T^*} \|u(t)\|_{H^{\gamma/2}} = \infty$ if $T^* < \infty$. For the reader's convenience, we give the local well-posedness for general $\alpha > 0$ in the appendix.

Let us define a Sobolev index α^* by $\alpha^* = (2k)^2$, where k is the least integer satisfying $k \geqslant \alpha/2$. We separately state our results for the low order case, $1 \leqslant \alpha < 2$, and the high-order case, $2 < \alpha < n/2 + 1$.

THEOREM 1.1. *Let $\gamma = \alpha$, $1 \leqslant \alpha < 2$ and $n \geqslant 4$. Assume that ψ is a non-negative smooth decreasing and radial function with $|\psi'(\rho)| \leqslant C\rho^{-1}$ for some $C > 0$. Additionally, assume that the initial datum satisfies $\varphi \in H^{\alpha^*}_{\mathrm{rad}}$ and $|x|^\ell \partial^{\mathfrak{j}} \varphi \in L^2(\mathbb{R})$ for $1 \leqslant \ell \leqslant 2$, $0 \leqslant |\mathfrak{j}| \leqslant 4 - 2\ell$. Then, if $E(\varphi) < 0$, the solution to (1.1) blows up in a finite time.*

THEOREM 1.2. *Let $\gamma = \alpha$, $2 < \alpha < 1 + n/2$ and $n \geqslant 4$. Assume that ψ is a non-negative smooth decreasing and radial function. Additionally, assume that the initial datum satisfies $\varphi \in H^{\alpha^*}_{\mathrm{rad}}$ and $|x|^\ell \partial^{\mathfrak{j}} \varphi \in L^2(\mathbb{R})$ for $1 \leqslant \ell \leqslant 2k$, $0 \leqslant |\mathfrak{j}| \leqslant 2k(2k - \ell)$. Then, if $E(\varphi) < 0$, the solution to (1.1) blows up in a finite time.*

The restriction $n \geqslant 4$ is due to the use of the Stein–Weiss inequality. The technical condition $\alpha < 1 + n/2$ is imposed because we make use of the convolution estimate (2.10) (lemma 2.3) and $n \geqslant 4$. For the proof of the theorems we show that the mean dilation is decreasing when $E(\varphi) < 0$. Clearly, this follows from

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle u, Au \rangle \leqslant 2\alpha E(\varphi), \tag{1.3}$$

which holds whenever $\gamma \geqslant \alpha$ and $\psi' \leqslant 0$. If $\gamma = \alpha$, from the estimate (2.7) we have, for $t \in [0, T^*)$,

$$\mathcal{M}(u) \leqslant 2\alpha^2 t^2 E(\varphi) + 2\alpha t (\langle \varphi, A\varphi \rangle + C\|\varphi\|^4_{L^2}) + \mathcal{M}(\varphi). \tag{1.4}$$

In order to validate (1.3) and (1.4), we need estimates for the moments $\|xu\|_{L^2}$ and $\||x|\nabla u\|_{L^2}$ on the time-interval $[0, T^*)$.

We finally remark that the argument of this paper does not readily work for the power-type nonlinearity. Since our argument relies on an H^{α^*} regularity assumption and the Stein–Weiss inequality, a different approach seems to be necessary in order to control the commutators.

[1]Such an assumption is not necessary for the usual Schrödinger equation.

The rest of paper is organized as follows. In § 2 we show the finite time blow-up while assuming proposition 3.1. In § 3 we provide the proof of proposition 3.1. The last section is devoted to the local well-posedness.

Notation

We use the following notation: $\partial^{\mathrm{j}} = \prod_{1\leqslant i\leqslant n}\partial_i^{j_i}$ for multi-index $\mathrm{j} = (j_1,\ldots,j_n)$ and $|\mathrm{j}| = \sum_i j_i$. $|\nabla| = \sqrt{-\Delta}$, $\dot H^s_r = |\nabla|^{-s}L^r$, $\dot H^s = \dot H^s_2$ and $H^s_r = (1-\Delta)^{-s/2}L^r$, $H^s = H^s_2$. $A \lesssim B$ and $A \gtrsim B$ means that $A \leqslant CB$ and $A \geqslant C^{-1}B$, respectively, for some $C > 0$. As usual, C denotes a positive constant, possibly depending on n, α and γ, which may differ at each occurrence.

2. Finite time blow-up

In this section we consider finite time blow-up of solutions to the Cauchy problem (1.1) of the mass-critical potentials. We begin with the dilation operator A. With more general assumptions for ψ and γ we obtain an estimate for the time evolution of the average of A.

Lemma 2.1. *Let ψ be a radially symmetric smooth function such that $\psi' = \partial_r\psi \leqslant 0$. Suppose that $u \in H^\alpha$ and $xu(t), |x|\nabla u(t) \in L^2$ for $t \in [0,T^*)$, where T^* is the maximal existence time. Then, for $\gamma \geqslant \alpha$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u, Au\rangle \leqslant 2\alpha E(\varphi). \tag{2.1}$$

Proof. Since $u \in H^\alpha$ and $|x|u, x\cdot\nabla u \in L^2$, $\langle u, Au\rangle$ is well defined and so is

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u, Au\rangle = \mathrm{i}\langle u, [H,A]u\rangle, \tag{2.2}$$

where $H = |\nabla|^\alpha - V_\gamma$. Here $[H,A]$ denotes the commutator $HA - AH$. Using the identity $[|\nabla|^\alpha, x] = -\alpha|\nabla|^{\alpha-2}\nabla$, we have

$$[|\nabla|^\alpha, A] = -\mathrm{i}\alpha|\nabla|^\alpha. \tag{2.3}$$

Similarly,

$$[-V_\gamma, A] = -\mathrm{i}(x\cdot\nabla)V_\gamma. \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.2), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u, Au\rangle = \alpha\langle u, |\nabla|^\alpha u\rangle + \langle u, (x\cdot\nabla)V_\gamma u\rangle. \tag{2.5}$$

