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A context in which finite or unique ergodicity is generic

ANDY Q. YINGST

University of South Carolina Lancaster, PO Box 889, Lancaster, SC 29721, USA
(e-mail: andy.yingst@gmail.com)

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Abstract. We show that for good measures, the set of homeomorphisms of Cantor space which preserve that measure and which have no invariant clopen sets contains a residual set of homeomorphisms which are uniquely ergodic. Additionally, we show that for refinable Bernoulli trial measures, the same set of homeomorphisms contains a residual set of homeomorphisms which admit only finitely many ergodic measures.

Key words: group actions, smooth dynamics, good measures, unique ergodicity
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1. *Stating the main theorem*

In their fundamental 1941 paper, [8], Oxtoby and Ulam showed that for $n \geq 2$, among homeomorphisms of the cube $[0, 1]^n$ which preserve Lebesgue measure, the property of being ergodic was generic. That is, the collection of such homeomorphisms which are ergodic is residual in the space. This result was extended in many ways and replicated in many spaces; the book of Alpern and Prasad [2] contains a wide range of results of this form, primarily focused on generic behavior in spaces of homeomorphisms of manifolds.

More recently there have been similar investigations for groups of homeomorphisms of Cantor space. One surprising result in this field due to Kechris and Rosendal [7] is that in the space of homeomorphisms of Cantor space, there exists a single homeomorphism whose isomorphism class is generic. This homeomorphism was then constructed more explicitly by Akin, Glasner and Weiss [4] providing a fairly complete understanding of its dynamical properties, which then are immediately understood to be generic properties in the homeomorphism space of the Cantor set.

In the present paper, we focus our attention on the group of homeomorphisms of Cantor space which preserve a single given measure. Let $\mathcal{C} = \{0, 1\}^{\mathbb{N}}$ denote Cantor space in two symbols, and let μ_r denote the Bernoulli trial measure on the Borel subsets of $\mathcal{C}$ so that the symbol '1' occurs with probability r independently in each coordinate. Suppose that r is an algebraic number in $(0, 1)$ with an algebraic conjugate $\bar{r}$ in the interval. If H is any homeomorphism of $\mathcal{C}$ which preserves μ_r, then, for any clopen set C, the property that $\mu_r(H(C)) = \mu_r(C)$ is equivalent to an integer polynomial equation about r. This equation must therefore also be true for $\bar{r}$, and so H must also preserve the measure $\mu_{\bar{r}}$, on the clopen sets and therefore on the Borel sets. A homeomorphism of $\mathcal{C}$ cannot distinguish between preserving μ_r and preserving $\mu_{\bar{r}}$ for any algebraic conjugate of r: the measures are glued together in a certain sense.

We show in this paper that for certain choices of r (those for which μ_r has a homogeneity property called refinability) and with a suitable restriction on the space of homeomorphisms considered, the generic behavior is that the set of ergodic measures are precisely those $\mu_{\bar{r}}$ where $\bar{r}$ is an algebraic conjugate of r in $(0, 1)$. Prior to this, we will use a simpler version of the same techniques to show that if μ is a measure on $\mathcal{C}$ (not necessarily a Bernoulli trial measure) with another homogeneity property called goodness, then unique ergodicity is generic in the same restricted sense.

Let $\text{hom}(\mathcal{C})$ denote the class of all homeomorphisms of $\mathcal{C}$. Fixing a complete metric d on $\mathcal{C}$, let $\hat{d}$ be the metric on $\text{hom}(\mathcal{C})$ defined by

$$\hat{d}(G, H) = \max_{x \in \mathcal{C}}\{\max\{d(G(x), H(x)), d(G^{-1}(x), H^{-1}(x))\}\}.$$

This metric is compatible with the usual topology of uniform convergence on $\text{hom}(\mathcal{C})$ and, since a limit in this metric will have an inverse, $(\text{hom}(\mathcal{C}), \hat{d})$ is a complete metric space. More commonly, however, we will understand this topology through the following notation: when $H \in \text{hom}(\mathcal{C})$ and $\mathcal{P}$ is a partition of $\mathcal{C}$ into clopen sets, let $N(H, \mathcal{P})$ denote the set of all homeomorphisms $G \in \text{hom}(\mathcal{C})$ with the property that for $C_1, C_2 \in \mathcal{P}$, $G^{-1}(C_2) \cap C_1$ is empty if and only if $H^{-1}(C_2) \cap C_1$ is empty. It is easy to verify that the sets of the form $N(H, \mathcal{P})$ are open and constitute a system of neighborhoods for the uniform topology on $\text{hom}(\mathcal{C})$.

Because we are interested in ergodicity, we would like to restrict our attention to the class of homeomorphisms with no invariant proper non-empty clopen sets, which we will call $\text{hom}^*(\mathcal{C})$. On $\mathcal{C}$, this is equivalent to the class of homeomorphisms with no proper attractors, or the class of homeomorphisms which are chain transitive. If $H \in \text{hom}(\mathcal{C})$ leaves the clopen set E invariant, and if E is distance ϵ away from the remainder of $\mathcal{C}$, then every homeomorphism whose $\hat{d}$ distance to H is within ϵ also leaves E invariant. So, the set of all homeomorphisms having an invariant clopen set is open in $\text{hom}(\mathcal{C})$ and removing them leaves $\text{hom}^*(\mathcal{C})$ as a complete metric space in its own right. (Note that $\text{hom}^*(\mathcal{C})$ is non-empty, for instance, because it includes the two-sided shift.)

For a measure m on $\mathcal{C}$, we let $\text{hom}(m)$ be the set of all $h \in \text{hom}(\mathcal{C})$ that preserve the measure m, and we let $\text{hom}^*(m)$ be the set of all $h \in \text{hom}^*(\mathcal{C})$ which preserve m. It is easy to verify that $\text{hom}(m)$ and $\text{hom}^*(m)$ are each closed in $\text{hom}(\mathcal{C})$, so each of these is also a complete metric space. One of our two main theorems is the following, which demonstrates a large class of uniquely or finitely ergodic homeomorphisms.

THEOREM 1.1. *For an irreducible integer polynomial $p(x)$ with $p(0) = \pm 1$ and $p(1) = \pm 1$, let R be the set of roots of $p(x)$ which lie in $(0, 1)$ and let $r \in R$. Let E be the set of all finitely ergodic homeomorphisms in $\text{hom}^*(\mu_r)$ whose ergodic measures are precisely those μ_s where $s \in R$. Then E is residual in $\text{hom}^*(\mu_r)$.*

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2. *Definitions and preliminaries*

A measure on Cantor space is *full* when open sets have positive measure and is *non-atomic* when points have measure zero. In [1], Akin defined a full, non-atomic probability measure m on $\mathcal{C}$ to be *good* when for any two clopen sets E and F with $m(E) \leq m(F)$, there exists a clopen set $\hat{E} \subseteq F$ with $m(\hat{E}) = m(E)$. That is, a measure on Cantor space is good if whenever a clopen set with a certain measure exists, such clopen sets exist in any clopen set large enough to hold it. One motivation for this definition is as follows.

Given two full, non-atomic probability measures on $\mathcal{C}$, there is not necessarily a homeomorphism of $\mathcal{C}$ that maps one measure to the other. This can be seen because the *clopen values set* of a measure m, $\{m(E) : E \text{ is clopen in } \mathcal{C}\}$, is a countable dense subset of the interval which is invariant under homeomorphism. Two measures on $\mathcal{C}$ will typically not have the same clopen values set and so will not be homeomorphic and, as it turns out, even two measures with the same clopen values set will sometimes not be homeomorphic. But, in [1], Akin showed that if two measures with the same clopen values set are both good, then they are homeomorphic.

A full non-atomic measure m was defined by Dougherty, Mauldin and Yingst [9] to be *refinable* when given three clopen sets E_1, E_2 and F with $m(E_1) + m(E_2) = m(F)$, there exists a decomposition of F into clopen sets $\hat{E}_1$, $\hat{E}_2$ with $m(\hat{E}_1) = m(E_1)$. (And, hence, $m(\hat{E}_2) = m(E_2)$.) It is clear that any good measure is refinable. The term refinable came about while trying to generalize Akin's homeomorphism theorem. In [9], Dougherty, Mauldin and Yingst showed the following result, with a proof almost identical to the proof of the same result for a good measure by Akin in [1].

THEOREM 2.1. (Dougherty, Mauldin and Yingst, and Akin) *Let μ and ν be good or refinable measures on Cantor spaces $\mathcal{C}_1$ and $\mathcal{C}_2$. Suppose that μ and ν have the same clopen values set. Then there is a homeomorphism $T : \mathcal{C}_1 \to \mathcal{C}_2$ with $\mu \circ T = \nu$.*

Slightly further, suppose that $\mathcal{P}_1$ and $\mathcal{P}_2$ are (finite) partitions of $\mathcal{C}_1$ and $\mathcal{C}_2$ into clopen sets, and that $\pi : \mathcal{C}_1 \to \mathcal{C}_2$ is a bijection so that $\mu(C) = \nu(\pi(C))$ for every $C \in \mathcal{P}_1$. Then there is a homeomorphism $T : \mathcal{C}_1 \to \mathcal{C}_2$ with $\mu \circ T = \nu$ such that $T(C) = \pi(C)$ for each $C \in \mathcal{P}_1$.

The notions of goodness and refinability were characterized for a Bernoulli trial measure in [9].

THEOREM 2.2. (Dougherty, Mauldin and Yingst) *The Bernoulli trial measure μ_r is refinable if and only if there is an integer polynomial $p(x)$ with $p(0) = \pm 1$, $p(1) = \pm 1$ and $p(r) = 0$. Further, μ_r is good when it is refinable and also r is the only one of its conjugates in* $[0, 1]$.

In the above theorem, we may assume that $p(x)$ is the irreducible polynomial of r over $\mathbb{Z}[x]$. With this characterization, Theorem 1.1 is discussing exactly those r for which μ_r is refinable. When μ_r is refinable or good, we sometimes say that the number r is refinable or good as well. As an example, $\mu_{1/2}$ is good, since $1/2$ is the unique root of $2x - 1$, an integer polynomial which equals ± 1 at 0 and 1. Meanwhile, $\alpha = (2 + \sqrt{2})/4$ is one of two roots of $8x^2 - 8x + 1$ in the interval, so μ_α is refinable but not good.

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Additionally, the following characterizing property was shown by Akin [1] and Glasner and Weiss [5]. (Glasner and Weiss showed that uniquely ergodic homeomorphisms yield good measures, and Akin showed the converse.)

THEOREM 2.3. (Akin, and Glasner and Weiss) *A measure on Cantor space is good if and only if there is a uniquely ergodic homeomorphism for which that measure is the unique ergodic measure.*

This result was extended from the case of a single measure to a finite-dimensional Choquet simplex of measures in an unpublished paper [3] of Dahl.

