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Transition thresholds and transport properties of the H state in the light of the magnetic entropy concept

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Abstract. The concept of magnetic entropy, the entropy pertaining to a magnetic configuration, is introduced and illustrated through its application to the description of the magnetic states assumed preferentially by a tokamak and of the transition between them. The magnetic equilibria associated with stationary magnetic entropy admit a bifurcation, and a new state can arise that can be attained spontaneously with an increase in the magnetic entropy. The thermodynamic relations between entropy production, heat transport and plasma energy allow one to express quantitatively the modifications of these quantities generated by the magnetic transition to the new state and to identify it with the L–H transition. The power and temperature thresholds of the transition are expressed in terms of the confinement time of the L state, and definite scalings are derived that compare favourably with observations. The theory implies the hysteresis and the upper density limit of the H state.

1. Introduction

In this paper, the point of view that considers the L–H transition as a bifurcation process involving the global magnetic configuration of a tokamak, entailing also a global change of the transport properties of the plasma, is developed with the aim of arriving at a quantitative formulation that could allow a comparison with observations. The concept underlying this approach is that of magnetic entropy, a quantity associated with the magnetic configuration, which, through the requirement that it should increase in an isolated system, determines the branch of the new bifurcated state that the plasma should reach spontaneously, and also, through the relations inherent to its thermodynamic meaning, determines the general transport properties of the new state.

In view of its relevance for this work, the meaning of the magnetic entropy and its domain of application are illustrated in an extended summary of the results of previous papers (especially in Secs 2 and 3) in order to allow the reader to understand the essential aspects of the concept without direct reference to the literature.

In Sec. 2, the framework of information theory is presented, in which the magnetic entropy is derived and is interpreted as a measure of the probability of coarse-grained magnetic configurations under given constraints that express the testable information available on the system. In Sec. 3, the magnetic entropy is applied to a tokamak, considered as an open system in which an equilibrium exists between the entropy produced in the system and the entropy injected into it by auxiliary power and by an Ohmic transformer. In this way, a family of magnetic equilibria is defined that corresponds to stationary entropy and to a stationary thermal state. However, when the condition of stationary entropy is relaxed, a bifurcation of the equilibrium can arise, and a new magnetic state appears that can be reached with a spontaneous entropy-increasing transition (Sec. 4). The transport properties of this state are discussed on the basis of the thermodynamic relations between the entropy production in the transition and the power balance equation (Sec. 5). This state, which has the physical characters of the H state, is not in thermal equilibrium, and an expression for the increase of the plasma energy per unit time is explicitly given.

A property inherent to the physics of the magnetic transition is the increase in the current density at the edge in a constant-total-current regime, possibly giving rise to peeling modes and to the phenomenon of ELMs.

The bifurcation condition or transition threshold is expressed by a relation between the auxiliary power (or the temperature) and the confinement time of the L state (Sec. 6). Assuming that the confinement time is given by the empirical scaling ITER89-P, one obtains from the bifurcation condition a scaling law for the threshold values of the auxiliary power and of the temperature. The theoretical results compare favourably with observations. Also, the bifurcation condition implies a hysteresis of the threshold, which is a consequence of the better confinement of the H state, and the occurrence of a reverse transition at an upper density limit.

2. The thermodynamics of magnetic plasma equilibria

2.1. The statistical model

The thermodynamics of magnetic plasma equilibria is based on the concept of magnetic entropy, a quantity that measures, in the framework of information theory, the probability of coarse-grained current density configurations in a suitably constrained possibility space (or information space) of this variable.

The information space is the site where the information available on the system can be specified. Let us subdivide the plasma volume V into N volume elements ΔV (sufficiently small that the macroscopic equilibrium can be considered as homogeneous in ΔV, but large enough to contain many particles). Let us consider the assembly of N volume elements, each in a six-dimensional space ($\mathbf{j}$, $\mathbf{x}$) of current density and position, where they can be arranged at random (taking care that their positions are all different, because the volume elements cannot overlap in V). The information space Γ is the joint $6N$-dimensional space $(\mathbf{j}_1, \dots \mathbf{j}_N; \mathbf{x}_1, \dots \mathbf{x}_N)$ of the N volume elements. A single point of Γ represents a particular current density configuration, reconstructed in the volume V with a coarse-graining ΔV.

Considering a large number of copies of the assembly of N volume elements, an equilibrium ensemble is set up by assigning the probability distribution

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$p(\mathbf{j}_1, \ldots \mathbf{j}_N; \mathbf{x}_1, \ldots \mathbf{x}_1)$ for the assembly to occupy any given volume element $d\Gamma$ in the space Γ. The distribution is assigned by maximizing the entropy

$$S = -\int_\Gamma p \ln p \, d\Gamma \tag{1}$$

under given constraints that express the testable information available on the system. The degree of adherence of the statistical predictions to observations provides a test of the reliability of the information (this point of view is called Bayesian; see Jaynes (1989) and, for a very readable exposition, see Garrett (1991) and Gull (1991)).

The constraints involved in our thermodynamic model of magnetic equilibria are as follows.

(i) There is a fixed (but unspecified) value of the current density dispersion generated by the underlying particle structure.

(ii) The fluctuations of the current density are superimposed on a fixed (but a priori unspecified) macroscopic configuration of the current density localized in a subdomain $\Omega \ll V$ (where V tends eventually to infinity with fixed V/N, according to the thermodynamic limit). The volume outside Ω is filled with a neutral fluctuating background of ions and electrons. The macroscopic equilibrium in Ω can exchange electric charges and energy with the background without modifying it. The background plays formally the same role of the heat bath in Gibbsian statistics.

(iii) However, the electromagnetic energy exchanged with the background can be finite on average, and is given by a fixed value Φ_{int}. The energy Φ_{int} simulates the interaction of the macroscopic magnetic equilibrium with the external world, and depends on the particular physical situation at hand. For instance, Φ_{int} can describe a Poynting flux of energy (as can be shown explicitly in the appropriate cases). In situations where the macroscopic equilibrium is isolated, one has $\Phi_{\text{int}} = 0$.

The entropy S, maximized under these constraints, is expressed by the following functional of the macroscopic current density distribution $\mathbf{j}(\mathbf{x})$ in Ω and of the related vector potential $\mathbf{A}(\mathbf{x})$ (Minardi, 1992a):

$$S = \frac{3}{2\,\Delta V \langle \Delta \tilde{j}^2 \rangle} \left\{ -\int_\Omega j^2 \, dV + \frac{\mu^2 c}{4\pi} \int_\Omega \mathbf{j} \cdot \mathbf{A} \, dV \right\}, \tag{2}$$

where $\langle \Delta \tilde{j}^2 \rangle$ is the variance of the current density fluctuations in each ΔV and an inessential additive constant has been omitted. μ^2 is a parameter (positive or negative) related to the Lagrange multiplier involved in the constraint (iii), and is given by the expression

$$\mu^{-2} = \frac{1}{8\pi} \frac{\int_\Omega A^2 \, dV}{\Phi + \Phi_{\text{int}}}, \tag{3}$$

where

$$\Phi = \frac{1}{2c} \int_\Omega \mathbf{j} \cdot \mathbf{A} \, dV. \tag{4}$$

2.2. Isothermal variations

In an isolated system, one has $S \leqslant 0$, as can be seen by substituting μ^2 given by (3) (where $\Phi_{\text{int}} = 0$) into (2) and applying the Schwartz inequality. The maximum $S = 0$ corresponds to $\mathbf{j} = (\mu^2 c/4\pi)\mathbf{A}$ and to the relaxed states of the Bessel function model of the pinch (Taylor 1974), the relaxation now being related to the increase in the magnetic entropy in a dissipative plasma totally isolated from the external world by a perfectly conducting shell.

However, a system is never rigorously isolated, and a point of view in which Φ_{int} is allowed to be non-zero is then appropriate. This is obtained by considering the meaning of the expression (3) for μ^2 and of the Lagrange multiplier τ, to which μ^2 is related by

$$\tau = -\frac{4\pi\,\Delta V\,\langle \Delta \tilde{j}^2 \rangle}{3\mu^2 c^2} \tag{5}$$

(τ is negative when $\mu^2 > 0$). The parameter τ plays in our formalism the role of a generalized temperature. When infinitesimal variations of $\mathbf{j}$ and $\mathbf{A}$ are considered (compatible with d'Alembert's equation) starting from an equilibrium with $\Phi_{\text{int}} = 0$ and taking τ (and μ^2) fixed (isothermal variations), Φ_{int} is forced to change in order to be consistent with (3), and the macroscopic system undergoes an infinitesimal exchange of energy with the external world. One can then proceed to study the variational properties of S with respect to the isothermal variations above. For instance, in the case of the pinch, one finds that S is maximum precisely for those values of μ^2 for which the equilibrium is stable according to the Bessel function model. Conversely, if S is not maximum, the increase in the magnetic entropy in the perturbed state implies the development of an instability and the appearance of large macroscopic motions in the plasma (Minardi 1989, 1992a).

Stability with respect to localized interchange and tearing modes can also be expressed by the variational properties of a suitable local version of the functional S, independently of specific dynamical considerations.

In general, S is not maximum when neighbouring equilibria exist, and τ is necessarily negative in this case.

The point of view above where $\Phi_{\text{int}} \neq 0$ will be particularly relevant in the treatment of an open system like a tokamak, which interacts with the external world through an Ohmic transformer and additional heating (see Sec. 3).

2.3. The tokamak

The entropy of an open (non-isolated) system is not required to be maximum in general, but when the system is stationary one can require that equilibrium exists between the entropy produced by the system and the entropy injected into it. In order to introduce this condition for a tokamak, we need an expression for the magnetic entropy of the tokamak states.

