

## NOTES

# PERSISTENT CATASTROPHIC SHOCKS AND EQUITY PREMIUMS: A NOTE

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This note demonstrates analytically that a persistent catastrophic shock on endowment growth, even if moderate, yields negative equity premiums when a representative agent is relatively prudent. In particular, it derives the minimum persistence necessary to have zero equity premiums.

**Keywords:** Catastrophic Shocks, Persistence, Equity Premiums

## 1. INTRODUCTION

Several papers, including Rietz (1988) and Barro (2006), claim that a catastrophic shock on endowment growth generates large equity premiums in an exchange economy with a representative agent. However, the following intuitive argument anticipates that this claim may not hold in cases in which shocks are persistent. When the realized catastrophic shock lowers endowment growth substantially, equity prices may soar due to strong precautionary savings by prudent consumers against the possibility of another catastrophic shock. Consequently, such a negative correlation between consumption growth and equity returns may generate negative equity premiums.

As pointed out by Campbell (1999) and others, the same logic is applicable to the case in which positively correlated productivity growth shocks work to reduce equity premiums. However, when shocks on endowment growth are catastrophic, persistent shocks may yield not only lower, but also negative equity premiums. Barro et al. (2009) provide numerical examples, thereby showing that persistent catastrophic shocks lead to negative equity premiums under empirically plausible setups.

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This note analytically confirms the above conjecture in an exchange economy of a representative agent with constant relative risk aversion.<sup>1</sup> In particular, we derive precisely the minimum persistence necessary to have zero equity premiums.

2. SETUP

2.1. A Simple Exchange Economy with a Representative Agent

Following Mehra and Prescott (1985) and Rietz (1988), we construct a single goods exchange economy with a representative agent. There are two financial assets, risk-free assets and Lucas trees, which are assigned claims on stochastic endowment. In this paper, the Lucas tree and equity are interchangeable.

Here,  $p_t^f$  and  $p_t^e$  denote the time- $t$  prices of risk-free assets and Lucas trees, respectively.  $Y_t$  is dividends on Lucas trees at time  $t$ , and is regarded as endowment.  $f_t$  and  $e_t$  indicate the outstanding positions of the two financial assets. A representative agent has a time-additive utility  $u(C)$  on consumption at time  $t$  ( $C_t$ ) with a discount factor  $\beta$ .

Given the above setup, the representative agent maximizes the following lifetime utility:

$$\max E_0 \left[ \sum_{t=0}^{\infty} \beta^t u(C_t) \right],$$

subject to  $C_t = Y_t e_{t-1} + p_t^e (e_{t-1} - e_t) + f_{t-1} - p_t^f f_t$ , where  $E_0$  denotes the expectation operator conditional on time-0 information.

The first-order conditions or Euler equations are

$$p_t^e = E_t \left[ \beta \frac{u'(C_{t+1})}{u'(C_t)} \{Y_{t+1} + p_{t+1}^e\} \right], \tag{1}$$

$$p_t^f = E_t \left[ \beta \frac{u'(C_{t+1})}{u'(C_t)} \right]. \tag{2}$$

The market clearing conditions consist of  $C_t = Y_t$ ,  $e_t = 1$ , and  $f_t = 0 \forall t$ . To obtain concrete analytical results, we assume below a preference with constant relative risk aversion (CRRA), or  $u(C_t) = \frac{C_t^{1-\gamma}-1}{1-\gamma}$  with  $0 < \gamma$ , where  $\gamma$  indicates the coefficient of relative risk aversion, or the reciprocal of the intertemporal elasticity of substitution.

