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Bifurcation values of families of real curves

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The detection of the bifurcation set of polynomial mapping $\mathbb{R}^n \to \mathbb{R}^p$, $n \geqslant p$, in more than two variables remains an unsolved problem. In this note we provide a solution for $n = p + 1 \geqslant 3$.

Keywords: bifurcation locus; polynomials map; fibration

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1. Introduction

The bifurcation locus of a polynomial mapping $F\colon \mathbb{R}^n \to \mathbb{R}^p$, $n \geqslant p$, is the minimal set of points $B(F) \subset \mathbb{R}^p$ outside which the mapping is a C^∞ locally trivial fibration. Unlike the local setting, the critical locus $\operatorname{Sing} F$ is not the only obstruction to the existence of fibrations in the global setting. The simplest evidence of such a phenomenon in the $p = 1$ case is in the example of $f(x, y) = x + x^2 y$, where $\operatorname{Sing} f = \emptyset$ but $B(F) = \{0\}$. In the $p > 1$ case, Pinchuk [9] provided an example of a polynomial mapping $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, where $\operatorname{Sing} F = \emptyset$ but $B(F) \neq \emptyset$, which is a negative answer to the Jacobian conjecture over the reals.

For the last 20 years, in more than two variables, one could only estimate $B(F)$ by supersets $A \supset B(F)$ according to certain *regularity conditions at infinity* [2, 5, 8, 10, 13, 14], etc. The bifurcation set $B(F)$ was shown to be detectable precisely only if $p = 1$ and $n = 2$ (see [3, 7, 15]). A similar situation holds over the complex field, with a large number of articles in recent decades (see, for example, [14] for references of work done before 2007).

We address here the problem of detecting the bifurcation set in algebraic families of real curves of more than one parameter, in particular the case when $n = p+1 \geqslant 3$. The methods developed in [3] or [7] cannot be extended beyond two variables, since they are based essentially on the use of the 'polar locus' or the 'Milnor set' (see definition 2.3), which are of dimension 1 only in the $n = 2$ case. Our task was to find a way to extend to higher dimensions the ideas established in [15] for $n = 2$.

As a matter of fact, we have to change the viewpoint of [15] and find completely new definitions for the *non-vanishing* condition and for the *non-splitting* condition. We then get the following extension of the main result of [15], while retaining its spirit and terminology.

THEOREM 1.1. *Let $X \subset \mathbb{R}^m$ be a real non-singular irreducible algebraic set of dimension $n \geqslant 3$ and let $F\colon X \to \mathbb{R}^{n-1}$ be an algebraic map. Let a be an interior point of the set $\operatorname{Im} F \setminus \overline{F(\operatorname{Sing} F)} \subset \mathbb{R}^{n-1}$ and let $X_t := F^{-1}(t)$. Then $a \notin B(F)$ if and only if the following two conditions are satisfied:*

- (i) *the Euler characteristic $\chi(X_t)$ is constant when t varies within some neighbourhood of a, and*
- (ii) *there is no component of X_t that vanishes at infinity as t tends to a.*

The above criterion (i) + (ii) *may be replaced by* (i′) + (ii′), *where*

- (i′) *the Betti numbers of X_b are constant for b in some neighbourhood of a, and*
- (ii′) *there is no splitting at infinity at a.*

Note that the Euler characteristic of regular fibres is given by the following simple formula:

$$\chi(X_t) = \frac{1}{2} \lim_{R \to \infty} \#[X_t \cap S_R],$$

where $S_R \subset \mathbb{R}^m$ denotes the sphere of radius R centred at the origin.

In order to situate our study in the mathematical landscape, we start by discussing the real counterparts of several well known results in the complex setting.

2. Real versus complex setting

2.1. The Abhyankar–Moh–Suzuki theorem

The famous example by Pinchuk [9] yields a polynomial mapping $\mathbb{R}^2 \to \mathbb{R}^2$ with no singularities but which is not a global diffeomorphism, thus providing a counterexample to the strong Jacobian conjecture over the reals. The Jacobian problem nevertheless remains open over $\mathbb{C}$.

We may then further ask what happens when a polynomial map is a component of a global diffeomorphism, since, over the complex field, one has the following well-known Abhyankar–Moh–Suzuki theorem [1, 12].