A convenient idealization of an Ohmic tokamak can be formulated by including the time-dependent current of the primary of the transformer in the current entering S. By considering a thin conducting shell at the edge of the plasma column where this current is localized and by including the shell in the integration domain of S, (2), the primary is treated as part of the system described by S. Then the entropy production of an Ohmic tokamak is calculated

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by taking the time derivative of S with respect to the time dependence of the localized current. The detailed calculation is given in the Appendix. One arrives at the expression

$$\frac{dS}{dt} = \frac{1}{\tau}\int_{V_p}\left(\frac{\mathbf{E}}{\mu^2}\cdot\nabla^2\mathbf{j}_p + \mathbf{E}\cdot\mathbf{j}_p\right)dV, \tag{6}$$

where V_p is the plasma volume (excluding the shell), $\mathbf{E} = \mathbf{e}_\phi E$, with $E = E_0 R_0/R$, is the toroidal electric field and $\mathbf{j}_p = \mathbf{e}_\phi j_\phi$ is the toroidal current density of the plasma.

In the presence of auxiliary heating with deposition profile $p_A(r)$, the total entropy production takes the following form (dropping the subscript p on $\mathbf{j}_p$):

$$\frac{dS}{dt} = \frac{1}{\tau}\int_{V_p}\left(\frac{\mathbf{E}}{\mu^2}\cdot\nabla^2\mathbf{j} + \mathbf{E}\cdot\mathbf{j} + p_A\right)dV. \tag{7}$$

This relation is the starting point of this paper.

2.4. *Connection with dynamics*

A relevant property of the entropy functional S should now be mentioned before concluding this brief summary. This functional has been constructed on the sole basis of the standard formalism of information theory applied to our particular information space Γ, independently of any specific dynamical assumption. Nevertheless, one finds that in the case of reversible processes (i.e. in the absence of dissipation) and for an isolated ergodic system, the functional S acquires a dynamical meaning (Minardi 1992b, 1993). Indeed, the vanishing of the first variation of S ($\delta S = 0$ with respect to the primary static variations $\delta\mathbf{j}$ and $\delta\mathbf{A}$ compatible with d'Alembert's equation) implies Hamilton's principle applied to a Lagrangian describing the motion of a system of independent electrons in a fixed ion background, compatible with the presence of the magnetic equilibrium described by the macroscopic current density $\mathbf{j}(r)$ and the related vector potential $\mathbf{A}(r)$ (assumed to be toroidally symmetric and in a small-Larmor-radius approximation).

The connection is lost in the presence of dissipative or radiative processes – a reflection of the fact that, in accordance with its definition in information theory, the entropy is basically not a dynamical concept.

3. Entropy production and heat transport

3.1. *Constraint on the thermal conductivity*

The relation (7) will be applied to the confinement zone of the tokamak, where the plasma is supposed to be quiescent enough to allow the use of our thermodynamic concepts. We write (7) in the form

$$\frac{dS}{dt} = \frac{1}{\mu^2\tau}\int_{\mathscr{S}}(E\boldsymbol{\nabla}j_\phi - j_\phi\boldsymbol{\nabla}E)\cdot d\mathscr{S} + \frac{1}{\tau}\int_{V_c}(\mathbf{E}\cdot\mathbf{j} + p_A)\,dV, \tag{8}$$

where V_c is the volume delimited by the two magnetic surfaces (denoted by $\mathscr{S}$) associated with the values $q = 1$ and $q \approx 2$ of the safety factor and $d\mathscr{S}$ is directed outside V_c.

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Let us suppose for a moment that the confinement region is the site of a reversible process. Then

$$\frac{dS}{dt} = \frac{1}{\tau}\frac{dQ}{dt}, \tag{9}$$

where dQ/dt is the net heat per unit time absorbed or emitted by the plasma in the confinement region:

$$\frac{dQ}{dt} = -\int_{\mathscr{S}} \mathbf{q}_h \cdot d\mathscr{S} + \int_{V_c} (\mathbf{E}\cdot\mathbf{j} + p_A)\, dV. \tag{10}$$

In a one-fluid approximation, $\mathbf{q}_h$ can be expressed in terms of a single diffusivity coefficient χ, $\mathbf{q}_h = -n\chi\nabla T$, and the power balance in the confinement zone takes the form

$$\frac{3}{2}\frac{d}{dt}\int_{V_c} nT\, dV = \int_{\mathscr{S}} n\chi\nabla T\cdot d\mathscr{S} + \int_{V_c} (\mathbf{E}\cdot\mathbf{j} + p_A)\, dV \tag{11}$$

Combining (8)–(11), one derives the relations

$$\int_S n\chi\nabla T\cdot d\mathscr{S} = \frac{1}{\mu^2}\int_{\mathscr{S}} (E\nabla j_\phi - j_\phi\nabla E)\cdot d\mathscr{S}, \tag{12a}$$

$$\frac{dS}{dt} = \frac{3}{2}\frac{d}{dt}\int_{V_c} nT\, dV. \tag{12b}$$

Equation (12a) is a definite relation between the current density profile and the heat flux, and implies stringent conditions on the transport properties of the plasma, whatever the scaling of the transport coefficients with respect to the fundamental parameters related to Larmor radius, collisionality and beta. We refer for a discussion of these conditions to a forthcoming paper (Lazzaro and Minardi, 1999), where the relation (12a) is applied to the description of the thermal transport of negative- or low-magnetic-shear states with stationary entropy, generated by strong auxiliary power.

The relations (12) will play an important role in our discussion.

3.2. States with stationary magnetic entropy

We require that the entropy be locally stationary so that the expression (7) will vanish when the integration is applied to any volume delimited by magnetic surfaces in the confinement zone of the plasma. Then the following equation should hold (in the cylindrical approximation):

$$\nabla^2 j_\phi + \mu^2 j_\phi = -\frac{\mu^2 p_A}{E_0} \quad (s\lambda \leqslant r \leqslant s), \tag{13}$$

where the range of r is delimited by $q_s \equiv q(s) \approx 2$ and $\hat{q} \equiv q(s\lambda) = 1$ (here and in the following, 'hats' denote quantities taken on the surface $\hat{q} = 1$). Equation (13) describes a family of magnetic states with stationary entropy labelled by the parameter μ^2.

In general j_ϕ is not Ohmically relaxed, but relaxation ($j_\phi \propto T^{3/2}$) can reasonably be assumed near the edge, where the temperature is low and the resistivity is

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higher. Thus, introducing the Spitzer resistivity $\eta = A(Z_{\mathrm{eff}})T^{-3/2}$ (where Z_{eff} is uniform), the following is a convenient choice of the relation between j_ϕ and T:

$$j_\phi(r) = \frac{E_0}{A(Z_{\mathrm{eff}})}T^{3/2}(r) + C_\phi \ln\frac{r}{s}, \tag{14}$$

where C_ϕ is a constant that will be determined from the power balance across the surface $\hat{q} = 1$. With the choice (14), the equation (13) for j_ϕ reduces to the power balance equation

$$-\frac{1}{r}\frac{\partial}{\partial r}\left[rn\chi\frac{\partial}{\partial r}T(r)\right] = E_0 j_\phi(r) + p_A(r) \quad (s\lambda \leqslant r \leqslant s), \tag{15}$$

provided that χ has the form

$$n(r)\chi(r) = \frac{3}{2}\frac{E_0^2}{A(Z_{\mathrm{eff}})\mu^2}T^{1/2}(r). \tag{16}$$

According to this relation, χ should scale as $T^{1/2}/n$, which agrees with models of transport in Ohmic tokamaks proposed some years ago by Ohkawa (1978), Inoue et al. (1980) and Itoh et al. (1986).

The power balance across the surface $\hat{q} = 1$ is expressed by the equality

$$-n\chi\frac{dT}{dr}\bigg|_{s\lambda}\hat{\mathscr{S}} = E_0\hat{j}\hat{V} + \int_{\hat{V}} p_A\,dV, \tag{17}$$

where $\hat{V} = 2\pi^2(s\lambda)^2R_0$ is the volume of the sawtooth region, $\hat{\mathscr{S}} = 4\pi^2 s\lambda R_0$ is the area of the surface $\hat{q} = 1$, and the current density $\hat{j}$ has been assumed to be uniform in the sawtooth region (by taking the average over sawtoothing) and equal to the value corresponding to $\hat{q} = 1$,

$$\hat{j} = \frac{cB}{2\pi R_0}. \tag{18}$$

The derivative of (14), using (16), gives

$$\frac{C_\phi}{s\lambda} = \frac{dj_\phi}{dA_\phi}\frac{dA_\phi}{dr}\bigg|_{s\lambda} - \frac{\mu^2}{E_0}n\chi\frac{dT}{dr}\bigg|_{s\lambda}, \tag{19}$$

where A_ϕ is the toroidal component of the vector potential and

$$-\left(\frac{dA_\phi}{dr}\right)_{s\lambda} = B_\theta(s\lambda) = \frac{2\pi\hat{j}s\lambda}{c}.$$

Putting

$$\hat{\nu}^2 \equiv \frac{4\pi}{c}\left(\frac{dj_\phi}{dA_\phi}\right)_{s\lambda}$$

and recalling (17), one obtains

$$C_\phi = \frac{(\mu s\lambda)^2\hat{j}}{2}\left(1 - \frac{\hat{\nu}^2}{\mu^2}\right) + \frac{\mu^2}{E_0}\int_0^{s\lambda} p_A\,r\,dr. \tag{20}$$

Once C_ϕ is given, the temperature profile is determined by (14), where $j_\phi(r)$ is

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fixed as a solution of (13) with the boundary conditions $j_\phi(s\lambda) = \hat{j}$ and $j_\phi(s) = j_s$. In particular, (14) fixes the temperature $\hat{T}$ on the surface $\hat{q} = 1$, while the temperature at the border $r = s$ is related to j_s by Ohmic relaxation.