THEOREM 2.4. (Dahl) *Let K be a Choquet simplex of non-atomic full probability measures on $\mathcal{C}$. Assume that K has finitely many extreme points which are mutually singular, and that when E, F are clopen sets in $\mathcal{C}$ with $m(E) < m(F)$ for every $m \in K$, there exists a clopen set $\hat{E} \subseteq F$ with $m(E) = m(\hat{E})$ for all $m \in K$. Then there is a homeomorphism T of $\mathcal{C}$ whose invariant measures are exactly the elements of K.*

This result was further extended by Ibarlucía and Melleray in [6], who gave a complete characterization of when a given simplex K of probability measures (not necessarily finite-dimensional) is the set of invariant measures of some minimal homeomorphism of Cantor space. We will return to this result at the end of our paper.

The present paper is motivated by a question asked by Mauldin in 2006: if the existence of a unique root r of $P(x)$ in the interval is equivalent to the existence of a homeomorphism with μ_r as a unique measure, does the refinable case which involves finitely many roots in the interval correspond with the existence of a finitely ergodic homeomorphism of $\mathcal{C}$ with the corresponding measures as its ergodic measures? Fixing a refinable number, we may let K be the convex hull of the set of μ_r, where r is an algebraic conjugate in (0, 1) of that refinable number. In §5, we will see that this set K does satisfy the conditions of Theorem 2.4 above, giving an affirmative answer to Mauldin's question. In this paper, Theorem 1.1 goes further, giving that such homeomorphisms are generic in $\hom^*(\mu_r)$ for any single refinable Bernoulli trial measure μ_r.

Note that in Theorem 1.1, the case when r has no other algebraic conjugate in the interval is exactly the case when μ_r is a good measure, and that in that case we have that homeomorphisms in $\hom^*(\mu_r)$ are generically uniquely ergodic. It turns out that in the good case, we do not need our measure to be a Bernoulli trial measure. A much simpler version of the proof of Theorem 1.1 applies, and we have the following, our second main theorem.

THEOREM 2.5. *Let μ be a good (full, non-atomic) measure on the Borel subsets of $\mathcal{C}$. The set of all uniquely ergodic homeomorphisms in* $\hom^*(\mu)$ *is residual in* $\hom^*(\mu)$. *(And, the unique ergodic measure for any such homeomorphism is μ.)*

Note that we defined $\mathcal{C} = \{0, 1\}^{\mathbb{N}}$, but the above theorem holds for any expression of Cantor space.

Continuing our example, the fair coin-tossing measure $\mu_{1/2}$ was noted above to be a good measure, and this theorem shows that among homeomorphisms of $\mathcal{C}$ which preserve $\mu_{1/2}$ and which have no invariant clopen subsets, unique ergodicity is generic.

In the present context, because we are restricting our attention to homeomorphisms with no invariant clopen sets, we need the following theorem, which is a mild strengthening of Theorem 2.1, which stated that refinable measures with the same clopen values set are homeomorphic. This time, we map $\mathcal{C}$ to itself so as to end up with no invariant clopen sets, allowing us to work in $\hom^*(\mu)$ rather than just in $\hom(\mu)$. Note that this theorem shows us that the transition from $\hom(\mu)$ to $\hom^*(\mu)$ still leaves us in a large space of homeomorphisms.

THEOREM 2.6. *Let μ be a measure on $\mathcal{C}$ which is either good or is a refinable Bernoulli trial measure, and let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of $\mathcal{C}$ into non-empty proper clopen sets with a bijection $\pi : \mathcal{P} \to \mathcal{Q}$ so that $\mu(\pi(C)) = \mu(C)$ for all $C \in \mathcal{P}$ and so that no proper non-empty clopen subset of $\mathcal{C}$ is left invariant by this correspondence. (That is, if $\emptyset \subsetneq S \subsetneq \mathcal{P}$, then $\cup_{C\in S} C \neq \cup_{C\in S}\pi(C)$.) Then there is a homeomorphism $H : \mathcal{C} \to \mathcal{C}$ with no invariant clopen subsets so that $H(C) = \pi(C)$ for each $C \in \mathcal{P}$.*

Proof. Since $\mathcal{C}$ is compact, $\mathcal{P}$ and $\mathcal{Q}$ are finite. As in the proofs of Theorem 2.1 in [**1**] and [**9**], for each $i \geq 0$, we construct $\mathcal{P}_i$ and $\mathcal{Q}_i$, partitions of $\mathcal{C}$ into non-empty proper clopen subsets of $\mathcal{C}$, and π_i a bijection from $\mathcal{P}_i$ to $\mathcal{Q}_i$ which preserves μ, so that $\mathcal{P}_{i+1}$ is a refinement of $\mathcal{P}_i$, $\mathcal{Q}_{i+1}$ is a refinement of $\mathcal{Q}_i$ and so that for $P \in \mathcal{P}_i$, π_{i+1} sends subsets of P to subsets of $\pi_i(P)$.

Let $\mathcal{P}_1 = \mathcal{P}$, $\mathcal{Q}_1 = \mathcal{Q}$ and $\pi_1 = \pi$. We go back and forth, at each step alternately refining either $\mathcal{P}_i$ or $\mathcal{Q}_i$ into basic clopen sets which separate any two points which differ in one of the first i coordinates, and we use refinability to generate the corresponding partition on the other side: if $\mathcal{P}_{2k}$, $\mathcal{Q}_{2k}$ and π_{2k} are constructed, let $\mathcal{P}_{2k+1}$ be any refinement of $\mathcal{P}_{2k}$ into basic clopen sets which separate points which differ in the first $2k + 1$ coordinates. For each $C \in \mathcal{P}_{2k}$, $\mathcal{P}_{2k+1}$ contains a partition of C into clopen sets. By refinability, there must be a partition of $\pi_{2k}(C)$ into clopen sets having the same measures as the refinement of C; we would like to choose such sets and let π_{2k+1} be this correspondence. The tricky part is that now we must ensure that the resulting correspondence leaves no clopen set fixed. When using refinability to choose elements of $\mathcal{Q}_{2k+1}$, if we wish to ensure that no clopen subset defined by π_{2k+1} is left invariant, there are finitely many clopen sets to consider: those which are expressible as a union of elements of $\mathcal{P}_{2k+1}$. Thus, we may choose elements of $\mathcal{Q}_{2k+1}$ one at a time. In each case, avoiding an invariant clopen set requires that we avoid one of at most finitely many choices, but for a good measure or a refinable Bernoulli trial measure, there are infinitely many options to choose from. (To be clear, when we choose an element of $\mathcal{Q}_{2k+1}$, we must consider any set which is a union of elements of $\mathcal{Q}_{2k+1}$ already chosen, or of elements of $\mathcal{Q}_{2k}$, as they will eventually be such a union after all choices have been made. When it comes time to 'choose' the final subset of an element of $\mathcal{Q}_{2k}$, we have no choice at all as we must use the remainder left from the pieces chosen already. However, any non-trivial invariant subset which includes this remainder set would, by complement, leave an invariant set which does not include the remainder and so this possibility will already have been avoided.)

With $\mathcal{P}_{2k+1}$, $\mathcal{Q}_{2k+1}$ and π_{2k+1} constructed, we build the $(2k + 2)$th stage similarly, choosing basic clopen sets on the $\mathcal{Q}$ side, and letting refinability induce the $\mathcal{P}$ side while avoiding invariant clopen sets.

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With the sequences $\{\mathcal{P}_n\}$ and $\{\mathcal{Q}_n\}$ constructed, let H be defined as follows: for x in $\mathcal{C}$, for each n there is a unique $E_n \in \mathcal{P}_n$ with $x \in E_n$. Because $\mathcal{P}_{2n+1}$ and $\mathcal{Q}_{2n}$ separate points in the first $2n-1$ coordinates, we have $\cap_{n=1}^{\infty} E_n = \{x\}$, and we may let $H(x)$ be the unique element of $\bigcap_{n=1}^{\infty} \pi_n(E_n)$. Any clopen set is a union of basic sets in $\mathcal{P}_n$ for some sufficiently large n, since by construction any basic open set depending on the first k elements is a union of elements of $\mathcal{P}_{k+1}$. A clopen set is a finite union of basic open sets, so is defined by some $\mathcal{P}_n$. Its image under H is the corresponding union of elements of $\mathcal{Q}_n$, and π_n witnesses that this clopen set is not invariant except trivially and that its measure is preserved. Our homeomorphism preserves measure for any clopen set and hence for any Borel set. □

Note that this theorem does not hold for a general refinable measure. As an example, if ν is any probability measure on $\mathcal{C}$ for which no two different clopen sets have the same measure, then ν is trivially refinable, but the only homeomorphism that preserves this measure is the identity.

3. *Reducing the main theorems to lemmas*

The bulk of the work proving Theorems 1.1 and 2.5 will come in proving the upcoming Lemmas 6.1 and 4.1. Proving the main theorems given their respective lemmas is almost identical however and we handle both cases simultaneously in this section. (The following is just a combined restatement of Theorems 1.1 and 2.5.)

THEOREM 3.1. *If μ is a good measure on $\mathcal{C}$, then the set of all uniquely ergodic homeomorphisms in* $\hom^*(\mu)$ *is residual in* $\hom^*(\mu)$. *(And, for each such measure, its unique ergodic measure is μ.)*

If $r \in (0, 1)$ is a root of an irreducible integer polynomial $p(x)$ with $p(0) = \pm 1$ and $p(1) = \pm 1$, if R is the set of roots of $p(x)$ which lie in $(0, 1)$ and if E is the set of all finitely ergodic homeomorphisms in $\hom^*(\mu_r)$ *whose ergodic measures are precisely those μ_s where $s \in R$, then E is a residual set in* $\hom^*(\mu_r)$.

Proof. Let either μ or μ_r be as above. (If in the second case, let R be as above as well.) We let K be the set of all measures of interest in either case. That is, $K = \{\mu\}$ if we are in the case of a good measure μ, or we let $K = \{\mu_r : r \in R\}$ for the refinable case.

Suppose that H is some homeomorphism of $\mathcal{C}$ in $\hom^*(\mu)$ which admits an ergodic measure not in K; call it m. Then m is supported on some Borel set with measure zero for each measure in K. Approximating that set by a clopen set D, we find the existence of D clopen with $\nu(D) < m(D)$ for all $\nu \in K$. By the ergodic theorem, for m-almost every x in $\mathcal{C}$ (and, in particular, for some x), we have that

$$\lim_{n\to\infty} \frac{1}{n+1} \#\{j = 0, \ldots, n : H^j(x) \in D\} = m(D) > \max_{\nu \in K} \nu(D).$$

For D clopen and for $\eta > 0$, let $V(D, \eta)$ be the set of all $H \in \hom^*(\mu)$ so that for some $x \in \mathcal{C}$, we have

$$\liminf_{n\to\infty} \frac{1}{n+1} \#\{j = 0, \ldots, n : H^j(x) \in D\} > \Big(\max_{\nu \in K} \nu(D)\Big) + \eta.$$

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By the preceding paragraph, we have that any H in $\hom^*(\mu)$ which has an ergodic measure not in K must be in $V(D, \eta)$ for some clopen D and some rational η. To show that the set of non-uniquely-ergodic homeomorphisms is meager in $\hom^*(\mu)$, it suffices to show that each $V(D, \eta)$ is nowhere dense.