THEOREM (Abhyankar–Moh–Suzuki). *A complex polynomial function $f\colon \mathbb{C}^2 \to \mathbb{C}$ which is a locally trivial fibration is actually equivalent to a linear function, modulo automorphisms of $\mathbb{C}^2$.*

This result again does not hold over $\mathbb{R}$ and it is actually not difficult to find examples, such as the following.

EXAMPLE 2.1. The polynomial function $g\colon \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = y(x^2 + 1)$, is a component of a diffeomorphism. One can see this by using the change of variables $(x, y) \mapsto (x, y/(x^2 + 1))$. Therefore, g is a globally trivial fibration. However, g cannot be linearized by a *polynomial* automorphism.

2.2. The Euler characteristic test

The following result was found in the 1970s [12]; see also [6].

THEOREM. *Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be a polynomial function and let $a \in \mathbb{C} \setminus f(\operatorname{Sing} f)$. Then $a \notin B(f)$ if and only if the Euler characteristic of the fibres, $\chi(f^{-1}(t))$, is constant for t varying in some neighbourhood of a.*

Its real counterpart emerged much later. It appears that, for polynomial functions $\mathbb{R}^2 \to \mathbb{R}$, the constancy of the Euler characteristic is not sufficient, and that other phenomena may occur at infinity, namely the 'splitting' or the 'vanishing' of components of fibres (see definition 3.1).

THEOREM 2.2 (Tibăr and Zaharia [15]). *Let X be a real algebraic non-singular surface and let $\tau\colon X \to \mathbb{R}$ be an algebraic map. Let $a \in \operatorname{Im} \tau$ be a regular value of τ and let $X_t := F^{-1}(t)$. Then $a \notin B(\tau)$ if and only if*

(i) *the Euler characteristic $\chi(X_t)$ is constant when t varies within some neighbourhood of a, and*

(ii) *there is no component of X_t that vanishes at infinity as t tends to a.*

Moreover, one can show that criterion (i) + (ii) is equivalent to (i′) + (ii′), where

(i′) the Betti numbers of X_t are constant for t in some neighbourhood of a, and

(ii′) there is no component of X_t that splits at infinity as t tends to a.

All the above conditions are necessary but none of them individually implies the local triviality of the map τ, as the examples in [15] show. Theorem 1.1 represents the extension of the above result to algebraic families of curves of more than one parameter.

2.3. Detecting bifurcation values by the Milnor set

It was shown in [5, 13] that, in the case of a polynomial map $F\colon \mathbb{R}^n \to \mathbb{R}^p$, the bifurcation non-critical locus $B(F)\setminus f(\operatorname{Sing} f)$ is included in the set of 'ρ-non-regular values at infinity'. The *ρ-regularity* is a 'Milnor type' condition that controls the transversality of the fibres of F to the spheres centred at $c \in \mathbb{R}^n$, more precisely, we have the following.

DEFINITION 2.3. Let $F\colon \mathbb{R}^n \to \mathbb{R}^p$, $n \geqslant p$, be a polynomial map. Let $\rho_c\colon \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ be the Euclidean distance function to the point $c \in \mathbb{R}^n$. We call the critical set of the mapping $(F, \rho_c)\colon \mathbb{R}^n \to \mathbb{R}^{p+1}$ the *Milnor set of* (F, ρ_c) and denote it by $M_c(F)$. We denote by

$$S_c(F) := \Big\{t_0 \in \mathbb{R}^p \mid \exists \{x_j\}_{j\in\mathbb{N}} \subset M_c(F),\ \lim_{j\to\infty} \|x_j\| = \infty \text{ and } \lim_{j\to\infty} F(x_j) = t_0\Big\}$$

the set of *ρ_c-non-regular values at infinity*. If $t_0 \notin S_c(F)$, we say that t_0 is *ρ_c-regular at infinity*. We set $S_\infty(F) := \bigcap_{c\in\mathbb{R}^n} S_c(F)$.