We now characterize persistent catastrophic shocks on endowment growth.<sup>2</sup> The aggregate endowment or consumption grows according to  $Y_{t+1} = x_{t+1} Y_t$  or  $C_{t+1} = x_{t+1} C_t$ . A gross growth rate  $x_{t+1}$  follows the Markov process:  $\text{Prob}(x_{t+1} = x_j | x_t = x_i) = q_{i,j}$ . There are two states, a normal state ( $n$ ) and a catastrophic state ( $c$ ). Here,  $x_c \equiv x_n - \kappa$ , where  $\kappa (> 0)$  denotes the size of catastrophic growth shocks. In a normal state at time  $t$ ,  $x_{t+1}$  takes  $x_n$  with probability  $q_{n,n} = 1 - \phi$ , and  $x_c$  with probability  $q_{n,c} = \phi$ . In a catastrophic state at time  $t$ ,  $x_{t+1}$  takes  $x_c$  with probability  $q_{c,c} = \psi$ , and  $x_n$  with probability  $q_{c,n} = 1 - \psi$ .

Given the above characterization, endowment growth shocks are named *catastrophic* with rather tiny  $\phi$  and extremely large  $\kappa$ . In addition, a catastrophic growth shock is called *persistent* if  $\phi < \psi$ , and it is regarded as *purely transitory* if  $\phi = \psi$ . The unconditional probability of a catastrophic state or  $\theta$  is equal to  $\frac{\phi}{1-\psi+\phi}$ .

Euler equations (1) and (2) can be rewritten in a recursive manner:

$$p^e(Y_t, x_t) = E_t \left[ \beta \frac{u'(Y_{t+1})}{u'(Y_t)} \{Y_{t+1} + p^e(Y_{t+1}, x_{t+1})\} \right], \tag{3}$$

$$p^f(Y_t, x_t) = E_t \left[ \beta \frac{u'(Y_{t+1})}{u'(Y_t)} \right]. \tag{4}$$

A price dividend ratio  $w_i$  is defined as  $p^e(C, i) = w_i C$ . Then, exploiting the property of CRRA preference, we can further rewrite equation (3) as follows:

$$\mathbf{w} \equiv \begin{bmatrix} w_n \\ w_c \end{bmatrix} = \begin{bmatrix} \beta(1-\phi)u'(x_n)x_n & \beta\phi u'(x_c)x_c \\ \beta(1-\psi)u'(x_n)x_n & \beta\psi u'(x_c)x_c \end{bmatrix} \begin{bmatrix} w_n + 1 \\ w_c + 1 \end{bmatrix} \tag{5}$$

A realized gross return on equity between state  $(C, i)$  and state  $(x_j C, j)$  is

$$r_{ij}^e = \frac{p^e(x_j C, j) + x_j C}{p^e(C, i)} = \frac{x_j(w_j + 1)}{w_i}. \tag{6}$$

In equation (6),  $x_j \frac{w_j}{w_i}$  corresponds to a capital gain, while  $\frac{x_j}{w_i}$  matches an income gain. Accordingly, the expected return on equity in state  $(C, i)$  is

$$R_i^e = \sum_{j=\{n,c\}} q_{ij} r_{ij}^e.$$

A risk-free rate  $R_i^f$  in state  $(C, i)$  is equal to  $1/p^f(C, i)$ . The unconditional expectations of equity returns and risk-free rates are  $R^e = (1 - \theta)R_n^e + \theta R_c^e$  and  $R^f = (1 - \theta)R_n^f + \theta R_c^f$ . Thus, the unconditional equity premium is defined as  $\Pi \equiv R^e - R^f$ .

### 2.2. The Minimum Persistence Necessary to Have Negative Equity Premiums

In this subsection, we derive analytically how small persistence  $\psi$  would yield negative equity premiums. As in a standard time-additive preference, the unconditional equity premium is

$$\Pi = - \left\{ (1 - \theta) \frac{\text{cov}_n[u'(x), R^e]}{E_n[u'(x)]} + \theta \frac{\text{cov}_c[u'(x), R^e]}{E_c[u'(x)]} \right\}. \tag{7}$$

With CRRA preferences, equation (7) implies that negative equity premiums emerge when equity returns are negatively correlated with consumption growth (or endowment growth in an exchange economy). First, we demonstrate below that

persistence ( $\psi > \phi$ ) works to reduce positive correlation between equity returns and consumption growth.

As implied by equation (6), when consumption growth declines by  $1 - x_c$  due to catastrophic shocks, the positive correlation between equity returns and consumption growth is weakened with  $w_c/w_n > 1$ . That is, when price dividend ratios are larger in a catastrophic state than in a normal state, the correlation is less positive.