The component A_ϕ of the vector potential is also fixed. Indeed, combining d'Alembert's equation $\nabla^2 A_\phi = -(4\pi/c) j_\phi$ with (13), one has

$$\frac{\mu^2 c}{4\pi} A_\phi = j_\phi - j_s - \hat{j}(\mu s)^2 \int_x^1 \frac{dx}{x} \int_0^x px\,dx - C_\phi \ln x, \tag{21}$$

where we have put $A_\phi(s) = 0$, $p \equiv p_A / E_0 \hat{j}$ and $x = r/s$. It follows from this relation that the completely Ohmically relaxed state ($C_\phi = 0$) of an Ohmic tokamak ($p_A = 0$) with vanishing current density at the boundary ($j_s = 0$) is an absolute maximum of the entropy (2) applied to a configuration with $\mathbf{j} = \mathbf{e}_\phi j_\phi$ (see Sec. 2.2).

It is convenient to divide the current density of (21) into two parts:

$$j_\phi = \frac{\mu^2 c}{4\pi} A_\phi + \quad , \tag{22}$$

where

$$= \hat{j}(\mu s)^2 \int_x^1 \frac{dx}{x} \int_0^x px\,dx + C_\phi \ln x. \tag{23}$$

Here we have omitted the constant term j_s because in the statistical procedure one has the freedom to define the coordinates of the information space Γ, as well as A_ϕ, apart from the constant.

The function   represents the factors related to the interaction of the plasma in the confinement zone with the external world, namely the auxiliary power p_A and the thermal interaction with the sawtooth zone described by the term involving C_ϕ. The part   of the current density makes the following contribution to the magnetic energy:

$$\begin{aligned} \frac{1}{2c} \int \quad A_\phi\,dV &= \frac{1}{2c} \int j_\phi A_\phi\,dV - \frac{\mu^2}{8\pi} \int A_\phi^2\,dV \\ &= \Phi - \frac{\mu^2}{8\pi} \int A_\phi^2\,dV. \end{aligned} \tag{24}$$

Comparing this expression with (3), one finds that it is just equal to $-\Phi_{\text{int}}$, where Φ_{int} is the interaction energy between the macroscopic equilibrium and the external world simulated by the infinite background. Consistently with its definition in the statistical procedure, Φ_{int} does not vanish when the macroscopic system is not isolated.

4. Magnetic transitions and irreversible transformations

In this section, we shall consider a certain transition between magnetic states that is associated with an increase in the magnetic entropy. In an isolated system, this transition should occur spontaneously. We shall see that the transition can be identified with the L–H transition.

4.1. Bifurcation of the magnetic equilibrium

We introduce the function $J(A_\phi(r)) = j_\phi(r)$ (with $A_\phi(r)$ and $j_\phi(r)$ monotonic) and write d'Alembert's equation in the form

$$\nabla^2 A_\phi = -\frac{4\pi}{c} J(A_\phi(r)). \tag{25}$$

Let us consider a neighbouring state $A_\phi + A_1$ defined in the confinement zone by the equation

$$\nabla^2 (A_\phi + A_1) = -\frac{4\pi}{c} J(A_\phi + A_1) \quad (s\lambda \leqslant r \leqslant s), \tag{26}$$

where $J(A_\phi + A_1)$ is the same function of its argument as in (25). Subtraction of (25) from (26) gives a nonlinear equation for A_1:

$$\nabla^2 A_1 = -\frac{4\pi}{c}[J(A_\phi + A_1) - J(A_\phi)]. \tag{27}$$

We impose the following boundary conditions:

$$\left.\frac{\partial A_1}{\partial r}\right|_{s\lambda} = 0, \tag{28a}$$

$$A_1(s) = 0. \tag{28b}$$

The condition (28a) implies continuity of the poloidal magnetic field on the surface $\hat{q} = 1$ (one could take $A_1(s\lambda) = 0$ instead, but this would imply a sheet current on this surface). The condition (28b) follows from conservation of the poloidal magnetic flux in the confinement zone.

Let us consider the eigenvalue equation

$$\nabla^2 A_1(x) + \frac{4\pi s^2}{c}\frac{dJ}{dA_\phi} A_1(x) = H A_1(x) \quad (\lambda \leqslant x \leqslant 1), \tag{29}$$

where H is its first eigenvalue. For $H = 0$, this equation can be obtained from (27) by neglecting the terms nonlinear in A_1. Thus, in this limit, the nonlinear solution for $A_\phi + A_1$ tends to the linear solution when A_1 is very small and in particular when $A_1 = 0$. The eigenvalue $H = 0$ is then the bifurcation point between the state A_ϕ and the nonlinear state $A_\phi + A_1$.

In order to study the nonlinear properties of (27), we expand the right-hand side up to second order in A_1. Putting

$$\nu^2(x) = \frac{4\pi}{c}\frac{dJ}{dA_\phi}, \tag{30a}$$

$$\beta(x) = -\frac{2\pi s^2}{c}\frac{d^2 J}{dA_\phi^2}, \tag{30b}$$

one obtains

$$\nabla^2 A_1 + (\nu s)^2 A_1 = \beta A_1^2. \tag{31}$$

Integration of both sides after multiplication by $x A_1$ gives

$$\int_\lambda^1 [-(\boldsymbol{\nabla} A_1)^2 + (\nu s)^2 A_1^2]\, x\, dx = \int_\lambda^1 \beta A_1^3 x\, dx. \tag{32}$$

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As is well known, the first eigenvalue is characterized by the following variational property:

$$-H = \inf\left\{\frac{\int_{\lambda}^{1}[(\nabla A_1)^2-(\nu s A_1)^2]x\,dx}{\int_{\lambda}^{1}A_1^2\,x\,dx}\right\}. \tag{33}$$

Thus, on comparing (32) and (33), one has

$$\frac{\int_{\lambda}^{1}\beta A_1^3\,x\,dx}{\int_{\lambda}^{1}A_1^2\,x\,dx} \leqslant H. \tag{34}$$

It follows from this inequality that, for $H < 0$, a non-trivial ($A_1 \neq 0$) solution of (31) can exist only if $\beta A_1 < 0$ in a sufficiently large range of r. Conversely, a nonlinear state with $\beta A_1 > 0$ exists only for $H > 0$.

It has been shown that $H < 0$ is sufficient for stability of the state A_ϕ with respect to the ϕ-independent collionsless modes A_1 (Laval et al. 1965; Schindler 1965). Thus only the nonlinear states $A_\phi + A_1$ associated with $H > 0$ are accessible from the state A_ϕ with a nonlinear perturbation A_1.

The sign of H depends on the parameters ν and β of (31). We assume that the state A_ϕ is the state of stationary entropy described by (21). One then obtains the relations

$$\begin{aligned}\nu^2(r) &= \frac{4\pi}{c}\frac{dJ}{dA_\phi}\\ &= \mu^2 + \frac{2\pi\hat{j}\mu^2}{I(r)}\int_{s\lambda}^{r} pr\,dr - \frac{\pi(\mu s\lambda)^2}{I(r)}\left(1-\frac{\hat{\nu}^2}{\mu^2}\right),\end{aligned} \tag{35}$$

$$\begin{aligned}\beta(r) &= -\frac{2\pi s^2}{c}\frac{d^2J}{dA_\phi^2}\\ &= \frac{2\pi\hat{j}(\mu s)^2}{cB_\theta^2(r)}\left[p(r) + \frac{\pi j(r)}{I(r)}(s\lambda)^2\left(1-\frac{\hat{\nu}^2}{\mu^2}\right) - \frac{2\pi j(r)}{I(r)}\int_{s\lambda}^{r} pr\,dr\right],\end{aligned} \tag{36}$$

where $I(r) = \frac{1}{2}cB_\theta r$. Numerical inspection of (36) shows that β can be negative or positive. Moreover, a numerical treatment of the eigenvalue equation (31) (assuming $p(r)$ to be uniform; Minardi 1992c) shows that when β is positive, H is always negative, and therefore the system is stable.

In the case of negative β, the nonlinear state with $A_1 < 0$ corresponds to $H > 0$, consistently with (34). The situation is sketched in Fig. 1, where $\hat{y} \equiv A_1(s\lambda)$ is plotted versus the control parameter H. The axis $\hat{A}_1 = 0$ corresponds to the state A_ϕ, while the branch $\hat{A}_1(H)$ corresponds to the neighbouring nonlinear state $A_\phi + A_1$ that bifurcates at $H = 0$. It follows from bifurcation theory, under certain conditions, that when the state A_ϕ is stable the neighbouring state $A_\phi + A_l$ is unstable and vice versa. Thus the transition to the neighbouring state with $A_1 < 0$, $H > 0$, leads to a stable state.

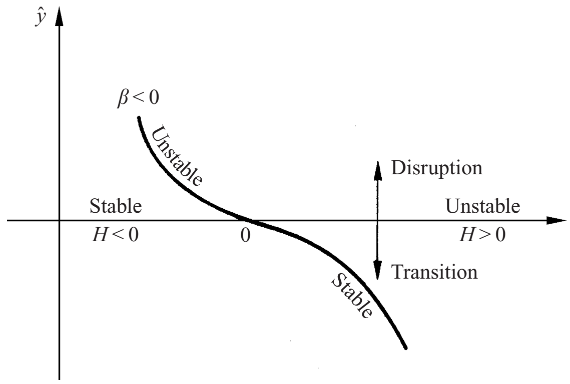

Figure 1. Sketch of the bifurcated states of (26) and of their stability properties for $\beta < 0$. The maximum amplitude $\hat{y} \equiv A_1(s\lambda)$ of the nonlinear perturbation is shown as a function of the control parameter H. The axis $\hat{y} = 0$ corresponds to the state A_ϕ. The states A_ϕ and $A_\phi + A_1$ bifurcate at $H = 0$, and a spontaneous transition $A_\phi \to A_\phi + A_1$ (indicated by the lower arrow) can occur for $H > 0$, leading to a stable state with $\hat{y} < 0$ and a higher magnetic entropy. The upward-pointing arrow corresponds to a decrease in the magnetic entropy, and does not lead to an equilibrium.