In the case of polynomials $f\colon \mathbb{C}^2 \to \mathbb{C}$ the following characterization has been proved (see [11, corollary 5.8]; [14, theorem 2.2.5]). *Let* $a \in \mathbb{C} \setminus f(\operatorname{Sing} f)$. *Then* $a \in B(f)$ *if and only if* $a \in S_0(f)$. This is no longer true over the reals, as shown by the following example from [15]: $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = y(2x^2y^2 - 9xy + 12)$, where $S_0(f)$ contains the origin of $\mathbb{R}$ but the bifurcation set $B(f)$ is empty.

However, with some more information along the branches of the Milnor set $M_c(f)$ that take into account the 'vanishing' and the 'splitting' phenomena at infinity (see definitions 3.1 and 4.3), one is able to produce a criterion. First, we note that there is some open dense set $\Omega_f \subset \mathbb{R}^2$ such that for $c \in \Omega_f$ the Milnor set $M_c(f)$ is a curve (or it is empty). For such a point $c \in \Omega_f$ one counts the number $\#[X_t^j \cap M_c(f)]$ of points of intersection of the connected components X_t^j of the fibres X_t with the curve $M_c(f)$. The following criterion holds. *Let* $a \in \mathbb{R} \setminus f(\operatorname{Sing} f)$. *Then* $a \in B(f)$ *if and only if* $a \in S_c(f)$ *and* $\lim_{t\to a} \#[X_t^j \cap M_c(f)] \not\equiv 0 (\operatorname{mod} 2)$ *for some sequence of connected components* X_t^j *of* X_t. This can easily be proved by using the results of our paper and is close to the main theorem of [7], which is proved for the larger class of polynomial functions defined on a smooth non-compact affine algebraic surface X. One significant difference between our approach and that of [7] is that we test *connected components* X_t^j of fibres and not just the fibres of f. This is because one may have vanishing and splitting at infinity in two different components of the same fibre, with one maximum and one minimum that would cancel in the framework of [7] but not in the above statement (see also [15, § 3, example 3.1] for the construction of such examples).

3. The non-vanishing condition

3.1. Non-vanishing at infinity

Let $X \subset \mathbb{R}^m$ be a real non-singular irreducible algebraic set of dimension n, and let $F\colon X \to \mathbb{R}^{n-1}$ be an algebraic map. Throughout this section the point a will denote an interior point of $\operatorname{Im} F \setminus \overline{F(\operatorname{Sing} F)}$.

As before, denote by X_b the fibre $F^{-1}(b)$. Then, let $X_b = \bigsqcup_j X_b^j$ be the decomposition of the fibre X_b into connected components. Define

$$\mu(b) := \max_j \inf_{x \in X_b^j} \|x\|.$$

Definition 3.1. We say that there is *vanishing at infinity at* $a \in \mathbb{R}^{n-1}$ if there exists a sequence of points $a_k \to a$ such that $\lim_{k\to\infty} \mu(a_k) = \infty$.

If there is no such sequence, we say that *there is no vanishing at* $a \in \mathbb{R}^{n-1}$ and we denote this situation by $\mathrm{NV}(a)$.

Remark 3.2. One can easily deduce from the above definition that NV is an open condition.

3.2. Proof of the first part of theorem 1.1

The regular fibres of F are one-dimensional manifolds. Hence, every such fibre is a finite union of connected components. Each such component is either compact and thus diffeomorphic to a circle, or non-compact and thus diffeomorphic to the affine line $\mathbb{R}$. Let us denote by $s(b)$ the number of compact components of the fibre

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$F^{-1}(b)$ and by $l(b)$ the number of non-compact components of this fibre. Let us note that these definitions make sense for a semi-algebraic set X; we shall occasionally use them in such a context in the proofs below.

Let $a \in \mathbb{R}^{n-1}$ be as in the statement of theorem 1.1 and let us assume $\mathrm{NV}(a)$. By remark 3.2, there exists a ball D centred at a, included in the interior of the set $\operatorname{Im} F \setminus \overline{F(\operatorname{Sing} F)} \subset \mathbb{R}^{n-1}$ such that $\mathrm{NV}(b)$ for any $b \in D$. For such a ball D, we show the following.