For Hartree type V_γ, we have

$$\begin{aligned}(x\cdot\nabla)V_\gamma = &-\gamma\int\frac{\psi(|x-y|)}{|x-y|^\gamma}|u(y)|^2\,\mathrm{d}y + \int\frac{\psi'(|x-y|)}{|x-y|^\gamma}|x-y|\,|u(y)|^2\,\mathrm{d}y\\ &-\int\left(\gamma\frac{\psi(|x-y|)}{|x-y|^{\gamma+1}} - \frac{\psi'(|x-y|)}{|x-y|^\gamma}\right)\frac{y\cdot(x-y)}{|x-y|}|u(y)|^2\,\mathrm{d}y,\end{aligned}$$

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$$\langle u, (x\cdot\nabla)V_\gamma u\rangle = 4\gamma V(u) + \iint \frac{|x-y|\psi'(|x-y|)}{|x-y|^\gamma}|u(x)|^2|u(y)|^2\,\mathrm{d}x\,\mathrm{d}y - \langle u, (x\cdot\nabla)V_\gamma u\rangle,$$

which implies that

$$\langle u, (x\cdot\nabla)V_\gamma u\rangle = 2\gamma V(u) + \frac{1}{2}\iint \frac{|x-y|\psi'(|x-y|)}{|x-y|^\gamma}|u(x)|^2|u(y)|^2\,\mathrm{d}x\,\mathrm{d}y.$$

Substituting this into (2.5) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u, Au\rangle \leqslant 2\alpha E(\varphi) + 2(\gamma-\alpha)V(u) + \frac{1}{2}\iint (|x-y|\psi'(|x-y|))\frac{|u(x)|^2|u(y)|^2}{|x-y|^\alpha}\,\mathrm{d}x\,\mathrm{d}y.$$

Since $\gamma \geqslant \alpha$ and $\psi'(|x|) \leqslant 0$, we obtain (2.1). This completes the proof of lemma 2.1. □

Next we consider the non-negative quantity $\mathcal{M}(u) = \langle u, Mu\rangle$ with the virial operator

$$M := x\cdot|\nabla|^{2-\alpha}x = \sum_{k=1}^{n} x_k|\nabla|^{2-\alpha}x_k.$$

Suppose that $u(t) \in H^{\alpha^*}$ and $|x|^{2k}u(t) \in L^2$ for $t \in [0, T^*)$. Then, since $\mathcal{M}(u) \lesssim \||x|\nabla u\|_{L^2}\|(1+|x|)^{2k}u\|_{L^2}$, from (3.5) the quantity $\mathcal{M}(u)$ is well defined and finite for all $t \in [0, T^*)$, and so is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(u) = \mathrm{i}\langle u, [H, M]u\rangle = \mathrm{i}\langle u, [|\nabla|^\alpha, M]u\rangle - \mathrm{i}\langle u, [V_\gamma, M]u\rangle. \tag{2.6}$$

LEMMA 2.2. *Suppose that $u(t) \in H^{\alpha^*}$ and $|x|^{2k}u(t) \in L^2$ for $t \in [0, T^*)$. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(u) \leqslant 2\alpha\langle u, Au\rangle + C\|\varphi\|_{L^2}^4 \tag{2.7}$$

for $t \in [0, T^*)$, where C is a positive constant depending on n and α, but not on u and φ.

Now, theorem 1.1 and theorem 1.2 immediately follow from lemmas 2.1 and 2.2 once we have proposition 3.1.

Proof. By the identity $|\nabla|^\alpha x = x|\nabla|^\alpha - \alpha|\nabla|^{\alpha-2}\nabla$, we have

$$[|\nabla|^\alpha, M] = |\nabla|^\alpha x\cdot|\nabla|^{2-\alpha}x - x\cdot|\nabla|^{2-\alpha}x|\nabla|^\alpha = -\alpha(x\cdot\nabla + \nabla\cdot x).$$

Hence, for a smooth function v we obtain

$$\begin{aligned}[v, M] &= vx\cdot|\nabla|^{2-\alpha}x - x\cdot|\nabla|^{2-\alpha}xv \\ &= v|x|^2|\nabla|^{2-\alpha} - (2-\alpha)vx\cdot\nabla|\nabla|^{-\alpha} - |\nabla|^{2-\alpha}|x|^2v - (2-\alpha)|\nabla|^{-\alpha}\nabla\cdot xv \\ &= [|x|^2v, |\nabla|^{2-\alpha}] + (\alpha-2)\left(vx\cdot\frac{\nabla}{|\nabla|}|\nabla|\,|\nabla|^{-\alpha} + |\nabla|\,|\nabla|^{-\alpha}\frac{\nabla}{|\nabla|}\cdot xv\right).\end{aligned}$$

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By a density argument, we may assume that $v = V_\alpha$ in the above identity. Thus, it suffices to show that

$$\|\varphi\|_{L^2}^4 \gtrsim |\langle u, [|x|^2 V_\alpha, |\nabla|^{2-\alpha}]u\rangle| + \left|\left\langle u, \left(V_\alpha x \cdot \frac{\nabla}{|\nabla|}|\nabla|\,|\nabla|^{-\alpha} + |\nabla|\,|\nabla|^{-\alpha}\frac{\nabla}{|\nabla|}\cdot x V_\alpha\right)u\right\rangle\right|. \tag{2.8}$$

Case 1 ($\alpha > 2$). We first consider the higher-order case $\alpha > 2$. The first term of the left-hand side of (2.8) is rewritten as

$$2|\mathrm{Im}\langle u, |x|^2 V_\alpha |\nabla|^{2-\alpha} u\rangle|. \tag{2.9}$$

To handle this we recall the following weighted convolution estimate (see [2, 3]).