From equation (5), we derive

$$\mathbf{w} = (\mathbf{I} - \mathbf{M})^{-1} \mathbf{M} \mathbf{i}, \tag{8}$$

where

$$\mathbf{M} \equiv \begin{bmatrix} \beta(1 - \phi)u'(x_n)x_n & \beta\phi u'(x_c)x_c \\ \beta(1 - \psi)u'(x_n)x_n & \beta\psi u'(x_c)x_c \end{bmatrix},$$

$\mathbf{I}$  is the identity matrix, and  $\mathbf{i}$  is the column vector whose elements are ones.

The solution to equation (8) is

$$w_n = \frac{\beta}{D(x_n, \kappa, \phi, \psi)} \left\{ \phi u'(x_c)x_c + u'(x_n)x_n [1 - \phi - \beta(\psi - \phi)u'(x_c)x_c] \right\}, \tag{9}$$

$$w_c = \frac{\beta}{D(x_n, \kappa, \phi, \psi)} \left\{ \psi u'(x_c)x_c + u'(x_n)x_n [1 - \psi - \beta(\psi - \phi)u'(x_c)x_c] \right\}, \tag{10}$$

where

$$D(x_n, \kappa, \phi, \psi) \equiv 1 - \beta\psi u'(x_c)x_c - \beta u'(x_n)x_n [1 - \phi - \beta(\psi - \phi)u'(x_c)x_c].$$

$D(x_n, \kappa, \phi, \psi)$  needs to be positive for a positive value of equity prices.

From equations (9) and (10), we have

$$w_c - w_n = \frac{\beta}{D(x_n, \kappa, \phi, \psi)} \delta(x_n, \kappa) (\psi - \phi), \tag{11}$$

where

$$\delta(x_n, \kappa) \equiv u'(x_c)x_c - u'(x_n)x_n.$$

$\delta(x_n, \kappa)$  is positive when  $\gamma$  is greater than one. Equation (11) implies that  $\frac{w_c}{w_n}$  is greater than one if  $\gamma$  is larger than one, and  $\psi$  is greater than  $\phi$ . Thus, the correlation between equity returns and consumption growth is less positive when growth shocks are persistent and consumers are prudent. That is, the persistence of growth shocks and the prudence of consumers are necessary for negative equity premiums, but are not sufficient. The following proposition demonstrates that

catastrophic growth shocks may lead to negative equity premiums with moderate persistence.

**PROPOSITION 1.** *If there exists a negative equity premium ( $\Pi < 0$ ), then  $\psi$  satisfies  $\phi < \psi_0 < \psi < 1$ , where  $\psi_0$  is defined as*

$$\psi_0 \equiv \phi + \frac{\kappa}{\beta x_n x_c [u'(x_c) - u'(x_n)]}. \tag{12}$$

See Appendix 1 for the proof.

Proposition 1 demonstrates that mild persistence of catastrophic shocks and the prudence of consumers are required to have negative equity premiums in the following respects. First, tiny  $\phi$  characteristic of catastrophic shocks itself reduces the minimum persistence  $\psi_0$  defined by equation (12).

Second, large  $\kappa$  also characteristic of catastrophic shocks lowers  $\psi_0$ . As shown in Appendix 2, by Taylor-expanding equation (12) with respect to  $\kappa$  in the neighborhood of  $x_c = x_n$  up to second order, we can prove that  $\frac{\partial \psi_0}{\partial \kappa} < 0$  and  $\frac{\partial^2 \psi_0}{\partial \kappa^2} > 0$  if  $\gamma > 1$ . That is,  $\psi_0$  is decreasing and convex with respect to  $\kappa$ .

Third, equation (12) indicates that more prudence or larger  $\gamma$  also contributes to reducing the minimum persistence  $\psi_0$ ; the denominator of the second term in its right-hand side  $\{u'(x_c) - u'(x_n)\}$  becomes larger with more sizable  $\gamma$ . Conversely, stronger persistence is required with less prudent consumers. For example, when a preference is logarithmic ( $\gamma = 1$ ),  $\psi_0 = \phi + \frac{1}{\beta}$  at the limit of  $\kappa = 0$ . Accordingly, it is impossible to have negative equity premiums with  $\beta < 1$  because  $\psi_0$  becomes larger than one.