LEMMA 3.3. *The numbers $s_X(b)$ and $l_X(b)$ are constant for $b \in D$.*

Proof. Let us fix some point $b \in D$ and let $L_{ab} \subset \mathbb{R}^{n-1}$ denote the unique line passing through the points a and b. The fibre X_t is a one-dimensional manifold for any $t \in D$; in particular, the inverse image $F^{-1}(L_{ab})$ is an algebraic family of non-singular real curves. It is known (as proved by Thom, Verdier and others; see, for example, [14, corollary 1.2.13], and the references therein) that the projection $\tau_{ab}\colon F^{-1}(L_{ab}) \to L_{ab}$ has a finite number of atypical values. In the hypotheses of theorem 1.1 and by remark 3.2, at each supposed atypical value of $L_{ab} \cap D$ one may apply theorem 2.2 for τ_{ab}. This leads to the conclusion that there are no atypical values of τ_{ab} on $L_{ab} \cap D$; in particular, the restriction of F is a locally trivial fibration over $L_{ab} \cap D$, and hence a trivial fibration. This implies $s_X(b) = s_X(a)$ and $l_X(b) = l_X(a)$. □

3.3. Compact components

Let us consider some compact connected component of the regular fibre X_a, if there is one. Then this compact component may be covered by finitely many open connected sets $B_i \subset X$ such that $B_i \cap X_a$ is connected and that the restriction $F_{|}\colon B_i \to F(B_i)$ is a trivial fibration. In particular, each fibre of this fibration is connected. There exists a sufficiently small closed ball $D \subset \mathbb{R}^{n-1}$ centred at a that is contained in all images $F(B_i)$. It then follows that the restriction $F_{|}\colon F^{-1}(D) \cap \bigcup_i B_i \to D$ is a proper submersion. Therefore, by Ehresmann's fibration theorem, this is a locally trivial fibration, and hence a trivial fibration (since D is contractible).

It follows that, for any $t \in \mathring{D}$, there is a unique connected component of the fibre X_t, which intersects the open set $F^{-1}(\mathring{D}) \cap \bigcup_i B_i$.

It also follows that $\mathcal{D} := F^{-1}(\mathring{D}) \cap \bigcup_i B_i$ is an open connected component of $F^{-1}(\mathring{D})$. Therefore, $F^{-1}(\mathring{D}) \setminus \mathcal{D}$ is an open subset of $F^{-1}(\mathring{D})$.

By lemma 3.3 and by taking an eventually smaller ball D, we have that, for any $t \in \mathring{D}$, $X_t \cap F^{-1}(\mathring{D}) \setminus \mathcal{D}$ has precisely $l_X(a)$ connected non-compact components and $s_X(a) - 1$ connected compact components.

In this way, we have produced a trivialisation on a connected component of $F^{-1}(\mathring{D})$ and we have reduced the problem to constructing a trivialisation within the set $F^{-1}(\mathring{D}) \setminus \mathcal{D}$, where the numbers are

$$s_{F^{-1}(\mathring{D})\setminus\mathcal{D}}(a) = s_X(a) - 1 \quad \text{and} \quad l_{F^{-1}(\mathring{D})\setminus\mathcal{D}}(a) = l_X(a).$$

We apply the above procedure until we eliminate one by one all the compact components. We may then assume from now on that the fibre X_t has no compact component, for any t in some neighbourhood of a.

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3.4. Line components

Consider a line component X_a^1 of X_a and fix some point $p \in X_a^1$. Since F is a submersion at p, there exists a small ball B_p at p such that $B_p \cap X_a$ is connected and that the restriction of F to $B_p \cap F^{-1}(D)$ is a trivial fibration over a sufficiently small disc $D \subset F(B_p)$ centred at p. It follows that, for any $t \in D$, the intersection $X_t \cap B_p$ is connected and thus included into a unique connected component of the fibre X_t.

Let $\mathcal{L}_1$ denote the union over all $t \in D$ of the connected components of the fibres X_t that intersect B_p. Note that each such connected component is a line component, since we have assumed that $s_X(a) = 0$. Thus, $s_X(t) = 0$ for all $t \in D$ (by reducing the radius of D, if needed), by lemma 3.3.

We have thus associated the connected set $\mathcal{L}_1$ to the chosen component X_a^1. Consider the similar construction for each other connected component of X_a. Namely, we start as above by choosing one point p_i on each component of X_a and some ball B_{p_i} at p_i. In this way we obtain the sets $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_{l_X(a)}$, where we recall that $l_X(a)$ denotes the number of connected components of X_a and that this number is a local invariant over the target set, by lemma 3.3. Without loss of generality, we may assume that the ball D in the target is common to all these constructions.

