Book reviews

Models and Games by Jouko Väänänen, Cambridge University Press, Cambridge Studies in Advanced Mathematics Series 132, 2011. Hardcover, ISBN 978-0-521-51812-3, 367 pp.
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1 Introduction

Models and Games by Jouko Väänänen is a book that examines logic from a game-theoretic perspective. Games are nowadays considered to be a standard tool in mathematical logic. They are easy to describe and use, and for this reason they have gained wide acceptance in logic during the second half of the 20th century. Proofs involving games are easier to state and understand and may lead to novel insights and to other unexpected results.

Models and Games gives a broad and detailed coverage of the use of games in mathematical logic. The logics that are studied in the book under the game-theoretic viewpoint are *first-order logic* (Chapters 1 to 6), *infinitary logics* (Chapters 7 to 9) and finally *logics with generalized quantifiers* (Chapter 10). There are three main games that are used throughout the book in order to study the above logics:

- The *Semantic Game*. This game is played by two players, namely Player I (the Doubter) and Player II (the Believer). This game captures the notion of *truth* in logic.
- The *Model Existence Game*. This game is again played by two players, but now Player I does not doubt the truth of a set of formulas but instead its *consistency* (which Player II believes).
- The *Ehrenfeucht–Fraïssé Game*. In this game Player I is trying to demonstrate that there is a difference between two structures, while Player II tries to establish that there is no such difference.

Apart from their simplicity and mathematical beauty, the above games are demonstrated to possess a significant power and versatility for the study of different types of logics.

2 Contents

The book consists of 10 chapters followed by an extensive bibliography and index.

Chapter 1 is a short (two-page) introduction to the main theme of the book, namely the three kinds of games that appear in logic: the *Semantic Game*, the *Model Existence Game* and the *Ehrenfeucht–Fraïssé Game*. A short discussion is given regarding the purpose of each game and their interconnections.

Chapter 2 gives a concise presentation of the basic set-theoretic notions that will be used throughout the book: countable and uncountable sets, equipollence of sets (namely the property of having the same cardinality), ordinals, cardinals and a short discussion on the axiom of choice. Of particular interest is the way that ordinals are introduced as the ranks of two-person games. This appears as a more entertaining way to introduce and motivate ordinals than the standard set-theoretic approach.

Chapter 3 introduces all the necessary material regarding two-person games of perfect information. The presentation starts with the case of finite games. The important notion of *determinacy* is defined, and Zermelo's Theorem (that every finite two-person game of perfect information is determined) is demonstrated. Infinite games are then considered and the important subclasses of open, closed and clopen games are introduced. Gale-Stewart's Theorem that every open (or closed) game is determined, is established. This is possibly one of the few points of the book that I felt that some more background material would be interesting to read. For example, it would be nice to state (without details) Martin's Theorem that every Borel game is determined, a result which generalizes Gale-Stewart's Theorem.

Chapter 4 gives an intuitive introduction to the three games of the book in the context of the first-order (FO) language of graphs. In this context, the Ehrenfeucht–Fraïssé Game is particularly easy to introduce, since graphs are very familiar objects for both mathematicians and computer scientists. Moreover, the proofs of the theorems are much easier to illustrate and understand. The most important result of this chapter is the Ehrenfeucht–Fraïssé theorem that two graphs satisfy the same sentences of quantifier rank $\leqslant n$ iff Player II has a winning strategy in the Ehrenfeucht–Fraïssé Game of n rounds.

Chapter 5 presents the extension of the Ehrenfeucht–Fraïssé Game to the context of arbitrary structures (and not just graphs as in Chapter 4). The main themes of this chapter are *isomorphism* and *partial isomorphism* of structures. The chapter starts by presenting the familiar notion of *structure* and then considers many examples of *isomorphic structures*. Two structures may not be exactly isomorphic but may be quite close to each other; for this reason, the notion of *partial isomorphism* is needed. In order to prove that two structures are partially isomorphic, one can use the technical tool of *back-and-forth sets*. Back and forth sets are actually equivalent to the infinite Ehrenfeucht–Fraïssé Game (Proposition 5.21 of the book). Therefore, it is argued that both the game and the back-and-forth sets can be used to establish partial isomorphism of two structures. Actually, the game is a more intuitive approach, while back-and-forth sets can be used to provide a rigorous and formal proof of partial isomorphism. Finally, *back-and-forth sequences* are defined and it is demonstrated that they correspond to the finite version of the Ehrenfeucht–Fraïssé Game.

Chapter 6 presents many well-known results regarding first-order logic through a game-theoretic perspective. The chapter starts by giving a short introduction to the syntax and semantics of first-order logic. It then discusses how *elementary equivalence* of two structures can be established based on back-and-forth sequences (and therefore by the Ehrenfeucht–Fraïssé Games). Back-and-forth sequences are also shown to be useful in demonstrating that certain properties of structures are not definable in FO (such as, for example, the property of a structure having an infinite universe). The Cub Game of D. Kueker is presented and used to derive the Löwenheim–Skolem Theorem. The Semantic Game and the Model Existence Game for FO are defined (it would be nice if the latter game was illustrated using a few examples, as is done for the Semantic Game). Then the Model Existence Game is used to establish certain well-known results of FO (such as the Compactness Theorem, the Omitting Types Theorem and so on). The chapter concludes with examining the case where the underlying vocabulary of structures is uncountable.

