

# Infinitely presented permutation stable groups and invariant random subgroups of metabelian groups

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(Received 26 November 2019 and accepted in revised form 27 January 2021)

*Abstract.* We prove that all invariant random subgroups of the lamplighter group  $L$  are co-sofic. It follows that  $L$  is permutation stable, providing an example of an infinitely presented such group. Our proof applies more generally to all permutational wreath products of finitely generated abelian groups. We rely on the pointwise ergodic theorem for amenable groups.

Key words: stability, invariant random groups, wreath products, metabelian groups, amenable groups

2020 Mathematics Subject Classification: 20F69, 20F16, 20P02, 37A99 (Primary)

## 1. Introduction

Let  $S(n)$  denote the symmetric group of degree  $n \in \mathbb{N}$  with the bi-invariant Hamming metric  $d_n$  given by

$$d_n(\sigma, \tau) = 1 - \frac{1}{n} |\text{Fix}(\sigma^{-1}\tau)|.$$

Let  $G$  be a finitely generated group. An *almost homomorphism* of  $G$  is a sequence of set-theoretic maps  $f_n : G \rightarrow S(n)$  satisfying

$$d_n(f_n(g)f_n(h), f_n(gh)) \xrightarrow{n \rightarrow \infty} 0, \quad \text{for all } g, h \in G.$$

The almost homomorphism  $f_n$  is *close to a homomorphism* if there is a sequence of group homomorphisms  $\rho_n : G \rightarrow S(n)$  satisfying

$$d_n(\rho_n(g), f_n(g)) \xrightarrow{n \rightarrow \infty} 0 \quad \text{for all } g \in G.$$

*Definition 1.1.* The group  $G$  is *permutation stable* (or *P-stable* for short) if every almost homomorphism of  $G$  is close to a homomorphism.

Various notions of stability have been considered in the literature; see [AP15, DCGLT17, Tho18]. In this paper we consider permutation stability. In recent years there has been a growing interest in this kind of stability, originating in the study of ‘almost solutions’ to group-theoretic equations [GR09]. Interestingly, permutation stability provides a possible approach to tackle the seminal problem of whether there are non-sofic groups; see [AP15, BLT19, BB19, GR09, Tho18] and the references therein.

Until quite recently only very few groups were known to be permutation stable: free groups (trivially), finite groups [GR09] and abelian groups [AP15]. The situation was dramatically changed in [BLT19]. That work established a connection between permutation stability and invariant random subgroups of a given amenable group  $G$ .

An invariant random subgroup is a conjugation invariant probability measure on the space of all subgroups of  $G$ . An invariant random subgroup  $\mu$  is called *co-sofic* if  $\mu$  is a limit of invariant random subgroups supported on finite index subgroups. (See §2 below for a more detailed discussion of invariant random subgroups.)

**THEOREM 1.2.** [BLT19] *A finitely generated amenable group  $G$  is permutation stable if and only if every invariant random subgroup of  $G$  is co-sofic.*

Proving that every invariant random subgroup is co-sofic is also of interest for general groups. For example, the Aldous–Lyons conjecture [AL+07] asserts that this is the case for free groups. Another example is the Stuck–Zimmer theorem [SZ94] proving this for all high rank lattices with property (T). See [Gel15] for details.

The significance of Theorem 1.2 is in transforming the question of permutation stability for amenable groups into the realm of invariant random subgroups. This new viewpoint enabled the authors of [BLT19] to give many examples of permutation stable groups, for example the polycyclic-by-finite ones as well as the Baumslag–Solitar groups  $B(1, n)$  for all  $n \in \mathbb{Z}$ . In these examples it is relatively straightforward to classify all invariant random subgroups and, in particular, show all are co-sofic.

If one considers slightly more complicated solvable groups, such as the lamplighter group  $\mathbb{Z}_2 \wr \mathbb{Z}$ , the answer is no longer immediately clear. A detailed study of the invariant random subgroups of  $\mathbb{Z}_2 \wr \mathbb{Z}$  was performed in [BGK15, GK14, HT15] but towards different sets of goals. The main goal of this paper is to prove the following theorem.

**THEOREM 1.3.** *Every invariant random subgroup of the lamplighter group  $\mathbb{Z}_2 \wr \mathbb{Z}$  is co-sofic and so the group  $\mathbb{Z}_2 \wr \mathbb{Z}$  is permutation stable.*

**1.1. Permutational wreath products.** Our results are in fact more general. Namely, let  $Q$  and  $B$  be finitely generated abelian groups. Let  $X$  be a set admitting an action of the group  $Q$  with finitely many orbits. The *permutational wreath product*  $B \wr_X Q$  corresponding to this  $Q$ -action and to the base group  $B$  is the semidirect product  $Q \ltimes \bigoplus_{x \in X} B$ . The group  $Q$  acts on the normal subgroup  $\bigoplus_{x \in X} B$  by group automorphisms permuting coordinates.

**THEOREM 1.4.** *Let  $G$  be a permutational wreath product of two finitely generated abelian groups. Every invariant random subgroup of  $G$  is co-sofic, and hence  $G$  is permutation stable.*

The standard wreath product  $G = B \wr Q$  of the two finitely generated abelian groups  $B$  and  $Q$  is finitely presented if and only if the group  $Q$  is finite [DC06]. Therefore Theorems 1.3 and 1.4 provide the first known examples of infinitely presented groups which are permutation stable†. In a subsequent paper [LL19] we show that there exist uncountably many non-solvable such groups.

*Remark 1.5.* The reader may find it easier to assume that  $G$  is the lamplighter group  $\mathbb{Z}_2 \wr \mathbb{Z}$  upon the first reading of this paper, and especially so in §10.

**1.2. An explicit example.** For the sake of illustrating Theorem 1.4 we provide an explicit application. Consider the wreath product

$$\mathbb{Z} \wr \mathbb{Z} \cong \langle b, t \mid [b, b^{t^i}] = \text{id} \text{ for all } i \in \mathbb{N} \rangle.$$

The group  $\mathbb{Z} \wr \mathbb{Z}$  is infinitely presented [LR04, pp. 241]. It is permutation stable by our Theorem 1.4. A hands-on interpretation of this can be formulated as follows.

**COROLLARY 1.6.** *For every  $\varepsilon > 0$  there are some  $\delta = \delta(\varepsilon) > 0$  and  $k = k(\varepsilon) \in \mathbb{N}$  with the following property: for every  $n \in \mathbb{N}$  and every pair of permutations  $\beta, \tau \in S(n)$  satisfying*

$$d_n([\beta, \beta^{\tau^i}], \text{id}) < \delta \text{ for all } i \in \{1, \dots, k\}$$

*there exist permutations  $b, t \in S(n)$  satisfying  $[b, b^{t^i}] = \text{id}$  for all  $i \in \mathbb{N}$  with*

$$d_n(b, \beta) < \varepsilon \text{ and } d_n(t, \tau) < \varepsilon.$$

This example illustrates that our results have a very concrete combinatorial meaning. It is perhaps of interest to point out that the proof, and in fact the entire rest of this paper, deals with invariant random subgroups studied via the methods of ergodic theory.

**1.3. Weiss approximated subgroups.** Let  $G$  be a discrete group and  $d$  be any compatible metric on the space of all probability measures on the space of all subgroups of  $G$ . A key idea of this paper is Weiss approximations.

*Definition 1.7.* A Weiss approximation for the subgroup  $H$  of  $G$  is a sequence  $(K_i, F_i)$  where for every  $i \in \mathbb{N}$ ,  $K_i \leq G$  is a finite index subgroup and  $F_i$  is a finite union of disjoint transversals of  $N_G(K_i)$  in  $G$  such that

$$\lim_{i \rightarrow \infty} d\left(\frac{1}{|F_i|} \sum_{f \in F_i} \delta_{fK_i f^{-1}}, \frac{1}{|F_i|} \sum_{f \in F_i} \delta_{fHf^{-1}}\right) = 0. \tag{1.1}$$

This notion is valid for any group  $G$ , but turns out to be especially useful in the amenable case. By using Weiss’s work on monotilable groups [Wei01] combined with

† After this paper was written and circulated, an updated version of [Zhe19] came out, showing that the Grigorchuk group (which is amenable and infinitely presented) is also permutation stable.

the Lindenstrauss point-wise ergodic theorem for amenable group [Lin01], we show the following theorem.

**THEOREM 1.8.** *Let  $G$  be a finitely generated residually finite amenable group. Then  $G$  admits a sequence of finite subsets  $F_i$  with the following property: an ergodic invariant random subgroup  $\mu$  of  $G$  is co-sofic if and only if  $\mu$ -almost every subgroup  $H$  is Weiss approximated by  $(K_i, F_i)$  for some sequence  $K_i$  of finite index subgroups.*

The above result follows as a combination of our two Theorems 3.10 and 3.14 below. We are then able to use Theorem 1.8 towards our main result by constructing suitable Weiss approximations.

**1.4. Outline of the paper.** Let us briefly outline the structure of the paper. In §2 we recall the notion of an invariant random subgroup of  $G$  and describe some of its properties. This is followed by §3 where we introduce and discuss Weiss approximations. In §4 we study Følner sequences and transversals in group extensions.

The rest of the paper is dedicated to a detailed analysis of the subgroups and invariant random subgroups of metabelian groups. The results of §5 and §7 hold true for all metabelian groups, while starting from §8 we restrict our attention to what we call ‘permutational metabelian groups’.

The construction of Weiss approximations is performed in §10 only for the split case. It amounts to the careful construction of Følner sequences of transversals that satisfy a certain statistical property of being ‘adapted’; see Definition 7.2.

**1.5. Open problems and questions.** We have enough evidence to believe that Theorem 1.4 could be true for all permutational metabelian groups. This would imply, for instance, that finitely generated free metabelian groups are permutation stable.

The classical result of Hall [Hal59] asserts that every metabelian group is residually finite. Wishful thinking seems to suggest that all metabelian groups are permutation stable. If true, this would be a far reaching strengthening of Hall’s theorem. Note that the smallest residually finite solvable group known to be non-permutation stable is the three-step solvable Abels group [BLT19].

More generally, a permutational wreath product  $B \wr_X Q$  of two arbitrary groups is residually finite if and only if both  $Q$  and  $B$  are residually finite and either  $B$  is abelian or  $Q$  is finite [Grü57]. A wreath product of two amenable groups is again amenable and hence sofic [Wei00]. A sofic group which is permutation stable must be residually finite [GR09]. Therefore if  $B \wr_X Q$  is sofic and permutation stable then, unless  $Q$  is finite, the base group  $B$  must be abelian. The moral is that while our Theorem 1.4 might not be as general as possible in the sense that  $Q$  could potentially be replaced by other amenable groups, at the very least the base group  $B$  has to be abelian.

The wreath product of two finitely generated abelian groups is known to be locally extended residually finite (LERF) by [Alp06]. Therefore our Theorem 1.4 gives a partial answer in the direction suggested by [BLT19, Question 8.6]: whether every LERF amenable group is permutation stable.

We follow the convention  $x^y = y^{-1}xy$  for conjugation and  $[x, y] = x^y x^{-1}$  for the commutator. The identities

$$[xy, z] = [x, z]^y [y, z] \quad \text{and} \quad [x, yz] = [x, z][x, y]^z \quad (1.2)$$

will be frequently used throughout this text.

## 2. Random subsets and subgroups

Let  $G$  be a countable discrete group. We recall the Chabauty space of all subgroups of  $G$  and define invariant random subgroups of  $G$ . We then investigate the notion of co-sofic invariant random subgroups in some detail.

2.1. *Space of subsets.* Consider the power set of  $G$ ,

$$\text{Pow}(G) = \{0, 1\}^\Gamma$$

equipped with the Tychonoff product topology. The space  $\text{Pow}(G)$  is compact and metrizable. Indeed, given an arbitrary enumeration  $G = \{g_1, g_2, \dots\}$ , the metric  $d_{\text{Pow}(G)}$  given by

$$d_{\text{Pow}(G)}(A, B) = \sum_{n=1}^{\infty} \frac{1_{A \Delta B}(g_n)}{2^n} \quad (2.1)$$

for any two subsets  $A, B \in \text{Pow}(G)$  induces the topology.

The group  $G$  acts on its power set  $\text{Pow}(G)$  by homeomorphisms via conjugation. We denote this action by  $c_g$ . Given an element  $g \in G$  and a subset  $A \subset G$ , denote

$$c_g A = A^{g^{-1}} = g A g^{-1}.$$

*Definition 2.1.* A sequence of subsets  $F_i \in \text{Pow}(G)$  is *exhausting* if  $F_i$  converges to  $G$  in the Tychonoff product topology on  $\text{Pow}(G)$ .

2.2. *Chabauty space of subgroups.* Consider the following subset of  $\text{Pow}(G)$ :

$$\text{Sub}(G) = \{H \leq G : H \text{ is a subgroup of } G\}.$$

The space  $\text{Sub}(G)$  is called the *Chabauty space* of the group  $G$ . The space  $\text{Sub}(G)$  is compact since it is a closed subset of  $\text{Pow}(G)$ . Moreover,  $\text{Sub}(G)$  is preserved by the conjugation action  $c_g$  of  $G$ .

Let  $\text{Sub}_{\leq d}(G)$  denote the subset of  $\text{Sub}(G)$  consisting of all subgroups of index at most  $d$  in  $G$ . Denote

$$\text{Sub}_{\text{fin}}(G) = \bigcup_{d \in \mathbb{N}} \text{Sub}_{\leq d}(G)$$

so that  $\text{Sub}_{\text{fin}}(G)$  consists of all the finite index subgroups of  $G$ . Of course, the subset  $\text{Sub}_{\text{fin}}(G)$  need not be closed in general.

2.3. *Spaces of probability measures.* Let  $\mathcal{P}(G)$  be the space of all Borel probability measures on the set  $\text{Sub}(G)$ . This is a compact space with the weak-\* topology according to the Banach–Alaoglu theorem. The conjugation action of  $G$  on its Chabauty space

Sub( $G$ ) gives rise to a corresponding push-forward action of  $G$  on the space  $\mathcal{P}(G)$ . We continue to use the notation  $c_g$  for this push-forward action.