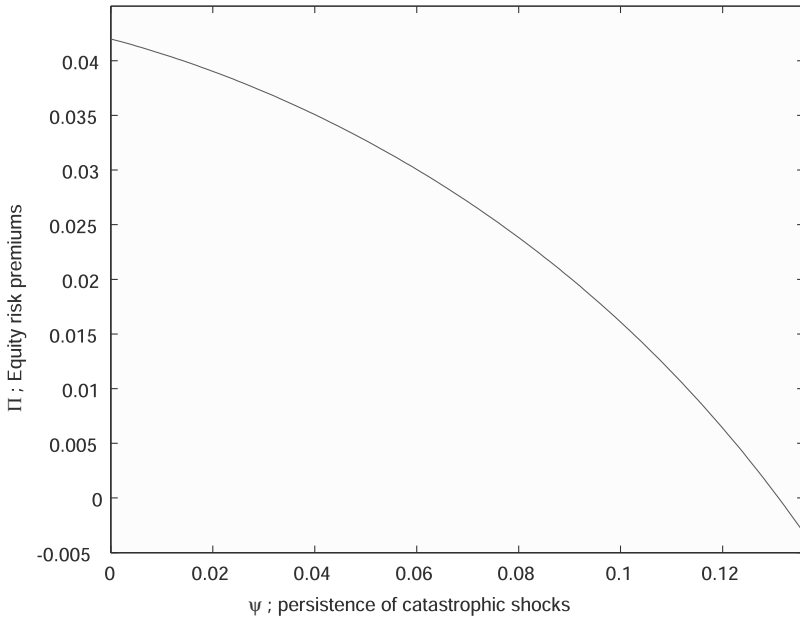
One caveat concerning Proposition 1 is that  $\psi_0$  is not sufficient, but is necessary to have zero equity premiums. Depending on a set of structural parameters, equilibrium may not exist at  $\psi = \psi_0$ .

### 2.3. Calibration

This subsection presents some results of simple calibration. Following Barro (2006), we assume that  $\beta = 0.97$ ,  $\gamma = 4$ ,  $x_n = 1.025$ , and  $\phi = 0.017$ . Next, we set  $\kappa$  equal to 0.4 such that the unconditional equity premium ( $\Pi$ ) may match the historical average (4% per year). Under this assumption,  $\psi_0 = 0.131$ . Then, a persistence parameter  $\psi$  is set at either 0.017 (purely transitory), 0.070, or 0.135 ( $> \psi_0 = 0.131$ ).

Figure 1 plots the unconditional equity premium given a persistence parameter  $\psi$ . As this figure shows, the unconditional equity premium declines quickly as catastrophic shocks are persistent. That is, large equity premiums disappear completely with mild persistence.

Table 1 reports some results of the above calibration exercise. For comparison, this table also includes the case of a logarithmic preference ( $\gamma = 1$ ). When  $\gamma = 4$ ,



**FIGURE 1.** The relationship between the unconditional equity premium ( $\Pi$ ) and persistence parameters ( $\psi$ ) ( $\beta = 0.97$ ,  $\gamma = 4$ ,  $\kappa = 0.4$ ,  $x_n = 1.025$ , and  $\phi = 0.017$ ).

the unconditional equity premium reduces from 0.04 ( $\psi = 0.017$ , or a purely transitory case) to 0.027 if  $\psi = 0.070$ , and to  $-0.003$  if  $\psi = 0.135$ .

A major reason for such substantial decreases in equity premiums is that the correlation between equity returns and consumption growth becomes negative as catastrophic shocks are relatively persistent. When catastrophic shocks are purely transitory, price dividend ratios do not differ between  $w_n$  and  $w_c$ , and negative (net) equity returns ( $0.656 - 1.000 = -0.344$ ) are realized upon the occurrence of catastrophic shocks. On the other hand, a net equity return increases from  $-0.344$  to  $-0.220$  if  $\psi = 0.070$ , and it becomes even positive (0.059) if  $\psi = 0.135$ .