4.2. Transition to a state with higher magnetic entropy

In this subsection, we shall show that the transition $A_\phi \to A_\phi + A_1$ with $A_1 < 0$ and $\beta < 0$ indicated by the downward-pointing arrow in Fig. 1 increases the magnetic entropy S, (2). Indeed, the change in S in this transition is given by the equality (applying (27))

$$S_1 = \frac{2\pi}{|\tau|\mu^2c^2}\left[\frac{c}{2\pi}\int_{V_c} J(A_\phi)\nabla^2 A_1\, dV - \left(\frac{\mu c}{4\pi}\right)^2 \int_{V_c} (A_\phi \nabla^2 A_1 + A_1 \nabla^2 A_\phi)\, dV\right]. \quad (37)$$

Noting that

$$\int_{V_c} J\nabla^2 A_1\, dV = \int_{V_c} A_1 \nabla^2 J\, dV + \int_{\mathscr{S}} (J\boldsymbol{\nabla} A_1 - A_1 \boldsymbol{\nabla} J)\cdot d\mathscr{S}, \quad (38)$$

$$\int_{V_c} A_\phi \nabla^2 A_1\, dV = \int_{V_c} A_1 \nabla^2 A_\phi\, dV + \int_{\mathscr{S}} (A_\phi \boldsymbol{\nabla} A_1 - A_1 \boldsymbol{\nabla} A_\phi)\cdot d\mathscr{S}, \quad (39)$$

where $\mathrm{d}\mathscr{S}$ is directed outwards from V_c, one obtains from (37)

$$S_1 = \frac{1}{|\tau|\mu^2 c}\left[\int_{V_c} A_1(\nabla^2 J + \mu^2 J)\, dV + \int_{\mathscr{S}} (J\boldsymbol{\nabla} A_1 - A_1 \boldsymbol{\nabla} J)\cdot d\mathscr{S} - \frac{\mu^2 c}{8\pi}\int_{\mathscr{S}} (A_\phi \boldsymbol{\nabla} A_1 - A_1 \boldsymbol{\nabla} A_\phi)\cdot d\mathscr{S}\right]. \quad (40)$$

Taking the boundary conditions (28) into account, we have

$$\int_{\mathscr{S}} A_1 \boldsymbol{\nabla} J \cdot d\mathscr{S} = \left(A_1 B_\theta \frac{dJ}{dA_\phi}\right)_{s\lambda} \hat{\mathscr{S}}, \quad (41)$$

$$\int_{\mathscr{S}} A_1 \boldsymbol{\nabla} A_\phi \cdot d\mathscr{S} = (A_1 B_\theta)_{s\lambda} \hat{\mathscr{S}}. \quad (42)$$

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We assume that the current density j_s on the outer surface is small enough not to influence the sign of S_1. Also, consistently with (21), we take $A_\phi(s) = 0$ (with this choice, the zeroth-order magnetic energy $(1/2c)\int JA_\phi \, dV$ is equal to the poloidal magnetic energy contained in $r \leqslant s$). Using (13) and (28), (40) becomes

$$S_1 = -\frac{1}{|\tau|\mu^2 c}\left[\mu^2 \int_{V_c} A_1 \frac{p_A}{E_0} dV + \left(A_1 B_\theta \left(\frac{dJ}{dA_\phi} - \frac{\mu^2 c}{8\pi}\right)\right)_{s\lambda} \hat{\mathscr{S}}\right] \tag{43}$$

One sees that S_1 is positive for $A_1 < 0$ when the following inequality holds:

$$\left(\frac{dJ}{dA_\phi}\right)_{s\lambda} > \frac{\mu^2 c}{8\pi}. \tag{44}$$

Now this inequality is a consequence of the fact that, in the case under consideration, β is negative in a sufficiently large region of r, in particular for $r = s\lambda$, where $|\beta|$ is larger. Thus one has from (36)

$$\beta(s\lambda) < 0: \quad 1 + \frac{\hat{p}_A}{E_0 \hat{j}} - \frac{\hat{\nu}^2}{\mu^2} < 0, \tag{45}$$

from which it follows that

$$\frac{c\hat{\nu}^2}{4\pi} \equiv \left(\frac{dJ}{dA_\phi}\right)_{s\lambda} > \frac{\mu^2 c}{4\pi}\left(1 + \frac{\hat{p}_A}{E_0 \hat{j}}\right) > \frac{\mu^2 c}{8\pi}. \tag{46}$$

In conclusion, for $H > 0$ and $A_1 < 0, \beta < 0$, a new magnetic equilibrium exists, associated with a poloidal flux-conserving rearrangement of the current density, which is accessible with an increase in the magnetic entropy in a spontaneous internal transition.

The opposite case, namely an evolution where A_1 is positive, is associated with a decrease in the magnetic entropy, and can only be realized through external perturbations. But when A_1 is positive, a nonlinear equilibrium solution of (31) in the case $H > 0$ does not exist (see Fig. 1), and the magnetic system cannot evolve toward a stationary situation. This case, which is possibly related to tokamak disruption, is not discussed in this paper.

5. Thermodynamics of the L–H transition

It is now essential for physical interpretation and for comparison with observations to identify the implications of the above magnetic transition on the transport properties of the plasma and on the power balance.

The tools for this discussion are the thermodynamic relations of Sec. 3 between the rate of change of the entropy and the change of the heat flux and of the plasma energy during the transition.

5.1. Magnetic entropy production in the transition

In the transition phase, the perturbation $A_1(t)$ is time-dependent and related to the electric field by $E_1 = -(1/c)\partial A_1/\partial t$. We suppose that the transition is adiabatic in the sense that the functional dependence of $J(A)$ on its argument is preserved and that (27) holds at least approximately also when A_1 is time-dependent. We note that the perturbations leading from a state $J(A_\phi)$ to a state $J(A_\phi + A_1)$ where A_1 is independent of ϕ) minimize the magnetic energy (Laval et al. 1965; Schindler 1965).

The entropy production associated with the change A_1 is then given by the time derivative of (40) (where (27) has been used implicitly):

$$\frac{dS_1}{dt} = \frac{1}{\mu^2\tau}\left[\int_{V_c} E_1(\nabla^2 J + \mu^2 J)\, dV + \int_{\mathscr{S}} (J\nabla E_1 - E_1 \nabla J)\cdot d\mathscr{S} + \frac{\mu^2 c}{8\pi}\int_{\mathscr{S}} E_1 \nabla A_\phi \cdot d\mathscr{S}\right] = \frac{1}{\mu^2\tau}\int_{V_c} J\nabla^2 E_1\, dV + \frac{1}{\tau}\int_{V_c} E_1 J\, dV + \frac{c}{8\pi\tau}\int_{\mathscr{S}} E_1 \nabla A_\phi \cdot d\mathscr{S} \tag{47}$$

Applying Ampère's law and the induction law

$$\frac{\partial E_1}{\partial r} = \frac{1}{c}\frac{\partial B_{1\theta}}{\partial t}, \tag{48}$$

the relation (47), after a partial integration, becomes

$$\frac{dS_1}{dt} = \frac{1}{\mu^2\tau}\int_{V_c} J\nabla^2 E_1\, dV - \frac{1}{4\pi\tau}\int_{V_c} \frac{\partial B_{1\theta}}{\partial t} B_\theta\, dV - \frac{c}{8\pi\tau}\hat{E}_1 \hat{B}_\theta \hat{\mathscr{S}}. \tag{49}$$

We now use the fact that, during the transition, the magnetic configuration in the confinement zone undergoes a poloidal-flux-conserving rearrangement of the current density. This means that the new current density $J(A_\phi + A_1)$ is obtained from $J(A_\phi)$ by shifting its value at the point r to the point ρ determined by the relation

$$A_\phi(r) = A_\phi(\rho) + A_1(\rho). \tag{50}$$

Putting $\rho = r + \xi$ and linearizing, one has the following expression for the shift ξ:

$$\xi = -\frac{A_1(r,t)}{\partial A_\phi/\partial r} = \frac{A_1(r,t)}{B_\theta(r)}. \tag{51}$$

The poloidal magnetic energy in the shifting confinement region is given by the relation

$$W_\theta = \frac{1}{8\pi}\int_{s\lambda+\hat{\xi}(t)}^{s} B_\theta^2\, r R_0\, 4\pi^2\, dr, \tag{52}$$

and its time derivative is

$$\frac{dW_\theta}{dt} = \frac{1}{4\pi}\int_{s\lambda+\hat{\xi}(t)}^{s} \frac{\partial B_{1\theta}}{\partial t} B_\theta\, r R_0\, 4\pi^2\, dr - \frac{1}{8\pi}\frac{d\hat{\xi}}{dt}\hat{B}_\theta^2 \hat{\mathscr{S}}. \tag{53}$$

Noting that $d\hat{\xi}/dt = -c\hat{E}_1/\hat{B}_\theta$, one obtains from (49) the production of magnetic entropy in the confinement region associated with the electromagnetic energy change in the transition phase:

$$\frac{dS_1}{dt} = \frac{1}{\mu^2\tau}\int_{V_c} J\nabla^2 E_1\, dV - \frac{1}{\tau}\frac{dW_\theta}{dt}. \tag{54}$$

5.2. Total entropy production and power balance

In order to obtain the total entropy production, one must add to (54) the contribution (7) associated with the equilibrium electric field E and with the auxiliary power p_A, taking into account that to the current density $J = J(A_\phi)$ one must now add the perturbation $j_1 = J(A_\phi + A_1) - J(A_\phi)$, i.e. $j = J + j_1$.