LEMMA 2.2. A sequence  $\mu_n \in \mathcal{P}(G)$  weak- $*$  converges to  $\mu \in \mathcal{P}(G)$  if and only if for every pair of finite subsets  $A, B \subset G$  we have

$$\mu_n(E_{A,B}) \xrightarrow{n \rightarrow \infty} \mu(E_{A,B}) \tag{2.2}$$

where

$$E_{A,B} = \{H \leq G : A \subset H \text{ and } B \subset G \setminus H\}. \tag{2.3}$$

*Proof.* Given a pair of finite subsets  $A, B \subset G$ , the subset  $E_{A,B}$  is the intersection of a ‘cylinder set’ in the Tychonoff product space  $\text{Pow}(G)$  with the Chabauty subspace  $\text{Sub}(G)$ . Therefore  $E_{A,B}$  is clopen. According to the Portmanteau theorem [Bil13, Theorem 2.1], the weak- $*$  convergence  $\mu_n \rightarrow \mu$  implies  $\mu_n(E_{A,B}) \rightarrow \mu(E_{A,B})$ .

The family of all subsets  $E_{A,B}$  with  $A$  and  $B$  ranging over finite subsets of the group  $G$  is a countable basis for the topology of  $\text{Sub}(G)$ . In addition, the family of subsets  $E_{A,B}$  is closed under finite intersections. These are precisely the two conditions needed for [Bil13, Theorem 2.2] which amounts to the converse direction of our lemma. The proof of that theorem is essentially an inclusion–exclusion formula.  $\square$

The space  $\mathcal{P}(G)$  is metrizable. In fact, it follows from Lemma 2.2 that if  $A_i$  and  $B_i$  is an arbitrary enumeration of all pairs of finite subsets of  $G$  then

$$d_{\mathcal{P}(G)}(\mu, \nu) = \sum_{i \in \mathbb{N}} \frac{|\mu(E_{A_i, B_i}) - \nu(E_{A_i, B_i})|}{2^i}, \quad \mu, \nu \in \mathcal{P}(G) \tag{2.4}$$

is a compatible metric for the topology of  $\mathcal{P}(G)$ .

2.4. *Invariant random subgroups.* Denote

$$\text{IRS}(G) = \{\mu \in \mathcal{P}(G) : c_g \mu = \mu \text{ for all } g \in G\}.$$

Note that  $\text{IRS}(G)$  is a weak- $*$  closed and hence a compact subset of  $\mathcal{P}(G)$ . An element  $\mu \in \text{IRS}(G)$  is called an *invariant random subgroup* [AGV+14].

An invariant random subgroup  $\mu \in \text{IRS}(G)$  is *ergodic* if every convex combination  $\mu = t\mu_1 + (1-t)\mu_2$  with  $\mu_1, \mu_2 \in \text{IRS}(G)$  and  $t \in (0, 1)$  is trivial in the sense that  $\mu_1 = \mu_2$ . In other words, the ergodic invariant random subgroups  $\text{IRS}_{\text{ergodic}}(G)$  are the extreme points of the compact convex set  $\text{IRS}(G)$ .

The compact convex space  $\text{IRS}(G)$  is a *Choquet simplex*. This means that every  $\mu \in \text{IRS}(G)$  admits an *ergodic decomposition*, namely there is a unique probability measure  $\nu_\mu$  on the Borel set  $\text{IRS}_{\text{ergodic}}(G)$  satisfying

$$\mu(f) = \int_{\text{IRS}_{\text{ergodic}}(G)} f \, d\nu_\mu \tag{2.5}$$

for every continuous function  $f$  on the compact space  $\text{Sub}(G)$ . To interpret this formula, regard  $f$  as a weak- $*$  continuous function on  $\text{IRS}(G)$ . The left-hand side of equation (2.5) now reads  $f(\mu)$  while the right-hand side reads  $\nu_\mu(f)$ . The reader is referred to [Phe01, §12] for details.

2.5. *Co-sofic invariant random subgroups.* Let  $\text{IRS}_{\text{fi}}(G)$  denote the subspace consisting of all  $\mu \in \text{IRS}(G)$  satisfying  $\mu(\text{Sub}_{\text{fi}}(G)) = 1$ . Clearly  $\text{IRS}_{\text{fi}}(G)$  is convex. Note that  $\mu \in \text{IRS}_{\text{fi}}(G)$  is ergodic if and only if  $\mu$  is a uniform atomic probability measure supported on the conjugacy class of some finite index subgroup  $H \leq G$ .

*Definition 2.3.* The invariant random subgroup  $\mu \in \text{IRS}(G)$  is *co-sofic* if

$$\mu \in \overline{\text{IRS}_{\text{fi}}(G)}^{\text{weak-*}}.$$

The collection of all co-sofic invariant random subgroups is convex and weak-\* closed, being the closure of a convex set.

The important notion of *sofic groups* was introduced by Gromov in [Gro99]. The name *sofic* was coined by Weiss in [Wei00]. For our purposes, it will suffice to state an equivalent definition, given in terms of invariant random subgroups. If  $G$  is a free group and  $N \triangleleft G$  is a normal subgroup then the quotient group  $G/N$  is *sofic* if and only if the invariant random subgroup  $\delta_N$  is co-sofic. See [Gel15, §2.5] for details about this fact. Every amenable as well as every residually finite group is sofic.

*Remark 2.4.* Let  $G$  be any group with normal subgroup  $N$ . One direction of the above statement holds true in general. Namely, if  $\delta_N$  is co-sofic then  $G/N$  is sofic. This can be seen by lifting to the free group. The converse implication is false in general. For example, the invariant random subgroup  $\delta_{\{e\}}$  is co-sofic if and only if  $G$  is residually finite, and some sofic groups are not residually finite. However, in the special case where  $G/N$  is residually finite,  $\delta_N$  is indeed co-sofic.

The following result shows that when studying co-sofic invariant random subgroups we may in some sense restrict our attention to ergodic ones.

**PROPOSITION 2.5.** *Let  $\mu \in \text{IRS}(G)$  be an invariant random subgroup with ergodic decomposition  $\nu_\mu \in \mathcal{P}(\text{IRS}_{\text{ergodic}}(G))$ . If  $\nu_\mu$ -almost every  $\lambda \in \text{IRS}_{\text{ergodic}}(G)$  is co-sofic then  $\mu$  is co-sofic.*

*Proof.* Assume that  $\nu_\mu$ -almost every  $\lambda \in \text{IRS}_{\text{ergodic}}(G)$  is co-sofic, so that

$$\text{supp}(\nu_\mu) \subset \overline{\text{IRS}_{\text{fi}}(G)}^{\text{weak-*}}.$$

Since the collection of all co-sofic invariant random subgroups is closed and convex we deduce that

$$\mu \in \overline{\text{conv}(\text{supp}(\nu_\mu))}^{\text{weak-*}} \subset \overline{\text{IRS}_{\text{fi}}(G)}^{\text{weak-*}}$$

as required. □

The following observation follows immediately from Proposition 2.5.

**COROLLARY 2.6.** *If every  $\mu \in \text{IRS}_{\text{ergodic}}(G)$  is co-sofic then every  $\mu \in \text{IRS}(G)$  is co-sofic as well.*

Ergodic co-sofic invariant random subgroups admit a very explicit description.

PROPOSITION 2.7. *Let  $\mu \in \text{IRS}(G)$  be ergodic and co-sofic. Then  $\mu = \lim v_i$  for some ergodic  $v_i \in \text{IRS}_{\text{fi}}(G)$ .*

*Proof.* For ease of notation, denote  $\text{IRS}_{\text{fe}}(G) = \text{IRS}_{\text{fi}}(G) \cap \text{IRS}_{\text{ergodic}}(G)$ . The invariant random subgroup  $\mu$  is co-sofic. Therefore we may write

$$\mu = \lim_i \mu_i \quad \text{where } \mu_i \in \text{IRS}_{\text{fi}}(G) = \text{conv}(\text{IRS}_{\text{fe}}(G)) \quad \text{for all } i \in \mathbb{N}. \tag{2.6}$$

Assume towards contradiction that

$$\mu \notin \overline{\text{IRS}_{\text{fe}}(G)}^{\text{weak-*}}.$$

By the definition of weak-\* convergence, this means that there is some real-valued continuous function  $f \in C(\text{Sub}(G))$  and some  $\varepsilon > 0$  such that

$$|\mu(f) - \lambda(f)| > \varepsilon \quad \text{for all } \lambda \in \text{IRS}_{\text{fe}}(G).$$

Consider the decomposition  $\text{IRS}_{\text{fe}}(G) = F_+ \cup F_-$  into two disjoint Borel sets with

$$\lambda(f) > \mu(f) + \varepsilon \quad \text{for all } \lambda \in F_+ \quad \text{and} \quad \lambda(f) < \mu(f) - \varepsilon \quad \text{for all } \lambda \in F_-.$$

Note that

$$\mu \notin \overline{\text{conv}(F_+)}^{\text{weak-*}} \cup \overline{\text{conv}(F_-)}^{\text{weak-*}}. \tag{2.7}$$

Equations (2.6) and (2.7) together imply that both subsets  $F_+$  and  $F_-$  are non-empty.

The above decomposition of the set  $\text{IRS}_{\text{fe}}(G)$  allows us to express each invariant random subgroup  $\mu_i$  as a convex combination in the following way:

$$\mu_i \in \text{conv}(F_+ \cup F_-) = \text{conv}(\text{conv}(F_+) \cup \text{conv}(F_-)) \quad \text{for all } i \in \mathbb{N}.$$

Taking the weak-\* limit  $\mu = \lim \mu_i$  gives

$$\mu \in \overline{\text{conv}(\text{conv}(F_+) \cup \text{conv}(F_-))}^{\text{weak-*}}. \tag{2.8}$$

Equations (2.7) and (2.8) together imply that  $\mu$  is necessarily a non-trivial convex combination. This is a contradiction to the ergodicity of  $\mu$ . □

We remark that Proposition 2.7 admits a more general formulation in terms of convex subsets of arbitrary Choquet simplices.

2.6. *Chabauty spaces of subgroups and quotients.* Let  $H$  be any subgroup of  $G$ . There is a continuous restriction map given by

$$\cdot|_H : \text{Pow}(G) \rightarrow \text{Pow}(H), \quad A \mapsto A|_H = A \cap H \quad \text{for all } A \subset G.$$

It is clear that  $\text{Sub}(G)|_H = \text{Sub}(H)$ . Similarly,

$$\text{Sub}_{\leq d}(G)|_H \subset \text{Sub}_{\leq d}(H) \quad \text{for all } d \in \mathbb{N} \quad \text{and} \quad \text{Sub}_{\text{fi}}(G)|_H \subset \text{Sub}_{\text{fi}}(H).$$

The restriction map is  $H$ -equivariant for the conjugation action of the subgroup  $H$ . Pushing forward via the restriction determines a map

$$\cdot|_H : \mathcal{P}(G) \rightarrow \mathcal{P}(H), \quad \mu \mapsto \mu|_H \quad \text{for all } \mu \in \mathcal{P}(G).$$

PROPOSITION 2.8. *If  $H$  is a subgroup of  $G$  then  $\text{IRS}(G)|_H \subset \text{IRS}(H)$ .*



*Proof.* This follows immediately from the fact that restriction to  $H$  is equivariant with respect to the conjugation action of  $H$ .  $\square$

Let  $Q$  be a quotient of  $G$  admitting a surjective homomorphism  $\pi : G \rightarrow Q$ . There is a corresponding map  $\pi : \text{Sub}(G) \rightarrow \text{Sub}(Q)$  of subgroups taking every subgroup  $H \leq G$  to its image  $\pi(H)$  in  $Q$ .

**PROPOSITION 2.9.** *The map  $\pi : \text{Sub}(G) \rightarrow \text{Sub}(Q)$  is  $G$ -equivariant and Borel measurable.*

*Proof.* The group  $G$  is acting on  $\text{Sub}(G)$  and on  $\text{Sub}(Q)$  by conjugation. The  $G$ -equivariance of the map  $\pi$  is clear. To see that  $\pi$  is Borel measurable observe that the Chabauty topology on  $\text{Sub}(Q)$  is generated by the subsets

$$S_q = \{L \leq Q : q \in L\}$$

and their complements. It is easy to verify that the preimage  $\pi^{-1}(S_q)$  is Borel in  $\text{Sub}(G)$  for every  $q \in Q$ .  $\square$

In particular, if  $\mu$  is an invariant random subgroup of  $G$  then  $\pi_*\mu$  is an invariant random subgroup of  $Q$ , and if  $\mu$  is ergodic then so is  $\pi_*\mu$ .

*Remark 2.10.* The map  $\pi : \text{Sub}(G) \rightarrow \text{Sub}(Q)$  is not Chabauty continuous in general. Consider, for example, the group  $\mathbb{Z} \times \mathbb{Z}$  and let  $\pi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$  be the projection to the first coordinate. Let  $H_n \in \text{Sub}(\mathbb{Z} \times \mathbb{Z})$  be the cyclic subgroup generated by the element  $(1, n)$  so that  $\pi(H_n) = \mathbb{Z}$  for all  $n \in \mathbb{N}$ . However, the subgroups  $H_n$  converge to the trivial subgroup in the Chabauty topology on  $\text{Sub}(\mathbb{Z} \times \mathbb{Z})$ .

### 3. Weiss approximable subgroups

We introduce a notion of Weiss approximable subgroups and relate this to co-soficity of invariant random subgroups. Weiss approximation is inspired by Benjamin Weiss's work [Wei01]. Our arguments in this section crucially rely on the pointwise ergodic theorem for amenable groups due to Lindenstrauss [Lin01].

**3.1. Transversals and finite index subgroups.** Let  $G$  be a discrete group and  $H$  be a fixed subgroup of  $G$ .

*Definition 3.1.* A (left) transversal for  $H$  in  $G$  is a subset  $T_0$  consisting of one element from each left coset  $tH$  of  $H$  in  $G$ . A finite-to-one (left) transversal  $T$  for  $H$  in  $G$  is a disjoint union of finitely many left transversals for  $H$  in  $G$ .

That is to say,  $T$  is a finite-to-one transversal for  $H$  in  $G$  if there is some  $k \in \mathbb{N}$  so that  $T$  consists of exactly  $k$  elements from each coset of  $H$ .