If  $\gamma$  increases from four, then the size of equity premiums ( $\Pi$ ) is more sensitive to a persistence parameter  $\psi$ . We again set  $\kappa$  to have 4% per year for  $\Pi$  given  $\gamma$ ;  $\kappa = 0.315$  for  $\gamma = 6$ ,  $\kappa = 0.263$  for  $\gamma = 8$ ,  $\kappa = 0.227$  for  $\gamma = 10$ , and  $\kappa = 0.171$  for  $\gamma = 15$ . Then, we compute the minimum persistence  $\psi_0$  for each case;  $\psi_0 = 0.081$  for  $\gamma = 6$ ,  $\psi_0 = 0.061$  for  $\gamma = 8$ ,  $\psi_0 = 0.050$  for  $\gamma = 10$ , and  $\psi_0 = 0.037$  for  $\gamma = 15$ . In all of these cases, equilibrium exists at  $\psi = \psi_0$ . Among more prudent consumers, large equity premiums disappear more quickly under less persistence ( $\psi_0$  closer to  $\phi = 0.017$ ).

The case with a logarithmic preference ( $\gamma = 1$ ) offers a contrasting result. The unconditional equity premium is quite small, and increases slightly with the persistence of catastrophic shocks; it is 0.004 in a purely transitory case, and it is 0.006 with  $\psi = 0.135$ . A reason for this consequence is that the correlation

**TABLE 1.** Calibration results ( $\beta = 0.97$ ,  $x_n = 1.025$ , and  $\phi = 0.017$ )

	(1)	(2)	(3)	(4)	(5)	(6)
$\gamma$	1	1	1	4	4	4
$\kappa$	0.4	0.4	0.4	0.4	0.4	0.4
$\psi$	0.017	0.07	0.135	0.017	0.07	0.135
$\Pi$	0.004	0.005	0.005	0.040	0.027	-0.003
$R^e$	1.050	1.049	1.049	1.068	1.052	1.018
$R^f$	1.045	1.045	1.044	1.029	1.025	1.021
$w$	32.3	32.3	32.3	20.3	29.3	1090.8
$w_n$	32.3	32.3	32.3	20.3	29.2	1076.6
$w_c$	32.3	32.3	32.3	20.3	35.4	1812.1
$r_{nn}^e$	1.057	1.057	1.057	1.076	1.060	1.026
$r_{nc}^e$	0.644	0.644	0.644	0.656	0.780	1.053
$Ex$	1.018	1.018	1.017	1.018	1.018	1.017

$\gamma$ : relative risk aversion;  $\psi$ : persistence of catastrophic shocks;  $\Pi$ : unconditional equity premiums;  $R^e$ : the unconditional expectation of equity returns;  $R^f$ : the unconditional expectation of risk-free rates;  $w$ : the unconditional expectation of price dividend ratios;  $w_i$ : the price dividend ratio conditional on either a normal state ( $i = n$ ) or a catastrophic state ( $i = c$ );  $r_{ni}^e$ : the realized gross equity return conditional on either of the two states; and  $Ex$ : the unconditional expectation of consumption growth.

between equity returns and consumption growth is always positive in the case of a logarithmic preference.