It then follows that the sets $\mathcal{L}_i$ are all *connected* (by definition) and *pairwise disjoint*. Indeed, if this is not true, then there is some $t \in D$ such that the fibre X_t has a connected component that belongs to more than one set $\mathcal{L}_i$. But, by the above construction, each $\mathcal{L}_i$ contains precisely one connected component of X_t, and the number of connected components of X_t is precisely $l_X(a)$ by lemma 3.3. We thus obtain a numerical contradiction.

Let us show that the sets $\mathcal{L}_i$ are also *open* and therefore they are manifolds. Let us fix i and fix some $q \in X_b \cap \mathcal{L}_i$ for some $b \in D$ as above. There exists a ball B_q, which has the same properties as the ball B_{p_i} considered above. This implies that a unique component of each fibre X_t intersects B_q for t in some sufficiently small ball $D' \subset D$ centred at b. We claim that the component of X_t intersecting B_q is precisely the component belonging to $\mathcal{L}_i$, as follows. Let $q_i \in X_b \cap B_{p_i}$. We consider a non-self-intersecting analytic path in X_b starting at q_i and ending at q. Being compact, this can be covered by finitely many small balls B_j with the same properties as B_q or B_{p_i}. We then apply the reasoning in § 3.3 to obtain that the restriction $F_| \colon F^{-1}(D) \cap \bigcup_j B_j \to D'$ for some sufficiently small D' is a proper submersion. Therefore, by Ehresmann's fibration theorem, this is a locally trivial, and hence trivial, fibration since D' is contractible. Since the fibres of this map are connected by our construction and since each of them intersects B_{p_i}, it follows that each fibre of $F_|$ is included in the corresponding fibre of $\mathcal{L}_i$. Since $F^{-1}(D') \cap \bigcup_j B_j$ is, in particular, a neighbourhood of the point $q \in \mathcal{L}_i$, this finishes the proof of our claim.

We conclude that the open sets $\mathcal{L}_i$ together provide a partition of $F^{-1}(D)$ into open manifolds. We may then apply [15, proposition 2.7], which we state below, in order to conclude that every restriction $F_| \colon \mathcal{L}_i \to D$ is a trivial fibration. This ends the proof of the first part of our theorem.

PROPOSITION 3.4 (Tibăr and Zaharia [15, proposition 2.7]). *Let $M \subseteq \mathbb{R}^n$ be a smooth submanifold of dimension $m+1$ and let $g \colon M \to \mathbb{R}^m$ be a smooth function without singularities and such that all its fibres $g^{-1}(t)$ are closed in $\mathbb{R}^n$ and diffeomorphic to $\mathbb{R}$. Then g is a C^∞ trivial fibration; in particular, $M \overset{\text{diffeo}}{\simeq} \mathbb{R}^{m+1}$.*

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Remark 3.5. Note that the sets $\mathcal{L}_i$ may be defined without the non-vanishing condition at a, but then the sets $\mathcal{L}_i$ may not exhaust $F^{-1}(D)$ or they may not be mutually disjoint. The first phenomenon is due to the vanishing of components and the second is due to the so-called 'splitting' phenomenon, which we present in the next section.

4. The non-splitting condition

We study here the phenomenon of splitting at infinity in families of curves of several parameters. The following definition of limit sets was used in a particular setting in [15] and corresponds to the 'lim sup' notation used in [4]. We have learned from [4] that such limits were considered classically by Painlevé and Kuratowski.

Definition 4.1. Let $\{M_k\}_k$ be a sequence of subsets of $\mathbb{R}^m$. A point $x \in \mathbb{R}^m$ is called a *limit point* of $\{M_k\}_k$ if there exists a sequence of points $\{x_i\}_{i\in\mathbb{N}}$ with $\lim_{i\to\infty} x_i = x$ and such that $x_i \in M_{k_i}$ for some integer sequence $\{k_i\}_i \subset \mathbb{N}$ with $\lim_{i\to\infty} k_i = \infty$.