For the good case, the upcoming Lemma 4.1 will give us that for any $T \in \hom^*(\mu)$ and any neighborhood $N(T, \mathcal{P})$ of T, there is a subneighborhood $N(H, \mathcal{Q})$ with $N(H, \mathcal{Q}) \cap V(D, \eta) = \emptyset$, while Lemma 6.1 will do the same for the refinable case. □

Our proof is largely inspired by the proof by Oxtoby and Ulam in [8] that the set of all ergodic measures in $\hom([0, 1]^n)$ is residual in $\hom([0, 1]^n)$ for $n \geq 2$. It may be worth noting how our argument differs from the classical case to give unique or finite ergodicity rather than just ergodicity. In our upcoming lemmas, we show that in any neighborhood of T, there is a homeomorphism H for which every point x enters the set D with frequency not too far from expected, and that this holds in a neighborhood of H as well. In the Oxtoby–Ulam proof, for a given T and D the equivalent lemma shows that for any neighborhood of T, there is a homeomorphism H (and a neighborhood of H) for which there is a periodic set of positive measure (a Cantor set) with the property that the number of its iterates inside D is approximately the correct proportion, and so that the union of the iterates has total measure 1/2. This gives that for a set of x of measure 1/2, the frequency with which they enter D is approximately the expected proportion.

A fundamental feature of our upcoming construction, then, is that rather than giving a large (by measure) set of points whose iterates under H are well behaved, in the current context we are able to do it for every point.

4. *The lemma for good measures*

We almost have all the tools necessary to state and prove Lemma 4.1, which will complete our proof of Theorem 2.5. To approximate a homeomorphism T on $\mathcal{C}$, it is sufficient to consider a fine partition $\mathcal{P}$ of $\mathcal{C}$ and the directed graph whose vertices are elements of $\mathcal{P}$ and whose edges are those $C_a \to C_b$ for which $T^{-1}(C_b) \cap C_a \neq \emptyset$. If μ is a measure on $\mathcal{C}$ which T preserves, we will also often consider the weighted directed graph generated by giving the edge $C_a \to C_b$ weight equal to $\mu(T^{-1}(C_b) \cap C_a)$, and considering an edge of weight zero to not exist. The resulting graph is *balanced*, meaning that the total weight of edges leaving a vertex equals the total weight of edges entering that vertex. While the weight of an edge is sometimes interpreted as the cost or difficulty to move along that edge, for us it represents *how much* movement happens along that edge, both by the given homeomorphism T and by the nearby homeomorphism we attempt to construct. In this paper, we treat any weight on an edge in a weighted graph as positive and, when altering a graph, if the weight of an edge becomes zero, that edge is removed.

We will make use of the following theorem, whose statement and proof are reminiscent of the well-known theorem that a connected graph for which every vertex has even degree admits an Eulerian circuit. The following was shown using slightly different terminology by Sun and Wang in [10].

THEOREM 4.1. (Sun and Wang) *Let G be a balanced weighted directed graph which is connected. Let $c_1, \ldots, c_L$ be the cycles in this graph. Then there exist positive values $w_1, \ldots, w_L$ such that for each edge $C_a \to C_b$ in G, the weight of that edge is $\sum_{j:(C_a \to C_b) \in c_j} w_i$.*

Here note that by *cycle* we mean simple cycle. That is, each c_j is a finite sequence of distinct edges $C_1 \to C_2 \to \cdots \to C_n = C_1$, where the vertices are distinct except that the initial vertex is the same as the final one. We do not need to repeat our cycles for every possible choice of starting vertex: whenever we discuss the set of all cycles of a graph, assume that one representative is chosen for each collection of cycles which are equivalent under shifting. Also note that the final sum above indicates a sum over all cycles for which the edge $C_a \to C_b$ appears in that cycle. We include a proof, as we will need to refer to its details later.

Proof. We move weight from the graph G to the values w_j in stages, labelling the tth stage w_j^t and G_t. Let G_0 be G and let $w_j^0 = 0$ for all j. Let $W_{G_t}(C_a \to C_b)$ denote the weight of the edge $C_a \to C_b$ in the graph G_t. As we make our construction, at every stage we will have for each edge that $W_G(C_a \to C_b) = (\sum_{j:(C_a \to C_b) \in c_j} w_j^t) + W_{G_t}(C_a \to C_b)$. At the start, all the weight of this sum is in the second term above, and as we proceed we move it out of the graph and into the w_j.

Let α be small. We would like all w_j to be positive, so we begin by removing a weight of α from each cycle and adding it to each w: let G_1 be obtained from G by subtracting α from the weight of each edge in G_0 once for each cycle which passes through that edge, and let each $w_j^1 = \alpha$ for each j. (Assume that α is sufficiently small that each edge in G_1 has some weight remaining.)

Note that G_1 is still balanced, and we have for each edge $C_a \to C_b$ that $W_G(C_a \to C_b) = (\sum_{j:(C_a \to C_b) \in c_j} w_j^1) + (W_{G_1}(C_a \to C_b))$. Choose an edge in G_1 with smallest positive weight. We argue that there must be a cycle (containing edges of positive G_1 weight) which contains this edge: consider the set of all vertices that can be reached from this edge in G_1. There can be no edge which leaves this set and, if the initial vertex of our chosen edge is not in this set, that edge witnesses that the set has some incoming weight, contradicting that G_1 is balanced. Thus, there is a path from our edge to the initial vertex of it and, by removing repetition, we find a cycle, c_m, containing our chosen edge.

Because the weight of our edge was minimal, every edge in this cycle has at least this much weight and so we remove this much weight from our graph and give it to w_m. Let $\{w_j^2\}$ be the same as $\{w_j^1\}$ except for w_m^2, to which we add the minimal weight described. Let G_2 be obtained from G_1 by subtracting this minimal weight from each edge of the cycle c_m and leaving all other weights unchanged. When this subtraction leaves an edge of weight zero, remove that edge from G_2.

The resulting G_2 is still balanced but has at least one fewer edge than G_1 had. We may continue in this way until some t for which all edges in G_t have weight zero. Letting $\{w_j\}$ denote this final state of the construction, these values satisfy $W_G(C_a \to C_b) = \sum_{j:(C_a \to C_b) \in c_j} w_j$, as desired. □

Recall that for a homeomorphism H of $\mathcal{C}$ and a partition $\mathcal{P}$ of $\mathcal{C}$ into clopen sets, $N(H, \mathcal{P})$ was defined to be the set of all homeomorphisms T of $\mathcal{C}$ with the property that for $C, D \in \mathcal{P}$, $T^{-1}(D) \cap C$ is empty if and only if $H^{-1}(D) \cap C$ is empty. In §3, we showed that Theorem 2.5 follows from the following lemma.

LEMMA 4.1. *Let μ be a good measure on $\mathcal{C}$, let $\eta > 0$, let $\mathcal{P}$ be a clopen partition of $\mathcal{C}$ and let $T \in \hom^*(\mu)$ be a homeomorphism. For any clopen set $D \subseteq \mathcal{C}$, there exist a homeomorphism $H_0 \in N(T, \mathcal{P})$ and a clopen refinement $\mathcal{Q}$ of $\mathcal{P}$ so that for any $H \in N(H_0, \mathcal{Q})$ and for any sufficiently large n, every $x \in \mathcal{C}$ has*

$$\frac{1}{n+1}\#\{j = 0, \ldots, n : H^j(x) \in D\} < \mu(D) + \eta.$$

Proof. Let $\mu, \mathcal{P}, T, D, \eta$ be as in the statement. By refining $\mathcal{P}$ if necessary, assume that D is a union of sets in $\mathcal{P}$. Let G be the weighted directed graph whose vertices are the sets in $\mathcal{P}$, whose edges are those $C_a \to C_b$ for which $T^{-1}(C_b) \cap C_a$ is non-empty and which gives the edge $C_a \to C_b$ weight equal to $\mu(T^{-1}(C_b) \cap C_a)$. This graph is balanced.

Let $c_1, \ldots, c_L$ be the cycles of our graph. By Theorem 4.1, there are positive values $w_1, \ldots, w_L$ such that $\mu(T^{-1}(C_b) \cap C_a) = \sum_{j:(C_a \to C_b) \in c_j} w_j$. Note that by adding over C_b, we also find that $\mu(C_a) = \sum_{j:C_a \in c_j} w_j$.

Our intent is now to build a closed path P in the graph G which for each $C \in \mathcal{P}$ passes through that vertex with frequency roughly proportional to $\mu(C)$. We will construct this path by repeatedly traversing cycles, and the values w_j give us the relative number of times to include the cycle c_j. To do this, we make these w_j discrete, finding integers N_j which are approximately proportional to w_j.

Suppose that N is a large integer. Let $N_j = \lfloor N w_j \rfloor = N w_j - \rho_j$. (Here and later we manage error by using ρ to indicate a quantity between zero and one.) Let Z be a clopen set whose measure is between $1/N$ and $1/(N+1)$, and let δ be $\mu(Z)$. (Since μ is full and non-atomic, the set of μ-measures of clopen sets is dense in $[0, 1]$, so this is possible.) Since we have $1/(N+1) < \delta < 1/N$ and thus $N_j\delta \leq N w_j \delta < w_j$, it follows that for each $C_a, C_b \in \mathcal{P}$, we have

$$\mu(T^{-1}(C_b) \cap C_a) = \sum_{j:(C_a \to C_b) \in c_j} w_j > \sum_{j:(C_a \to C_b) \in c_j} N_j \delta.$$

Using goodness, we will soon decompose the elements of $\mathcal{P}$ into sets of size δ. The above verifies that there is room in the set $T^{-1}(C_b) \cap C_a$ to find N_j of them, disjoint, for each cycle c_j which contains the edge $C_a \to C_b$, and this ability to remove clopen sets witnesses that the measure left over is the measure of some clopen set. Before finding and naming these removed sets, however, we examine what can be done with the leftovers.