Lemma 2.3. *Let $0 < \gamma < n-1$ and $n \geqslant 2$. Then, for any $f \in L^1_{\mathrm{rad}}$ and $x \neq 0$,*

$$\int |x-y|^{-\gamma}|f(y)|\,\mathrm{d}y \lesssim |x|^{-\gamma}\|f\|_{L^1}. \tag{2.10}$$

From lemma 2.3 and mass conservation, (2.9) is bounded by

$$C\|\psi\|_{L^\infty}\|\varphi\|_{L^2}^2 \int |u(x)|\,|x|^{-(\alpha-2)} \int |x-y|^{-(n-\alpha+2)}|u(y)|\,\mathrm{d}y\,\mathrm{d}x. \tag{2.11}$$

To estimate this, we make use of the following inequality due to Stein–Weiss [9].

For $f \in L^p$ with $1 < p < \infty$, $0 < \lambda < n$, $\beta < n/p$ and $n = \lambda + \beta$,

$$\||x|^{-\beta}(|\cdot|^{-\lambda} * f)\|_{L^p} \lesssim \|f\|_{L^p}. \tag{2.12}$$

Applying (2.12) with $p = 2$, $\beta = \alpha - 2$ and $\lambda = n - (\alpha - 2)$, (2.11) is bounded by $C\|\varphi\|_{L^2}^4$.

We write the second term of the right-hand side of (2.8) as

$$2\left|\mathrm{Im}\left\langle u, V_\alpha x \cdot \frac{\nabla}{|\nabla|}|\nabla|\,|\nabla|^{-\alpha}u\right\rangle\right|.$$

By using lemma 2.3 we see that this is bounded by

$$C\|\psi\|_{L^\infty}\|\varphi\|_{L^2}^2 \int |u(x)|\,|x|^{-(\alpha-1)} \int |x-y|^{-(n-(\alpha-1))}\left|\left(\frac{\nabla}{|\nabla|}u\right)(y)\right|\mathrm{d}y\,\mathrm{d}x.$$

Applying (2.12) with $p = 2$, $\beta = \alpha - 1$ and $\lambda = n - (\alpha - 1)$, and Plancherel's theorem, we get the desired bound (2.8).

Case 2 ($1 \leqslant \alpha < 2$). Now we consider the fractional case $1 \leqslant \alpha < 2$. The second term of the right-hand side of (2.8) can be treated in the same way as the high-order case and it is bounded by $C\|\varphi\|_{L^2}^4$. Hence, it suffices to consider the first term. Let us set $g = |x|^2 V_\alpha$. Then we need only to obtain

$$\|[|\nabla|^{2-\alpha}, g]u\|_{L^2} \leqslant C\|\varphi\|_{L^2}^3, \tag{2.13}$$

which gives $|\langle u, [|x|^2 V_\alpha, |\nabla|^{2-\alpha}]u\rangle| \lesssim \|\varphi\|_{L^2}^4$, and thus (2.8). The kernel $K(x, y)$ of the commutator $[|\nabla|^{2-\alpha}, g]$ can be written as $k(x-y)(g(y) - g(x))$, where k is the kernel of the pseudo-differential operator $|\nabla|^{2-\alpha}$. Let K^* be the kernel of the dual operator of $[|\nabla|^{2-\alpha}, g]$. Then, obviously, $K^*(x,y) = -K(x,y)$.

Suppose that

$$\|g\|_{\dot{\Lambda}^{2-\alpha}} = \sup_{x\neq y\in\mathbb{R}^n} \frac{|g(x)-g(y)|}{|x-y|^{2-\alpha}} < \infty.$$

Since $|k(x-y)| \lesssim |x-y|^{-n-(2-\alpha)}$, $|\nabla k(x-y)| \lesssim |x-y|^{-n-1-(2-\alpha)}$ and $0 < 2-\alpha \leqslant 1$, it is easy to see that

$$\begin{aligned} |K(x,y)| &\lesssim |x-y|^{-n}, \\ |K(x,y) - K(x',y)| &\lesssim \frac{|x-x'|^{2-\alpha}}{|x-y|^{n+2-\alpha}} \quad \text{if } |x-x'| \leqslant \frac{|x-y|}{2}, \\ |K(x,y) - K(x,y')| &\lesssim \frac{|y-y'|^{2-\alpha}}{|x-y|^{n+2-\alpha}} \quad \text{if } |y-y'| \leqslant \frac{|x-y|}{2} \end{aligned}$$

and we obtain similar expressions for K^* (because $K^*(x,y) = -K(x,y)$). Let ζ be a normalized bump function supported in the unit ball and set $\zeta^{x_0,N}(x) = \zeta((x-x_0)/N)$. By [8, theorem 3, p. 294], in order to prove (2.13), it is sufficient to show that

$$\|[|\nabla|^{2-\alpha}, g](\zeta^{x_0,N})\|_{L^2} \leqslant C\|\varphi\|_{L^2}^2 N^{n/2} \tag{2.14}$$

with C independent of x_0, N and ζ.