**PROPOSITION 3.2.** *If  $G$  is finitely generated then there is a countable family  $\{T_i\}_{i \in \mathbb{N}}$  of finite subsets such that every finite index subgroup  $H$  admits  $T_i$  as a finite-to-one transversal for all indices  $i \in \mathbb{N}$  sufficiently large.*

*Proof.* For every  $i \in \mathbb{N}$  let  $N_i$  be the intersection of all subgroups of  $G$  with index at most  $i$ . Since  $G$  is finitely generated there are only finitely many such subgroups and therefore  $N_i$  is a finite index normal subgroup of  $G$ . Let  $T_i$  be any transversal to the subgroup  $N_i$  in the group  $G$ . It follows that  $T_i$  is a finite-to-one transversal to any finite index subgroup  $H$  satisfying  $[G : H] \leq i$ . □

We say that such  $\{T_i\}_{i \in \mathbb{N}}$  is a *universal sequence of transversals* in the group  $G$ .

3.2. *Actions of finite sets.* Let  $F \subset G$  be a finite subset. Given any subgroup  $H \leq G$ , we will let  $F * H$  denote the probability measure

$$F * H = \frac{1}{|F|} \sum_{f \in F} c_f \delta_H \tag{3.1}$$

regarded as a point in  $\mathcal{P}(G)$ . Recall that  $c_f \delta_H = \delta_{fHf^{-1}} = \delta_{Hf^{-1}}$ .

PROPOSITION 3.3. *If  $[G : N_G(H)] < \infty$  and  $T$  is any finite-to-one transversal of the subgroup  $N_G(H)$  then  $T * H \in \text{IRS}_{\text{fi}}(G)$  and  $T * H$  is supported on the finite conjugacy class of  $H$  in  $G$ .*

*Proof.* Let  $T$  be any finite-to-one left transversal in  $G$  of the subgroup  $N_G(H)$ . Let  $k \in \mathbb{N}$  be such that  $T = \coprod_{j=1}^k T_j$  where each  $T_j$  is a left one-to-one transversal of  $N_G(H)$ . Let  $\mu_H \in \text{IRS}(G)$  be the invariant random subgroup of  $G$  supported on the finite family of subgroups conjugate to  $H$ . Note that

$$\mu_H = \frac{1}{k} \sum_{j=1}^k \mu_{H_j} = \frac{1}{k} \sum_{j=1}^k \frac{1}{[G : N_G(H)]} \sum_{g \in T_j} \delta_{Hg^{-1}} = \frac{1}{|T|} \sum_{g \in T} c_g \delta_H = T * H,$$

as required. □

3.3. *Weiss approximation.* Let  $H$  be a subgroup of the discrete group  $G$ .

Definition 3.4. The subgroup  $H$  is *Weiss approximable* in  $G$  if there are finite index subgroups  $K_i$  of  $G$  with finite-to-one transversals  $F_i$  to  $N_G(K_i)$  such that

$$d_{\mathcal{P}(G)}(F_i * K_i, F_i * H) \xrightarrow{i \rightarrow \infty} 0$$

where  $d_{\mathcal{P}(G)}$  is any compatible metric on the space  $\mathcal{P}(G)$ . We will say that the sequence  $(K_i, F_i)$  is a *Weiss approximation* for  $H$  in  $G$ .

The two sequences  $F_i * K_i$  and  $F_i * H$  are not required to converge. However, by the compactness of the space  $\mathcal{P}(G)$  it is always possible to pass to a subsequence such that the limits of  $F_i * K_i$  and of  $F_i * H$  do exist and coincide in  $\mathcal{P}(G)$ .

PROPOSITION 3.5. *Let  $(K_i, F_i)$  be a Weiss approximation for the subgroup  $H$  of  $G$ . If  $F_i * H$  weak-\* converges to some  $\mu \in \mathcal{P}(G)$  then  $\mu \in \text{IRS}(G)$  and  $\mu$  is co-sofic.*

*Proof.* Up to passing to a subsequence we may assume that the weak- $*$  limit of the sequence  $\mu_i = F_i * K_i$  in  $\mathcal{P}(G)$  exists and coincides with  $\mu = \lim F_i * H$ . In addition,  $\mu_i \in \text{IRS}_{\text{fi}}(G)$  by Proposition 3.3. As  $\text{IRS}(G)$  is weak- $*$  closed we have that  $\mu \in \text{IRS}(G)$ . We conclude that  $\mu$  is co-sofic.  $\square$

**COROLLARY 3.6.** *A normal subgroup  $N \triangleleft G$  is Weiss approximable if and only if the invariant random subgroup  $\delta_N \in \text{IRS}(G)$  is co-sofic.*

*Proof.* Let  $N \triangleleft G$  be a normal subgroup. It is clear that  $F * \delta_N = \delta_N$  for any finite subset  $F \subset G$ . If  $N$  is Weiss approximable then Proposition 3.5 shows that  $\delta_N$  is co-sofic. Conversely, if  $\delta_N$  is co-sofic then we obtain a Weiss approximation for  $N$  by combining Propositions 2.7 and 3.3.  $\square$

The question of co-soficity for invariant random subgroups of the form  $\delta_N$  was discussed in Remark 2.4. In particular, if  $N$  is a normal subgroup such that  $G/N$  is residually finite then  $N$  is Weiss approximable in  $G$ .

We now present an explicit condition for a given sequence  $(K_i, F_i)$  to constitute a Weiss approximation for the subgroup  $H$ .

**PROPOSITION 3.7.** *Let  $H$  be a subgroup of  $G$ . If a given sequence  $K_i$  of finite index subgroups of  $G$  with finite-to-one transversals  $F_i$  of  $N_G(K_i)$  satisfies*

$$p_i(g) = \frac{|\{f \in F_i : g^f \in K_i \Delta H\}|}{|F_i|} \xrightarrow{i \rightarrow \infty} 0$$

for all elements  $g \in G$  then  $(K_i, F_i)$  is a Weiss approximation for the subgroup  $H$ .

*Proof.* Consider a pair  $A$  and  $B$  of finite subsets of  $G$ . Denote as before

$$E_{A,B} = \{L \leq G : A \subset L \text{ and } B \subset G \setminus L\}$$

so that  $E_{A,B}$  is a clopen subset of the Chabauty space  $\text{Sub}(G)$ . By definition

$$(F_i * K_i - F_i * H)(E_{A,B}) = \frac{|\{f \in F_i : c_f K_i \in E_{A,B}\}| - |\{f \in F_i : c_f H \in E_{A,B}\}|}{|F_i|}.$$

Let  $\chi_{A,B}$  denote the characteristic function of the subset  $E_{A,B}$ . An element  $f \in F_i$  has a non-zero contribution to the numerator of the above expression if and only if  $\chi_{A,B}(c_f H) \neq \chi_{A,B}(c_f K_i)$ . This happens precisely whenever  $(A \cup B)^f \cap (K_i \Delta H) \neq \emptyset$ . We obtain the upper bound

$$\begin{aligned} |(F_i * K_i - F_i * H)(E_{A,B})| &\leq \frac{|\{f \in F_i : (A \cup B)^f \cap (K_i \Delta H) \neq \emptyset\}|}{|F_i|} \leq \\ &\leq \sum_{g \in A \cup B} \frac{|\{f \in F_i : g^f \in K_i \Delta H\}|}{|F_i|} = \sum_{g \in A \cup B} p_i(g). \end{aligned}$$

The assumption that  $p_i(g) \rightarrow 0$  for all elements  $g \in G$  gives

$$\lim_{i \rightarrow \infty} |(F_i * K_i - F_i * H)(E_{A,B})| = 0 \tag{3.2}$$

for every pair of finite subsets  $A$  and  $B$  of  $G$ .

Recall equation (2.4) introduced in the discussion following Lemma 2.2. This is an infinite series expression defining an explicit compatible metric  $d_{\mathcal{P}(G)}$ . All terms of that series are uniformly bounded in the range  $[0, 1]$ . Therefore equation (3.2) implies  $d_{\mathcal{P}(G)}(F_i * K_i, F_i * H) \rightarrow 0$ .  $\square$

3.4. *The pointwise ergodic theorem for amenable groups.*

*Definition 3.8.* A (left) Følner sequence for  $G$  is a sequence of finite subsets  $F_i$  of  $G$  such that

$$\frac{|gF_i \Delta F_i|}{|F_i|} \rightarrow 0 \tag{3.3}$$

for every element  $g \in G$ .

Recall that a Følner sequence  $F_i$  is *tempered* if there is a constant  $c > 0$  so that

$$\left| \bigcup_{j < i} F_j^{-1} F_i \right| \leq c|F_i|.$$

The tempered condition is needed for the ergodic theorem for amenable groups, at least in the general case. However, every Følner sequence admits a tempered subsequence [Lin01, 1.4]. For this reason the tempered condition will not be a major issue from our point of view.

**THEOREM 3.9.** (Lindenstrauss [Lin01]) *Let  $G$  be an amenable group acting by homeomorphisms on a Hausdorff second countable compact space  $X$  ergodically with invariant probability measure  $\mu$ . Let  $F_i$  be a tempered Følner sequence for  $G$ . Then for  $\mu$ -almost every point  $x \in X$  we have*

$$\frac{1}{|F_i|} \sum_{g \in F_i} \delta_{gx} \xrightarrow{i \rightarrow \infty} \mu \tag{3.4}$$

*in the weak- $*$  topology.*

For the reader’s convenience let us briefly recall the well-known argument to derive the above statement as an immediate consequence of [Lin01, Theorem 1.2].

*Proof of Theorem 3.9.* It follows from [Lin01] that for every function  $f \in L^1(X, \mu)$ ,

$$\frac{1}{|F_i|} \sum_{g \in F_i} f(gx) \xrightarrow{i \rightarrow \infty} \int f \, d\mu \tag{3.5}$$

holds true for every point  $x$  belonging to some  $\mu$ -conull subset  $X_f \subset X$ .

Let  $\mathcal{F}$  be a countable  $L^\infty$ -dense subset of  $C(X)$ . Such a family  $\mathcal{F}$  exists by the Stone–Weierstrass theorem. It is clear that  $\mathcal{F} \subset L^1(X, \mu)$ . So equation (3.5) holds for

every function  $f \in \mathcal{F}$  and every point in the  $\mu$ -conull subset  $\bigcap_{f \in \mathcal{F}} X_f$ . In particular, for every such point the sequence of probability measures  $(1/|F_i|) \sum_{g \in F_i} \delta_{gx}$  weak- $*$  converges to the ergodic probability measure  $\mu$ .  $\square$

3.5. *The ergodic theorem and co-sofic subgroups.* The following result will be our main tool in establishing that a given invariant random subgroup is co-sofic.

**THEOREM 3.10.** *Let  $G$  be an amenable group and  $\mu \in \text{IRS}(G)$ . Let  $F_i$  be a fixed Følner sequence in the group  $G$ . If  $\mu$ -almost every subgroup  $H$  admits a Weiss approximation  $(K_i, F_i)$  with some sequence of finite index subgroups  $K_i$  of  $G$  then  $\mu$  is co-sofic.*

*Proof.* In light of Proposition 2.5 we may assume without loss of generality that  $\mu$  is ergodic. Consider the probability measure preserving action of the amenable group  $G$  on the Borel space  $(\text{Sub}(G), \mu)$ .

We may assume, up to passing to a subsequence, that the Følner sequence  $F_i$  is tempered. Since a subsequence of Weiss approximation is again a Weiss approximation, the assumption of the theorem continues to hold.

Let  $X_1 \subset \text{Sub}(G)$  be the subset consisting of all subgroups  $H \in \text{Sub}(G)$  for which  $(1/|F_i|) \sum_{f \in F_i} \delta_{c_f H}$  converges to the invariant random subgroup  $\mu$  as  $i \rightarrow \infty$ . By Theorem 3.9 the subset  $X_1$  is  $\mu$ -measurable and satisfies  $\mu(X_1) = 1$ .

Let  $X_2 \subset \text{Sub}(G)$  be the subset consisting of all subgroups  $H \in \text{Sub}(G)$  which are Weiss approximable with respect to some sequence  $(K_i, F_i)$ . The subset  $X_2$  is  $\mu$ -measurable and satisfies  $\mu(X_2) = 1$  by our assumption. Therefore  $X_1 \cap X_2 \neq \emptyset$ . Let  $H \in \text{Sub}(G)$  be any subgroup belonging to  $X_1 \cap X_2$ . The result follows from Proposition 3.5.  $\square$

*Remark 3.11.* There is nothing special about amenable groups in Theorem 3.10. In principle the same argument applies to any group  $G$  and sequence of subsets  $F_i$  satisfying a pointwise ergodic theorem. Examples of such situations include the case where  $G$  is hyperbolic and  $F_i$  is a sequence of balls [BN13] and the case of  $S$ -arithmetic lattices [GN09].

For residually finite amenable groups Theorem 3.10 admits a converse. We include it below for the sake of completeness, even though it will not be used towards our main results. This will rely on the following interesting result due to Weiss [Wei01]; see also [AJZN11, Theorem 6].

**THEOREM 3.12.** (Weiss) *Let  $G$  be a residually finite amenable group. Let  $N_i$  be a descending sequence of finite index normal subgroups of  $G$  satisfying  $\bigcap_i N_i = \{e\}$ . Then there exists a Følner sequence  $F_i$  of transversals of the subgroups  $N_i$ .*

The notion of a universal sequence of transversals was introduced following Proposition 3.2.

**COROLLARY 3.13.** *If  $G$  is a finitely generated residually finite amenable group then  $G$  admits a Følner sequence of universal transversals.*

*Proof.* The corollary follows exactly as in the proof of Proposition 3.2 while relying on Theorem 3.12 in the choice of the transversals.  $\square$

We are ready to present the promised converse to Theorem 3.10.

**THEOREM 3.14.** *Let  $G$  be a finitely generated residually finite amenable group. There exists a Følner sequence in the group  $G$  such that if  $\mu \in \text{IRS}(G)$  is ergodic and co-sofic then  $\mu$ -almost every subgroup  $H$  of  $G$  admits a Weiss approximation  $(K_i, F_i)$  for some sequence  $K_i$  of finite index subgroups.*

*Proof.* Fix a Følner sequence  $F_i$  of universal transversals in the group  $G$  as provided by Corollary 3.13. Up to passing to a subsequence we may assume that the sequence  $F_i$  is tempered.

Let  $\mu$  be an ergodic and co-sofic invariant random subgroup of  $G$ . There exists a sequence of finite index subgroups  $K_i$  so that  $T_i * K_i \rightarrow \mu$  for any sequence of finite-to-one transversals  $T_i$ ; see Propositions 2.7 and 3.3. Up to ‘repeating’ each subgroup  $K_i$  finitely many times, we may assume that  $F_i$  is indeed a finite-to-one transversal to  $K_i$  for all  $i \in \mathbb{N}$ . In particular,  $F_i * K_i \rightarrow \mu$  with respect to our fixed universal Følner sequence of transversals.