Consider the graph $\bar{G}$ with sets in $\mathcal{P}$ as vertices and which gives the edge $C_a \to C_b$ weight equal to

$$\mu(T^{-1}(C_b) \cap C_a) - \bigg(\sum_{j:(C_a \to C_b) \in c_j} N_j \delta\bigg).$$

As observed above, each of these weights is the measure of a clopen set. By choice of w_j and N_j, this weight is

$$\sum_{j:(C_a\to C_b)\in c_j} (w_j - N_j\delta) = \sum_{j:(C_a\to C_b)\in c_j} (w_j(1-\delta N) + \delta\rho_j).$$

From $1/N > \delta > 1/(N+1)$, it follows that $0 < (1-\delta N) < \delta$ and so the weight of each edge in $\bar{G}$ is bounded by $2L\delta$. Again, this graph is balanced, and again we can decompose it into cycles.

We apply Theorem 4.1 again. Note that in the graph $\bar{G}$, the weight of each edge is the μ measure of some clopen set. In the proof of Theorem 4.1, if the first small positive value α was chosen to be the measure of a clopen set, then every edge weight involved in that construction will be derived as the difference of μ measures of clopen sets, and will by goodness again be the measure of a clopen set. Applying the theorem in this way, we find values e_t, each of which is the μ measure of a clopen set, so that for any $C_a, C_b \in \mathcal{P}$, we have

$$\mu(T^{-1}(C_b)\cap C_a) - \Bigg(\sum_{j:(C_a\to C_b)\in c_j} N_j\delta\Bigg) = \sum_{j:(C_a\to C_b)\in c_j} e_j.$$

Finally, we would like these e_j to have size at most δ. To do this, we will remove as many copies of δ as necessary. For each j, write $e_j = d_j\delta + \epsilon_j$, where $d_j \geq 0$ is an integer and $0 < \epsilon_j \leq \delta$. Note that the quantity on the left above was shown to be less than $2L\delta$, so we have each $d_j < 2L$. Also note that since ϵ_j can be produced by taking e_j and repeatedly subtracting δ, each ϵ_j is also the measure of a clopen set.

These extra copies of δ will now be absorbed into the N_j. Let $\hat{N}_j = N_j + d_j$ for each j. We now have for each C_a and C_b in $\mathcal{P}$ that

$$\mu(T^{-1}(C_b)\cap C_a) = \Bigg(\sum_{j:(C_a\to C_b)\in c_j} \hat{N}_j\delta\Bigg) + \Bigg(\sum_{j:(C_a\to C_b)\in c_j} \epsilon_j\Bigg),$$

and, in particular, adding over all C_b, we have for $C \in \mathcal{P}$ that

$$\mu(C) = \Bigg(\sum_{j:C\in c_j} \hat{N}_j\delta\Bigg) + \Bigg(\sum_{j:C\in c_j} \epsilon_j\Bigg). \tag{4.1}$$

Recall that our first graph G had elements of $\mathcal{P}$ as vertices, and had an edge $C_a \to C_b$ when $\mu(T^{-1}(C_b)\cap C_a) \neq 0$. We will now argue the existence of a closed path in G so that the number of times it visits the vertex C is exactly $\sum_{j:C\in c_j} \hat{N}_j$.

Begin by considering the closed path in G which travels the cycle $c_1 \hat{N}_1$ times. Because the map T has no invariant clopen sets, the union of those elements of $\mathcal{P}$ which occur in this path cannot be a proper T-invariant subset and so one of them must occur in another cycle, say c_{j_1}. By choosing an appearance of this vertex in our path as constructed so far, we can break our path and insert a copy of c_{j_1}, shifted so that it begins (and ends) with this overlapping vertex. We repeat, inserting a total of $\hat{N}_{j_1}$ copies of this cycle into our path. If these were the only cycles, we are done. If not, again one of the vertices used in our path as constructed so far must occur in one of the cycles not yet incorporated, as otherwise the

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union of vertices already used would be a non-trivial T-invariant set. Continue in this way until we have inserted $\hat{N}_j$ copies of the cycle c_j for each j. Name this resulting path P.

As desired, for any $C \in \mathcal{P}$, the total number of times it occurs in this path is $\sum_{j:C\in c_j} \hat{N}_j$. Since each ϵ_j is non-negative, equation (4.1) shows that the set C has room to contain this many clopen sets of measure δ. Using goodness, remove this many disjoint clopen sets from each $C \in \mathcal{P}$ and arrange them as ordered in the path P constructed above, naming them

$$F_1 \to F_2 \to F_3 \to \cdots F_z \to F_1,$$

where $z = \sum_{C\in\mathcal{P}} \sum_{j:C\in c_j} \hat{N}_j$ is the total length of the path and each set F_t is a disjoint subset of the tth element of our path P.

Lastly, we need to divide our leftovers into sets of size ϵ_j and work out how these will be incorporated into our path. From (4.1), we know that the part of each $C \in \mathcal{P}$ not used in the F_i has measure equal to $\sum_{j:(C)\in c_j} \epsilon_j$. Since ϵ_j is the measure of some clopen set, we can decompose these remainder sets into clopen sets of these sizes using goodness. For each $j = 1, \ldots, L$, let $l(c_j)$ denote the length of the cycle c_j and write

$$c_j = C_{j,1} \to C_{j,2} \to \cdots \to C_{j,l(c_j)} \to C_{j,1}.$$

For each j and each $i = 1, \ldots, l(c_j)$, let $E_{j,i}$ be the set chosen from the remainder of C_i^j of size equal to ϵ_j. (Or let it be a distinct one of those sets in the unlikely event some ϵ_j are equal.)

Note that at this point, we are close to building our homeomorphism as desired. We could here let $\mathcal{P}$ and $\mathcal{Q}$ be the partitions of $\mathcal{C}$ into the sets F_j and $E_{j,i}$, let π be the partition which maps $F_j \mapsto F_{j+1}$ (but $F_z \mapsto F_1$) and which for each j maps $E_{j,i} \mapsto E_{j,i+1}$ (but $E_{j,l(c_j)} \mapsto E_{j,1}$). We would then be able to use Theorem 2.1 to extend π to a homeomorphism which maps set to set along these paths. The resulting map would not satisfy our requirements for two reasons, however: for points in the F_j sets, the proportion of time they spend in D would be approximately the proportion of the F_j which are in D, which, since the F_j are almost all of $\mathcal{C}$, would be close the the correct proportion. But, for points in our leftover sets $E_{j,i}$, we would have no such control. Further, the resulting homeomorphism could not be in $\hom^*(\mu)$, since the union of the F_j would be a non-trivial invariant clopen set. We will fix both these problems by interrupting the F path, allowing points to sometimes take 'detours' along the E. This will stop the invariant clopen set from existing, but since a point can enter each one of the E detours at most once during its long journey down the F path, we will not significantly affect the proportion of time our point spends in D. We do this now.

For each $j = 1, \ldots, L$, let $b(j)$ be an index along our path so that $F_{b(j)} \in C_{j,l(c_j)}$ and so that $F_{b(j)+1} \in C_{j,1}$. This is possible because the edge $C_{j,l(c_j)} \to C_{j,1}$ appears in the cycle c_j, and many copies of this cycle were used in the construction of the path P which led to the path of the F_k. Further, having so many choices, we can assume that the indices $b(j)$ and $b(j) + 1$ are all distinct and that none of them is 1 or z. For each of these indices, let $A(j) \subseteq F_{b(j)}$ and $B(j) \subseteq F_{b(j)+1}$ be clopen sets of size ϵ_j. (This is possible because $\delta > \epsilon_j$, because ϵ_j is the measure of a clopen set and because μ is good.) These carved

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out sets will be the entry and exit from our detours out of the F and into the E, and then back into the F.

We are now ready to construct our map. Let $\mathcal{P}_0$ be the partition of $\mathcal{C}$ into all the sets $E_{j,i}$, all the sets $B(j)$, all the sets F_j which do not contain a set $B(j)$ and all the sets $F_{b(j)} \setminus B(j)$. Meanwhile, let $\mathcal{Q}_0$ be the partition of $\mathcal{C}$ into all the sets $E_{j,i}$, all the sets $A(j)$, all the sets F_j which do not contain a set $A(j)$ and all the sets $F_{b(j)+1} \setminus A(j)$. Let π_0 be the bijection from $\mathcal{P}_0 \to \mathcal{Q}_0$ defined by

$$F_i \mapsto F_{i+1} \text{ if } i \neq z \text{ and } i \neq b(j) \text{ for any } j,$$

$$F_z \mapsto F_1,$$

$$F_{b(j)} \setminus B(j) \mapsto F_{b(j)+1} \setminus A(j),$$

$$B(j) \mapsto E_{j,1},$$

$$E_{j,k} \mapsto E_{j,k+1} \text{ if } k \neq l(c_j) \text{ and}$$

$$E_{j,l(c_j)} \mapsto A(j).$$

An inspection of this map shows that it preserves μ. (Every set involved has measure equal to δ, ϵ_j or $\delta - \epsilon_j$ for some j.) Slightly more complicated is verifying that it allows no invariant clopen sets: first note that because our indices $b(j)$ and $b(j)+1$ were all distinct and different from 1 and z, that the set F_i for each i must appear unaltered in either $\mathcal{P}$ or $\mathcal{Q}$. Because an invariant set must be expressible both as a union of sets in $\mathcal{P}$ and as a union of sets in $\mathcal{Q}$, it must be expressible as a union of sets F_i and $E_{j,i}$. Since at least part of F_i maps to at least part of F_{i+1}, if an invariant set contains any part of any set F_i, it must contain all of all sets F_i. A similar statement applies for each group of sets $\{E_{j,i}\}_{i=1}^{l(c_j)}$. Finally, since the set $B(j)$ is a part of an F_i set which maps into one of the $E_{j,i}$ sets, an invariant set which contains part of an F_i set or part of an $E_{j,i}$ set must contain all of both and hence must contain all $E_{j,i}$ sets and hence must be all of $\mathcal{C}$.

We may therefore apply Theorem 2.6. There is a homeomorphism $H_0 \in \hom^*(\mu)$ which maps set to set according to π_0. Let $\mathcal{Q}$ be the refinement of $\mathcal{P}_0$ and $\mathcal{Q}_0$. Note that any $H \in N(H_0, \mathcal{Q})$ must also map set to set according to π_0.

Fix $H \in N(H_0, \mathcal{Q})$, let n be a large integer and consider the expression

$$\frac{1}{n+1}\#\{j = 0, \ldots, n : H^j(x) \in D\}.$$

There is a number A so that a point can be iterated under H at most A times before it enters the set F_1. (We may take A to be the total number of sets in $\mathcal{P}_0$, for example.) Once a point enters F_1, it will then pass through every set F_t, possibly including some detours along a path $\{E_{j,i}\}_{i=1}^{l(c_j)}$ for some j values. Prior to its re-entering F_1 again, the proportion of time it will have spent in D will be exactly

$$\frac{\sum_{C \in \mathcal{P}: C \subseteq D} \sum_{j: C \in c_j} (\hat{N}_j + I_j)}{\sum_{E \in \mathcal{P}} \sum_{j: E \in c_j} (\hat{N}_j + I_j))}, \tag{4.2}$$

where I_j is 1 if our point entered the E_j detour or is 0 if it did not.