The set of all limit points of $\{M_k\}_k$ will be denoted by $\lim M_k$.

In the remainder of this paper the point a will be an interior point of $\operatorname{Im} F \setminus \overline{F(\operatorname{Sing} F)} \subset \mathbb{R}^{n-1}$, as in the statement of theorem 1.1.

Remark 4.2. Let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence of points in $\operatorname{Im} F \setminus \overline{F(\operatorname{Sing} F)}$ such that $b_k \to a$ and such that, for each k, $X^j_{b_k}$ is a fixed connected component of X_{b_k}. Then $\lim X^j_{b_k}$ is either empty or a union of connected components of X_a. This is a more precise version of [15, lemma 2.3(i)] and follows from the definition of the limit and from the fact that a is a regular value of F.

Definition 4.3. We say that there is *no splitting at infinity at* $a \in \mathbb{R}^{n-1}$ (which we abbreviate as $\mathrm{NS}(a)$) if the following holds: let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^{n-1}$ such that $b_k \to a$ and let $\{p_k\}_{k\in\mathbb{N}}$ be a convergent sequence in X such that $F(p_k) = b_k$. If $X^j_{b_k}$ denotes the connected component of X_{b_k} that contains p_k, then the limit set $\lim X^j_{b_k}$ is connected.

We say that there is *strong non-splitting at infinity at* $a \in \mathbb{R}^{n-1}$ (which we abbreviate by $\mathrm{SNS}(a)$) if, in addition to the definition of $\mathrm{NS}(a)$, we require the following: if all the components $X^j_{b_k}$ are compact, then the limit $\lim X^j_{b_k}$ is compact too.

This notion of 'non-splitting' (NS) extends that introduced in [15] for $n = 2$.

Remark 4.4.

(a) For two sequences $\{b_k\}_{k\in\mathbb{N}}$ and $\{p_k\}_{k\in\mathbb{N}}$ as above, denoting by X^j_a the connected component of X_a that contains $p := \lim p_k$ and by $X^j_{b_k}$ the connected component of X_{b_k} that contains p_k, by remark 4.2 we have the inclusion $X^j_a \subset \lim X^j_{b_k}$. Therefore, $\mathrm{NS}(a)$ means that $\lim X^j_{b_k} = X^j_a$.

(b) We do not know whether $\mathrm{NS}(a)$ implies $\mathrm{NS}(b)$ for b in a sufficiently small neighbourhood of a. However, this implication holds whenever the Betti numbers of X_b are constant for b in a neighbourhood of a. This follows from the second part of the proof of theorem 1.1.

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4.1. Proof of the second part of theorem 1.1

Conditions (i′) and (ii′) are obviously necessary for $a \notin B(F)$. Let us prove that they imply the conditions (i) and (ii) of theorem 1.1. Since condition (i) is obviously implied by condition (i′), the rest of the proof will be devoted to show condition (ii).

Let us denote by $X_a^1, \dots, X_a^l$ the connected components of X_a. For each $j = 1, \dots, l$, we choose a point $z_j \in X_a^j$ and, as in § 3.3, we fix a sufficiently small ball B_j at z_j such that $B_j \cap X_a$ is connected and that the restriction of F to $B_j \cap F^{-1}(D_j)$ is a trivial fibration over a sufficiently small disc $D_j \subset F(B_j)$ centred at a. We may assume that the small balls $B_1, \dots, B_l$ are pairwise disjoint. In particular, for each $b \in \bigcap_j D_j$ we have that B_j intersects exactly one connected component of X_b. We therefore may define a function Φ_b on the set $\{1, \dots, l\}$ with values in the set of connected components $X_b^1, \dots, X_b^{s_b}$ of X_b by setting $\Phi_b(j)$ to be the unique component of X_b that intersects B_j.