The ergodic theorem for amenable groups (Theorem 3.9) implies that  $\mu$ -almost every subgroup  $H$  satisfies  $F_i * H \rightarrow \mu$ . We deduce that  $\mu$ -almost every subgroup  $H$  admits a Weiss approximation of the form  $(K_i, F_i)$ .  $\square$

Since the invariant random subgroup  $\mu$  appearing in Theorem 3.14 is ergodic the same sequence  $(K_i, F_i)$  is a Weiss approximation to  $\mu$ -almost every subgroup.

#### 4. Group extensions and Goursat triples

Let  $G$  be a discrete group. Assume that  $G$  is an extension of the group  $Q$  by the group  $N$ . This means that there is a short exact sequence

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1.$$

It will be useful to fix an arbitrary lift  $\hat{q} \in G$  for every element  $q \in Q$ . This means that  $\hat{q}N = q \in Q$ . In particular, if  $I \subset Q$  is any subset we will use the notation

$$\hat{I} = \{\hat{q} : q \in I\}.$$

4.1. *Goursat’s triplets.* Given a subgroup  $H$  of  $G$ , consider the pair of subgroups

$$N_H = H \cap N \leq N \quad \text{and} \quad Q_H = HN/N \leq Q.$$

Let  $\alpha_H$  denote the natural group isomorphism

$$\alpha_H : Q_H \rightarrow H/N_H.$$

For every element  $q \in Q_H$  we will regard the image  $\alpha_H(q)$  as a subset of the group  $G$ . Namely,  $\alpha_H(q) \subset H \subset G$  is a coset of the subgroup  $N_H$ . Indeed,

$$\alpha_H(q) = H \cap \hat{q}N \quad \text{for all } q \in Q_H.$$

PROPOSITION 4.1. (Goursat lemma for group extensions) *There is a bijective correspondence between subgroups  $H$  of  $G$  and triples  $[Q_H, N_H, \alpha_H]$  where:*

- $Q_H$  is a subgroup of  $Q$ ;
- $N_H$  is a subgroup of  $N$ ; and
- $\alpha_H : Q_H \rightarrow N_G(N_H)/N_H$  is a homomorphism satisfying  $\alpha_H(q)N_H \subset \widehat{q}N$  for every element  $q \in Q_H$ .

We will denote  $[H] = [Q_H, N_H, \alpha_H]$  and call this the *Goursat triplet* associated to the subgroup  $H$ . The subgroup  $H$  is uniquely determined by its Goursat triplet. Every homomorphism  $\alpha_H$  participating in a Goursat triplet is necessarily injective.

We caution the reader that the notation  $N_G(N_H)$  stands for the normalizer in the group  $G$  of the subgroup  $N_H$ .

*Proof of Proposition 4.1.* The discussion preceding the statement demonstrates how to associate to every subgroup  $H \leq G$  a triple  $[N_H, Q_H, \alpha_H]$  as above.

For the converse, assume that we are given a triple  $[Q_H, N_H, \alpha_H]$  with the above properties. Consider the subgroup  $H$  of  $G$  given by

$$H = \alpha_H(Q_H)N_H.$$

In other words,  $H$  is the image of the homomorphism  $\alpha_H$  regarded as a subset of the group  $G$ . It is easy to verify that the resulting subgroup  $H$  corresponds to the given Goursat triplet.

We conclude that the triple  $[Q_H, N_H, \alpha_H]$  uniquely determines the subgroup  $H$  and that this correspondence is bijective. □

The above makes it clear that the index of a subgroup  $H$  of  $G$  is equal to

$$[G : H] = [N : N_H][Q : Q_H]$$

with the understanding that  $\infty \cdot n = \infty \cdot \infty = \infty$ .

PROPOSITION 4.2. *Let  $H$  be a subgroup of  $G$  and  $g \in G$  be an element with  $q = gN \in Q$ . Then the Goursat triplet of the conjugate subgroup  $H^g$  is*

$$[H^g] = [N_H^g, Q_H^g, \alpha_H^g]$$

where the isomorphism  $\alpha_H^g : Q_H^g \rightarrow H^g/N_H^g$  is given by

$$\alpha_H^g(r^q)N_{H^g} = (\alpha_H(r)N_H)^g$$

for every  $r \in Q_H$ .

*Proof.* We leave the straightforward verification to the reader. □

4.2. *Transversals in group extensions.* Recall that  $\widehat{q} \in G$  is some fixed and arbitrary lift for each element  $q \in Q$ .

PROPOSITION 4.3. *Let  $I$  and  $P$  be left transversals, respectively, for the subgroups  $R \leq Q$  and  $L \leq N$ . Then  $F = \widehat{I}P$  is a left transversal for any subgroup  $H$  of  $G$  with  $Q_H = R$  and  $N_H = L$ .*

*Proof.* Let  $H \leq G$  be any subgroup with Goursat triplet  $[H] = [R, L, \alpha]$  where  $\alpha : R \rightarrow N_G(L)/L$  is any homomorphism with  $\alpha(r)L \subset \widehat{r}N$  for every element  $r \in R$ . We claim that for every element  $g \in G$  there is a unique element  $h \in H$  such that  $gh \in F$ .

Consider some fixed element  $g \in G$  with  $gN = q \in Q$ . Let  $r \in R$  be the unique element so that  $qr = s \in I$ , and let  $h_1 \in \alpha_H(r) \subset H$  be an arbitrary element. As  $r = h_1N$  and  $s = \widehat{s}N$  we have

$$gh_1 = \widehat{s}n$$

for some element  $n \in N$ . There a unique element  $h_2 \in L$  such that  $nh_2 = p \in P$ . Denote  $h = h_1h_2 \in H$  so that

$$gh = gh_1h_2 = \widehat{s}nh_2 = \widehat{s}p \in F.$$

The uniqueness of the element  $h \in H$  satisfying the above condition follows from the uniqueness of  $r \in R$  combined with the uniqueness of the element  $h_2 \in L$ . □

4.3. *Følner sets in group extensions.* Følner sets for  $G$  can be constructed by combining Følner sets for  $Q$  and for  $N$ . Recall that following result of Weiss from [Wei01, §2].

PROPOSITION 4.4. [Wei01] *Let  $I_i$  be a Følner sequence in  $Q$  and  $P_i$  be a Følner sequence in  $N$ . Then there is a subsequence  $k_i$  such that  $F_i = \widehat{I}_iP_{k_i}$  is a Følner sequence in  $G$ .*

Unfortunately Proposition 4.4 will not suffice for our needs, as we will usually not be able to pass to a subsequence of the Følner sequence  $P_i$  in the group  $N$ . We will have to rely on the more precise but specialized Lemma 10.11.

5. *Subgroups and invariant random subgroups of metabelian groups*

Let  $G$  be a finitely generated metabelian group. Therefore the group  $G$  is the intermediate term in a short exact sequence

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

such that the normal subgroup  $N$  as well as the finitely generated quotient subgroup  $Q$  are abelian. The conjugation action of  $G$  on  $N$  factors through the quotient  $Q$ . Given an element  $q \in Q$ , we will frequently use the notation  $n \mapsto n^q \in N$ . The abelian group  $N$  can be regarded as a finitely generated  $\mathbb{Z}[Q]$ -module with respect to this conjugation action of  $Q$ .

From now on, when studying metabelian groups we will use multiplicative notation for the quotient group  $Q$  and additive notation for the normal subgroup  $N$ .

5.1. *Residual finiteness in metabelian groups.* In our work we crucially rely on the following classical theorem [Hal59].

THEOREM 5.1. (Hall) *Finitely generated metabelian groups are residually finite.*



Hall’s theorem allows us to find ‘Chabauty approximations from above’ inside the subgroup  $N$ , in the following sense.

PROPOSITION 5.2. *Let  $H$  be a subgroup of  $G$  with Goursat triplet*

$$[H] = [Q_H, N_H, \alpha_H].$$

*Assume that the metabelian subgroup  $HN$  is finitely generated. Then for every finite subset  $T \subset N$  there is a finite index subgroup  $M \leq N$  with*

$$M \cap T = N_H \cap T, \quad N_H \leq M \leq N, \quad \text{and} \quad H \leq N_G(M).$$

*Proof.* The subgroup  $N_H$  is normalized by the subgroup  $H$  as well as by the abelian group  $N$ . Therefore  $N_H$  is a normal subgroup of  $HN$ . Let  $\pi_H$  denote the quotient homomorphism  $\pi_H : HN \rightarrow HN/N_H$ .

The finitely generated metabelian group  $HN/N_H$  is residually finite by Hall’s theorem. So there is a finite index normal subgroup  $\bar{M} \triangleleft HN/N_H$  such that  $\pi_H(n) \notin \bar{M}$  for every element  $n \in T \setminus N_H$ .

Consider the subgroup  $M = \pi_H^{-1}(\bar{M}) \cap N$ . Since  $\pi_H^{-1}(\bar{M})$  has finite index in  $HN$ , the subgroup  $M$  has finite index in  $N$ . It is clear that  $N_H = \ker \pi_H \leq M$ . The condition  $\bar{M} \cap \pi_H(T) = N_H$  implies that  $M \cap T = N_H \cap T$ . Finally, since  $\bar{M}$  is normal in  $HN/N_H$  it follows that  $M$  is normalized by  $H$ . □

5.2. *Invariant random subgroups of metabelian groups.* Let  $\pi$  be the natural quotient map from  $\text{Sub}(G)$  to  $\text{Sub}(Q)$ . The map  $\pi$  is Borel and  $G$ -equivariant according to Proposition 2.9. Let  $\rho$  be the restriction map from  $\text{Sub}(G)$  to  $\text{Sub}(N)$ . The map  $\rho$  is continuous and  $N$ -equivariant. In light of the Goursat triplet description established in Proposition 4.1 the fiber of the product map

$$\pi \times \rho : \text{Sub}(G) \rightarrow \text{Sub}(Q) \times \text{Sub}(N)$$

over a given pair of subgroups  $(R, L)$  with  $R \leq Q$  and  $L \leq N$  naturally corresponds to the set of all possible homomorphisms  $\alpha$  of the form

$$\alpha : R \rightarrow N_G(L)/L$$

where  $\alpha(r)L \subset \widehat{r}N$  for all elements  $r \in R$ .

Recall that  $Q$  is a finitely generated abelian group. In particular,  $Q$  is Noetherian in the sense that every subgroup of  $Q$  is finitely generated. Therefore there at most countably many homomorphisms whose domain is a subgroup of  $Q$ . By the previous paragraph, the fibers of the product map  $\pi \times \rho$  are countable as well. Let us apply these observations to the study of invariant random subgroups of  $G$ .

PROPOSITION 5.3. *Let  $\mu$  an invariant random subgroup of the finitely generated metabelian group  $G$ . Then  $\mu$ -almost every subgroup  $H$  satisfies  $[N : N_N(H)] < \infty$ .*

*Proof.* The ergodic decomposition allows us to assume without loss of generality that  $\mu$  is ergodic. In particular,  $\pi_*\mu$  is an ergodic invariant random subgroup of the abelian group

$Q$  and as such must be atomic. Therefore there is a subgroup  $Q_\mu$  of  $Q$  such that  $\mu$ -almost every subgroup  $H$  satisfies  $\pi(H) = Q_\mu$ .

Consider the push-forward  $\nu = \rho_*\mu \in \text{IRS}(N)$  where  $\rho : \text{Sub}(G) \rightarrow \text{Sub}(N)$  is the restriction map as above. By the discussion preceding this proposition, the fibers of the map  $\rho$  are  $\mu$ -almost surely countable. We would like to conclude that the  $N$ -orbit by conjugation of  $\mu$ -almost every subgroup  $H$  of  $G$  is finite.

With the above goal in mind, consider the disintegration of the probability measure  $\mu$  along the Borel map  $\rho$ . This is a  $\nu$ -measurable map

$$\lambda : \text{Sub}(N) \rightarrow \mathcal{P}(G), \quad \lambda : L \mapsto \lambda_L$$

with the following properties:

- $\lambda_L(\rho^{-1}(L)) = 1$  for  $\nu$ -almost every subgroup  $L \in \text{Sub}(N)$ ; and
- $\mu = \int_{\text{Sub}(N)} \lambda_L d\nu(L)$ .

This means that  $\lambda_L$  is a probability measure giving total measure 1 to the countable fiber  $\rho^{-1}(L)$  for  $\nu$ -almost every subgroup  $L$  of  $N$ .

The conjugation action of  $N$  preserves the fiber  $\rho^{-1}(L)$  as well as the probability measure  $\lambda_L$  for  $\nu$ -almost every subgroup  $L$ ; see Proposition 4.2. Recall that a probability measure preserving action on a countable set has finite orbits. Therefore the  $N$ -orbit of  $\lambda_L$ -almost every point of the fiber  $\rho^{-1}(L)$  is finite  $\nu$ -almost surely.

The conclusion follows by relying on the integral formula  $\mu = \int \lambda_L d\nu$  to verify that the condition  $[N : N_N(H)] < \infty$  is satisfied by  $\mu$ -almost every subgroup  $H \leq G$ . □

5.3. *On subgroups with finite orbit for  $N$ -conjugation.* Let  $H$  be a subgroup of the metabelian group  $G$ . We now discuss certain consequences of the algebraic information  $[N : N_N(H)] < \infty$  to be used in §7 below.

PROPOSITION 5.4. *If  $[N : N_N(H)] < \infty$  then the subset  $[h, N]N_H$  of the coset space  $N/N_H$  is finite for every  $h \in H$ .*

*Proof.* Let  $h \in H$  be fixed. Consider the map  $f_h$  given by

$$f_h : N \rightarrow N, \quad f_h(n) = [h, n].$$

We claim that the map  $f_h$  is a group homomorphism. Indeed, given any pair of elements  $n, m \in N$ , it follows from equation (1.2) that

$$f_h(n + m) = [h, n + m] + N_H = [h, n] + [h, m]^n + N_H = f_h(n) + f_h(m).$$

Observe that  $f_h(N_N(H)) \subset N_H$ . The assumption  $[N : N_N(H)] < \infty$  implies that  $f_h(N) = [h, N]$  is finite in the coset space  $N/N_H$ , as required. □

The following corollary is a more sophisticated variant of Proposition 5.4 where we take into account an ascending sequence  $M_i$  of subgroups of  $N$ . Recall that a sequence of subsets or subgroups of  $G$  is exhausting if it converges to  $G$  in the Chabauty topology; see Definition 2.1.