Taking the form (47) for dS_1/dt, one has, up to first order,

$$\frac{dS_{\text{tot}}}{dt} = \frac{dS}{dt} + \frac{dS_1}{dt} = \frac{1}{\mu^2\tau}\int_{V_c} J\nabla^2 E_1\, dV + \frac{1}{\mu^2\tau}\int_{\mathscr{S}} (E\nabla j - j\nabla E)\cdot d\mathscr{S}$$
$$+ \frac{c}{8\pi\tau}\int_{\mathscr{S}} E_1 \nabla A_\phi \cdot d\mathscr{S} + \frac{1}{\tau}\int_{V_c} (E + E_1) j\, dV + \frac{1}{\tau}\int_{V_c} p_A\, dV. \qquad (55)$$

The physical meaning of the terms in this relation follows from comparison with the power balance in the confinement zone. The net heat dQ/dt absorbed or emitted per unit time by the plasma is expressed by the equality

$$\frac{dQ}{dt} = -\int_{\mathscr{S}} \mathbf{q}_h \cdot d\mathscr{S} + \int_{V_c} (E + E_1) j\, dV + \int_{V_c} p_A\, dV \qquad (56)$$

which reduces to (10) of section 3.1 in the case $E_1 = j_1 = 0$. Just as in this section we identify the surface terms of the power balance and of the entropy production and obtain

$$-\int_{\mathscr{S}} \mathbf{q}_h \cdot d\mathscr{S} = \frac{1}{\mu^2}\int_{\mathscr{S}} (E\nabla j - j\nabla E)\cdot d\mathscr{S} + \frac{c}{8\pi}\int_{\mathscr{S}} E_1 \nabla A_\phi \cdot d\mathscr{S}. \qquad (57)$$

Thus the expression (55) for the total entropy production becomes

$$\frac{dS_{\text{tot}}}{dt} = \frac{1}{\mu^2\tau}\int_{V_c} J\nabla^2 E_1\, dV + \frac{1}{\tau}\frac{dQ}{dt}. \qquad (58)$$

Let us introduce the entropy S_r related to the reversible process:

$$\frac{dS_r}{dt} = \frac{1}{\tau}\frac{dQ}{dt}. \qquad (59)$$

We show that

$$\frac{dS_{\text{tot}}}{dt} > \frac{dS_r}{dt}. \qquad (60)$$

Indeed, one has $E_1 = -(1/c)\partial A_1/\partial t$, where A_1, which in a linear approximation is the first eigenfunction of (29) and satisfies (28), is monotonic and negative. It follows that the space derivative of E_1 is negative for $r > s\lambda$ (it vanishes for $r = s\lambda$). Then, recalling that τ is negative, we have

$$\frac{1}{\tau}\int_{V_c} J\nabla^2 E_1\, dV = \frac{4\pi^2 R_0}{\tau}\int_{s\lambda}^{s} dr\, J\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}E_1\right)$$
$$= \frac{4\pi^2 R_0}{\tau}\left(\left| Jr\frac{\partial E_1}{\partial r}\right|_s - \int_{s\lambda}^{s} \frac{\partial E_1}{\partial r}\frac{\partial J}{\partial r}\, r\, dr\right) > 0. \qquad (61)$$

The inequality (60) then follows from (58).

5.3. The basic phenomenology of the magnetic transition

We now express (56) in a form that allows better physical insight into the change in the power balance involved in the transition. Combining (47) and (54), one obtains

$$\int_{V_c} E_1 J\, dV = -\frac{dW_\theta}{dt} - \frac{c}{8\pi}\int_{\mathscr{S}} E_1 \nabla A_\phi \cdot d\boldsymbol{\mathscr{S}}. \tag{62}$$

Substitution of (62) into (56) gives

$$\frac{dQ}{dt} = -\int_{\mathscr{S}} \mathbf{q}_h \cdot d\boldsymbol{\mathscr{S}} + \int_{V_c} Ej\, dV + \int_{V_c} p_A\, dV - \frac{dW_\theta}{dt} - \frac{c}{8\pi}\int_{\mathscr{S}} E_1 \nabla A_\phi \cdot d\boldsymbol{\mathscr{S}}. \tag{63}$$

Noting that $E_1(s) = 0$, one has

$$-\int_{\mathscr{S}} E_1 \nabla A_\phi \cdot d\boldsymbol{\mathscr{S}} = -\hat{E}_1 \hat{B}_\theta \hat{\mathscr{S}} = \int_{\hat{\mathscr{S}}} \mathbf{E}_1 \times \mathbf{B}_\theta \cdot d\boldsymbol{\mathscr{S}} \tag{64}$$

where

$$d\boldsymbol{\mathscr{S}} = \mathbf{e}_r\, d\mathscr{S}, \qquad \mathbf{E}_1 = \mathbf{e}_\phi E_1, \qquad \mathbf{B}_\theta = \mathbf{e}_\theta B_\theta.$$

Introducing (53) and (64) into (63), the final form of the power balance equation in the confinement region is obtained:

$$\frac{dQ}{dt} = -\int_{\mathscr{S}} \mathbf{q}_h \cdot d\mathscr{S} + \int_{V_c} Ej\, dV - \frac{1}{4\pi}\int_{V_c} B_\theta \frac{\partial B_{1\theta}}{\partial t} dV$$
$$+ \frac{c}{4\pi}\int_{\mathscr{S}} \mathbf{E}_1 \times \mathbf{B}_\theta \cdot d\boldsymbol{\mathscr{S}} + \int_{V_c} p_A\, dV. \tag{65}$$

Before discussing the consequences of this relation for thermal transport, some observations on the electromagnetic aspects of the transition are in order.

Noting that $B_{1\theta} = -\partial A_1/\partial r$ is negative, the poloidal magnetic energy decreases in the transition phase. This phase is characterized by a Poynting flux of energy towards the sawtooth region and an inward shift of the surface $\hat{q} = 1$ (see (51)).

Now it is essential to note that the global current of the tokamak is kept constant by an appropriate feedback of the external poloidal system. This has the relevant consequence that the decrease of the current in the confinement region must be associated with the formation of a sheet current at the edge, induced by the feedback system in order to keep the global current constant. Subsequently, the current density near the edge increases by resistive diffusion, and contributes, together with other factors, to the formation of a temperature pedestal.

At this point, our picture makes contact with those models that consider the increase of the edge current density as one of the main factors responsible for the excitation of the edge modes (i.e. the peeling modes) and for the complex phenomenon of ELMs (Manickam 1992; Huysmans et al. 1992; Connor 1998; Connor et al. 1998).

Although our thermodynamic theory is unable, by itself, to say something about the detailed dynamics, we have nevertheless reached a point of view from which, through the insight that the theory provides, a connection can be established with definite dynamical processes.

5.4. *Effect of the magnetic transition on thermal transport*

Let us now discuss the change in the transport properties of the plasma generated by the transition. We consider the asymptotic situation when the final magnetic state is reached and the electromagnetic quantities are time-independent (e.g. $E_1 \propto \partial A_1/\partial t = 0$). Taking into account the boundary condition of the heat flux on the surface $\hat{q} = 1$,

$$\int_{\hat{\mathscr{S}}} \mathbf{q}_h \cdot d\mathscr{S} = \int_{\hat{V}} p_A \, dV + \int_{\hat{V}} E\hat{j} \, dV, \tag{66}$$

(65) becomes

$$\frac{dQ}{dt} = -\int_{\mathscr{S}_s} \mathbf{q}_h \cdot d\mathscr{S} + \int_{V_p} Ej \, dV + \int_{V_p} p_A \, dV, \tag{67}$$

where, according to (57),

$$\int_{\mathscr{S}_s} \mathbf{q}_h \cdot d\mathscr{S} = -\frac{1}{\mu^2} \int_{\mathscr{S}_s} [E\boldsymbol{\nabla}(j_\phi + j_1) - (j_\phi + j_1)\boldsymbol{\nabla}E] \cdot d\mathscr{S}. \tag{68}$$

Here $\mathscr{S}_s$ is the magnetic surface at $r = s$, and $V_p = \hat{V} + V_c$. From this relation, one obtains the first-order change of the heat flux after the magnetic transition. In the cylindrical approximation, one has (recalling that $A_1(s) = 0$)

$$\begin{aligned} \left(\int_{\mathscr{S}_s} \mathbf{q}_h \cdot d\mathscr{S}\right)_1 &= -\frac{E_0}{\mu^2}\frac{dj_1}{ds}\mathscr{S}_s \approx -\frac{c}{4\pi}\frac{\nu^2(s)}{\mu^2} E_0 \frac{dA_1}{ds}\mathscr{S}_s \\ &= -\frac{c}{4\pi}\frac{\nu^2(s)}{\mu^2} E_0 |B_{1\theta}(s)| \, \mathscr{S}_s < 0, \end{aligned} \tag{69}$$

which shows that the heat flux across the outer edge of the confinement region decreases after the transition.

The conservation of energy implies that the net heat absorbed or emitted per unit time by the plasma is identified with the rate of change of the plasma energy

$$\frac{3}{2}\frac{d}{dt}\int_{V_p} nT \, dV = \frac{dQ}{dt}. \tag{70}$$

If the system is in thermal equilibrium before the transition, the rate of change of the plasma energy after the transition is given by the first-order terms of the power balance equation. We have

$$\begin{aligned} \frac{3}{2}\frac{d}{dt}\int_{V_p} nT \, dV &= -\left(\int_{\mathscr{S}_s} \mathbf{q}h \cdot d\mathscr{S}\right)_1 + \int_{V_p} Ej_1 \, dV \\ &\approx -\left(\int_{\mathscr{S}_s} \mathbf{q}_h \cdot d\mathscr{S}\right)_1 - \frac{c}{2}\pi R_0 E_0 s \frac{dA_1}{ds}, \end{aligned} \tag{71}$$

where j_1 has been expressed with (27).

The following inequality can be seen to hold using (35):

$$\frac{\nu^2(s)}{\mu^2} > 1 + \frac{q_s}{s_2}\frac{1}{\pi c B E_0}[P_A(s) - P_A(s\lambda)] \gg 1. \tag{72}$$

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The term involving the heat flux in (71) is then the leading term, and one has

$$\frac{3}{2}\frac{d}{dt}\int_{V_p} nT\,dV \approx \frac{c}{4\pi}\frac{\nu^2(s)}{\mu^2}E_0|B_{1\theta}(s)|S_s$$

$$> \frac{R_0}{s}q_s\frac{|B_{1\theta}(s)|}{B}[P_A(s)-P_A(s\lambda)]$$

$$= \frac{|B_{1\theta}(s)|}{B_\theta(s)}[P_A(s)-P_A(s\lambda)]. \tag{73}$$

For the validity of the theory, $B_{1\theta}$ should be at least one order of magnitude lower than B_θ, but still the rate of change of the plasma energy, generated purely by the electromagnetic transition, may be a practically significant fraction of the deposited power.