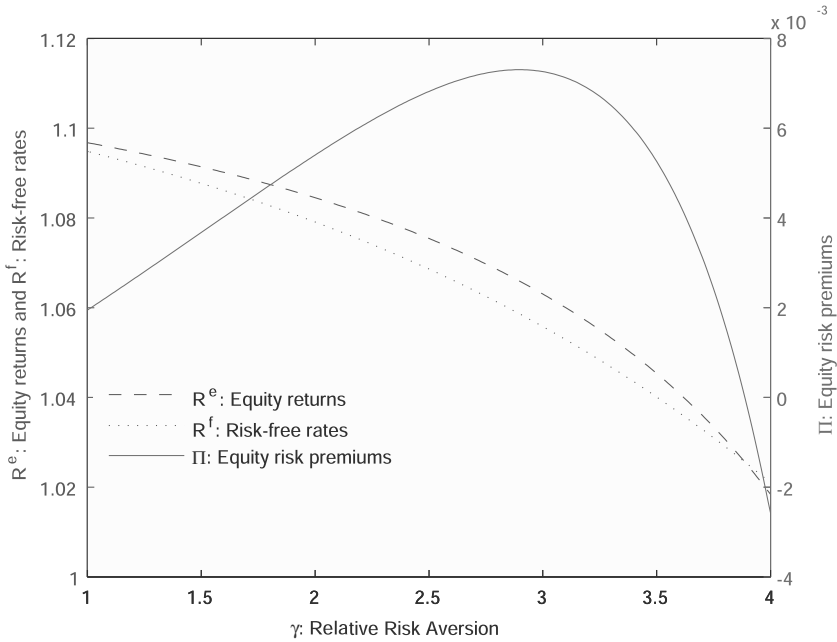
To illuminate further the impact of persistent catastrophic shocks, we switch a stochastic discount factor from the time-additive expected utility model [ $\beta(\frac{C_{t+1}}{C_t})^{-\gamma}$ ] to the nonexpected utility model proposed by Epstein and Zin (1991) and others, or

$$\beta \frac{\epsilon - \gamma \epsilon}{\epsilon - 1} \left( \frac{C_{t+1}}{C_t} \right)^{\frac{\gamma - 1}{\epsilon - 1}} \left( \frac{C_{t+1} + p_{t+1}^e}{p_t^e} \right)^{\frac{1 - \gamma \epsilon}{\epsilon - 1}},$$

where  $\epsilon$  represents the intertemporal elasticity of substitution. If  $\gamma = \frac{1}{\epsilon}$ , then the latter stochastic discount factor reduces to the former. Under the above nonexpected utility, the coefficient of relative prudence is equal to  $\gamma + \gamma \epsilon$ ; that is, an increase in  $\gamma$  enhances not only relative risk aversion, but also relative prudence.

One of analytical merits in the nonexpected utility specification is that unlike in the expected utility preference, it is possible to explore pure effects of changes in relative risk aversion ( $\gamma$ ) and relative prudence ( $\gamma + \gamma \epsilon$ ) without any variation in intertemporal substitution ( $\epsilon$ ). By exploiting this advantage, we numerically investigate how asset pricing behaves with an increase in  $\gamma$  when  $\epsilon$  is fixed.<sup>3</sup>

When  $\gamma$  increases, the unconditional average of risk-free rates ( $R^f$ ) always decline because a precautionary saving motive is stronger with an increase in  $\gamma$ . With respect to effects on risk premiums, however, an increase in  $\gamma$  yields two



**FIGURE 2.** The effect of changes in relative risk aversion ( $\gamma$ ) on asset pricing ( $R^e$ ,  $R^f$ , and  $\Pi$ ) given intertemporal elasticity of substitution constant ( $\epsilon = 0.25$ ,  $\beta = 0.97$ ,  $\kappa = 0.4$ ,  $\psi = 0.135$ ,  $x_n = 1.025$ , and  $\phi = 0.017$ ).

opposite effects in the presence of persistent catastrophic shocks. On the one hand, risk premiums tend to enlarge with relative risk aversion by a usual mechanism (the positive effect). On the other hand, stronger precautionary savings, motivated by an increase in  $\gamma$ , tend to reduce equity premiums, because they weaken the positive correlation between consumption growth and equity returns immediately after the realization of catastrophic shocks (the negative effect).