COROLLARY 5.5. *Let  $M_i$  be an exhausting sequence of subgroups of  $N$  such that  $H \leq N_G(M_i)$  for all  $i \in \mathbb{N}$ . If  $[N : N_N(H)] < \infty$  then for every element  $h \in H$  there is a finite subset  $\Phi_h \subset N$  such that*

$$[h, M_i] \subset \Phi_h + (H \cap M_i)$$

for all  $i \in \mathbb{N}$  sufficiently large.

*Proof.* Fix an element  $h \in H$ . Consider the map  $f_h : N \rightarrow N$  given by  $f_h(n) = [h, n]$ . It is a group homomorphism, as verified in the proof of Proposition 5.4. It follows that there is a finite subset  $\Phi_h \subset N$  such that

$$f_h(N) = \Phi_h + N_H \subset N.$$

The family  $M_i$  exhausts  $N$  and we may assume that  $i$  is sufficiently large so that  $\Phi_h \subset M_i$ . Since  $h \in H \leq N_G(M_i)$  we have  $f_h(M_i) \subset M_i$ .

Observe that every element  $x \in f_h(M_i)$  satisfies  $x \in M_i$  as well as  $x = f + n$  for a pair of elements  $f \in \Phi_h \subset M_i$  and  $n \in N_H$ . Therefore  $n = x - f \in M_i$  and hence

$$x = f + n \in \Phi_h + (N_H \cap M_i).$$

We conclude that

$$f_h(M_i) \subset \Phi_h + (H \cap M_i)$$

as required. □

While it is possible to prove Corollary 5.5 directly and then deduce Proposition 5.4 as a special case, the current outline seems to be more transparent.

5.4. *Formulas for conjugates in metabelian groups.* The following is an elementary commutator computation that can be used to decide which elements in a given conjugacy class belong to a given subgroup.

PROPOSITION 5.6. *Let  $H$  be a subgroup of  $G$ . Consider a pair of elements  $g, f \in G$  with*

$$g = \widehat{q}n, \quad f = \widehat{r}m, \quad q, r \in Q, \quad \text{and} \quad n, m \in N.$$

*Then  $g^f \in H$  if and only if both  $q \in Q_H$  and*

$$[g, f] + (n - a_q) \in N_H$$

*where  $a_q \in N$  is any element satisfying*

$$\alpha_H(q)N_H = H \cap \widehat{q}N = \widehat{q}(a_q + N_H).$$

*Proof.* We may use equation (1.2) to express the conjugate  $g^f$  as

$$g^f = g[g, f] = \widehat{q}(n + [g, f]).$$

In particular,  $g^f \in H$  if and only if  $q \in Q_H$  as well as

$$n + [g, f] \in a_q + N_H.$$

The result follows. □

6. Consistent homomorphisms

Let  $A$  and  $B$  be discrete groups. Let  $C$  as well as  $C_i$  for all  $i \in \mathbb{N}$  be normal subgroups of the group  $B$ . Let  $f : A \rightarrow B/C$  be a homomorphism.

*Definition 6.1.* A family of homomorphisms  $f_i : A \rightarrow B/C_i$  is *consistent* with the homomorphism  $f$  if for every  $a \in A$  there is an element  $b_a \in B$  such that  $f_i(a) = b_a C_i$  for all  $i$ .

Chabauty convergence can be used to get a consistent family of maps.

**LEMMA 6.2.** *Assume that  $A$  is finitely presented. If the normal subgroups  $C_i$  Chabauty converge to the subgroup  $C$  then there is an index  $i_0 \in \mathbb{N}$  and a family of homomorphisms  $f_i : A \rightarrow B/C_i$  defined for all  $i > i_0$  and consistent with the homomorphism  $f$ .*

*Proof.* Let  $\Sigma$  be a finite generating set and  $R$  be a finite set of defining relations for the group  $A$ . Let  $F_\Sigma$  be the free group on the set  $\Sigma$  with quotient map  $p : F_\Sigma \rightarrow A$  so that  $N = \ker p$  is normally generated by  $R$  regarded as a subset of  $F_\Sigma$ .

Consider the quotient maps

$$q : B \rightarrow B/C \quad \text{and} \quad q_i : B \rightarrow B/C_i \quad \text{for all } i \in \mathbb{N}.$$

Choose an element  $b_\sigma \in B$  for each generator  $\sigma \in \Sigma$  such that

$$b_\sigma C = q(b_\sigma) = f \circ p(\sigma) = f(\sigma N).$$

Let  $\bar{f} : F_\Sigma \rightarrow B$  be the homomorphism defined by  $\bar{f}(\sigma) = b_\sigma$  for each generator  $\sigma \in \Sigma$  of the free group  $F_\Sigma$ . Note that  $f \circ p = q \circ \bar{f}$ . Therefore  $\bar{f}(R) \subset C$ . The sequence  $C_i$  Chabauty converges to  $C$  by assumption. Let  $i_0 \in \mathbb{N}$  be a sufficiently large index so that  $\bar{f}(R) \subset C_i$  and therefore  $\bar{f}(N) \leq C_i$  for all  $i > i_0$ .

Let  $s : A \rightarrow F_\Sigma$  be an arbitrary section to the quotient map  $p$ , namely  $p \circ s(a) = a$  for all elements  $a \in A$ . For each  $i \in \mathbb{N}$  consider the map  $f_i : A \rightarrow B/C_i$  defined by

$$f_i = q_i \circ \bar{f} \circ s : A \rightarrow B/C_i.$$

We claim that  $f_i$  is a group homomorphism for all indices  $i > i_0$ . Indeed, for each pair of elements  $a_1, a_2 \in A$  the section  $s$  satisfies  $s(a_1)s(a_2)N = s(a_1 a_2)N$ . Therefore

$$(\bar{f} \circ s)(a_1)(\bar{f} \circ s)(a_2)C_i = (\bar{f} \circ s)(a_1 a_2)C_i \quad \text{for all } i > i_0. \tag{6.1}$$

The claim follows by observing that equation (6.1) implies  $f_i(a_1)f_i(a_2) = f_i(a_1 a_2)$ .

It remains to verify that the family of homomorphisms  $f_i$  is consistent with  $f$ . Precomposing the equation  $f \circ p = q \circ \bar{f}$  with the section  $s$  gives

$$f = f \circ p \circ s = q \circ \bar{f} \circ s : A \rightarrow B/C.$$

We conclude that all elements  $a \in A$  satisfy

$$f(a) = (\bar{f} \circ s)(a)C \quad \text{and} \quad f_i(a) = (\bar{f} \circ s)(a)C_i$$

for all  $i > i_0$ . This is exactly the requirement in Definition 6.1. □

Let  $G$  be a finitely generated metabelian group with normal abelian subgroup  $N$  and abelian quotient  $Q \cong G/N$ . Let  $H$  be a subgroup of  $G$  with Goursat triplet  $[H] = [Q_H, N_H, \alpha_H]$ .

**COROLLARY 6.3.** *Let  $N_i$  be a sequence of subgroups of  $N$  satisfying  $H \leq N_G(N_i)$  and Chabauty converging to  $N_H$ . Then there exists a family of homomorphisms*

$$\alpha_i : Q_H \rightarrow N_G(N_i)/N_i$$

*defined for all  $i \in \mathbb{N}$  sufficiently large and consistent with the isomorphism  $\alpha_H$ .*

The fact that the homomorphisms  $\alpha_i$  are consistent with the homomorphism  $\alpha_H$  automatically implies that  $\alpha_i(q)N_i \subset \widehat{q}N$  for all elements  $q \in Q_H$  and all  $i \in \mathbb{N}$ .

*Proof of Corollary 6.3.* Recall that the map  $\alpha_H$  is an isomorphism from  $Q_H$  to  $H/N_H$ . Since  $Q$  is a finitely generated abelian group, its subgroup  $Q_H$  is finitely generated as well. The image of  $\alpha_H$  can be regarded as being contained in the larger group  $HN/N_H$ . We may therefore apply Lemma 6.2 with respect to the groups

$$A = Q_H, \quad B = HN, \quad C = N_H, \quad \text{and} \quad C_i = N_i$$

to obtain family of homomorphisms  $\alpha_i : Q_H \rightarrow HN/N_i$  defined for all  $i \in \mathbb{N}$  sufficiently large and consistent with the homomorphism  $\alpha_H$ . As  $N_i \triangleleft N$  and  $H \leq N_G(N_i)$  it follows that the image of each  $\alpha_i$  lies in  $N_G(N_i)/N_i$ . □

### 7. Controlled approximations

Let  $G$  be a finitely generated metabelian group with a normal abelian subgroup  $N$  and abelian quotient group  $Q \cong G/N$ .

Let  $H \leq G$  be any subgroup. We introduce an auxiliary notion of controlled approximations for  $H$ . We also define an accompanying notion of a sequence of finite-to-one transversals being adapted to a given controlled approximation.

Under certain favorable algebraic circumstances, that always apply if  $H$  lies in the support of an invariant random subgroup of  $G$ , a controlled approximation with its adapted sequence of transversals gives rise to a Weiss approximation; see Theorem 7.5.

**7.1. Controlled approximations.** Let  $[H] = [Q_H, N_H, \alpha_H]$  be the Goursat triplet of the subgroup  $H$  of the finitely generated metabelian group  $G$ .

We briefly discuss the underlying idea of a controlled approximation prior to stating the formal definition. Roughly speaking, this is a sequence of finite index subgroups  $K_i$  of  $G$  with Goursat triplets  $[K_i] = [Q_i, N_i, \alpha_i]$ .

We require that the sequences  $Q_i$  and  $N_i$  Chabauty converge to the subgroups  $Q_H$  and  $N_H$ , respectively, and that the homomorphisms  $\alpha_i$  are consistent with  $\alpha_H$  in the sense of Definition 6.1.

While we may assume without loss of generality that the  $Q_i$  approximate  $Q_H$  ‘from above’ in the sense that  $Q_H \leq Q_i$  for all  $i \in \mathbb{N}$ , this is not the case for the approximation of  $N_H$  by the  $N_i$ . The main point is that the Chabauty convergence of the  $N_i$  to  $N_H$  is ‘controlled’ by a sequence of subgroups  $M_i \leq N$ . The restriction of the approximation  $N_i$  to each subgroup  $M_i$  is ‘from above’, namely  $N_H \cap M_i \leq N_i \cap M_i$ .

Finally, the ‘speed’ of the Chabauty convergence of the  $N_i$  towards the subgroup  $N_H$  is ‘controlled’ by a sequence of subsets  $T_i \subset N$ .

*Definition 7.1.* A controlled approximation for  $H$  is a sequence  $(K_i, M_i, T_i)$  where

- $K_i \leq G$  is a finite index subgroup with Goursat triplet  $[K_i] = [Q_i, N_i, \alpha_i]$ ,
- $M_i \leq N$  is a subgroup with  $H \leq N_G(M_i)$ , and
- $T_i \subset M_i$  is a subset

such that

- (1) the subgroups  $Q_i$  Chabauty converge to  $Q_H$ ,
- (2) the maps  $\alpha_i|_{Q_H}$  are consistent with  $\alpha_H$ ,
- (3) the sequence  $T_i$  exhausts the subgroup  $N$ ,
- (4) the subgroups  $N_i$  satisfy

$$\text{for all } i \in \mathbb{N} \quad N_H \cap T_i = N_i \cap T_i \quad \text{and} \quad N_H \cap M_i \leq N_i \cap M_i. \quad (7.1)$$

If the sequence  $K_i$  is a controlled approximation for the subgroup  $H$  then by items (3) and (4) the sequence  $N_i = K_i \cap N$  Chabauty converges to the subgroup  $N_H$ . Since  $Q_H$  is finitely generated we may assume without loss of generality that  $Q_H \leq Q_i$  for all  $i \in \mathbb{N}$  and the restrictions  $\alpha_i|_{Q_H}$  are well defined. Note that if the subgroup  $Q_H$  has infinite index in the group  $Q$  then  $Q_H$  will be strictly contained in  $Q_i$  for all  $i \in \mathbb{N}$ .

We will frequently abuse notation and simply refer to the sequence  $K_i$  of finite index subgroups as a controlled approximation, without an explicit mention of the remaining data (that is, the subgroups  $M_i$  and the subsets  $T_i$ ).

*7.2. Adapted sequences of transversals.* The following two definitions introduce a companion notion to that of controlled approximations.

*Definition 7.2.* The sequence  $A_i$  of finite subsets of  $G$  is adapted to the sequence  $B_i$  of finite subsets of the group  $G$  if for any finite subset  $\Phi \subset N$  and for every element  $g \in G$ ,

$$\frac{|\{b \in B_i : [g, b] + \Phi \subset A_i\}|}{|B_i|} \rightarrow 1. \quad (7.2)$$

*Definition 7.3.* A sequence of finite-to-one transversals  $F_i \subset G$  of the subgroups  $N_G(K_i)$  is adapted to the controlled approximation  $(K_i, M_i, T_i)$  if:

- $F_i = \widehat{I}_i P_i$  with  $I_i \subset Q$  and  $P_i \subset N$  for all  $i \in \mathbb{N}$ ;
- the sequences  $I_i$  and  $P_i$  exhaust the groups  $Q$  and  $N$  respectively; and
- the sequence of subsets  $T_i$  is adapted to the sequence  $\widehat{I}_i$ .

In our application it will so happen that  $T_i = P_i$ . This coincidence is not mandatory for the proof of Theorem 7.5 given below.

*7.3. From controlled approximations to Weiss approximations.* A controlled approximation is in some sense an algebraic notion, while a Weiss approximation is statistical in nature. For metabelian groups it is nevertheless possible to go from one notion to the other.

LEMMA 7.4. *If  $(K_i, M_i, T_i)$  is a controlled approximation of the subgroup  $H$  then*

$$T_i + (N_H \cap M_i) \subset N \setminus (N_H \Delta N_i)$$

where  $N_i = K_i \cap N$  for all  $i \in \mathbb{N}$ .

*Proof.* Consider an element  $x \in N$  equal to  $x = t + n$  where  $t \in T_i$  and  $n \in N_H \cap M_i$ . From equation (7.1) of Definition 7.1 we have that  $t \notin N_H \Delta N_i$  and  $n \in N_H \cap N_i$  for all  $i \in \mathbb{N}$ . There are two cases to consider.