Clearly the plasma is not in thermal equilibrium immediately after the transition, although a magnetic equilibrium has been reached.

6. Transition thresholds

In this section, we shall extend the results obtained in a previous paper (Minardi 1997) on the power threshold by deriving the scaling of the average temperature in the confinement region at the threshold point, the hysteresis of the inverse transition and the upper density limit of the H state.

In order to proceed with our discussion, we recall, for the convenience of the reader, the procedure followed in the 1997 paper for deriving the power transition threshold.

6.1. Transformations leaving H invariant

The threshold is the bifurcation point $H = 0$ of the magnetic equilibrium. A scaling law for the threshold values of physical quantities is a relation between the quantities such that H is invariant when the values change consistently with the relation.

The eigenvalue H is fixed by (29) with the boundary conditions (28), and is then controlled by the function $[s\nu(r)]^2$ given by (35). Any relation that leaves $[s\nu(r)]^2$ and the boundary conditions invariant is then a scaling law of the threshold values because it leaves $H = 0$ invariant.

In order to simplify the discussion, we consider the physical case where the auxiliary power largely exceeds the Ohmic power, $p_A \gg E\hat{j}$. In this case, the equation (13) for the initial magnetic equilibrium becomes

$$\nabla^2 j_\phi = -\frac{\mu^2 p_A}{E_0\hat{j}}, \tag{74}$$

whose solution is

$$j_\phi - j_s = \frac{\mu^2}{E_0}\int_r^s \frac{dr}{r}\int_0^r p_A\, r\,dr + C_\phi \ln\frac{r}{s}, \tag{75}$$

where C_ϕ is determined by the condition $j_\phi(s\lambda) = \hat{j}$:

$$C_\phi = \frac{\hat{j}}{\ln\lambda}\left(1 - \frac{j_s}{\hat{j}} - \frac{\mu^2}{E_0\hat{j}}\int_{s\lambda}^s \frac{dr}{r}\int_0^r p_A\, r\,dr\right\}. \tag{76}$$

Taking as before

$$j_\phi = \frac{E_0}{A(Z_{\text{eff}})} T^{3/2} + C_\phi \ln\frac{r}{s}, \tag{77}$$

(74) reduces to the power balance equation with the thermal conductivity (16) and T is expressed as follows:

$$\frac{E_0}{A(Z_{\text{eff}})} T^{3/2} = j_s + \frac{\mu^2}{E_0}\int_r^s \frac{dr}{r}\int_0^r p_A\, r\, dr. \tag{78}$$

From this, one derives the relations

$$\left(\frac{\hat{T}}{T_s}\right)^{3/2} = \frac{\hat{j}}{j_s}\delta, \tag{79a}$$

$$\frac{E_0}{A(Z_{\text{eff}})}\hat{T}^{3/2} = \hat{j}\delta = \frac{cB}{2\pi R_0}\delta. \tag{79b}$$

Here $T_s = T(s)$ and δ, given by the equality

$$\delta = \frac{j_s}{\hat{j}} + \frac{\mu^2}{E_0\hat{j}}\int_{s\lambda}^s \frac{dr}{r}\int_0^r p_A\, r\, dr, \tag{80}$$

is a measure of the deviation from Ohmic relaxation on the surface $\hat{q} = 1$.

Recalling (75), $(s\nu)^2$ takes the form

$$(s\nu)^2 = \frac{I_s q_s}{I(x)}\left[\frac{2}{\ln\lambda}\quad\left(\frac{j_s}{\hat{j}} - 1\right) + \frac{\mu^2 P_A(1)}{cBE_0\pi}G(x)\right], \tag{81}$$

where

$$G(x) = \frac{1}{P_A(1)}\left[P_A(x) + \int_\lambda^1 \frac{dx}{x}P_A(x)\right], \tag{82}$$

$$I(x) = 2\pi s^2\hat{j}\left(\frac{\lambda^2}{2} + \int_\lambda^x \frac{j_\phi}{\hat{j}}x\, dx\right). \tag{83}$$

Here $x = r/s$, $P_A(x)$ is the total auxiliary power injected in the region $r \leqslant xs$, $I(x)$ is the toroidal current flowing in this region, $I_s = I(x=1)$ and $q_s = cs^2B/2R_0 I_s$.

Inspection of (81) shows that $(\nu s)^2$ is invariant when P_A, B, E_0, I and s change in such a way as to leave invariant q_s and the combination

$$\frac{\mu^2 P_A}{cBE_0} = C_0, \tag{84}$$

where C_0 is a dimensionless constant that corresponds to the bifurcation point $H = 0$ for assigned values of q_s, λ and $j_s/\hat{j}$, compatible with the initial equilibrium (in the 1997 paper, we considered an example with $C_0 = 5.786$, $\mu s = 0.45$ and $j_s/\hat{j} = 0.18$).

The quantities in (84) are not independent. Indeed, on one hand, one has the condition (79b), which relates E_0, B and T for fixed $j_s/\hat{j}$, where, at the threshold point, δ takes the form

$$\delta = \frac{j_s}{\hat{j}} + \frac{C_0}{2\pi}\int_\lambda^1 \frac{dx}{x}\frac{P_A(x)}{P_A(1)} \tag{85}$$

(δ is of order unity). On the other hand, we have to take into account the condition imposed on T by the energy balance. Applying the procedure of the 1997 paper, taking into account the conditions above, (84) can be converted into the following relation:

$$\frac{\mu^4 P_A^5 \tau_L^3}{B^4 R A^2 (Z_{\text{eff}}) a^6 n^3} = c^4 C_m \delta^{(2m+1)/(1+m)}, \tag{86}$$

where $m(\frac{1}{2} \leqslant m \leqslant 3)$ is the exponent of the power dependence T^m of the thermal conductivity on the temperature, τ_L is the energy confinement time of the L state, n is the average plasma density, a is the minor radius and C_m is given by

$$C_m = [C_0^{5+2m} \pi^{4m+1} 3^{3m} (\tfrac{1}{4})^3]^{1/(1+m)}.$$

6.2. Similarity transformations and transition thresholds

Let us express the confinement time of the L state with the empirical scaling law conventionally known as ITER89-P (Yushmanov et al. 1990):

$$\tau_L = 0.048 A_i^{0.5} I^{0.85} R^{1.2} a^{0.3} n_{20}^{0.1} B^{0.2} P_A^{-0.5}, \tag{87}$$

where A_i is the mass number, and I is expressed in MA, R and a in m, the average plasma density n_{20} in 10^{-20} m^{-3}, B in T, P_A in MW and τ_L in s. Introducing (87) into (86), one obtains

$$P_A = P_{20} n_{20}^{0.77} B^{0.97}, \tag{88}$$

where P_{20} can be written in the following two forms:

$$P_{20} = 3.62 \times 10^{-3} \frac{a^{1.46} Z_{\text{eff}}^{0.57}}{\mu^{1.14} A_i^{0.43} I^{0.73} R^{0.74}} C_m^{0.285} \delta^{2(2m-1)/7(m+1)} \tag{89a}$$

$$= 0.27 \frac{a^{1.46} Z_{\text{eff}}^{0.57}}{(\mu s)^{1.14} A_i^{0.43} I_s^{0.16} R^{0.17}} \left(\frac{q_s}{B}\right)^{0.57} C_m^{0.285} \delta^{2(2m-1)/7(m+1)}. \tag{89b}$$

Here $q_s = 5Bs^2/I_s R$ in the engineering units used above. The first form (89a) (where μ is measured in cm^{-1}) shows that P_{20} is independent of B and n. The scaling of P_A with respect to these quantities is then entirely contained in (88) and is consistent with observations (Takizuka et al. 1997; Ryter et al. 1996). The second form (89b) shows that P_{20} depends very slowly on the current I provided that I can be practically identified with I_s and the change of the current $I_s \to I_{s'}$ is associated:

(i) with a shift of the invariant value q_s of the safety factor from the surface $r = s$ to the surface $r = s'$ related to s by $s^2/I_s = s'^2/I_{s'}$; and
(ii) with a change of μ such that μs is fixed.

This is a similarity transformation that leaves (81) and (84) (and then H) invariant and which has a significant physical implication, as we shall see presently. Indeed, the bifurcation condition (84) can be written in the form

$$\frac{(\mu s)^2 P_A}{c E_0 s^2 B} = \frac{(\mu s)^2 \pi}{q_s} \frac{P_A}{P_\Omega} = C_0, \tag{90}$$

where $q_s = cBs^2/2I_s R$ and $P_\Omega = 2\pi R E_0 I_s$ is the Ohmic power. It follows from this relation that the ratio P_A/P_Ω between the auxiliary power and the Ohmic power in the region $q \leqslant q_s$ (defined with the invariant value q_s) remains constant at the bifurcation point while the current is changing under the above transformation that preserves the values of q_s and μs. It must be noted that the electric field E_0 is not invariant but depends on μ, as can be seen after elimination of $\hat{T}^{3/2}$ from (79b) using the power balance equation (see (5.7) of the Minardi (1997), where δ has been put equal to 1).

In accordance with the above considerations, a transformation in which both B and I_s change should be performed in two steps. First, B is changed keeping I_s and μ fixed while s is changed in such a way that Bs^2 remains constant, namely the invariant value q_s is shifted from s to $s' = s(B/B')^{1/2}$. In this first step, the first form (89a) of P_{20} must be applied. Secondly, B, s^2/I_s and μs are kept fixed and I_s is changed using the second form (89b).

The preceding considerations entail natural implications on the behaviour of the temperature at the threshold point. First we observe that the threshold value of $\hat{T}/T_s$, given by (79a) where δ is given by (85), is anchored only to the ratio $\hat{j}/j_s$ between the boundary values of the current density (the power deposition profile involved in (85) is smoothed out by a double integration). This fact may be related to the stiffness of the T profile observed in the H state (Gruber et al. 1997).