Chapter 7 is devoted to the study of infinitary logic through the use of games. The logic $L_{\infty\omega}$ that is presented allows the use of infinite conjunctions and disjunctions (while the rest of its syntax is similar to that of FO). The Semantic Game for $L_{\infty\omega}$ is almost identical to that for FO. On the other hand, the Ehrenfeucht–Fraïssé Game for $L_{\infty\omega}$ is more complicated than the one for FO since it involves the use of ordinals (the basic difference is that during a play of the former game, Player I "moves down an ordinal"). It is demonstrated that both the Semantic and the Ehrenfeucht–Fraïssé Games can be employed as useful tools for the study and analysis of $L_{\infty\omega}$.

Chapter 8 presents the model theory of infinitary logics. Initially, the Löwenheim–Skolem Theorem for $L_{\infty\omega}$ is presented based on the Cub Game. Subsequently, the infinitary logic $L_{\omega_1\omega}$ is considered, namely the extension of first-order logic, which is obtained by allowing countable disjunctions and conjunctions. The Model Existence Game for $L_{\omega_1\omega}$ is defined and the Model Existence Theorem is proved. Based on the Model Existence Theorem, the undefinability of well-order and the Craig Interpolation Theorem for $L_{\omega_1\omega}$ are derived. Then the study of the model theory of the infinitary logic $L_{\kappa^+\omega}$ is initiated, where κ^+ is a successor cardinal with $\kappa > \omega$. The chapter concludes with the presentation and study of Game Logic, which involves formulas that have an alternation of quantifiers that can best be explained through the use of an appropriate Semantic Game.

Chapter 9 considers stronger infinitary logics that permit the use of infinite quantifiers. In particular, the logic $L_{\infty\lambda}$ is presented, where λ is a cardinal. Roughly speaking, $L_{\infty\lambda}$ allows quantification over sets of variables of cardinality less than λ; in addition, it also allows conjunctions and disjunctions of sets of formulas over a fixed set of free variables of cardinality less than λ. Variations of the Ehrenfeucht–Fraïssé Game are defined which are appropriate for this type of logics.

Chapter 10 considers and studies extensions of first-order logic that use generalized quantifiers. In many cases in mathematics, but also in computer science and linguistics, the need for a more general form of quantification arises. Examples of such generalized quantifiers include the *finiteness quantifier* $\exists^{<\omega}$ ("there exist finitely many"), the *infinity quantifier* $\exists^{\geqslant\omega}$ ("there exist infinitely many"), the *counting quantifier* $\exists^{\geqslant n}$ ("there exist at least n") and so on. For every generalized quantifier Q one can create a corresponding extension of first-order logic. It turns out that the Ehrenfeucht–Fraïssé Games can be used to study such extensions of FO. The chapter concludes by presenting various results regarding generalized quantifier logics (such as, for example, the Compactness Theorem for the logic $L_{\omega\omega}(\exists^{>2^\omega})$ whose generalized quantifier allows us to express "there exist more than the reals").

3 Assessment and conclusions

The *Models and Games* by Jouko Väänänen is an interesting book that demonstrates the deep connections between logic and game theory. To my knowledge, there are no other similar books that cover these connections to such a breadth and depth. The book is carefully written, it has many examples and a wealth of exercises (more than 550!). In the first few chapters, many of the basic proofs are illustrated by examples or by first considering and discussing easier special cases; the full proof is then much easier to grasp.

As a computer scientist, I mostly enjoyed reading Chapters 1 to 6 as well as Chapter 10. Chapters 7, 8 and 9 are more technical than the other ones; especially, Chapter 9 reflects current research directions in the area of infinitary logics (and possibly the recent personal research interests of the author himself).

I find the book ideal for a computer science graduate course on logic from a game-theoretic perspective. In such a course, the instructor could use the material in Chapters 1 through 6, which cover most of the mainstream topics in mathematical logic from a game theory viewpoint; possibly, such a course could also cover parts of Chapter 7 for a brief introduction to infinitary logics as well as part of Chapter 10, which covers generalized quantifier logics (that appear to have immediate applications in both computer science and linguistics). Chapters 8 and 9 seem more appropriate as part of a graduate course in mathematical logic (with emphasis on infinitary logics).

Finally, I believe that the book is especially interesting for researchers that work in the area of logic programming. Game theory is nowadays an important semantic tool for functional and imperative programming languages (Abramsky and McCusker 1999). Despite the fact that certain (sparse) works in the theory of logic programming have advocated a game-theoretic approach (see, for example, van Emden 1986), I believe that game theory still has to play an important role in the future development of logic programming.

In conclusion, the *Models and Games* by Jouko Väänänen is an interesting book that, while written by a logician, has lots of things to offer to computer scientists with theoretical inclinations.

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