We now show (2.14). The commutator $[|\nabla|^{2-\alpha}, g]$ can be written as

$$\sum_{j=1}^n [T_j, g]\partial_j + \sum_{j=1}^n T_j(\partial_j g), \tag{2.15}$$

where $T_j = -|\nabla|^{2-\alpha}(-\Delta)^{-1}\partial_j$. For the first sum of (2.15) we obtain

$$\|[T_j, g]\partial_j(\zeta^{x_0,N})\|_{L^2} \leqslant C\|\varphi\|_{L^2}^2 N^{n/2}. \tag{2.16}$$

Indeed, let k_j be the kernel of T_j. If $\alpha = 1$, k_j is the kernel of the Riesz transform. If $1 < \alpha < 2$, it is easy to see that $|k_j(x,y)| \lesssim |x-y|^{-n+\alpha-1}$ (note that $|\hat{k}_j(\xi)| \lesssim |\xi|^{-(\alpha-1)}$). Thus, it follows that

$$|K_j(x,y)| = |k_j(x-y)|\,|g(y)-g(x)| \lesssim \|g\|_{\dot{\Lambda}^{2-\alpha}} |x-y|^{-(n-1)}.$$

Hence, for $|x-x_0| < 2N$, $|[T_i, g]\partial_i(\zeta^{x_0,N})(x)| \lesssim \|g\|_{\dot{\Lambda}^{2-\alpha}}$. This gives

$$\|[T_i, g]\partial_i(\zeta^{x_0,N})\|_{L^2(\{|x-x_0|<2N\})} \lesssim \|g\|_{\dot{\Lambda}^{2-\alpha}} N^{n/2}.$$

If $|x-x_0| \geqslant 2N$, we have $|[T_i, g]\partial_i(\zeta^{x_0,N})(x)| \lesssim \|g\|_{\dot{\Lambda}^{2-\alpha}} N^{n-1}|x-x_0|^{-(n-1)}$. Hence,

$$\begin{aligned} \|[T_i, g]\partial_i(\zeta^{x_0,N})\|_{L^2(\{|x-x_0|\geqslant 2N\})} &\lesssim \|g\|_{\dot{\Lambda}^{2-\alpha}} N^{n-1} \left(\int_{|x|>2N} |x|^{-2(n-1)}\,\mathrm{d}x \right)^{1/2} \\ &\lesssim \|g\|_{\dot{\Lambda}^{2-\alpha}} N^{n/2}. \end{aligned}$$

We now show that $\|g\|_{\dot{\Lambda}^{2-\alpha}} \leqslant C\|\varphi\|_{L^2}^2$, which gives (2.16). If $x \neq y$, then

$$|g(x)-g(y)| \leqslant |x-y| \int_0^1 |\nabla g(z_s)|\,\mathrm{d}s, \quad z_s = x + s(y-x).$$

Since $|\psi'(\rho)| \leqslant C\rho^{-1}$ for $\rho > 0$, from lemma 2.3 and mass conservation it follows that

$$|\nabla g(z_s)| \lesssim |z_s|^{1-\alpha}\|u\|_{L^2}^2 = ||x| - s|x-y||^{1-\alpha}\|\varphi\|_{L^2}^2,$$

provided that $\alpha < n-2$. Since

$$\sup_{a>0}\int_0^1 |a-s|^{-\theta}\,\mathrm{d}s \leqslant C_\theta \quad \text{for } 0<\theta<1,$$

we have

$$|g(x) - g(y)| \lesssim |x-y|^{2-\alpha}\|\varphi\|_{L^2}^2.$$

Thus, we obtain (2.16).

Finally, we need to handle the second sum of (2.15). If $\alpha = 1$, T_j is a Riesz transform. Thus,

$$\|T_j((\partial_j g)\zeta^{x_0,N})\|_{L^2} \leqslant C\|\partial_j g\|_{L^\infty} N^{n/2}.$$

By lemma 2.3 for $\alpha = 1$, we obtain $|\partial_j g(x)| \leqslant |x|V_1 + |x|^2|\partial_j V_1| \lesssim \|\varphi\|_{L^2}^2$. Hence,

$$\|T_j((\partial_j g)\zeta^{x_0,N})\|_{L^2} \leqslant C\|\varphi\|_{L^2}^2 N^{n/2}. \tag{2.17}$$

For $1 < \alpha < 2$, the kernel $k_j(x)$ of T_j is bounded by $C|x|^{-(n-\alpha+1)}$. So, from the duality and lemma 2.3 with $\alpha < n-2$, we have, for any $\psi \in L^2$,

$$\begin{aligned}
|\langle \psi, T_j((\partial_j g)\zeta^{x_0,N})\rangle| &= |\langle T_j^*\psi, (\partial_j g)\zeta^{x_0,N}\rangle| \\
&\leqslant CN^{n/2}\left\| |\partial_j g(\cdot)| \int |\cdot - y|^{-(n-\alpha+1)}|\psi(y)|\,\mathrm{d}y \right\|_{L^2} \\
&\leqslant CN^{n/2}\|\varphi\|_{L^2}^2 \left\| |\cdot|^{1-\alpha} \int |\cdot - y|^{-(n-\alpha+1)}|\psi(y)|\,\mathrm{d}y \right\|_{L^2},
\end{aligned}$$

where T_j^* is the dual operator of T_j. Using (2.12) with $\beta = \alpha - 1$, $\lambda = n - \alpha + 1$ and $p = 2$, we obtain

$$|\langle \psi, T_j(\partial_j g\zeta^{x_0,N})\rangle| \leqslant C\|\psi\|_{L^2}\|\varphi\|_{L^2}^2 N^{n/2}.$$

Thus, it follows that

$$\|T_j(\partial_j g\zeta^{x_0,N})\|_{L^2} \leqslant C\|\varphi\|_{L^2}^2 N^{n/2}. \tag{2.18}$$

Therefore, combining the estimates (2.16)–(2.18) yields (2.14). This completes the proof of lemma 2.2. □