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Recall that $\hat{N}_j = N_j + d_j$, where $d_j < 2L$, and that $N_j = Nw_j - \rho_j$, where $0 \le \rho_j \le 1$. Expanding, the above becomes

$$\frac{\sum_{C\in\mathcal{P}:C\subseteq D}\sum_{j:C\in c_j}(Nw_j - \rho_j + d_j + I_j)}{\sum_{E\in\mathcal{P}}\sum_{j:E\in c_j}(Nw_j - \rho_j + d_j + I_j))}$$

$$\le \frac{\sum_{C\in\mathcal{P}:C\subseteq D}\sum_{j:C\in c_j}(w_j + (2L+1)/N)}{\sum_{E\in\mathcal{P}}\sum_{j:E\in c_j}(w_j - 1/N))}.$$

As N becomes large, this expression approaches

$$\frac{\sum_{C\in\mathcal{P}:C\subseteq D}\sum_{j:C\in c_j} w_j}{\sum_{E\in\mathcal{P}}\sum_{j:E\in c_j} w_j} = \frac{\sum_{C\in\mathcal{P}:C\subseteq D}\mu(C)}{\sum_{E\in\mathcal{P}}\mu(E)} = \frac{\mu(D)}{1}.$$

Because the w_j were fixed before choosing N, we may assume that N was chosen sufficiently large that the expression in (4.2) is less than $\mu(D) + \eta/2$. Thus, for any point x, the proportion of time it spends in D in its first n iterates under any map $H_2 \in N(H, \mathcal{Q})$ is at most $\mu(D) + \eta/2$ for the time between the first time it enters F_1 and the last time it enters F_z. This amount of time includes all but at most $2A$ iterates. It is elementary to verify that for sufficiently large n, the total proportion of time will be less than $\mu(D) + \eta$. □

With Lemma 4.1 now proved, our proof of Theorem 2.5 is complete. □

5. *Partition polynomials*

We now prepare to prove Lemma 6.1, which is proved similarly to Lemma 4.1 but for a refinable Bernoulli trial measure rather than any good measure. When working with Bernoulli trial measures, the following terminology is useful. A *partition polynomial* $p(x)$ is any polynomial for which there is some (sufficiently) large n so that $p(x)$ can be expressed in the form $p(x) = \sum_{k=0}^{n} c_k x^k (1-x)^{n-k}$, where each c_k is an integer between 0 and $\binom{n}{k}$. Any clopen set $E \subseteq \mathcal{C}$ can be written as a finite disjoint union of basic open sets which depend on the first n symbols. Letting c_k be the number of these which have exactly k 1's in those n symbols shows the existence of a partition polynomial p with $p(x) = \mu_x(E)$. Likewise, given a partition polynomial, the coefficients c_k can be taken as instructions for how many basic clopen sets of length n having k 1's to use, giving a clopen set E such that $p(x) = \mu_x(E)$. In either situation we say that p is the partition polynomial *associated with* E, or that E is a clopen set *associated with* p.

We need the following three known results about partition polynomials and refinable measures, which were shown by Dougherty, Mauldin and Yingst in [9].

THEOREM 5.1. (Dougherty, Mauldin and Yingst) *The set of partition polynomials consists of the constant polynomials zero and one, and those integer polynomials which map the open interval* (0, 1) *into itself.*

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(It is clear that the partition polynomials 0 and 1 correspond with $\emptyset$ and $\mathcal{C}$. For any other partition polynomial, the equation $p(x) = \mu_x(E)$ for some proper non-empty clopen E means that we must have $0 < p(x) < 1$ when $0 < x < 1$. The above theorem gives that this property is also sufficient for an integer polynomial to be a partition polynomial.)

THEOREM 5.2. (Dougherty, Mauldin and Yingst) *Given a clopen set E whose associated partition polynomial is p, there is a clopen subset of E associated with the partition polynomial q if and only if* $0 < q < p$ *on* $(0, 1)$.

THEOREM 5.3. (Dougherty, Mauldin and Yingst) *Given a refinable number r in* $(0, 1)$*, R the set of algebraic conjugates of r in* $(0, 1)$*, and given an integer polynomial $f(x)$, the set of all integer polynomials which agree with f at r is dense (in the $L^\infty([0, 1])$ sense) in the set of all continuous functions which are integer-valued at* 0 *and* 1 *and which agree with f at every r in R.*

Slightly further, if g is a polynomial which is positive on $(0, 1)$*, and f is an integer polynomial which is positive and less than g and less than* 1 *at each r in R, then there is a partition polynomial $\hat{f}$ which agrees with f at each r in R and with $0 < \hat{f} < g$ on* $(0, 1)$.

Combining these, we find the following, which lays bare the connection between a good measure and a refinable Bernoulli trial measure. This theorem was never stated clearly in [9], but follows easily from the results above.

THEOREM 5.4. *Let μ_r be a refinable Bernoulli trial measure, let R be the set of algebraic conjugates of r in* $(0, 1)$ *and let E and F be clopen sets in $\mathcal{C}$ with $\mu_r(E) < \mu_r(F)$. Then there exists a clopen set $\hat{E} \subseteq F$ with $\mu_r(E) = \mu_r(\hat{E})$ if and only if we have $\mu_s(E) < \mu_s(F)$ for every $s \in R$.*

Proof. Suppose such an $\hat{E}$ exists. Letting p_E and $p_{\hat{E}}$ be the partition polynomials associated with E and $\hat{E}$, we have that $p_E(r) = p_{\hat{E}}(r)$. This is a polynomial equation satisfied by r and so is satisfied by each $s \in R$, meaning that $\mu_s(E) = p_E(s) = p_{\hat{E}}(s) = \mu_s(\hat{E}) < \mu_s(F)$.

On the other hand, if $\mu_s(E) < \mu_s(F)$ for all $s \in R$, let p_E and p_F be the partition polynomials associated with E and F. By Theorem 5.3, there is an integer polynomial f which agrees with p_E at each $s \in R$ with $0 < f < p_F$ on $(0, 1)$. By Theorem 5.1, f is a partition polynomial and by Theorem 5.2 there is a clopen subset $\hat{E} \subseteq F$ associated with f. This subset has $\mu_r(\hat{E}) = f(r) = p_E(r) = \mu_r(E)$. □

In [6], Ibarlucía and Melleray defined a collection of measures K on a Cantor space to be *good* when given two clopen sets A and B with $m(A) < m(B)$ for all $m \in K$, there exists a subset $\hat{A}$ of B with $m(\hat{A}) = m(A)$ for all m in K. (This definition follows Glasner and Weiss, who showed in [5] that if K is the set of all T-invariant measures for some minimal homeomorphism on Cantor space, that K is good in this sense, and also follows Akin, who in [1] used the word ‘good’ to define this property for a single measure.) We note that the above Theorem 5.4 shows that the set $\{\mu_r\}_{r\in R}$ is good in this sense and this property extends to the convex hull of the set.

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6. *Proving the lemma for refinable Bernoulli trial measures*

We already showed in §3 how Theorem 1.1 follows from Lemma 6.1 below. We prove this lemma now.

LEMMA 6.1. *Let μ_r be a refinable Bernoulli trial measure, let $\{r = r_1, r_2, \ldots, r_R\}$ be the set of algebraic conjugates of r in $(0, 1)$, let $T \in \hom^*(\mathcal{C})$, let $\mathcal{P}$ be a clopen partition of $\mathcal{C}$, let D be a clopen set in $\mathcal{C}$ and let $\eta > 0$. There exist a homeomorphism $H_0 \in N(T, \mathcal{P})$, a clopen refinement $\mathcal{Q}$ of $\mathcal{P}$, and $M > 0$, so that for any homeomorphism $H \in N(H_0, \mathcal{Q})$, any $x \in \mathcal{C}$ and any $n > M$, we have that*

$$\frac{1}{n+1}\#\{j = 0, \ldots, n : H^j(x) \in D\} < \max_{s \in R}\{\mu_s(D)\} + \eta.$$

Before moving into the proof, we discuss how this is more difficult than the proof of Lemma 4.1, which applied to a single good measure. In that construction, we built a closed path in $\mathcal{P}$ so that for each $C \in \mathcal{P}$ our path spent an amount of time in C proportional to the measure of C. In this way, we decomposed elements of $\mathcal{P}$ into small sets and mapped these set to set according to our path. Because these proportions lined up, these sets could absorb almost all of the measure of $\mathcal{C}$. The small remainder was then decomposed into cycles and then integrated into the path without significantly affecting the proportion of time a point spent in any C when iterated under a homeomorphism constructed to respect that path.

For the refinable case, we have several measures to consider at once, requiring three major modifications. One is that we will need a different path for each measure: a path P_i will be made so that the proportion of time it spends in a set in $\mathcal{P}$ will be approximately the μ_{r_i} measure of that set. The small sets which our homeomorphism will map along that path can be chosen to take up most of the μ_{r_i} measure of $\mathcal{C}$ while using very little μ_{r_j} measure for $i \neq j$. In this way, the total measure will be consumed for each μ_{r_i}, meaning that we will have used up almost all of $\mathcal{C}$ in the construction under any relevant measure.

The second modification comes when trying to integrate the remainder left after the paths are made. In order to decompose the remainder into cycles considering all measures simultaneously, we will want our remainder sets to have almost the exact same measure for each μ_{r_i}. To enforce this, we will need our main paths to not only occupy most of the space under their respective measure, but to occupy almost exactly the correct amount. Thirdly, the question of what the μ_{r_i} measure of the small sets that move along our paths should be for each measure and for each path so that all combined they take up almost exactly the correct amount of space eventually leads an overdetermined system of linear equations, which we cannot solve. To correct this, we use not one path per measure, but several paths per measure. These different paths used by each measure will be similar enough to each other that they all spend an approximately correct proportion of time in each set in $\mathcal{P}$, but will be different enough from each other to impact the solvability of the resulting system of equations. With this outline in mind, we begin the proof.

Proof. We begin as in the proof of Lemma 4.1 for a good measure. By refining $\mathcal{P}$ by D if necessary, we may assume that D is a union of elements of $\mathcal{P}$. Consider the directed graph G whose vertices are elements of $\mathcal{P}$ and so that there is an edge from C_a to C_b exactly when $C_a \cap T^{-1}(C_b)$ is non-empty, and let $c_1, \ldots, c_L$ be the cycles in this graph. (Again, we do not need repetition for different shifts of a cycle.) As before, G is connected and each edge appears in some cycle.