Claim 4.5. $\mathrm{NS}(a)$ *implies that there exists a ball* $D \subset \bigcap_j D_j$ *centred at* a *such that, for any* $b \in D$, Φ_b *is a bijection.*

Proof of claim 4.5. Since $b_0(X_t)$ is constant at a, there is a sufficiently small disc D' centred at a (which we may assume is included in $\bigcap_j D_j$) such that $s_b = l$ for all $b \in D'$. It is therefore enough to prove that Φ_b is injective on some sufficiently small disc $D \subset D'$ centred at a. By *reductio ad absurdum*, suppose that there exists a sequence of points $\{b_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^{n-1}$ such that $b_k \to a$, and $i_k, j_k \in \{1, \dots, l\}$, $i_k \neq j_k$, such that $\Phi_{b_k}(i_k) = \Phi_{b_k}(j_k)$. Since the set of all subsets with exactly two elements of $\{1, \dots, l\}$ is finite, by passing to a subsequence we may assume that there exist $i, j \in \{1, 2 \dots, l\}$, $i \neq j$, such that $\Phi_{b_k}(i) = \Phi_{b_k}(j)$ for every k. We get that the limits $\lim \Phi_{b_k}(i)$ and $\lim \Phi_{b_k}(j)$ coincide and, by remark 4.4(a), that they are equal to some connected component of X_a.

On the other hand, since $F_{|B_i \cap F^{-1}(D_i)}$ and $F_{|B_j \cap F^{-1}(D_j)}$ are trivial fibrations it follows that the sets $B_i \cap F^{-1}(D_i) \cap \lim_k \Phi_{b_k}(i)$ and $B_j \cap F^{-1}(D_j) \cap \lim_k \Phi_{b_k}(j)$ are non-empty and they are contained in different components of X_a. This yields a contradiction. Our claim is proved. □

Finally, let us show that we have $\mathrm{NV}(a)$. If this were not the case, then there would exist a sequence $\{b_k\}_{k\in\mathbb{N}}$ converging to a such that $\lim_{k\to\infty} \mu(b_k) = \infty$ (see the definition of μ in § 3.1). This implies that there is a connected component $X_{b_k}^j$ of X_{b_k} such that $X_{b_k}^j \cap (\bigcup_{j=1}^l B_j) = \emptyset$, and this contradicts the surjectivity of Φ_{b_k}.

This ends the proof of the reduction of the second part of theorem 1.1 to its first part.

Remark 4.6. In the above proof we need to assume the constancy of the Betti number $b_1(X_t)$, since this condition is *not implied* by the constancy of the Betti number $b_0(X_t)$, by $\mathrm{NS}(a)$ and by $\mathrm{NV}(a)$ together. The cause of this behaviour, which can be seen in [15, example 3.2], is the ‘breaking’ of oval components at infinity. Nevertheless, such a loss of points at infinity can be avoided if instead of $\mathrm{NS}(a)$ we require the $\mathrm{SNS}(a)$ condition of definition 4.3, as shown by the following result.

Corollary 4.7. *In the conditions of theorem 1.1, the following equivalence holds:*

$$a \notin B(F) \iff \mathrm{SNS}(a) \text{ and } \mathrm{NV}(a).$$

Proof. Conditions SNS(a) and NV(a) are obviously necessary for $a \notin B(F)$. Let us show the sufficiency. By remark 4.4(a), NS(a) implies $b_0(X_t) \geqslant b_0(X_a)$ for t in some sufficiently small disc centred at a. Next, NS(a) together with NV(a) imply that $b_0(X_t) = b_0(X_a)$. What we only need in order to conclude is the constancy of $b_1(X_t)$ for t in some neighbourhood of a, but this is exactly what the condition SNS(a) ensures. □

The conditions NV(a), NS(a) (and hence also SNS(a)) are conditions 'at infinity'. More precisely, one can prove the following statement in a similar way to that above.

THEOREM 4.8. *Let $X \subset \mathbb{R}^m$ be a real non-singular irreducible algebraic set of dimension n and let $F\colon X \to \mathbb{R}^{n-1}$ be an algebraic map. Let $a \in \operatorname{Im} F$ be a regular value of F and let $R \gg 1$ be large enough that X_a is transversal to the sphere $X \cap S_R^{m-1}$. Let us denote by G the restriction of F to $X \setminus B_R^m$ and by X_t its fibres.*

If a is an interior point of the set $\operatorname{Im} G \setminus \overline{G(\operatorname{Sing} G)} \subset \mathbb{R}^{n-1}$, then $a \notin B(G)$ if and only if either conditions (i) and (ii) or conditions (i′) and (ii′) of theorem 1.1 hold.

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