3. Propagation of the moment

We now discuss estimates for the propagation of moments $\||x|^{2k}u\|_{L^2}$ when $|x|^{2k}\varphi \in L^2$ and the solution $u \in C([0,T^*); H^{\alpha^*})$. For $\alpha < 2k$, we use the kernel estimate of Bessel potentials. Let us denote, respectively, the kernels of Bessel potentials $D^{-\beta}$ and $|\nabla|^\alpha D^{-2k}$ ($\beta = \alpha - 2k$) by $G_\beta(x)$ and $K(x)$, where $D = \sqrt{1-\Delta}$. Then

$$K(x) = \sum_{j=0}^{\infty} A_j G_{2j+\beta}(x),$$

where the coefficients A_j are given by the expansion $(1-t)^{\alpha/2} = \sum_{j=0}^{\infty} A_j t^j$ for $|t| < 1$ with $\sum_{j\geqslant 0} |A_j| < \infty$. One can show that $(1+|x|)^\ell K \in L^1$ for $\ell \geqslant 1$ and that it has decreasing radial and integrable majorant. In fact, from the integral representation

$$G_{2j+\beta}(x) = \frac{1}{(4\pi)^{n/2}\Gamma(j+\beta/2)} \int_0^\infty \lambda^{(2j+\beta-n)/2-1} \mathrm{e}^{-|x|^2/4\lambda} \mathrm{e}^{-\lambda}\,\mathrm{d}\lambda,$$

it follows that, for j with $2j+\beta < n$,

$$G_{2j+\beta}(x) \leqslant C(|x|^{-n+2j+\beta}\chi_{\{|x|\leqslant 1\}}(x) + \mathrm{e}^{-c|x|}\chi_{\{|x|>1\}}(x)) \tag{3.1}$$

and, for j with $2j+\beta \geqslant n$,

$$G_{2j+\beta}(x) \leqslant C(\chi_{\{|x|\leqslant 1\}}(x) + \mathrm{e}^{-c|x|}\chi_{\{|x|>1\}}(x)). \tag{3.2}$$

Here, the constants c and C of (3.1) and (3.2) are independent of j. So, the function $(1+|x|)^\ell G_{2j+\beta}$ has a decreasing radial and integrable majorant, which is chosen uniformly on j, and so does K. For details see [7, pp. 132–135].

PROPOSITION 3.1. *Let T^* be the maximal time of solution $u \in C([0,T^*); H^{\alpha^*})$ to (1.1). If $|x|^\ell \partial^{\mathfrak{j}}\varphi \in L^2(\mathbb{R})$ for $1 \leqslant \ell \leqslant 2k$, $0 \leqslant |\mathfrak{j}| \leqslant 2k(2k-\ell)$, then $|x|^\ell\partial^{\mathfrak{j}} u(t) \in L^2(\mathbb{R})$ for all $t \in [0,T^*)$.*

Let us set $\psi_\varepsilon(x) = \mathrm{e}^{-\varepsilon|x|^2}$. For the proof of proposition 3.1 we use the following bootstrapping lemma.

LEMMA 3.2. *Let ℓ and m be integers such that $2 \leqslant \ell \leqslant 2k$ and $0 \leqslant m \leqslant \alpha^* - 2k$. Suppose that $\sup_{0\leqslant t'\leqslant t}(\|u(t')\|_{H^{2k+m}} + \||x|^j\partial^{\mathfrak{j}} u(t')\|_{L^2}) < \infty$ for all $t \in [0,T^*)$ and $0 \leqslant j \leqslant \ell-1$, $|\mathfrak{j}| \leqslant 2k+m$. Then $\sup_{0\leqslant t'\leqslant t}\||x|^\ell \partial^{\mathfrak{m}} u(t')\|_{L^2} < \infty$ for all $t \in [0,T^*)$ and $|\mathfrak{m}| = m$.*

Proof. Let $v = \partial^{\mathfrak{m}} u$ and let

$$\boldsymbol{m}_\varepsilon(t) = \langle v(t), |x|^{2\ell}\psi_\varepsilon^2 v(t)\rangle.$$

From the regularity of the solution u, it follows that

$$\boldsymbol{m}_\varepsilon'(t) = 2\,\mathrm{Im}\langle v, [|\nabla|^\alpha, |x|^{2\ell}\psi_\varepsilon^2]v\rangle + 2\,\mathrm{Im}\langle |x|^\ell\psi_\varepsilon v, |x|^\ell\psi_\varepsilon \partial^{\mathfrak{m}}(V_\alpha u)\rangle =: 2(\mathrm{I}+\mathrm{II}).$$

We first prove the case $\alpha < 2k$. We rewrite I as

$$\mathrm{I} = \mathrm{Im}\langle |x|^\ell\psi_\varepsilon v, [|\nabla|^\alpha D^{-2k}, |x|^\ell\psi_\varepsilon]D^{2k}u\rangle + \mathrm{Im}\langle |\nabla|^\alpha D^{-2k}(|x|^\ell\psi_\varepsilon v), [D^{2k}, |x|^\ell\psi_\varepsilon]v\rangle =: I_1 + I_2.$$

By the kernel representation of $|\nabla|^\alpha D^{-2k}$, we have

$$\begin{aligned} &|[|\nabla D^{-2k}, |x|^\ell\psi_\varepsilon|]D^{2k}u(x)| \\ &\quad\leqslant \int K(x-y)||x|^\ell\psi_\varepsilon(x) - |y|^\ell\psi_\varepsilon(y)|\,|D^{2k}u(y)|\,\mathrm{d}y \\ &\quad\lesssim \int K(x-y)|x-y|(|x|^{\ell-1}+|y|^{\ell-1})|D^{2k}u(y)|\,\mathrm{d}y \\ &\quad\lesssim \int K(x-y)|x-y|^\ell|D^{2k}u(y)|\,\mathrm{d}y + \int K(x-y)|x-y|\,|y|^{\ell-1}|D^{2k}u(y)|\,\mathrm{d}y. \end{aligned}$$