For each r_i, we can consider the weighted directed graph G_i which has the same edges as G and which assigns to the edge $C_a \to C_b$ a weight equal to $\mu_{r_i}(T^{-1}(C_b) \cap C_a)$. Applying Theorem 4.1 for each i, we find positive values $w_{i,j}$ for $i = 1, \ldots, R$ and $j = 1, \ldots, L$ so that for all $C_a, C_b \in \mathcal{P}$, $\mu_{r_i}(C_a \cap T^{-1}(C_b)) = \sum_{j:(C_a \to C_b) \in c_j} w_{i,j}$. Note that we additionally have for $C \in \mathcal{P}$ that $\mu_{r_i}(C) = \sum_{j:C \in c_j} w_{i,j}$.

Suppose that N is a large integer. Let $N_{i,j} = \lfloor N w_{i,j} \rfloor = N w_{i,j} - \rho_{i,j}$. (The symbol ρ continues to indicate a quantity between zero and one.)

Building a path that traverses the cycle $c_j N_{i,j}$ times will lead to a path that spends time in each set roughly equal to its μ_{r_i} measure. But, as noted before this proof, this will eventually lead to an overdetermined system of equations and so we now instead replace $N_{i,j}$ with several similar values, $N_{i,j,t}$.

Let $\beta = 1/\sqrt{N}$ and, for $t = 0, \ldots, L$, let $N_{i,j,t} = N_{i,j}$ if $j \neq t$, but let $N_{i,j,j} = \lfloor (1 - \beta) N_{i,j} \rfloor = (1 - \beta) N_{i,j} - \hat{\rho}_{i,j}$.

Note that we now have that $N_{i,j,t}$ is always equal to either $N w_{i,j} - \rho_{i,j}$ or to $(1 - \beta)(N w_{i,j} - \rho_{i,j}) - \hat{\rho}_{i,j}$. In either case, we always have that $N_{i,j,t}/N \to w_{i,j}$ as $N \to \infty$ and we have the following.

$$\frac{\sum_{C:C \subseteq D} \sum_{j:C \in c_j} N_{i,j,t}}{\sum_{E \in \mathcal{P}} \sum_{j:E \in \mathcal{P}} N_{i,j,t}} \approx \frac{\sum_{C:C \subseteq D} \sum_{j:C \in c_j} w_{i,j}}{\sum_{E \in \mathcal{P}} \sum_{j:E \in \mathcal{P}} w_{i,j}} = \frac{\sum_{C:C \subseteq D} \mu_{r_i}(C)}{\sum_{E \in \mathcal{P}} \mu_{r_i}(E)} = \mu_{r_i}(D).$$

Assume that N is sufficiently large that we have

$$\frac{\sum_{C:C \subseteq D} \sum_{j:C \in c_j} N_{i,j,t}}{\sum_{E \in \mathcal{P}} \sum_{j:E \in \mathcal{P}} N_{i,j,t}} < \mu_{r_i}(D) + \frac{\eta}{2}$$

for all i and t. Slightly further, to accommodate detours at the end, assume that this inequality holds even if some of the $N_{i,j,t}$ values were increased by one. Along the way, we will also need to assume that N is sufficiently large that $w_{i,j} > 1/N^3$, and that $8L/(N^{3/2} w_{i,j}) < 1/(2(L+1)N)$ for each i, j. We may now fix N to be so large that all of these hold.

Fixing k and t, we build a closed path $P_{k,t}$ as in the proof of Lemma 4.1 by beginning with the cycle c_1 repeated $N_{k,1,t}$ times, finding a cycle c_a which intersects it and inserting that cycle $N_{k,a,t}$ times, continuing until we have used all cycles. (We will always have a cycle which intersects, as otherwise the cycles used before we ran out of options will create an invariant set.) Construct such a path $P_{k,t}$ for each k and t. By cycling if necessary, we may assume that each path $P_{k,t}$ begins with the same vertex C_1. The path $P_{k,t}$ will include an edge $C_a \to C_b$ a number of times equal to $\sum_{j:C_a \to C_b \in c_j} N_{k,j,t}$. For each path, let $n(k, t)$ be the number of edges in $P_{k,t}$.

In order to be able to decompose our final remainder sets at the end, we will need the remainder sets to be almost exactly the same size. We now find adjusted $w_{i,j}$ values to reserve this remainder size. Let $\{g_j\}_{j=1}^{L}$ be a set of real numbers between 1/2 and 1 which are linearly independent over the rationals. Let $\epsilon = 1/N^3$ and, for all i and j, let $w^*_{i,j} = w_{i,j} - \epsilon g_j$. We chose N so that $w_{i,j} > 1/N^3$, giving that these $w^*_{i,j}$ are all positive.

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Soon we will find small clopen sets to move along the paths so that the total measure occupied by the sets approximately fills up the space. We now need to work out what size these sets should be. We have already decomposed the measure of the space into cycles: the values $w^*_{i,j}$ represent the total μ_{r_i} measure of motion along each edge in the cycle c_j we wish to use, and this should be the total contribution along the cycle by all paths. We now let $x_{i,k,t}$ denote the desired μ_{r_i} measure of a typical small set which moves according to the path $P_{k,t}$. For each k and t, the number of copies of c_j used in the path $P_{k,t}$ was $N_{k,j,t}$ and so we expect that the values $x_{i,k,t}$ should satisfy

$$w^*_{i,j} = \sum_{k=1}^{R} \sum_{t=0}^{L} N_{k,j,t} x_{i,k,t} \tag{6.3}$$

for all $i = 1, \ldots, R$ and $j = 1, \ldots, L$. We will now show that the above system of equations has a solution $\{x_{i,k,t}\}$ with each $x_{i,k,t} > 0$. (The extra parameter t was introduced so that this system would not be overdetermined.) Note that the value $t = 0$ gives us an additional variable $x_{i,k,0}$ to work with, but that $j = 0$ does not correspond with an additional requirement.

For $k \neq i$, let $x_{i,k,t} = \delta$, where $\delta = 1/R(L+1)N^2$, and, for $k = i$, let $x_{i,k,t} = 1/(L+1-\beta)N + \phi_{i,t}$, where $\phi_{i,t}$ will be defined shortly. (The $P_{k,t}$ paths are supposed to take up most of the μ_{r_i} measure when $r = i$ and almost no measure when $r \neq i$, as reflected by δ being much smaller than the $x_{i,k,t}$ for $i = k$.) With this, we split off the $k = i$ term to find

$$\sum_{k=1}^{R} \sum_{t=0}^{L} N_{k,j,t} x_{i,k,t} = \bigg(\sum_{k=1, k\neq i}^{R} \sum_{t=0}^{L} N_{k,j,t} \delta \bigg) + \sum_{t=0}^{L} N_{i,j,t} \bigg(\frac{1}{(L+1-\beta)N} + \phi_{i,t} \bigg).$$

We let $F(i,j) = \sum_{k=1,k\neq i}^{R} \sum_{t=0}^{L} N_{k,j,t}$, so the first sum above can be written as $\delta F(i,j)$. For most t values, $N_{i,j,t} = N_{i,j}$, with the exception being that $N_{i,j,j} = N_{i,j} - \beta N_{i,j} - \hat{\rho}_{i,j}$. If we pull out the extra contribution from the $t = j$ term, we find that the above equals

$$\delta F(i,j) + \bigg(\sum_{t=0}^{L} N_{i,j} \bigg(\frac{1}{(L+1-\beta)N} + \phi_{i,t} \bigg) \bigg) - (\beta N_{i,j} + \hat{\rho}_{i,j}) \bigg(\frac{1}{(L+1-\beta)N} + \phi_{i,t} \bigg).$$

Note that most of the first sum has no dependence on t. If we assume that our $\phi_{i,t}$ values satisfy $\sum_{t=0}^{L} \phi_{i,t} = 0$, then this becomes

$$\delta F(i,j) + \bigg(\frac{(L+1)N_{i,j}}{(L+1-\beta)N} \bigg) - \frac{\beta N_{i,j} + \hat{\rho}_{i,j}}{(L+1-\beta)N} - (\beta N_{i,j} + \hat{\rho}_{i,j}) \phi_{i,t}.$$

Moving the $\beta N_{i,j}$ from the second fraction to the first reduces that fraction to $N_{i,j}/N$. Then, expanding $N_{i,j} = N w_{i,j} - \rho_{i,j}$, the above becomes

$$w_{i,j} + \delta F(i,j) - \frac{\rho_{i,j}}{N} - \frac{\hat{\rho}_{i,j}}{(L+1-\beta)N} - (\beta N_{i,j} - \hat{\rho}_{i,j}) \phi_{i,t}.$$

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The above will equal $w^*_{i,j} = w_{i,j} - \epsilon g_j$ if for $t = 1, \ldots, L$ we take

$$\phi_{i,t} = \frac{1}{\beta N_{i,j} + \hat{\rho}_{i,j}}\bigg(\delta F(i, j) - \frac{\rho_{i,j}}{N} - \frac{\hat{\rho}_{i,j}}{(L+1-\beta)N} + \epsilon g_j\bigg),$$

while our above assumption that $\sum_{t=0}^{L} \phi_{i,t} = 0$ forces us to take $\phi_{i,0} = -\sum_{t=1}^{L} \phi_{i,t}$.

We now show that $\phi_{i,j}$ is sufficiently small to ensure that our $x_{i,j,t}$ are positive. From $\beta = 1/\sqrt{N}$ and $N_{i,j} \approx N w_{i,j}$, we find that the initial factor of $\phi_{i,t}$ above is approximately $1/\sqrt{N} w_{i,j}$ and can be no greater than $2/\sqrt{N} w_{i,j}$. The definition of $F(i, j)$ is as a sum of $(L+1)(R-1)$ terms each no greater than N, and the factor of $\delta = 1/R(L+1)N^2$ gives us that $\delta F(i, j) < 1/N$. Each ρ is at most one and $\epsilon = 1/N^3$, so we find that for $t \neq 0$, we have

$$|\phi_{i,t}| < \frac{2}{\sqrt{N} w_{i,j}}\bigg(\frac{1}{N} + \frac{1}{N} + \frac{1}{(L+1-\beta)N} + \frac{1}{N^3}\bigg) < \frac{8}{N^{3/2} w_{i,j}},$$

while $\phi_{i,0}$, being a sum of L terms of at most this size, has $|\phi_{i,0}| < 8L/N^{3/2} w_{i,j}$. When we fixed N we assumed that it was sufficiently large that this gives $|\phi_{i,t}| < 8L/N^{3/2} w_{i,j} < 1/2(L+1-\beta)N$ for all t. Since $x_{i,k,t}$ was defined to be either δ or $1/(L+1-\beta)N + \phi_{i,t}$, we find that $x_{i,k,t} > 0$. (For later, it is convenient to note that $\delta < 1/N^2 < 1/2(L+1-\beta)N$ means that δ is the smallest of all $x_{i,k,t}$ values.)