Figure 2 depicts how asset pricing behaves with an increase in  $\gamma$  between one and four<sup>4</sup> and fixed  $\epsilon (= 0.25)$  when persistence in catastrophic shocks is moderately high, or  $\psi = 0.135$ . We set a set of structural parameters such that there emerges a negative risk premium under the expected utility specification ( $\gamma = \frac{1}{\epsilon} = 4$ ). As this figure demonstrates, risk-free rates decrease monotonically with relative risk aversion. In terms of impacts on equity premiums, as long as  $\gamma$  is small, the positive effect dominates the negative effect, thereby enlarging  $\Pi$ . As  $\gamma$  increases further, however, the negative effect dominates the positive effect, thereby reducing  $\Pi$ . In particular, the negative effect is more dominant in the neighborhood where there emerges a negative risk premium. It is hard to obtain such a dominant negative impact of increasing  $\gamma$  on  $\Pi$  without any moderate persistence in catastrophic shocks.<sup>5</sup>



### 3. DISCUSSION

This note has derived analytically the minimum persistence of catastrophic shocks on endowment growth to have zero equity premiums under CRRA preferences. In addition, it has demonstrated numerically that under a reasonable set of structural parameters, large equity premiums quickly disappear when catastrophic shocks are mildly persistent.

As demonstrated in Maddison (2003), Barro (2006), Barro et al. (2009), and others, catastrophic shocks on economic growth are often observed to be persistent. Given the extreme sensitivity of equity premiums to the persistence of catastrophic shocks, we should be deeply cautious of attributing the emergence of large equity premiums to the presence of catastrophic shocks.

As one remark on an alternative way to incorporate dynamic nature, Gourio (2010) introduces time-varying disaster risks<sup>6</sup> into the real business cycle model. In his specification, investment opportunities change over time even during a normal regime, while our persistence specification yields a difference in investment opportunities between before and after the realization of catastrophic shocks. In this regard, Gourio (2010) can consider high frequency effects of disaster risks, which cannot be addressed in our model.

#### NOTES

1. Barro et al. (2009) suggest that this conjecture may not hold under nonexpected preferences with elastic intertemporal substitution. On the other hand, using the same class of nonexpected preferences, Gourio (2008) shows that equity premiums become not negative, but rather low given quick recoveries in the aftermath.

2. Our formulation of catastrophic shocks is rather different from that of Cecchetti, Lam, and Mark (1990), who adopt a Markov regime switching model to characterize the normal and depression regimes. In their formulation, a regime, either of the two regimes, is fixed prior to the realization of aggregate shocks.

3. We basically follow the numerical computation method proposed by Weil (1989).

4. If  $\gamma$  is greater than 4.01, then there never exists any equilibrium.

5. In all cases presented by Weil (1989), for example, average equity premiums increase with relative risk aversion even under the nonexpected utility specification.

6. Following Beaudry and Portier (2006), Gourio (2010) interprets such time-varying risks as a sort of news shock by which only future investment opportunities are influenced.

#### REFERENCES

- Barro, R.J. (2006) Rare disasters and asset markets in the twentieth century. *Quarterly Journal of Economics* 121(3), 823–866.
- Barro, R.J., E. Nakamura, J. Steinsson, and J. Ursúa (2009) Crises and Recoveries in an Empirical Model of Consumption Disasters. NBER Working Paper No. 15920.
- Beaudry, P., and F. Portier (2006) Stock prices, news, and economic fluctuations. *American Economic Review* 96, 1293–1307.
- Campbell, J.Y. (1999) Asset prices, consumption, and the business cycle. In J.B. Taylor and M. Woodford (eds.), *Handbook of Macroeconomics I*, pp. 1231–1301. Amsterdam: Elsevier.
- Cecchetti, S.G., P. Lam, and N.C. Mark (1990) Mean reversion in equilibrium asset prices. *American Economic Review* 80(3), 398–418.

Epstein, L.G., and S.E. Zin (1991) Substitution, risk aversion, and the temporal behavior of consumption and asset returns: An empirical analysis. *Journal of Political Economy* 99, 263–286.

Gourio, F. (2008) Disasters and recoveries: A note on the Barro-Rietz explanation of the equity-risk premium puzzle. *American Economic Review* 98(2), 68–73.

Gourio, F. (2010) Disaster Risk and Business Cycles. NBER Working Paper No. 15399.

Maddison, A. (2003) *The World Economy: Historical Statistics*. Paris: OECD.

Mehra, R., and E.C. Prescott (1985) The equity-risk premium: A puzzle. *Journal of Monetary Economics* 15, 145–161.