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Since $|x|^\ell K$ is integrable, the Cauchy–Schwarz inequality gives

$$I_1 \lesssim \sqrt{\boldsymbol{m}_\varepsilon}(\|u\|_{H^{2k}} + \||x|^{\ell-1}D^{2k}u\|_{L^2}).$$

As for I_2, we have

$$\begin{aligned} I_2 &= \sum_{1\leqslant j\leqslant k} c_j \operatorname{Im}\langle |\nabla|^\alpha D^{-2k}(|x|^\ell \psi_\varepsilon v), [\Delta^j, |x|^\ell \psi_\varepsilon]v\rangle \\ &= \sum_{1\leqslant j\leqslant k} c_j \operatorname{Im}\Big\langle |\nabla|^\alpha D^{-2k}(|x|^\ell\psi_\varepsilon v), \sum_{\substack{|\mathrm{j}_1|+|\mathrm{j}_2|+|\mathrm{j}_3|=2j\\ 0\leqslant|\mathrm{j}_3|\leqslant 2j-1}} c_{\mathrm{j}_1,\mathrm{j}_2,\mathrm{j}_3}\partial^{\mathrm{j}_1}(|x|^\ell)\partial^{\mathrm{j}_2}\psi_\varepsilon\partial^{\mathrm{j}_3}v\Big\rangle. \end{aligned}$$

Note that $|\partial^{\mathrm{j}_1}(|x|^\ell)| \lesssim |x|^{\ell-|\mathrm{j}_1|}$ and $|\partial^{\mathrm{j}_2}\psi_\varepsilon(x)| \lesssim \varepsilon^{|\mathrm{j}_2|/2}(1+\varepsilon|x|^2)^{|\mathrm{j}_2|/2}\psi_\varepsilon(x)$. Hence, it follows that

$$\begin{aligned} I_2 &\lesssim \||\nabla|^\alpha D^{-2k}(|x|^\ell\psi_\varepsilon v)\|_{L^2} \\ &\quad\times \sum_{1\leqslant j\leqslant k}\Bigg(\sum_{\substack{|\mathrm{j}_1|+|\mathrm{j}_2|+|\mathrm{j}_3|=2j\\ 0\leqslant|\mathrm{j}_3|\leqslant j-\ell}} + \sum_{\substack{|\mathrm{j}_1|+|\mathrm{j}_2|+|\mathrm{j}_3|=j\\ j-\ell\leqslant|\mathrm{j}_3|\leqslant 2j-1}}\Bigg)\||x|^{\ell-|\mathrm{j}_1|-|\mathrm{j}_2|}\partial^{\mathrm{j}_3}v\|_{L^2} \\ &\lesssim \sqrt{\boldsymbol{m}_\varepsilon}\sum_{1\leqslant j\leqslant k}\Bigg(\sum_{\substack{|\mathrm{j}_1|+|\mathrm{j}_2|+|\mathrm{j}_3|=2j\\ 0\leqslant|\mathrm{j}_3|\leqslant j-\ell}} + \sum_{\substack{|\mathrm{j}_1|+|\mathrm{j}_2|+|\mathrm{j}_3|=2j\\ j-\ell\leqslant|\mathrm{j}_3|\leqslant 2j-1}}\Bigg)\||x|^{|\mathrm{j}_3|-(j-\ell)}\partial^{\mathrm{j}_3}v\|_{L^2}. \end{aligned}$$

Here we used the fact that the kernel of $|\nabla|^\alpha D^{-2k}$ is integrable. Conventionally, the summand is zero if $j-\ell<0$. By the Hardy–Sobolev inequality, we obtain, for $0\leqslant|\mathrm{j}_3|\leqslant j-\ell$,

$$\||x|^{|\mathrm{j}_3|-(j-\ell)}\partial^{\mathrm{j}_3}v\|_{L^2} \lesssim \|\partial^{\mathrm{j}_3}v\|_{H^{j-\ell-|\mathrm{j}_3|}} \lesssim \|v\|_{H^{j-\ell}} \lesssim \|u\|_{H^{j-\ell+m}}.$$

If $j-\ell\leqslant|\mathrm{j}_3|\leqslant 2j-1$, then

$$\||x|^{|\mathrm{j}_3|-(j-\ell)}\partial^{\mathrm{j}_3}v\|_{L^2} = \||x|^{|\mathrm{j}_3|-(j-\ell)}\partial^{\mathrm{j}_3+\mathfrak{m}}u\|_{L^2}.$$

Thus, we finally obtain

$$I \lesssim \sqrt{\boldsymbol{m}_\varepsilon(t)}\Bigg(\|u(t)\|_{H^{2k+m}} + \sum_{0\leqslant|\mathrm{j}|\leqslant 2k+m}\|(1+|x|)^{\ell-1}\partial^{\mathrm{j}}u(t)\|_{L^2}\Bigg). \tag{3.3}$$

For the case in which $\alpha=2k$, we do not need the estimate for I_1. For the estimate of $I_2 = \operatorname{Im}\langle|x|^\ell\psi_\varepsilon v, [\Delta^k, |x|^\ell\psi_\varepsilon]v\rangle$, we estimate similarly to obtain (3.3).