We have now found positive values $x_{i,k,t}$ which satisfy $\sum_{k=1}^{R} \sum_{t=0}^{L} N_{k,j,t} x_{i,k,t} = w^*_{i,j}$. Using that the integer polynomials are dense in $L^\infty[\alpha, 1-\alpha]$ for any small positive α, for any k and t we can find an integer polynomial $q(x)$ with $q(r_i) \approx x_{i,k,t}$ and then, by using Theorem 5.3, we may then replace $q_{k,t}$ with a partition polynomial which takes the same value at each r_i. For each k and t, we find a partition polynomial $p_{k,t}$ with $p_{k,t}(r_i) \approx x_{i,k,t}$ for each i with $p_{k,t}(r_i) < x_{i,k,t}$. Write $p_{k,t}(r_i) = x_{i,k,t} - \epsilon^{\#}_{i,k,t}$ and note that we can assume that the $\epsilon^{\#}_{i,k,t}$ are as small as needed in the following paragraphs. We have for each $C_a, C_b \in \mathcal{P}$ that

$$\begin{aligned}\mu_{r_i}(T^{-1}(C_b) \cap C_a) &= \sum_{j:(C_a \to C_b) \in c_j} w_{i,j} > \sum_{j:(C_a \to C_b) \in c_j} w^*_{i,j} \\ &= \sum_{j:(C_a \to C_b) \in c_j} \sum_{k=1}^{R} \sum_{t=0}^{L} N_{k,j,t} x_{i,k,t} > \sum_{j:(C_a \to C_b) \in c_j} \sum_{k=1}^{R} \sum_{t=0}^{L} N_{k,j,t} p_{k,t}(r_i).\end{aligned}$$

Using Theorem 5.2 repeatedly, we can then find disjoint clopen subsets of $T^{-1}(C_b) \cap C_a$ associated with the partition polynomial $p_{k,t}$. In particular, for all k and t we can find $\sum_{j:(C_a \to C_b) \in c_j} N_{k,j,t}$ such sets, all disjoint. This means that for each $C \in \mathcal{P}$, we have found a total of $\sum_{j:C \in c_j} N_{k,j,t}$ such sets. The path $P_{k,t}$ was constructed so that it entered the set C exactly this many times. For each vertex C on the path, choose a distinct one of these subsets of C associated with the partition polynomial $p_{k,t}$. (For the first and last vertices of our closed path, use the same choice.) Name these sets

$$F_{k,t,1} \to F_{k,t,2} \to F_{k,t,3} \to \cdots \to F_{k,t,n(k,t)-1} \to F_{k,t,n(k,t)} \to F_{k,t,1},$$

so that the set $F_{k,t,u}$ is a subset of the uth set in the path $P_{k,t}$.

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When these sets are removed from $T^{-1}(C_b) \cap C_a$, the remainder is a clopen set, call it $L_{a,b}$, having μ_{r_i} measure

$$\begin{aligned}
\mu_{r_i}(L_{a,b}) &= \mu_{r_i}(T^{-1}(C_b) \cap C_a) - \sum_{j:(C_a \to C_b) \in c_j} \sum_{k=1}^{R} \sum_{t=0}^{L} N_{k,j,t} p_{k,t}(r_i) \\
&= \sum_{j:(C_a \to C_b) \in c_j} \left(w_{i,j} - \sum_{k=1}^{R} \sum_{t=0}^{L} N_{k,j,t}(x_{i,k,t} + \epsilon^{\#}_{i,k,t}) \right) \\
&= \sum_{j:(C_a \to C_b) \in c_j} \left(w_{i,j} - w^{*}_{i,j} - \sum_{k=1}^{R} \sum_{t=0}^{L} N_{k,j,t} \epsilon^{\#}_{i,k,t} \right) \\
&= \sum_{j:(C_a \to C_b) \in c_j} \epsilon g_j - \sum_{j:(C_a \to C_b) \in c_j} \bar{\epsilon}_{i,j},
\end{aligned} \tag{6.4}$$

where we let $\bar{\epsilon}_{i,j} = \sum_{k=1}^{R} \sum_{t=0}^{L} N_{k,j,t} \epsilon^{\#}_{i,k,t}$.

To be able to incorporate these remaining sets into our homeomorphism H, we will need to decompose these leftovers into cycles. We do this as in the proof of Theorem 4.1 but with the added difficulty that instead of finding values for each i separately, we need to find actual clopen sets, meaning that we must work with all i values at the same time.

For each i, consider the weighted directed graph G_i^1 which has elements of $\mathcal{P}$ as vertices and which has an edge $C_a \to C_b$ when the edge $C_a \to C_b$ appears in our original graph G, and which gives that edge weight equal to the value in (6.4). Note that this is positive if either sum has any terms, which is exactly when the edge exists in G_i^1. Also note that for each i the sum can also be viewed as an integer linear combination (over j) of terms of the form $(\epsilon g_j - \bar{\epsilon}_{i,j})$. Because each set in $\mathcal{P}$ is an initial vertex of an edge in some cycle exactly when it is a terminal vertex of an edge in that cycle, this graph is balanced. We consider the question, which edge has the smallest weight? Recall that $\epsilon = 1/N^3$ was chosen early in the proof, while the $\epsilon^{\#}_{i,k,t}$ were chosen to be small more recently, meaning that we may assume that the $\bar{\epsilon}_{i,j}$ are much smaller than ϵ. When the $\bar{\epsilon}_{i,j}$ are small enough, the edge with the smallest weight is an edge for which the first sum, $\sum_{(C_a \to C_b) \in c_j} \epsilon g_j$, is smallest.

Because the g_j are linearly independent over $\mathbb{Q}$, no two of these sums can be equal except trivially. That is, the only way two edges can have the same (smallest) value of the first sum is if those two sums use the same coefficients in all ϵg_j terms. But, rewriting the value as $\sum_{j:(C_a \to C_b) \in c_j} (\epsilon g_j - \bar{\epsilon}_{i,j})$, we see that in that case, the entire expression is the same, and any such edges all have the same weight. Furthermore, since the first sum has no dependence on i, we find that these edges of minimal weight are actually minimal for each i, provided again that the $\epsilon^{\#}$ are sufficiently small to prevent interference.

This means that for some choice of a_1 and b_1, the clopen set L_{a_1,b_1} has the smallest μ_{r_i} measure among non-empty choices for all i and, furthermore, that if some $L_{a,b}$ has the same measure as L_{a_1,b_1} for some μ_{r_i}, then it has equal measure for all i. By Theorem 5.4, this means that each nonempty $L_{a,b}$ contains (or *is*) a clopen set which for all i has the same μ_{r_i} measure as L_{a_1,b_1}.

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For any i, our graph G_i^1 balanced, so any edge must appear in a cycle, and that must be a cycle in $G_{i,1}$ for each i. Let $e(1)$ be an index so that $c_{e(1)}$ is a cycle in $G_{i,1}$ containing the edge $C_{a_1} \to C_{b_1}$, and let $n(1)$ be its length. Write $c_{e(1)} = C_{d_1} \to C_{d_2} \to \cdots \to C_{d_{n(1)}} \to C_{d_1}$. For each set $L_{d_i,d_{i+1}}$, remove a clopen set with μ_{r_i} measure equal to $\mu_{r_i}(L_{a_1,b_1})$ for each i. Let $L^2_{d_i,d_{i+1}}$ be the portion remaining (or $\emptyset$ if none remains). Name the removed sets $\{E_t^1\}_{t=1}^{n(1)}$, so that E_t^1 is a subset of C_{d_t} for each t. (When $L_{a,b}$ did not contain a E_t^1 set, let $L^2_{a,b} = L_{a,b}$.) Letting G_i^2 denote the graph which gives the edge $C_a \to C_b$ weight equal to the μ_{r_i} measure of $L^2_{a,b}$, removing edges which now have weight zero, we again have a balanced weighted directed graph, but with at least one fewer edge. The weight of a typical edge in this graph will now either be the weight of an edge from the previous graph, or else will be the difference of two weights of edges of the previous graph. In either case, we again have that the weight of each edge is an integer linear combination (over j) of terms $(\epsilon g_j - \bar{\epsilon}_{i,j})$.

By the same reasoning, if the $\epsilon^\#$ are small enough, then the question of which edge in G_i^2 has the smallest weight will depend only on which combination of the ϵg_j is smallest. Because the $\bar{\epsilon}_{i,j}$ occur only in the same linear combinations as the ϵg_j, if two edges have the same minimal combination of the ϵg_j, they must be completely the same, and be the same for all i. As before, we may choose a cycle containing this edge of minimal weight, remove a set of this measure from each $L^2_{a,b}$ in that cycle, calling the removed sets E_i^2 and the remainder sets $L^3_{a,b}$, and consider new graphs G_i^3 with edges of weight zero removed. We may continue in this way, removing clopen sets arranged into cycles until all edges have weight zero, at which point all of $\mathcal{C}$ will have been broken into cycles.

(During the above paragraph, we assumed that $\epsilon^\#$ was sufficiently small without an explicit way of saying what 'sufficiently small' would mean that made sense at the time we chose $\epsilon^\#$. Since the procedure will repeat at most L times, and since taking the difference of two integer linear combinations will produce another integer linear combination whose maximum coefficient has increased by a factor of at most two, it is sufficient to choose $\epsilon^\#$ sufficiently small that for any comparison of two of the finitely many integer linear combinations of $(\epsilon g_j - \bar{\epsilon}_{i,j})$ having coefficients of size at most 2^L, the $\bar{\epsilon}_{i,j}$ do not impact which is smaller.)

Let f be the number of cycles needed to decompose the error completely in this way, so that our error sets are arranged into cycles $\{E_t^s\}_{t=1}^{n(s)}$, for $s = 1, \ldots, f$, with $e(s)$ being the index such that the sets $\{E_t^s\}_{t=1}^{n(s)}$ correspond with the cycle $c_{e(s)}$, meaning that E_t^s is a subset of the tth vertex of the cycle $c_{e(s)}$.

Let g be a partition polynomial which is smaller than any of the $p_{k,t}$ at any algebraic conjugate of r in $(0, 1)$. (We may for example take $g(x) = x^m(1 - x)^m$ for any sufficiently large m.) We would like to use Theorem 2.6 to construct our homeomorphism to map the sets $F_{k,t,1} \to F_{k,t,2} \to \cdots \to F_{k,t,n(k,t)} \to F_{k,t,1}$ as described above, but this will lead to an invariant clopen set. To fix this, we carve out small clopen 'transfer sets' which allow a small set from the end of $P_{k,t}$ to move to the beginning of $P_{k,t+1}$, or from the end of $P_{k,L}$ to the beginning of $P_{k+1,0}$, or from the end of $P_{R,L}$ to the beginning of $P_{1,0}$. For each k and t, use Theorem 5.2 to find $A_{k,t}$, a subset of $F_{k,t,1}$ associated with g, and $B_{k,t}$, a subset of $F_{k,t,n(k,t)}$ associated with g.