- If  $t \in N_H \cap N_i$  then  $x = t + n \in N_H \cap N_i$ .
- If  $t \notin N_H \cup N_i$  then  $x \notin N_H \cup N_i$  since  $t = x - n$  and  $n \in N_H \cap N_i$ .

In either case  $x \notin N_H \Delta N_i$ . □

THEOREM 7.5. *Let  $(K_i, M_i, T_i)$  be a controlled approximation to the subgroup  $H$ , and let  $F_i$  be an adapted sequence of finite-to-one transversals of  $N_G(K_i)$  in  $G$ . If  $[N : N_N(H)] < \infty$ , then the sequence  $(K_i, F_i)$  is a Weiss approximation for  $H$ .*

*Proof.* We wish to prove that the sequence  $(K_i, F_i)$  is indeed a Weiss approximation for the subgroup  $H$ . With this goal in mind fix an element  $g = \widehat{q}n \in G$  for some pair of elements  $q \in Q$  and  $n \in N$ . Relying on the explicit condition given in Proposition 3.7, we need to verify that

$$p_i(g) = \frac{|\{f \in F_i : g^f \in K_i \Delta H\}|}{|F_i|} \xrightarrow{i \rightarrow \infty} 0. \tag{7.3}$$

There are two separate cases to deal with, depending on whether the element  $q$  belongs to the subgroup  $Q_H$  or not.

The first case to deal with is where  $q \notin Q_H$ . Then by item (1) of Definition 7.1 we deduce that  $q \notin Q_i$  as well for all  $i \in \mathbb{N}$  sufficiently large. As the group  $Q$  is abelian, this means that  $g^f \notin K_i \Delta H$  for all elements  $f \in G$  whatsoever and certainly for all  $f \in F_i$ . In particular,  $p_i(g) = 0$  for all  $i \in \mathbb{N}$  sufficiently large.

The second and more interesting case is where  $q \in Q_H$ . By item (2) of Definition 7.1, the restrictions  $\alpha_i|_{Q_H}$  are consistent with the homomorphism  $\alpha_H$ . Therefore there is an element  $a_q \in N$  such that for all  $i \in \mathbb{N}$  we have

$$\alpha_H(q) = \widehat{q}(a_q + N_H) \quad \text{and} \quad \alpha_i(q) = \widehat{q}(a_q + N_i).$$

To estimate the quantity  $p_i(g)$  consider a given element  $f \in F_i = \widehat{I}_i P_i$  where

$$f = \widehat{r}m, \quad r \in I_i, \quad \text{and} \quad m \in P_i.$$

Recall that Proposition 5.6 gives an explicit criterion for the conjugate  $g^f$  to be contained in either subgroup  $H$  or  $K_i$  of  $G$ . As we assumed that  $q \in Q_H$  and  $i$  is sufficiently large so that  $q \in Q_i$ , this criterion implies that

$$g^f \in H \Delta K_i \quad \text{if and only if} \quad [g, f] + \delta \in N_H \Delta N_i. \tag{7.4}$$

where  $\delta = n - a_q \in N$  depends only on  $g$ .

Since  $q \in Q_H$  there some element  $h \in H$  such that  $hN = gN$ . Equation (1.2) implies that  $[g, m] = [h, m]$ . Therefore Corollary 5.5 supplies a finite subset  $\Phi_g \subset N$  so that

$$[g, f] = [g, m] + [g, \widehat{r}] = [h, m] + [g, \widehat{r}] \in [g, \widehat{r}] + \Phi_g + (N_H \cap M_i). \tag{7.5}$$

On the other hand, Lemma 7.4 tells us that

$$T_i + (N_H \cap M_i) \subset N \setminus (N_H \Delta N_i). \tag{7.6}$$

By putting together equations (7.4), (7.5) and (7.6) we infer that

$$[g, \widehat{r}] + (\Phi_g + \delta) \subset T_i \text{ implies } g^f \in G \setminus (H \Delta K_i). \tag{7.7}$$

To conclude the proof of the case where  $q \in Q_H$ , we are required to show that  $p_i(g) \xrightarrow{i \rightarrow \infty} 0$ . Relying on equation (7.7), this would follow from

$$\frac{|\{r \in I_i : [g, \widehat{r}] + (\Phi_g + \delta) \subset T_i\}|}{|I_i|} \xrightarrow{i \rightarrow \infty} 1. \tag{7.8}$$

By assumption, the sequence  $F_i$  of finite-to-one transversals is adapted to the controlled approximation  $(K_i, M_i, T_i)$  in the sense of Definition 7.3. This means that the sequence  $T_i$  is adapted to the sequence of lifts  $\widehat{T}_i$  in the sense of Definition 7.2. Finally, observe that equation (7.8) follows as a special case of equation (7.2). □

### 8. Permutational metabelian groups

Let  $Q$  be a finitely generated abelian group. A  $Q$ -set is a set admitting an action of the group  $Q$ . Let  $X$  be a fixed  $Q$ -set.

8.1. *Permutational modules.* Let  $B$  be a finitely generated abelian group. The *permutational module* corresponding to the  $Q$ -set  $X$  and with base group  $B$  is the abelian group

$$B^X = \bigoplus_{x \in X} B^x \tag{8.1}$$

where for every  $x \in X$  the direct summand  $B^x$  is an isomorphic copy of  $B$ . The group  $Q$  is acting on  $B^X$  by group automorphisms by permuting coordinates. A permutational module is a module in the standard sense over the group ring  $\mathbb{Z}[Q]$ . The  $\mathbb{Z}[Q]$ -module  $B^X$  is finitely generated if and only if the  $Q$ -set  $X$  admits finitely many  $Q$ -orbits.

For every subset  $Y \subset X$  we will denote

$$B^Y = \bigoplus_{x \in Y} B^x. \tag{8.2}$$

The group  $B^Y$  is a direct summand of the permutational module  $B^X$  with its additive group structure. Similarly, given a subset  $C$  of  $B$  and a subset  $Y$  of  $X$ , we will denote

$$C^Y = \{b \in B^Y : b^x \in C \text{ for all } x \in Y\}. \tag{8.3}$$

*Remark 8.1.* We caution the reader that, opposite to what is customary, we use the notation  $B^X$  for a direct sum rather than a direct product.



PROPOSITION 8.2. *Let  $K \triangleleft Q$  be the kernel of the  $Q$ -action on  $X$ . Then the subgroup  $K$  acts trivially on the permutational module  $B^X$ .*

*Proof.* Immediate from the definitions. □

PROPOSITION 8.3. *Let  $R$  be a subgroup of  $Q$ . If the subset  $Y \subset X$  is  $R$ -invariant then the subgroup  $B^Y$  of the permutational module  $B^X$  is  $R$ -invariant.*

The above means that  $B^Y$  is a  $\mathbb{Z}[R]$ -submodule of the module  $B^X$ .

*Proof of Proposition 8.3.* Note that an element  $b \in B^X$  lies in  $B^Y$  if and only if  $b_i = 0$  for all  $i \in X \setminus Y$ . If the subset  $Y \subset X$  is  $R$ -invariant then this condition is  $R$ -invariant as well. □

A  $Q$ -factor of the  $Q$ -set  $X$  is a  $Q$ -set  $\bar{X}$  admitting a  $Q$ -equivariant surjective map  $f : X \rightarrow \bar{X}$ . A transversal to a  $Q$ -factor map  $f$  is a subset  $Y \subset X$  such that the restriction of  $f$  to  $Y$  is a bijection onto  $\bar{X}$ .

LEMMA 8.4. *Let  $f : X \rightarrow \bar{X}$  be a  $Q$ -factor map with transversal  $Y \subset X$ .*

- (1) *There is a surjective  $\mathbb{Z}[Q]$ -module homomorphism  $\pi_f : B^X \rightarrow B^{\bar{X}}$ .*
- (2) *The restriction of  $\pi_f$  to the subgroup  $B^Y$  is an isomorphism onto  $B^{\bar{X}}$ .*
- (3) *Let  $L$  be a subgroup of  $B^{\bar{X}}$ . Then  $T \subset B^Y$  is a finite-to-one transversal for  $\pi_f^{-1}(L)$  in  $B^X$  if and only if  $\pi_f(T)$  is a finite-to-one transversal for  $L$  in  $B^{\bar{X}}$ .*

This elementary lemma crucially relies on the base group  $B$  being abelian. Note the great similarity to [Grü57, Lemma 3.2].

*Proof.* The surjective homomorphism  $\pi_f$  is given by the natural isomorphisms

$$\pi|_{B^x} : B^x \xrightarrow{\cong} B^{f(x)}$$

on each coordinate. Since the base group  $B$  is abelian,  $\pi_f$  is well defined. Moreover, the restriction of  $\pi_f$  to  $B^Y$  is clearly an isomorphism onto  $B^{\bar{X}}$ . This shows (1) and (2). Item (3) immediately follows as the subgroup  $B^Y$  is a complement to the subgroup  $\ker \pi_f$  in  $B^X$ , and relying on the isomorphism theorem for groups. □

For example, if  $R$  is a (normal) subgroup of  $Q$  then the space of  $R$ -orbits  $R \backslash X$  is a  $Q$ -factor of  $X$ .

8.2. *Permutational wreath products.* A permutational metabelian group is a metabelian group  $G$  admitting a short exact sequence

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

with both groups  $N$  and  $Q$  being abelian and such that the action of  $Q$  on  $N$  is a permutational module with respect to some  $Q$ -set  $X$ . A permutational metabelian group  $G$  is finitely generated if and only if  $Q$  is finitely generated and the  $Q$ -set  $X$  admits finitely many orbits.

A *permutational wreath product* with respect to the  $Q$ -set  $X$  and with base group  $B$  is a split permutational metabelian group, namely

$$G = B \wr_X Q \cong Q \ltimes N$$

where  $N$  is the permutational module  $B^X$  with its additive group structure. We will regard the permutational wreath product  $G$  as an extension of the group  $Q$  by the group  $N$ .

A special case of the above class of groups is the *wreath product*  $G = B \wr Q$ . This is simply the permutational wreath product  $B \wr_Q Q$  where the group  $Q$  is acting on itself by translations. This group is of course split as well.

The *free metabelian group* is an example of a non-split permutational metabelian group. In fact, if  $x_1, \dots, x_d$  is an ordered generating set for the free metabelian group  $\Phi$  then its derived subgroup  $[\Phi, \Phi]$  is a free  $\mathbb{Z}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ -module with basis given by the set of commutators  $\{[x_i, x_j] : i < j\}$ . This follows from the *commutator collecting process* introduced by Philip Hall in [Hal34].

8.3. *Commutator identities.* Let  $G$  be a split metabelian group. The standard commutator identity as in equation (1.2) implies that

$$[qn, rm] = [q, rm]^n + [n, rm] = [q, r]^m + [q, m] + [n, r]^m + [n, m] = [q, m] - [r, n] \tag{8.4}$$

for any choice of elements  $q, r \in Q$  and  $n, m \in N$ .

### 9. Finitely generated abelian groups

Let  $Q$  be a finitely generated abelian group. The structure theorem for finitely generated abelian groups allows us to decompose  $Q$  as a direct sum  $Q = A \oplus T$  where  $A$  is a free abelian group of rank  $d$  for some  $d \in \mathbb{N} \cup \{0\}$  and  $T$  is torsion. Fix an arbitrary basis  $\Sigma \subset Q$  for the free abelian group  $A$ .

*Definition 9.1.* Given an element  $q \in Q$  with

$$q = t \prod_{\sigma \in \Sigma} \sigma^{n_\sigma}, \quad n_\sigma \in \mathbb{Z}, \quad t \in T,$$

define the *seminorm*  $\|q\|_\Sigma$  of  $q$  with respect to the basis  $\Sigma$  as

$$\|q\|_\Sigma = \max_{\sigma \in \Sigma} |n_\sigma|.$$

Note that  $\|\cdot\|_\Sigma$  is a well-defined norm on the torsion-free part  $A \leq Q$  while  $\|t\|_\Sigma = 0$  for all  $t \in T$ . The definition of  $\|\cdot\|_\Sigma$  clearly depends on the choice of the particular basis  $\Sigma$ .

Roughly speaking, balls with respect to  $\|\cdot\|_\Sigma$  look like ‘cubes’ in the group  $Q$ . The ball  $\{q \in Q : \|q\|_\Sigma \leq k\}$  has ‘sides’ of length  $2k + 1$ , which is always an odd number. For our purposes, we will require ‘cubes’ which look almost like  $\|\cdot\|_\Sigma$ -balls but allow for arbitrary ‘sides’.

*Definition 9.2.* The subset  $B_Q(k, \Sigma)$  of the group  $Q$  is given by

$$B_Q(k, \Sigma) = \left\{ t \prod_{\sigma \in \Sigma} \sigma^{n_\sigma} : -\lfloor k/2 \rfloor < n_\sigma \leq \lfloor k/2 \rfloor \text{ for all } \sigma \in \Sigma, t \in T \right\}$$

for every  $k \in \mathbb{N}$ .

The subset  $B_Q(k, \Sigma)$  coincides with a  $\| \cdot \|_\Sigma$ -ball with for  $k$  odd, while for  $k$  even it is ‘off by 1’. The torsion part  $T$  is contained in  $B_Q(k, \Sigma)$  for every  $k \in \mathbb{N}$ .

For every  $k \in \mathbb{N}$ , let  $Q[k]$  denote the subgroup generated by the  $k$ th powers of all elements of  $Q$ . The subgroup  $Q[k]$  has finite index and is characteristic in  $Q$ . The following is essentially an immediate consequence of the above definitions.

**PROPOSITION 9.3.** *Let  $k \in \mathbb{N}$ . The subset  $B_Q(k, \Sigma)$  is a finite-to-one transversal to the subgroup  $Q[k]$  for every  $k \in \mathbb{N}$ . In fact  $B_A(k, \Sigma)$  is a (one-to-one) transversal to  $A[k]$  where  $A$  is the torsion-free part of the group  $Q$ .*

Universal sequences of transversals were introduced in Proposition 3.2 above.

**COROLLARY 9.4.** *The sequence  $B_Q(k!, \Sigma)$  is a universal sequence of transversals in the group  $Q$ .*

*Proof.* This follows immediately from the observation that any finite index subgroup  $R \leq Q$  with  $d = [Q : R]$  satisfies  $Q[d] \leq R$ . □

The subsets  $B_Q(k, \Sigma)$  are all centered at the identity element of the group  $Q$ . This behavior is captured in the following definition that will play an important role in the following section.