Now we proceed to estimate II. For this let us observe that

$$\begin{aligned} \mathrm{II} &= \sum_{\substack{\mathfrak{m}_1+\mathfrak{m}_2=\mathfrak{m}\\ 0\leqslant|\mathfrak{m}_2|\leqslant m-1}} c_{\mathfrak{m}_1,\mathfrak{m}_2}\operatorname{Im}\langle|x|^\ell\psi_\varepsilon v, |x|^\ell\psi_\varepsilon\partial^{\mathfrak{m}_1}V_\alpha\partial^{\mathfrak{m}_2}u\rangle \\ &\lesssim \sqrt{\boldsymbol{m}_\varepsilon}\sum_{\substack{\mathfrak{m}_1+\mathfrak{m}_2=\mathfrak{m}\\ 0\leqslant|\mathfrak{m}_2|\leqslant m-1}} \||x|\partial^{\mathfrak{m}_1}V_\alpha\|_{L^\infty}\||x|^{\ell-1}|\partial^{\mathfrak{m}_2}u\|_{L^2}. \end{aligned}$$

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By Young's inequality, we estimate

$$|x|\,|\partial^{\mathfrak{m}_1}V_\alpha| \lesssim \sum_{\mathfrak{m}_1^1+\mathfrak{m}_1^2=\mathfrak{m}_1} \int |x-y|^{-\alpha}(|x-y|+|y|)|\partial^{\mathfrak{m}_1^1}u(y)|\,|\partial^{\mathfrak{m}_1^2}u(y)|\,\mathrm{d}y$$
$$\lesssim \sum_{\mathfrak{m}_1^1+\mathfrak{m}_1^2=\mathfrak{m}_1} \bigg(\int |x-y|^{-(\alpha-1)}(|\partial^{\mathfrak{m}_1^1}u(y)|^2+|\partial^{\mathfrak{m}_1^2}u(y)|^2)\,\mathrm{d}y$$
$$+\int |x-y|^{-\alpha}(|y|^2|\partial^{\mathfrak{m}_1^1}u(y)|^2+|\partial^{\mathfrak{m}_1^2}u(y)|^2)\,\mathrm{d}y\bigg).$$

Using the Hardy–Sobolev inequality, we get

$$\mathrm{II} \lesssim \sqrt{\boldsymbol{m}_\varepsilon} \sum_{0\leqslant|\mathfrak{j}|\leqslant k+m} \|(1+|x|)^{\ell-1}\partial^{\mathfrak{j}}u\|_{L^2}^3. \tag{3.4}$$

From (3.3) and (3.4) it follows that

$$\boldsymbol{m}_\varepsilon'(t) \leqslant \sqrt{\boldsymbol{m}_\varepsilon(t)}\bigg(\|u(t)\|_{H^{2k+m}} + \sum_{0\leqslant|\mathfrak{j}|\leqslant 2k+m}(1+\|(1+|x|)^{\ell-1}\partial^{\mathfrak{j}}u(t)\|_{L^2})^3\bigg),$$

which implies

$$\sqrt{\boldsymbol{m}_\varepsilon(t)} \lesssim \sqrt{\boldsymbol{m}_\varepsilon(0)}$$
$$+\int_0^t\bigg(\|u(t')\|_{H^{2k+m}} + \sum_{0\leqslant|\mathfrak{j}|\leqslant 2k+m}(1+\|(1+|x|)^{\ell-1}\partial^{\mathfrak{j}}u(t')\|_{L^2})^3\bigg)\,\mathrm{d}t'.$$

Letting $\varepsilon \to 0$, by Fatou's lemma we obtain $\sup_{0\leqslant t'\leqslant t}\||x|^\ell\partial^{\mathfrak{m}}u\|_{L^2} < \infty$ for all $t \in [0,T^*)$. □

Proof of proposition 3.1. In view of lemma 3.2 it suffices to show that

$$\sup_{0\leqslant t'\leqslant t}\||x|\partial^{\mathfrak{j}}u(t')\|_{L^2} < \infty \quad \text{for all } |\mathfrak{j}| \leqslant \alpha^*-2k \text{ and } t\in[0,T^*), \tag{3.5}$$

provided that $u \in C([0,T^*);H^{\alpha^*})$. In fact, we can use the same estimates of $\boldsymbol{m}_\varepsilon$ as in (3.3) and (3.4) for the case $\ell = 1$, to obtain

$$\sqrt{\boldsymbol{m}_\varepsilon(t)} \lesssim \sqrt{\boldsymbol{m}_\varepsilon(0)} + \int_0^t(\|u(t)\|_{H^{\alpha^*}} + \|u(t)\|_{H^{\alpha^*}}^3)\,\mathrm{d}t'.$$

A limiting argument implies (3.5). This completes the proof of proposition 3.1. □

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Appendix A.

In this section we provide a proof of the local well-posedness of the Hartree equation (1.1). Here we only assume that $\alpha > 0$ and $\psi \in L^\infty$.

PROPOSITION A.1. *Let $\psi \in L^\infty$. Let $\alpha > 0$, $0 < \gamma < n$ and $n \geqslant 1$. Suppose that $\varphi \in H^s(\mathbb{R}^n)$ with $s \geqslant \gamma/2$. There then exists a positive time T such that the Hartree equation (1.1) has a unique solution $u \in C([0,T];H^s) \cap C^1([0,T];H^{s-\alpha})$. Moreover, if T^* is the maximal existence time and is finite, then $\lim_{t\nearrow T^*}\|u(t)\|_{H^{\gamma/2}} = \infty$.*

Proof. We use the standard contraction-mapping argument, so we shall be brief.