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Lastly, we need to clarify where exactly our error sets will be incorporated into the paths. We will fit them in as detours along the path $P_{1,0}$. For each v, let $b(v)$ be an index for which the set $F_{1,0,b(v)}$ is a subset of the same element of $\mathcal{P}$ as $E^v_{n(v)}$ is and for which $F_{1,0,b(v)+1}$ is a subset of the same element of $\mathcal{P}$ as E^v_1 is. Because we have only at most one v for each cycle, and because the path $P_{1,0}$ was built using many copies of each cycle, we can assume that the choices $b(v)$ and $b(v)+1$ are all distinct and are not 1 or $n(1,0)$. Noting that the error sets $\{E^s_t\}$ have $\mu_{r_i}(E^s_t) < \epsilon < \delta$ while each F has measure at least as large as δ for each i, we may use Theorem 5.4 to find a subset of $F_{1,0,b(v)}$ with the same μ_{r_i} measure as the sets from $\{E^v_{a_k}\}_{k=1,\ldots,n(v)}$. Name this set $B(v)$. Also, find such a subset of $F_{1,0,b(v)+1}$, naming it $A(v)$.

We are now ready to apply Theorem 2.6. Let $\mathcal{P}_0$ be the partition of $\mathcal{C}$ consisting of all the sets $B(v)$ and $F_{1,0,b(v)} \setminus B(v)$, all sets $B_{k,t}$ and $F_{k,t,n(k,t)} \setminus B_{k,t}$, all sets $F_{k,t,\alpha}$ which do not contain a set $B(v)$ or a $B_{k,t}$ and all sets E^v_i. Similarly, let $\mathcal{Q}_0$ be the partition of $\mathcal{C}$ consisting of all sets $A(v)$ and $A_{k,t}$, all sets $F_{1,0,b(v)+1} \setminus A(v)$ and $F_{k,t,1} \setminus A_{k,t}$, all other $F_{k,t,\alpha}$ sets and all sets E^v_i.

Let $\pi_0 : \mathcal{P}_0 \to \mathcal{Q}_0$ be the bijection defined by

$$F_{1,0,b(v)} \setminus B(v) \mapsto F_{1,0,b(v)+1} \setminus A(v);\;\; F_{k,t,n(k,t)} \setminus B_{k,t} \mapsto F_{k,t,1} \setminus A_{k,t};$$

$$F_{k,t,\alpha} \mapsto F_{k,t,\alpha+1} \text{ for those } F_{k,t,\alpha} \text{ not containing a } B \text{ set};$$

$$B_{k,t} \mapsto A_{k,t+1} \text{ (if } t < L);\;\; B_{k,L} \mapsto A_{k+1,0} \text{ (if } k < R);\;\; B_{R,L} \mapsto A_{1,0};$$

$$B(v) \mapsto E^v_1;\;\; E^v_i \mapsto E^v_{i+1} \text{ if } i < n(v);\;\; E^v_{n(v)} \mapsto A(v)$$

for all appropriate values of k, i, t or v.

The bijection π_0 preserves μ_{r_1}. It has no non-empty proper invariant sets: for each k, t, α, the set $F_{k,t,\alpha}$ appears unaltered in either $\mathcal{P}_0$ or $\mathcal{Q}_0$ and, since an invariant set must be expressible both as a union of sets in $\mathcal{P}_0$ and as a union of sets in $\mathcal{Q}_0$, it must be a union of sets $F_{k,t,\alpha}$ and sets $E^v_{n(v)}$. But, for each set $F_{k,t,\alpha}$, π_0 maps at least part of that set to $F_{k,t,\alpha+1}$ if $\alpha < n(k,t)$, or to $F_{k,t+1,1}$ if $\alpha = n(k,t)$ and $t < L$, or to $F_{1,0,1}$ if $\alpha = n(k,t)$ and $t = L$. So, an invariant set which contains part of one $F_{k,t,\alpha}$ set must contain all of the $F_{k,t,\alpha}$ sets. Similarly, part of an $F_{k,t,\alpha}$ maps to E^v_1 and on to $E^v_{n(v)}$, which maps to part of an $F_{k,t,\alpha}$, so a set which contains the $F_{k,t,\alpha}$ must contain all of the E^v_i and vice versa. So, π_0 has no non-trivial invariant set.

Using Theorem 2.6, there is a homeomorphism $H_0 \in \hom^*(\mu_r)$ which maps set to set according to π_0. Let $\mathcal{Q}$ be the refinement of $\mathcal{P}$ by the partitions $\mathcal{P}_0$ and $\mathcal{Q}_0$, and let $H \in N(H_0, \mathcal{Q}_0)$. Note that this means that H, like H_0, maps any set $U \in \mathcal{P}_0$ to $\pi_0(U)$. We consider a large integer n, a point $x \in \mathcal{C}$ and we examine the expression

$$\frac{1}{n+1}\#\{j = 0, \ldots, n : H^j(x) \in D\}.$$

There is a fixed number A so that a point can be iterated at most A steps under H without entering a set of the form $F_{k,t,1}$ for some k and some t: we may take A to be the largest of the $n(k,t)$ plus the total number of the sets E^v_i. (See Figure 1.)

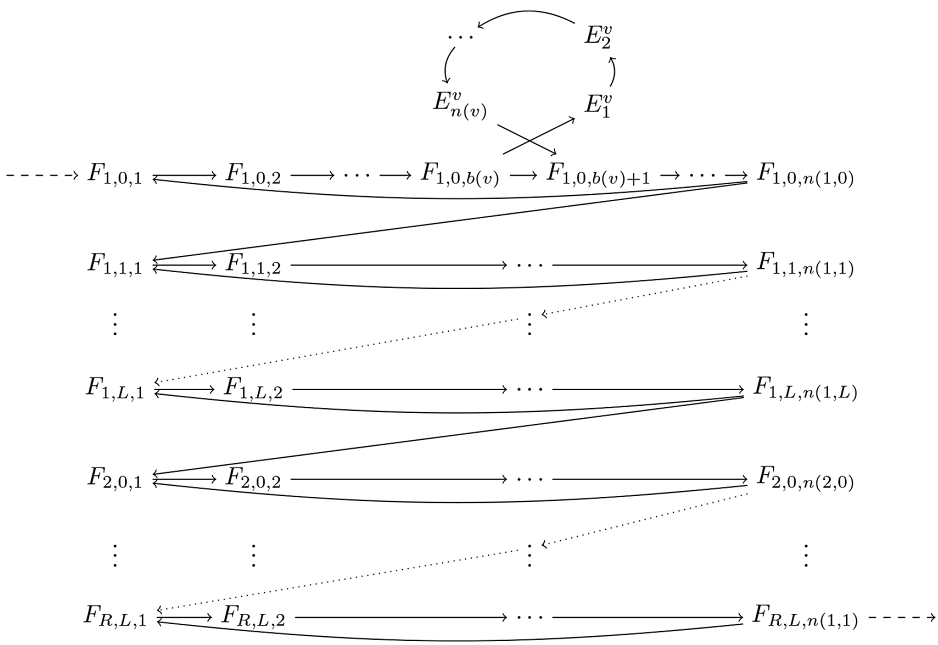

FIGURE 1. The action of π_0: a point must travel an entire path $P_{k,t}$ (a horizontal row), after which it either returns to the start of the path or transfers to the next path below. Only the top path $P_{1,0}$ incorporates detours into the error cycles, of which only one is displayed.

Let n be large. As in the proof of Lemma 4.1, the points $\{H^j(x)\}_{j=0}^n$ can be grouped into at most $2A$ points from the beginning and end, and groups which travel along a path $P_{k,t}$ with possible detours into the E_i^v sets if $(k, t) = (0, 1)$. When there are no detours, the number of times the path $P_{k,t}$ enters a set $C \in \mathcal{P}$ is $\sum_{j:C\in c_j} N_{k,j,t}$ and so for such a group of iterates, the proportion of time spent in D is then

$$\frac{\sum_{C\in\mathcal{P}:C\subseteq D}\sum_{j:C\in c_j} N_{k,j,t}}{\sum_{E\in\mathcal{P}}\sum_{j:E\in c_j} N_{k,j,t}}.$$

By our first assumption on N, the above proportion is less than $\mu_{r_k}(D) + \eta/2$ and this will remain true if some of the $N_{k,j,t}$ are increased by one, as they may if the point takes a detour into a cycle of error sets corresponding with the cycle c_j.

So, for all but $2A$ of the first n iterates of x under H, the proportion of time in D is a weighted average of proportions which are each at most $\max_k\{\mu_{r_k}(D)\} + \eta/2$. For large n, the $2A$ uncontrolled points will have total weight less than $\eta/2$ and so the total proportion of time in D will be less than $\max\{\mu_{r_k}(D)\} + \eta$. □

With Lemma 6.1 proved, our proof of Theorem 1.1 is complete. □

Note that instead of doing the work of using transfer sets to ensure that our bijection π_0 had no invariant clopen sets, we could have used Theorem 2.1 to construct a homeomorphism which has an invariant clopen set. This shows that the set of homeomorphisms with an invariant clopen set is dense (and open) in $\mathrm{hom}(\mu_r)$.

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It is worth viewing this result in the context of Ibarlucía and Melleray's paper [6]. There, a collection K of full, non-atomic probability measures on a Cantor space is defined to be a *dynamical simplex* when it is compact, convex, good and when for any clopen set A, any integer n and any $\epsilon > 0$, there is a clopen set $B \subseteq A$ with $n\mu(B) \in [\mu(A) - \epsilon, \mu(A)]$ for all $\mu \in K$. The main theorem presented in [6] is the following.

THEOREM 6.1. (Ibarlucía and Melleray) *A simplex K of probability measures on a Cantor space is the set of invariant measures of some minimal homeomorphism if and only if K is a dynamical simplex.*

Suppose that r is a refinable number and R is the set of its conjugates in (0, 1), and let K be the convex hull of $\{\mu_r\}_r \in R$. Then, as noted earlier, K is a good set of measures, and is clearly compact and convex. That it satisfies the final requirement above also follows from Theorems 5.3 and 5.2, giving us that K is a dynamical simplex, and the above theorem applies. Further, the requirements above are very nearly exactly what were used in our proof of Theorem 1.1 and so it seems likely that the construction used here can be applied to any finite-dimensional dynamical simplex, showing for a dynamical simplex K, that finite ergodicity is generic among homeomorphisms which preserve each measure in K. Could it work in an infinite-dimensional case?

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