Rietz, T.A. (1988) The equity risk premium: A solution. *Journal of Monetary Economics* 22, 117–131.

Weil, P. (1989) The equity premium puzzle and the risk-free rate puzzle. *Journal of Monetary Economics* 24, 401–421.

## APPENDIX

### Appendix 1: Proof of Proposition 2

As equation (7) implies, the unconditional equity premiums are negative when both  $\text{cov}_n[u'(x), R^e]$  and  $\text{cov}_c[u'(x), R^e]$  are positive. After some manipulation, we have

$$\text{cov}_n[u'(x), R^e] = \left\{ (1 - \phi)\phi \left[ u'(x_c) - u'(x_n) \right] \left( r_{n,c}^e - r_{n,n}^e \right) \right\}.$$

Because  $u'(x_c) > u'(x_n)$ ,  $r_{n,c}^e - r_{n,n}^e > 0$  implies  $\text{cov}_n[\beta u'(x), R^e] > 0$ .

$r_{n,c}^e - r_{n,n}^e > 0$  is rewritten as  $r_{n,c}^e - r_{n,n}^e = (x - \kappa) \frac{w_c + 1}{w_n} - x \frac{w_n + 1}{w_n}$ . Substituting equations (9) and (10), we have

$$\psi > \phi + \frac{\kappa}{\beta x_n x_c [u'(x_c) - u'(x_n)]}.$$

Accordingly,  $\psi_0 \equiv \phi + \frac{\kappa}{\beta x_n x_c [u'(x_c) - u'(x_n)]}$ .

Similarly,  $\text{cov}_c[\beta u'(x_n), R^e]$  implies  $\psi_0 \equiv \phi + \frac{\kappa}{\beta x_n x_c [u'(x_c) - u'(x_n)]}$ .

(Q.E.D.)

### Appendix 2: Derivation of first and second derivatives of equation (12)

Equation (12) is rewritten as

$$\psi_0 = \phi + \frac{\kappa}{\beta x_n \delta(x_n, \kappa) + \beta u'(x_n) x_n \kappa},$$

where  $\delta(x_n, \kappa) \equiv u'(x_c) x_c - u'(x_n) x_n$ .

Taylor-expanding  $\delta(x_n, \kappa)$  with respect to  $\kappa$  at the neighborhood of  $x_c = x_n$  up to second order leads to

$$\delta(x_n, \kappa) \simeq u'(x_n) (\gamma - 1) \kappa - \frac{u''(x_n)}{2} (\xi - 2) \kappa^2,$$

where  $\gamma \equiv -\frac{u''(x_n)}{u'(x_n)} x_n$  (relative risk aversion), and  $\xi \equiv -\frac{u'''(x_n)}{u''(x_n)} x_n$  (relative prudence). Note that  $\xi = \gamma + 1$  under CRRA preference. Then,  $\beta x_n \delta(x_n, \kappa) + \beta u'(x_n) x_n \kappa$  can be rewritten as  $\beta u'(x_n) x_n \gamma \kappa - \beta \frac{u''(x_n) x_n}{2} (\xi - 2) \kappa^2$ .

Given the above preparation, we have

$$\psi_0 = \phi + \frac{1}{g\kappa + h}$$

where  $g = -\frac{1}{2}\beta u''(x_n)x_n(\xi - 2)$ , and  $h = \beta u'(x_n)x_n\gamma$ .

Then, we have the following derivatives with respect to  $\kappa$ :

$$\frac{\partial \psi_0}{\partial \kappa} = -\frac{g}{(g\kappa + h)^2} = \frac{\beta u''(x_n)x_n(\xi - 2)}{2\{\beta x_n(x_n - \kappa)[u'(x_c) - u'(x_n)]\}^2} < 0,$$

and

$$\frac{\partial^2 \psi_0}{\partial \kappa^2} = \frac{2g^2}{(g\kappa + h)^3} = \frac{[\beta u''(x_n)x_n(\xi - 2)]^2}{2\{\beta x_n(x_n - \kappa)[u'(x_c) - u'(x_n)]\}^3} > 0.$$