The scaling and the value of the average temperature at the threshold point can be estimated by substituting the threshold value of the power into the expression for the confinement time

$$\tau_L = \frac{3nTV}{2P_A}, \tag{91}$$

where $V = 2\pi a^2 R$ and τ_L is given by (87). Here we assume, among other things, that the profiles do not change when n and T vary, otherwise τ_L would depend on the form factors. This profile resiliency is validated implicitly by the very existence of substantially universal scaling laws that involve only global or averaged quantities.

The following expressions are obtained from (91) for a given machine, i.e. for fixed values of R and a. For transformations of B with fixed I,

$$T_{ev} = 6 \times 10^{-3} n_{20}^{-0.52} B^{0.69} I^{0.49} R^{-0.17} a^{-0.97} Z_{\mathrm{eff}}^{0.28} A_i^{0.29} C_m^{0.14} \delta^{(2m-1)/7(m+1)} \mu^{-0.57} \tag{92}$$

(with μ in cm^{-1}). For transformations of I with fixed B,

$$T_{ev} = 53 n_{20}^{-0.52} B^{0.69} I^{0.77} R^{0.12} a^{-0.97} Z_{\mathrm{eff}}^{0.28} A_i^{0.29} C_m^{0.14} \delta^{(2m-1)/7(m+1)} \left(\frac{q_s}{B}\right)^{0.28} (\mu s)^{-0.57}. \tag{93}$$

The average density n_{20} can be expressed in terms of the edge density n_{ea} through the relation $n_{20} \propto n_{ea}^{0.48} B^{0.05}$ (Righi et al. 1998). One gets $T \propto n_{ea}^{-0.25} B^{0.66} I^{0.77}$, which should be compared with the observations in Asdex-U ($T \propto n_{ea}^{-0.30} B^{0.80} I^{0.50}$; Ryter et al. 1998). Note that these observations concern the values of the temperature at the top of the pedestal rather than the average temperature. However, the two values should be related, in view of the stiffness of the temperature profile.

It is worth stressing the fact that the scaling laws derived above are a consequence of the confinement time combined with the bifurcation condition

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(86) (or (84)). Thus, given the bifurcation condition, a theory that explains the confinement time is also an explanation of the temperature and power thresholds, and vice versa.

6.3. Hysteresis and upper density limit

According to the bifurcation condition (86), the threshold power decreases when the confinement time increases at constant density. Let us indicate with subscripts L and H the quantities in the L and H states respectively. One has from (86)

$$\frac{P_{AH}}{P_{AL}} = \left(\frac{\tau_L n_H}{\tau_H n_L}\right)^{3/5}. \tag{94}$$

In the H state, the confinement time is larger, $\tau_H > \tau_L$, and since the density is the same immediately after the transition, the power threshold in the H state, P_{AH}, is lower than that in the L state, P_{AL}. However, the density n_H increases in the H state, and so does P_{AH}. The critical density value n_{Hc} above which the H state reverses to the L state is reached when $P_{AH} = P_{AL}$:

$$n_{Hc} = n_L \frac{\tau_H}{\tau_L}. \tag{95}$$

An estimate of n_{Hc} can be obtained from the known empirical scalings of τ_H.

The above considerations are consistent with the general belief that confinement plays a role in the phenomenology of the H state. At this point, we are confronted with the conventional view that the L–H transition is related to some important change in the transport properties at the edge. An adaptation process of the thermal transport at the edge is undoubtedly necessary for sustaining the increasing temperature of the H state in the confinement region. But the process seems a consequence of the evolution in this region imposed by the transition to a higher-entropy state, rather than the cause of it. Of course, it may happen that the conditions at the outer boundary of the confinement region are unable to sustain the transition in the interior. The determination of the edge conditions compatible with the transition is still an open problem, but much experimental and theoretical progress has been made in this field.

7. Conclusions

Among the infinite number of possible magnetic configurations of a tokamak, considered as an open system in equilibrium with external energy interactions, the configurations that can be preferentially realized are associated with a stationary magnetic entropy, a concept that measures the probability of the magnetic configurations.

When a bifurcation between magnetic states arises, the branch spontaneously accessible to the plasma is associated with a transition toward a higher magnetic entropy.

The thermodynamic relations between the entropy produced in the transition, the heat transport and the plasma energy allow the formulation of a quantitative scheme in which the most typical aspects of the phenomenology of the L–H transition and the relevant properties of the H state can be considered from a unitary point of view.

The H state, in comparison with the L state, has a higher entropy and a lower poloidal magnetic energy in the confinement region. In a constant-current tokamak regime, the magnetic transition in the interior involves an increase in the current density at the edge. After the transition, the state is in magnetic equilibrium but not in thermal equilibrium, and the plasma energy content increases according to an explicitly given relation. The transition thresholds, the hysteresis and the upper density limit are essentially determined by the confinement time through the bifurcation condition.

The general quantitative relations that we have derived in the framework of thermodynamics are independent of specific dynamical assumptions on the mechanism underlying the transport properties of the L and H states and the occurrence of the transition. However, while a detailed dynamical description is beyond the scope of thermodynamics, a general scheme of the transition process has been provided in which a dynamical approach should be inscribed. In this scheme, the processes at the edge cannot be considered independently from those in the confinement region, which enact the transition, but rather they should be interpreted as the reaction of the edge to these processes, directed to the preservation of the higher-entropy magnetic configuration formed in the interior.

Appendix. Entropy production in an Ohmic tokamak

We consider a plasma in contact with a very thin conducting shell of effective radius r_s. In the expression (2) for the magnetic entropy, the integration domain Ω includes the shell. The current density $\mathbf{j} = \mathbf{j}_p + \mathbf{j}_s$ is formed by a stationary current density $\mathbf{j}_p$ in the plasma volume V_p and by a time-dependent current $\mathbf{j}_s$ localized in the shell, which simulates the current in the primary of the Ohmic transformer. The time-dependent magnetic field $\mathbf{B}_e$ created by $\mathbf{j}_s$ outside the shell induces an electric field $\mathbf{E}$ inside the plasma. $\mathbf{E}$ vanishes in the conducting shell, while $\mathbf{E} \neq 0$ with $\nabla \times \mathbf{E} = 0$ in the plasma. The surface of radius r_s is then a surface of discontinuity for $\mathbf{E}$ and $\partial \mathbf{B}_e/\partial t$. The time derivative of the shell current satisfies the relation

$$\frac{4\pi}{c}\frac{\partial \mathbf{j}_s}{\partial t} = \delta(r-r_s)\mathbf{e}_n \times \frac{\partial \mathbf{B}_e}{\partial t}, \tag{A 1}$$

where $\mathbf{e}_n$ is the surface unit vector directed outwards. Then the time derivative of S, (2), is given by the expression

$$\begin{aligned}\frac{dS}{dt} &= \frac{2\pi}{(\mu c)^2 \tau}\left(\int_\Omega 2\mathbf{j}_p \cdot \frac{\partial \mathbf{j}_s}{\partial t} dV - \frac{\mu^2 c}{4\pi}\int_\Omega \mathbf{j}_p \cdot \frac{\partial \mathbf{A}_s}{\partial t} dV - \frac{\mu^2 c}{4\pi}\int_\Omega \frac{\partial \mathbf{j}_s}{\partial t}\cdot \mathbf{A}_p \, dV\right) \\ &= \frac{2\pi}{(\mu c)^2 \tau}\left[\frac{c}{2\pi}\int_{\mathscr{S}} \mathbf{j}_p \cdot d\mathscr{S} \times \frac{\partial \mathbf{B}_e}{\partial t} + \frac{(\mu c)^2}{4\pi}\int_{V_p} \mathbf{j}_p \cdot \mathbf{E}\, dV \right. \\ &\quad \left. - \left(\frac{\mu c}{4\pi}\right)^2 \int_{\mathscr{S}} d\mathscr{S} \times \frac{\partial \mathbf{B}_e}{\partial t}\cdot \mathbf{A}_p\right],\end{aligned} \tag{A 2}$$

where $\mathbf{A}_p$ and $\mathbf{A}_s$ are the vector potentials related to $\mathbf{j}_p$ and $\mathbf{j}_s$,

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}_s}{\partial t}, \qquad d\mathscr{S} = \mathbf{e}_n \, d\mathscr{S}, \qquad dV = dr\, d\mathscr{S}.$$

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The volume integral is extended to the plasma volume V_p, excluding the shell because $\mathbf{E}$ vanishes there (a quadratic term in $\mathbf{j}_s$, which involves the square of a delta function, can be shown to vanish when calculated properly with the theory of distributions).

The surface $\mathscr{S}$ must be thought as infinitesimally close to the outer side of the discontinuity surface $\mathscr{S}_s$ with radius r_s. The electric field $\mathbf{E}$ vanishes on $\mathscr{S}$ (since $\mathscr{S}$ is inside the conducting shell) but not on $\mathscr{S}_s$, and jumps to zero across $\mathscr{S}_s$ in the direction d$\mathscr{S}$. Thus we can write

$$\int_{\mathscr{S}} (\mathbf{E}\times\mathbf{B}_p)\cdot d\mathscr{S} = \int_{\mathscr{S}_s} (\mathbf{E}\times\mathbf{B}_p)\cdot d\mathscr{S} + \int_{\mathscr{S}} [\![(\mathbf{E}\times\mathbf{B}_p)]\!]\cdot d\mathscr{S} = 0, \tag{A 3a}$$

$$\int_{\mathscr{S}} [\![(\mathbf{E}\times\mathbf{B}_p)]\!]\cdot d\mathscr{S} = -\int_{\mathscr{S}} \mathbf{B}_p\cdot d\mathscr{S}\times[\![\mathbf{E}]\!] = -\int_{\mathscr{S}} (\nabla\times\mathbf{A}_p)\cdot d\mathscr{S}\times[\![\mathbf{E}]\!], \tag{A 3b}$$

where $[\![\;]\!]$ denotes the jump; $[\![\mathbf{E}]\!] = -\mathbf{E}$.