Let $(X(T,\rho),d)$ be a complete metric space with metric d defined by

$$X(T,\rho) = \{u \in L_T^\infty(H^s(\mathbb{R}^n))\colon \|u\|_{L_T^\infty H^s} \leqslant \rho\}, \quad d_X(u,v) = \|u-v\|_{L_T^\infty L^2}.$$

We define a mapping $\mathcal{N} : u \mapsto \mathcal{N}(u)$ on $X(T,\rho)$ by

$$\mathcal{N}(u)(t) = U(t)\varphi - \mathrm{i}\int_0^t U(t-t')F(u)(t')\,\mathrm{d}t', \tag{A 1}$$

where $U(t) = \mathrm{e}^{-\mathrm{i}t|\nabla|^\alpha}$. For $u \in X(T,\rho)$ and $s \geqslant \gamma/2$ we estimate

$$\begin{aligned}
\|\mathcal{N}(u)\|_{L_T^\infty H^s} &\leqslant \|\varphi\|_{H^s} + T\|F(u)\|_{L_T^\infty H^s}\\
&\lesssim \|\varphi\|_{H^s} + T(\|V_\gamma(|u|^2)\|_{L_T^\infty L^\infty}\|u\|_{L_T^\infty H^s}\\
&\qquad\qquad + \|V_\gamma(|u|^2)\|^s_{L_T^\infty H_{2n/\gamma}}\|u\|_{L_T^\infty L^{2n/(n-\gamma)}})\\
&\lesssim \|\varphi\|_{H^s} + T(\|V_\gamma(|u|^2)\|_{L_T^\infty L^\infty}\|u\|_{L_T^\infty H^s}\\
&\qquad\qquad + \|V_\gamma(\langle\nabla\rangle^s(|u|^2))\|_{L_T^\infty L^{2n/\gamma}}\|u\|_{L_T^\infty L^{2n/(n-\gamma)}})\\
&\lesssim \|\varphi\|_{H^s} + T(\|V_\gamma(|u|^2)\|_{L_T^\infty L^\infty}\|u\|_{L_T^\infty H^s}\\
&\qquad\qquad + \|\langle\nabla\rangle^s(|u|^2)\|_{L_T^\infty L^{(2n-\gamma)/2n}}\|u\|_{L_T^\infty L^{2n/(n-\gamma)}})\\
&\lesssim \|\varphi\|_{H^s} + T(\|u\|^2_{L_T^\infty H^{\gamma/2}}\|u\|_{L_T^\infty H^s} + \|u\|^2_{L_T^\infty L^{2n/(n-\gamma)}}\|u\|_{L_T^\infty H^s})\\
&\lesssim \|\varphi\|_{H^s} + T\|u\|^2_{L_T^\infty H^{\gamma/2}}\|u\|_{L_T^\infty H^s}\\
&\lesssim \|\varphi\|_{H^s} + T\rho^3.
\end{aligned} \tag{A 2}$$

Here we used: the generalized Leibniz rule (see [4, lemmas A1–A4, appendix]) for the second and fifth inequalities; the fractional integration for the fourth inequality; and the trivial inequality

$$V_\gamma = \int_{\mathbb{R}^n} \frac{\psi(x-y)}{|x-y|^\gamma}|u(y)|^2\,\mathrm{d}y \leqslant \|\psi\|_{L^\infty}\int_{\mathbb{R}^n}|x-y|^{-\gamma}|u(y)|^2\,\mathrm{d}y,$$

the Hardy–Sobolev inequality

$$\sup_{x\in\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\frac{|u(x-y)|^2}{|y|^\gamma}\,\mathrm{d}y\right| \lesssim \|u\|^2_{\dot{H}^{\gamma/2}}$$

and the Sobolev embedding $H^{\gamma/2} \hookrightarrow L^{2n/(n-\gamma)}$ for the last one. If we choose ρ and T such that $\|\varphi\|_{H^s} \leqslant \rho/2$ and $CT\rho^3 \leqslant \rho/2$, then $\mathcal{N}$ maps $X(T,\rho)$ to itself.

Now we show that $\mathcal{N}$ is a Lipschitz map with a sufficiently small T. Let $u, v \in X(T, \rho)$. Then we have

$$\begin{aligned} d_X(\mathcal{N}(u), \mathcal{N}(v)) &\lesssim T\|V_\gamma(|u|^2)u - V_\gamma(|v|^2)v\|_{L_T^\infty L^2} \\ &\lesssim T(\|V_\gamma(|u|^2)(u-v)\|_{L_T^\infty L^2} + \|V_\gamma(|u|^2 - |v|^2)v\|_{L_T^\infty L^2}) \\ &\lesssim T(\|u\|^2_{L_T^\infty H^{\gamma/2}} d_X(u, v) \\ &\qquad + \|V_\gamma(|u|^2 - |v|^2)\|_{L_T^\infty L^{2n/\gamma}} \|v\|_{L_T^\infty L^{2n/(n-\gamma)}}) \\ &\lesssim T(\rho^2 d_X(u, v) + \rho\||u|^2 - |v|^2\|_{L_T^\infty L^{2n/(2n-\gamma)}}) \\ &\lesssim T(\rho^2 + \rho(\|u\|_{L_T^\infty L^{2n/(n-\gamma)}} + \|v\|_{L_T^\infty L^{2n/(n-\gamma)}})) d_X(u, v) \\ &\lesssim T\rho^2 d_X(u, v). \end{aligned}$$

The above estimate implies that the mapping $\mathcal{N}$ is a contraction if T is sufficiently small. The uniqueness and time regularity follow easily from (1.1) and a similar contraction argument.

Finally, let T^* be the maximal existence time. If $T^* < \infty$, then it is obvious from the estimate (A 2) and the standard local well-posedness theory that $\lim_{t \nearrow T^*} \|u(t)\|_{H^{\gamma/2}} = \infty$. This completes the proof of proposition A.1. □

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