*Definition 9.5.* A sequence  $I_i$  of finite subsets of the group  $Q$  is *centered* if for any element  $r \in Q$ ,

$$\frac{|\{q \in I_i : r \in qI_i\}|}{|I_i|} \xrightarrow{i \rightarrow \infty} 1. \tag{9.1}$$

We leave the easy verification of the following fact to the reader.

**PROPOSITION 9.6.** *Let  $k_i \in \mathbb{N}$  be any sequence with  $\lim k_i = \infty$ . Then  $B_Q(k_i, \Sigma)$  is a centered exhausting Følner sequence in the group  $Q$ .*

### 10. Approximations in permutational wreath products

Let  $G$  be a finitely generated permutational wreath product

$$G = B \wr_X Q$$

where  $B$  and  $Q$  are finitely generated abelian groups and  $X$  is a  $Q$ -set. Since the group  $G$  is finitely generated,  $X$  admits finitely many  $Q$ -orbits; see §8. We emphasize that  $G$  is split. In fact,

$$G = Q \ltimes N \quad \text{where } N = B^X.$$

Our current goal is the following construction.

**THEOREM 10.1.** *Let  $R$  be a fixed subgroup of  $Q$ . Then  $G$  admits a Følner sequence  $F_i$  with the following property. If  $H$  is any subgroup of  $G$  with  $R = Q_H$  and  $[N : N_N(H)] < \infty$  then  $H$  has a controlled approximation  $(K_i, M_i, T_i)$  such that  $F_i$  is an adapted sequence of finite-to-one transversals of  $N_G(K_i)$  for all  $i \in \mathbb{N}$  sufficiently large.*

Controlled approximations and adapted sequences of finite-to-one transversals were introduced in Definition 7.1 and 7.3, respectively.

The proof of Theorem 10.1 will consist of several consecutive lemmas. Every lemma will build on the previous ones. We will allow ourselves to use any notation and objects introduced in this section without explicit mention.

**10.1. Chabauty approximations.** Let  $R$  be a fixed subgroup of  $Q$ . Choose an arbitrary basis  $\Sigma$  for the torsion-free part of  $Q$ .

Let  $r \in \mathbb{N}$  denote the number of the  $Q$ -orbits in the  $Q$ -set  $X$  and  $X = \coprod_{l=1}^r X_l$  be the corresponding orbit decomposition. Choose an arbitrary point  $x_l \in X_l$  for every  $l \in \{1, \dots, r\}$ . In particular,  $X_l \cong Q/S_l$  where  $S_l = \text{Stab}_Q(x_l) \leq Q$ .

**LEMMA 10.2.** *There is a monotone sequence  $n_i \in \mathbb{N}$  such that*

- (1) *the finite index subgroups  $Q_i = R + Q[n_i]$  Chabauty converge to  $R$ ,*
- (2) *the subgroups  $Q_i$  satisfy  $Q_i = R \oplus V_i$  for some subgroups  $V_i \leq Q$ ,*
- (3) *the subsets  $I_i = B_Q(\Sigma, n_i)$  form a centered exhausting Følner sequence of finite-to-one transversals of the subgroups  $Q_i$ , and*
- (4) *the  $Q$ -factor maps  $f_i : X \rightarrow V_i \backslash X$  admit  $R$ -invariant transversals  $Y_i \subset X$  for all  $i \in \mathbb{N}$ .*

Roughly speaking,  $B_Q(\Sigma, n)$  is a ball of diameter  $n$  based at the identity in the seminorm  $\|\cdot\|_\Sigma$  with respect to the basis  $\Sigma$ . See Definition 9.2 for details.

*Proof of Lemma 10.2.* By the structure theorem for finitely generated abelian groups there is a direct sum decomposition

$$Q = U \oplus V$$

such that  $R$  is a finite index subgroup of  $U$ , the torsion subgroup  $T$  is contained in  $U$ , and the subgroup  $V$  is torsion-free. In particular, there is some  $m \in \mathbb{N}$  such that  $U[m] \leq R$ . The same theorem allows us to find direct sum decompositions

$$V = W_l \oplus W'_l$$

such that  $R + S_l$  is a finite index subgroup of  $U \oplus W_l$  for every  $l \in \{1, \dots, r\}$ . So there are integers  $m_l \in \mathbb{N}$  such that  $(U \oplus W_l)[m_l] \leq R + S_l$ . Take

$$k = \text{lcm}(m, m_1, \dots, m_r).$$

Define  $n_i = ik$  and  $Q_i = R + Q[n_i]$  for all  $i \in \mathbb{N}$ . Note that

$$Q_i = R \oplus V_i \quad \text{where } V_i = V[n_i].$$

The subgroups  $Q_i$  Chabauty converge to the subgroup  $R$ . This concludes the proof of items (1) and (2).

Item (3) concerning the subsets  $I_i = B_Q(\Sigma, n_i)$  follows immediately by combining Propositions 9.3 and 9.6.

It remains to establish item (4). Note that

$$R + S_l + Q[n_i] = Q_i + S_l = (R + S_l) \oplus W'_{i,l}$$

where  $W'_{i,l} = W'_l[n_i]$  for every  $i \in \mathbb{N}$  and  $l \in \{1, \dots, r\}$ . Note that  $S_l + V_i = S_l \oplus W'_{i,l}$ . Moreover,  $X_l = Q/S_l$  and

$$V_i \setminus X_l = V_i \setminus Q/S_l \cong Q/(S_l + V_i) \cong Q/(S_l \oplus W'_{i,l})$$

for all  $l \in \{1, \dots, r\}$ . Consider the subsets

$$Y_i = \prod_{l=1}^r Y_{i,l} \quad \text{where } Y_{i,l} = (U + W_l + I_i)x_l.$$

Since  $R \leq U$ , it is clear that all subsets  $Y_{i,l}$  as well as the subset  $Y_i$  are  $R$ -invariant. Moreover, every subset  $Y_{i,l}$  is a transversal to the restricted  $Q$ -factor map

$$X_l \cong Q/S_l \rightarrow Q/(S_l \oplus W'_{i,l}) \cong V_i \setminus X_l.$$

This concludes the proof. □

Denote  $Z_i = \prod_{j=1}^r I_i x_j$ . As  $I_i$  exhausts  $Q$ , it follows that  $X = \bigcup_i Z_i$ . The inclusions  $Z_i \subset Y_i \subset X$  hold for all  $i \in \mathbb{N}$ . The subsets  $Z_i$  are always finite. The subsets  $Y_i$  are infinite if and only if  $R$  admits infinite orbits.

10.2. *Adapted sequences.* The metabelian group  $G$  is split. In particular, we may identify each  $I_i$  with its lift, regarding it as a subset of  $G$  for every  $i \in \mathbb{N}$ .

Let  $\Delta$  be an arbitrary basis for the torsion-free part of the base group  $B$ . Denote

$$E_i = B_B(i!, \Delta)$$

for all  $i \in \mathbb{N}$ . Therefore  $E_i$  is a centered exhausting Følner sequence in the finitely generated abelian group  $B$ ; see Proposition 9.6. The reason for using  $i!$  instead of just  $i$  will become clear in the proof of Lemma 10.9 below; cf. Corollary 9.4.

Recall that the subset  $E^Z \subset B^X$  was defined in the obvious way for a pair of subsets  $E \subset B$  and  $Z \subset X$ ; see equation (8.3) and Remark 8.1.

LEMMA 10.3. *The sequence  $T_i = E_i^{Z_i}$  is adapted to the sequence  $I_i$ .*

*Proof.* Let  $g \in G$  be a fixed element with  $g = qn$  where  $q \in Q$  and  $n \in N$ . Let  $\Phi \subset N$  be any finite subset. We are required to show that

$$\frac{|\{r \in I_i : [g, r] + \Phi \subset T_i\}|}{|I_i|} \xrightarrow{i \rightarrow \infty} 1. \tag{10.1}$$

There is a finite subset  $D \subset B$  and an index  $j \in \mathbb{N}$  such that  $\{n, -n\} \cup \Phi \subset D^{Z_j}$ . Recall that  $Z_i = \prod_{j=1}^r I_i x_j$ . Combined with the fact that  $I_i$  is centered, this gives

$$\frac{|\{r \in I_i : rZ_j \subset Z_i\}|}{|I_i|} \xrightarrow{i \rightarrow \infty} 1. \tag{10.2}$$

Moreover,  $Z_j \subset Z_i$  for all  $i \in \mathbb{N}$  sufficiently large (in fact, this happens for all  $i \geq j$ ). Since  $E_i$  is an exhausting sequence in the group  $B$  we have  $D + D + D \subset E_i$  for all  $i \in \mathbb{N}$  sufficiently large.

It follows from equation (8.4) that the commutator  $[g, r]$  is equal to

$$[g, r] = [qn, r] = [n, r] = n - n^r \tag{10.3}$$

for every element  $r \in I_i$ . Therefore equation (10.3) implies

$$[g, r] + \Phi \subset D^{Z_j} + D^{rZ_j} + D^{Z_j} \subset (D + D + D)^{Z_j \cup rZ_j}. \tag{10.4}$$

Observe that the two conditions  $D + D + D \subset E_i$  and  $Z_j \cup rZ_j \subset Z_i$  imply that  $[g, r] + \Phi \subset T_i = B_i^{Z_i}$  according to equation (10.4). The result follows from this observation and making use of equation (10.2). □

10.3. *Controlled approximations.* Let  $H \leq G$  be a subgroup with Goursat triplet  $[H] = [R, N_H, \alpha_H]$ . From now on we will denote  $M_i = B^{Y_i}$ .

LEMMA 10.4. *There is a sequence of finite index subgroups  $N_i \leq N$  satisfying*

$$Q_i \leq N_G(N_i), \quad N_H \cap T_i = N_i \cap T_i, \quad \text{and} \quad N_H \cap M_i \leq N_i \cap M_i \tag{10.5}$$

for all  $i \in \mathbb{N}$ .

*Proof.* Consider the  $Q$ -sets  $\bar{X}_i = V_i \setminus X$  with the associated  $Q$ -factor maps  $f_i : X \rightarrow \bar{X}_i$ . These give rise to surjective  $\mathbb{Z}[Q]$ -module homomorphisms  $\pi_i : B^X \rightarrow B^{\bar{X}_i}$  as in Lemma 8.4. Consider the auxiliary permutational wreath product group

$$\Gamma_i = Q_i \ltimes B^{\bar{X}_i}$$

and its subgroup

$$\bar{H}_i = Q_i \pi_i(H \cap M_i) \leq \Gamma_i,$$

defined for every  $i \in \mathbb{N}$ . The subgroups  $\Gamma_i$  are finitely generated, as each  $Q_i$ , being a finite index subgroup of  $Q$ , acts on  $\bar{X}_i$  with finitely many orbits.

Since the subsets  $Y_i \subset X$  are  $R$ -invariant, we have  $R \leq N_G(H \cap M_i)$  for all  $i \in \mathbb{N}$  according to Proposition 8.3. In addition,  $V_i$  is acting trivially on the  $Q$ -factor set  $\bar{X}_i$  and on  $B^{\bar{X}_i}$ ; see Proposition 8.2. As  $Q_i = R \oplus V_i$  by item (2) of Lemma 10.2, this shows that the subsets  $\bar{H}_i$  are indeed subgroups.

Take  $\bar{T}_i = \pi_i(T_i)$ . We may apply Proposition 5.2 with respect to the subgroup  $\bar{H}_i$  and the finite subset  $\bar{T}_i$  of the finitely generated metabelian group  $\Gamma_i$ . This results in a finite index subgroup  $\bar{N}_i$  of  $B^{\bar{X}_i}$  normalized by  $Q_i$ , containing the subgroup  $\pi_i(H \cap M_i)$  and satisfying  $\bar{H}_i \cap \bar{T}_i = \bar{N}_i \cap \bar{T}_i$ .

We now pull back the subgroup  $\bar{N}_i$  by letting  $N_i = \pi_i^{-1}(\bar{N}_i)$  so that the subgroup  $N_i$  has finite index in  $N$ . It follows from Lemma 8.4 that the subgroups  $N_i$  satisfy equation (10.5) as required. □

LEMMA 10.5. *Let  $g \in G$  be any element. Then  $[v, g] \in N_i$  for all  $i \in \mathbb{N}$  and for all elements  $v \in V_i$ .*

*Proof.* We may assume that  $g = qm$  for some  $q \in Q$  and  $m \in N$ . Fix some index  $i \in \mathbb{N}$  and let  $v \in V_i$  be any element. It follows from equation (8.4) that

$$[v, g] = [v, qm] = [v, m].$$

Since  $\ker \pi_i \leq N_i$ , the subgroup  $V_i$  lies in the kernel of the  $Q$ -action on the quotient  $N/N_i$ ; see Proposition 8.2. We conclude that  $[v, m] \in N_i$  as required.  $\square$

LEMMA 10.6. *The subgroup  $H$  admits a controlled approximation  $(K_i, M_i, T_i)$  so that the finite index subgroups  $K_i \leq G$  admit Goursat triplets  $[K_i] = [Q_i, N_i, \alpha_i]$ .*

*Proof.* In light of equation (10.5), the last remaining step in constructing the controlled approximation  $(K_i, M_i, T_i)$  for the subgroup  $H$  is to determine the injective homomorphisms  $\alpha_i$ . We may apply Corollary 6.3 and obtain a sequence of injective homomorphisms

$$\alpha'_i : Q_H \rightarrow N_G(N_i)/N_i$$

defined for all  $i \in \mathbb{N}$  sufficiently large and consistent with  $\alpha_H$ . Up to discarding finitely many terms and reindexing, we may assume that  $\alpha'_i$  is defined for all  $i \in \mathbb{N}$ . Let  $\alpha_i$  be the extension of each homomorphism  $\alpha'_i$  to  $Q_i = Q_H \oplus V_i$  determined by the condition

$$\alpha_i(q) = qN_i$$

for all  $q \in V_i$ . The fact that  $V_i \leq Q_i \leq N_G(N_i)$  for all  $i \in \mathbb{N}$  implies that the image of each map  $\alpha_i$  belongs to  $N_G(N_i)$ .

To see that every  $\alpha_i$  is indeed a group homomorphism, we are required to verify that  $\alpha_i(r)$  and  $\alpha_i(v)$  commute in the group  $N_G(N_i)/N_i$  for every pair of elements  $r \in R$  and  $v \in V_i$ . This is an immediate consequence of Lemma 10.5.  $\square$

From now on we will refer to the sequence  $(K_i, M_i, T_i)$  constructed above as the *standard controlled approximation* to the subgroup  $H$ .