Let us consider an infinitesimal volume element $\Delta\tau$ with vanishing thickness and bases of area $d\mathscr{S}$ situated at opposite sides of the discontinuity surface $\mathscr{S}_s$ and parallel to it. Applying the general coordinate-free definition of the curl, one has at the discontinuity surface

$$\frac{1}{c}\frac{\partial\mathbf{B}_e}{\partial t} = -\nabla\times\mathbf{E} = -\frac{1}{\Delta\tau}\oint_{\mathscr{S}_\tau} d\mathscr{S}\times\mathbf{E} = -\frac{1}{\Delta\tau}d\mathscr{S}\times[\![\mathbf{E}]\!], \tag{A 4}$$

where $\mathscr{S}_\tau$ is the surface of the volume element $\Delta\tau$.

The magnetic field $\mathbf{B}_p$ created by the current in the plasma does not penetrate into the conducting shell. Then $\mathbf{A}_p$ jumps to zero across $\mathscr{S}_s$, and we have

$$\nabla\times\mathbf{A}_p = \frac{1}{\Delta\tau}\oint_{\mathscr{S}_\tau} d\mathscr{S}\times\mathbf{A}_p = \frac{1}{\Delta\tau}d\mathscr{S}\times[\![\mathbf{A}_p]\!] = -\frac{1}{\Delta\tau}d\mathscr{S}\times\mathbf{A}_p. \tag{A 5}$$

Combining (A 4) and (A 5), (A 3) becomes

$$\int_{\mathscr{S}_s} (\mathbf{E}\times\mathbf{B}_p)\cdot d\mathscr{S} - \frac{1}{c}\int_{\mathscr{S}} \mathbf{A}_p\cdot d\mathscr{S}\times\frac{\partial\mathbf{B}_e}{\partial t}. \tag{A 6}$$

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Noting that (recalling that $\nabla \times \mathbf{E} = 0$ in V_p)

$$\int_{\mathscr{S}_s} (\mathbf{E} \times \mathbf{B}_p) \cdot d\mathscr{S} = -\frac{4\pi}{c} \int_{V_p} \mathbf{E} \cdot \mathbf{j}_p \, dV, \tag{A 7}$$

we obtain from (A 6)

$$\frac{1}{c} \int_{\mathscr{S}} \mathbf{A}_p \cdot d\mathscr{S} \times \frac{\partial \mathbf{B}_e}{\partial t} = -\frac{4\pi}{c} \int_{V_p} \mathbf{E} \cdot \mathbf{j}_p \, dV. \tag{A 8}$$

Furthermore, we want to show that

$$\frac{1}{c} \int_{\mathscr{S}} \mathbf{j}_p \cdot d\mathscr{S} \times \frac{\partial \mathbf{B}_e}{\partial t} = \int_{V_p} \mathbf{E} \cdot \nabla^2 \mathbf{j}_p \, dV. \tag{A 9}$$

We start from the relation

$$\begin{aligned} \int_{V_p} \mathbf{E} \cdot \nabla^2 \mathbf{j}_p \, dV &= -\int_{V_p} \mathbf{E} \cdot \nabla \times \nabla \times \mathbf{j}_p \, dV \\ &= \int_{\mathscr{S}_s} (\mathbf{E} \times \nabla \times \mathbf{j}_p) \cdot d\mathscr{S}, \end{aligned} \tag{A 10}$$

and observe that, similarly to (A 3), one can write

$$\begin{aligned} \int_{\mathscr{S}} (\mathbf{E} \times \nabla \times \mathbf{j}_p) \cdot d\mathscr{S} &= \int_{\mathscr{S}_s} (\mathbf{E} \times \nabla \times \mathbf{j}_p) \cdot d\mathscr{S} + \int_{\mathscr{S}} [\![\mathbf{E} \times \nabla \times \mathbf{j}_p]\!] \cdot d\mathscr{S} \\ &= 0, \end{aligned} \tag{A 11a}$$

$$\int_{\mathscr{S}} [\![\mathbf{E} \times \nabla \times \mathbf{j}_p]\!] \cdot d\mathscr{S} = \int_{\mathscr{S}} (\nabla \times \mathbf{j}_p) \cdot d\mathscr{S} \times [\![\mathbf{E}]\!] \tag{A 11b}$$

But $\mathbf{j}_p$ jumps to zero on crossing $\mathscr{S}_s$ in the direction of $d\mathscr{S}$:

$$\begin{aligned} \nabla \times \mathbf{j}_p &= \frac{1}{\Delta\tau} \oint_{\mathscr{S}_\tau} d\mathscr{S} \times \mathbf{j}_p \\ &= \frac{1}{\Delta\tau} d\mathscr{S} \times [\![\mathbf{j}_p]\!] \\ &= -\frac{1}{\Delta\tau} d\mathscr{S} \times \mathbf{j}_p. \end{aligned} \tag{A 12}$$

Combining (A 10)–(A 12) and (A 4), one obtains the desired relation (A 9):

$$\begin{aligned} \int_{V_p} \mathbf{E} \cdot \nabla^2 \mathbf{j}_p \, dV &= \int_{\mathscr{S}_s} (\mathbf{E} \times \nabla \times \mathbf{j}_p) \cdot d\mathscr{S} \\ &= -\int_{\mathscr{S}} (\nabla \times \mathbf{j}_p) \cdot d\mathscr{S} \times [\![\mathbf{E}]\!] \\ &= \frac{1}{c} \int_{\mathscr{S}} \mathbf{j}_p \cdot d\mathscr{S} \times \frac{\partial \mathbf{B}_e}{\partial t}. \end{aligned} \tag{A 13}$$

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Finally, inserting (A 8) and (A 9) into (A 2), we recover the relation (6) of the text:

$$\frac{dS}{dt} = \frac{1}{\tau}\int_{V_p} \frac{\mathbf{E}}{\mu^2}(\nabla^2 \mathbf{j}_p + \mu^2 \mathbf{j}_p)\, dV. \qquad \text{(A 14)}$$

References

Connor, J. W. 1998 *Plasma Phys. Contr. Fusion* **40**, 191.

Connor, J. W., Hastie, R. J., Wilson, H. R. and Miller, R. L. 1998 *Phys. Plasmas* **5**, 2687.

Garrett, A. J. M. 1991 In: *Maximum Entropy in Action* (ed. B. Buck and V. A. Macaulay), p. 139. Clarendon Press, Oxford.

Gruber, O. and ASDEX-Upgrade Team 1997 *Plasma Phys. Contr. Fusion* **39**, B19.

Gull, S. F. 1991 In: *Maximum Entropy in Action* (ed. B. Buck and V. A. Macaulay), p. 171. Clarendon Press, Oxford.

Huysmans, G. T. A., de Blanck, H. J., Kerner, W., Goedbloed, J. P. and Nave M. F. F. 1992 In: *Proceedings of 19th EPS Conference on Controlled Fusion and Plasma Physics* (ed. W. Freysinger, K. Lackner, R. Schrittwieser and W. Lindinger), Part I, p. 247. European Physical Society, Geneva.

Inoue, S., Itoh, K. and Terashima, Y. 1980 *J. Phys. Soc. Jpn* **48**, 264.

Itoh, K., Itoh, S.-I. and Perkins, F. W. 1986 *Comments Plasma Phys. Contr. Fusion* **10**, 103.

Jaynes, E. T. 1989 In: *Maximum Entropy and Bayesian Methods* (ed. J. Skilling), p. 1. Kluwer, Dordrecht.

Laval, G., Pellat, R. and Vuillemin, M. 1965 In: *Proceedings of International IAEA Fusion Energy Conference, Culham, 1965*, Vol. 2, p. 259. International Atomic Energy Agency, Vienna.

Lazzaro, E. and Minardi, E. 1999 Privileged negative or low magnetic shear equilibria in auxiliary heated tokamaks. Submitted to *J. Plasma Phys.*

Manickam, J. 1992 *Phys. Fluids* **B4**, 1901.

Minardi, E. 1989 *Plasma Phys. Contr. Fusion* **31**, 229.

Minardi, E. 1992a *J. Plasma Phys.* **48**, 281.

Minardi, E. 1992b *J. Plasma Phys.* **48**, 309.

Minardi, E. 1992c *Plasma Phys. Contr. Fusion* **34**, 989.

Minardi, E. 1993 *J. Plasma Phys.* **50**, 505.

Minardi, E. 1997 *J. Plasma Phys.* **57**, 449.

Ohkawa, T. 1978 *Phys. Lett.* **67A**, 35.

Righi, E. and H-Mode Data Base Working Group 1998 *Plasma Phys. Contr. Fusion* **40**, 857.

Ryter, F. and H-Mode Data Base Working Group 1996 *Nucl. Fusion* **36**, 1217.

Ryter, F., Suttrop, W., Bruesenhaber, B., Kaufmann, M., Mertens, V., Murmann, H., Peeters, A. G., Stober, J., Schweinzer, J., Zohm, H. and ASDEX-Upgrade Team 1998 *Plasma Phys. Contr. Fusion* **40**, 725.

Schindler, K. 1965 In: *Proceedings of International Conference on Phenomena in Ionized Gases* (ed. B. Perović and D. Tosić), Vol. 2, p. 736 Gradevinska Kniga, Beograd.

Takizuka, T. and ITER Confinement Database and Modelling Expert Group 1997 In: *Proceedings of 19th International IAEA Fusion Energy Conference, Montreal, 1996*, Vol. 2, p. 795.

Taylor, J. B. 1974 *Phys. Rev. Lett.* **33**, 1139.

Yushmanov, P. N., Takizuka, T., Reidel, K. S., Kardaun, O. J. W. F., Cordey, J. C., Kaye, S. M. and Post, D. E. 1990 *Nucl. Fusion* **30**, 1999.