LEMMA 10.7. *Consider the standard controlled approximation  $K_i$ . Let  $m \in N$  be any element. If  $[q, m] \in N_i$  for all  $q \in Q_i$  then  $m \in N_N(K_i)$ .*

*Proof.* Fix an index  $i \in \mathbb{N}$  and an element  $g \in K_i$ . We are required to show that  $g^m \in K_i$  with respect to the element  $m \in N$  as in the statement of lemma.

The element  $g$  can be written as  $g = q(a_q + n)$  where the elements  $q \in Q_i$ ,  $n \in N_i$ , and  $a_q \in N$  satisfy  $\alpha_i(q) = q(a_q + N_i)$ . It follows from equation (8.4) that

$$g^m = g[g, m] = g[q(a_q + n), m] = g[q, m] = q(a_q + n + [q, m]);$$

cf. Proposition 5.6. If  $[q, m] \in N_i$  then  $g^m \in K_i$  as required.  $\square$

LEMMA 10.8. *The standard controlled approximation  $(K_i, M_i, T_i)$  to the subgroup  $H$  satisfies  $N_N(H) \cap M_i \leq N_N(K_i)$  for all  $i \in \mathbb{N}$ .*

*Proof.* Fix an  $i \in \mathbb{N}$  and consider any element  $m \in N_N(H) \cap M_i$ . Taking into account Lemma 10.7, it suffices to show that  $[q, m] \in N_i$  for all  $q \in Q_i$ .

Consider some fixed element  $q \in Q_i$ . Write  $q = rv$  for some uniquely determined elements  $r \in R = Q_H$  and  $v \in V_i$  (we are using a multiplicative notation for the group  $Q$ ).

The fact that  $R \leq N_G(M_i)$  combined with  $m \in M_i$  implies  $[r, m] \in M_i$ . In addition, as  $R = Q_H$  there exists an element  $h \in H$  with  $h = rl$  for some  $l \in N$ . As  $m \in N_N(H)$  we have from equation (8.4) that

$$[r, m] = [rl, m] = [h, m] \in H \cap N = N_H.$$

It follows from equation (7.1) that  $[r, m] \in M_i \cap N_H \leq N_i$ .

We know that  $[v, m] \in N_i$  from Lemma 10.5. Therefore both commutators  $[v, m]$  and  $[r, m]$  belong to  $N_i$ . We may apply equation (1.2) and obtain

$$[q, m] = [vr, m] = [v, m]^r + [r, m] \in N_i \leq K_i \leq N_N(K_i), \tag{10.6}$$

as required. We used the fact that  $R \leq Q_i \leq N_G(N_i)$  to deduce  $[v, m]^r \in N_i$  in the above equation (10.6). □

10.4. *Følner sequences of transversals.* Recall that  $H$  is a fixed subgroup of  $G$  with  $R = Q_H$ , and  $(K_i, M_i, T_i)$  is the standard controlled approximation for  $H$ .

LEMMA 10.9. *If  $[N : N_N(H)] < \infty$  then  $F_i = I_i T_i = I_i E_i^{Z_i}$  are finite-to-one transversals of  $N_G(K_i)$  in  $G$  for all  $i \in \mathbb{N}$  sufficiently large.*

*Proof.* Since  $X = \bigcup_i Z_i$  the subgroups  $B^{Z_i}$  exhaust the subgroup  $N$ . Therefore

$$d = [N : N_N(H)] = [B^{Z_i} : N_N(H) \cap B^{Z_i}] \tag{10.7}$$

for all  $i \in \mathbb{N}$  sufficiently large. From this point and until the end of the this proof, let  $i \in \mathbb{N}$  be any index satisfying equation (10.7) and with  $i \geq d$ .

Recall that  $E_i = B_B(i!, \Delta)$  is a universal sequence of transversals in the abelian group  $B$ . In fact, the proof of Corollary 9.4 shows that  $T_i = E_i^{Z_i}$  is a finite-to-one transversal to any subgroup of  $B^{Z_i}$  whose index divides  $i!$ . It is therefore a consequence of equation (10.7) that the subset  $T_i$  is a finite-to-one transversal to  $N_N(H)$  in  $N$ . As  $T_i \subset M_i$ , an elementary group-theoretic argument shows that  $T_i$  is at the same time a finite-to-one transversal to  $N_N(H) \cap M_i$  in  $M_i$ .

We have previously established that  $N_N(H) \cap M_i \leq N_N(K_i)$ ; see Lemma 10.8. Therefore  $T_i$  is a finite-to-one transversal to  $N_N(K_i) \cap M_i$  in  $M_i$ . The normalizer  $N_N(K_i)$  satisfies

$$\ker \pi_i \leq N_i \leq N_N(K_i),$$

where  $\pi_i : B^X \rightarrow B^{\bar{X}_i}$  is the  $\mathbb{Z}[Q]$ -module homomorphism appearing in the proof of Lemma 10.4. Item (3) of Lemma 8.4 shows therefore that  $T_i$  is a finite-to-one transversal to the subgroup  $N_N(K_i)$  in  $N$ .

As  $K_i \leq N_G(K_i)$ , the subset  $I_i$  is a finite-to-one transversal to the projection  $Q_i$  of  $N_G(K_i)$  to the quotient group  $Q$ . We conclude, relying on Proposition 4.3, that  $F_i = I_i T_i = I_i E_i^{Z_i}$  is indeed a finite-to-one transversal to  $N_G(K_i)$  in  $G$ . □



LEMMA 10.10. For any element  $n \in N$  and any  $\varepsilon > 0$  there is an index  $j = j(n, \varepsilon) \in \mathbb{N}$  so that  $T_i = E_i^{Z_i}$  is  $(n^q, \varepsilon)$ -invariant for all indices  $i > j$  and all elements  $q \in Q$  satisfying  $n^q \in T_i$ .

*Proof.* Fix some  $\varepsilon > 0$ . Let  $n = \bigoplus_{x \in X} b_x$  be any element, where  $b_x \in B$ . Recall that  $X = \bigcup_i Z_i$ . Therefore there is an index  $i_0 \in \mathbb{N}$  such that  $b_x = 0$  for all  $x \in X \setminus Z_{i_0}$ . Since  $E_i$  is a Følner sequence in the group  $B$ , we may choose  $j = j(n, \varepsilon)$  such that  $E_i$  is  $(b_x, \varepsilon/|Z_{i_0}|)$ -invariant for all indices  $i > j$  and all elements  $b_x \in B$  with  $x \in Z_{i_0}$ .

Consider an element  $q \in Q$  satisfying  $n^q \in T_i$  for some index  $i > j$  as in the statement of the lemma. In particular,  $qZ_{i_0} \subset Z_i$ . We claim that the set  $T_i$  is  $(n^q, \varepsilon)$ -invariant. Indeed, the size of the symmetric difference  $n^q T_i \Delta T_i$  satisfies

$$|n^q T_i \Delta T_i| \leq |E_i|^{|Z_i|-1} \sum_{x \in Z_{i_0}} |b_{qx} E_i \Delta E_i| \leq |Z_{i_0}| \cdot \frac{\varepsilon}{|Z_{i_0}|} \cdot |T_i| = \varepsilon |T_i|$$

as required. □

LEMMA 10.11.  $F_i = I_i T_i = I_i E_i^{Z_i}$  is a Følner sequence in the group  $G$ .

*Proof.* Fix a constant  $\varepsilon > 0$  and an element  $g \in G$ . Write  $g = qn$  for some elements  $q \in Q$  and  $n \in N$ . We will show that the subsets  $F_i$  are  $(g, \varepsilon)$ -invariant for all  $i \in \mathbb{N}$  larger than some  $i_0 = i_0(g, \varepsilon) \in \mathbb{N}$ .

Let  $\varepsilon' > 0$  be a sufficiently small constant so that  $(1 - 2\varepsilon')^2 > 1 - \varepsilon$ . Since  $I_i$  is a Følner sequence in the group  $Q$  there is an index  $i_1 \in \mathbb{N}$  such that the subset  $I_i$  is  $(q, \varepsilon')$ -invariant for all  $i > i_1$ . Since the sequence  $I_i$  is centered, we may argue exactly as in Lemma 10.3 to find an index  $i_2 \in \mathbb{N}$  such that

$$|J_i| > (1 - \varepsilon')|I_i| \quad \text{where } J_i = \{r \in I_i : n^r \in T_i\}$$

for all  $i > i_2$ . Finally, by Lemma 10.10 there is some index  $i_3 \in \mathbb{N}$  such that  $T_i$  is  $(n^r, \varepsilon')$ -invariant provided  $n^r \in T_i$  and for all  $i > i_3$ . Take  $i_0 = \max\{i_1, i_2, i_3\}$ .

We claim that the subset  $F_i = I_i T_i$  is indeed  $(g, \varepsilon)$ -invariant for all indices  $i > i_0$ . Fix such an index  $i \in \mathbb{N}$ . There is a subset  $I'_i \subset I_i$  such that

$$I'_i \subset qI_i \cap I_i \quad \text{and} \quad |I'_i| > (1 - \varepsilon')|I_i|.$$

For every element  $r \in J_i$  there is a subset  $T'_i(r) \subset T_i$  satisfying

$$T'_i(r) \subset (n^r + T_i) \cap T_i \quad \text{and} \quad |T'_i(r)| > (1 - \varepsilon')|T_i|.$$

The translate  $gF_i$  can be expressed as

$$gF_i = gI_i T_i = qn \bigcup_{r \in I_i} rT_i = q \bigcup_{r \in I_i} r(n^r + T_i) = \bigcup_{r \in I_i} qr(n^r + T_i). \tag{10.8}$$

It follows from equation (10.8) and from the above discussion that

$$\bigcup_{r \in I'_i \cap J_i} rT'_i(r) \subset gF_i \cap F_i. \tag{10.9}$$

Using equations (10.8) and (10.9), we can estimate the size of  $gF_i \cap F_i$  to be at least

$$|gF_i \cap F_i| \geq (1 - 2\varepsilon')|I_i|(1 - \varepsilon')|T_i| > (1 - 2\varepsilon')^2|F_i| > (1 - \varepsilon)|F_i|,$$

as required. □

*Remark 10.12.* Lemmas 10.9 and 10.11 are to be compared with [Wei01]; see also Corollary 3.13 and Proposition 4.4 above. Weiss’s results are of a similar nature but more general as they apply to arbitrary group extensions and provide a universal sequence of transversals.

However, Lemmas 10.9 and 10.11 are more precise, in the sense that we do not need to pass to any subsequences whatsoever. This issue is crucial for our purposes, since passing to a subsequence would violate Lemma 10.3.

10.5. *Summary.* Let  $R$  be a fixed subgroup of  $Q$ , and let  $H \leq G$  be any subgroup with  $R = Q_H$  and with  $[N : N_N(H)] < \infty$ . We put together all of the above propositions, culminating in a proof of Theorem 10.1 stated in the beginning of §10.

The main part of the argument is contained in Lemmas 10.4 and 10.6. This is where we construct the controlled approximation  $(K_i, M_i, T_i)$  for the subgroup  $H$ . Here the  $K_i \leq G$  are finite index subgroups with Goursat triplets  $[K_i] = [Q_i, N_i, \alpha_i]$ , the subgroups  $M_i \leq N$  are given by  $M_i = B^{Y_i}$ , and the subsets  $T_i \subset M_i$  are given by  $T_i = E_i^{Z_i} \subset M_i$ .

The sequence  $F_i = I_i T_i$  is Følner in the group  $G$  and consists of finite-to-one transversals of the subgroups  $K_i$ . This is established in Lemmas 10.9 and 10.11. The sequence of subsets  $T_i$  is adapted to the sequence  $I_i$  by Lemma 10.3.

We have verified all of the necessary requirements for  $(K_i, M_i, T_i)$  to be a controlled approximation to the subgroup  $H$  with an adapted sequence of finite-to-one transversals  $F_i$ ; see Definitions 7.1 and 7.3. In our case the subsets  $T_i$  and  $P_i$  that appear in these two definitions happen to coincide. This completes the proof of Theorem 10.1. □

### 11. Invariant random subgroups of permutational wreath products

We are ready to present a proof of our main result, Theorem 1.4. It is restated below for the reader’s convenience.

*Theorem.* Let  $G$  be a permutational wreath product of two finitely generated abelian groups. Then every invariant random subgroup of  $G$  is co-sofic.

*Proof.* In light of Corollary 2.6 it will suffice to prove the theorem for ergodic invariant random subgroups.

Let  $\mu \in \text{IRS}_{\text{ergodic}}(G)$  be any ergodic invariant random subgroup of  $G$ . Consider the map  $\pi_* : \text{IRS}_{\text{ergodic}}(G) \rightarrow \text{IRS}_{\text{ergodic}}(Q)$ ; see Proposition 2.9. Since  $Q$  is abelian,  $\pi_*\mu$  is atomic. In other words, there is some fixed subgroup  $R \leq Q$  such that  $\mu$ -almost every subgroup  $H \leq G$  satisfies  $R = Q_H$ . Moreover,  $[N : N_N(H)] < \infty$  holds for  $\mu$ -almost every subgroup  $H \leq G$  according to Proposition 5.3.

The group  $G$  admits a Følner sequence  $F_i$  such that  $\mu$ -almost every subgroup  $H \leq G$  has a controlled approximation with finite index subgroups  $K_i$  and with  $F_i$  being an

adapted sequence of finite-to-one transversals of  $N_G(K_i)$ . This is precisely the content of Theorem 10.1 applied with respect to the subgroup  $R \leq Q$ .

The existence of these controlled approximations implies that  $\mu$ -almost every subgroup  $H$  of  $G$  is Weiss approximable, as was established in Theorem 7.5.

We conclude that the invariant random subgroup  $\mu$  is co-sofic, relying on the ergodic theorem for amenable groups; see Theorem 3.10.  $\square$

The permutation stability of a finitely generated permutational wreath product of two finitely generated abelian groups follows from the fact that all invariant random subgroups in such a group are co-sofic and relying on Theorem 1.2.

*Acknowledgements.* The authors would like to thank Oren Becker for stimulating discussions and, in particular, for bringing to our attention Corollary 3.13.

The first named author is indebted to the NSF for support. The second named author is indebted to the NSF, the ISF and the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement No. 692854) for support.

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