J. Inst. Math. Jussieu (2021) **20**(3), 859–891
doi:10.1017/S1474748019000355

ON LOCAL ENERGY DECAY ESTIMATE OF THE OSEEN SEMIGROUP IN TWO DIMENSIONS AND ITS APPLICATION

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(Received 11 March 2019; revised 21 May 2019; accepted 24 May 2019; first published online 1 July 2019)

Abstract We study the temporal decay estimate of the Oseen semigroup in a two-dimensional exterior domain. We establish the local energy decay estimate with a suitable dependence on the small translation speed, which is a significant improvement of Hishida's result in 2016. As an application, we prove the L^q-L^r estimates of the Oseen semigroup uniformly in the small translation speed.

Keywords: Oseen semigroup; two-dimensional flows past an obstacle; local energy decay; resolvent analysis

2010 *Mathematics subject classification*: Primary 76D07
Secondary 35B35

1. Introduction

In this paper, we study the Oseen equation in a two-dimensional exterior domain:

$$\begin{cases} \partial_t u - \Delta u + \alpha \partial_1 u + \nabla p_u = 0, & \int_{\Omega_4} p_u \, dx = 0, \quad t > 0, \ x \in \Omega, \\ \operatorname{div} u = 0, & t \geqslant 0, \ x \in \Omega, \\ u|_{\partial\Omega} = 0, & t > 0, \\ u|_{t=0} = \mathbb{P}_\Omega f, & x \in \Omega. \end{cases} \tag{1.1}$$

Here, the fluid domain Ω is assumed to be an unbounded domain in $\mathbb{R}^2$ with a smooth and compact boundary, $u = u(t,x) = (u_1(t,x), u_2(t,x))$ and $p_u = p_u(t,x)$, $x = (x_1, x_2)$, are respectively the unknown velocity field and the pressure field of the fluid, $f = (f_1(x), f_2(x))$ is a given vector field, and $\mathbb{P}_\Omega$ is the Helmholtz–Leray projection to the space of solenoidal vector fields, which is defined precisely later. The complement of the domain Ω represents the obstacle and is normalized in the following sense: $\operatorname{diam}(\mathbb{R}^2 \setminus \Omega) = 1$ and the origin of the coordinates is located interior to $\mathbb{R}^2 \setminus \Omega$. The set Ω_4 is defined as $\Omega_4 = \Omega \cap \{|x| \leqslant 4\}$. The number α is a positive constant, which represents a background constant flow in x_1 direction. Physically, it also represents a translation speed of the obstacle. We use the standard notation for derivatives: $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, $\Delta = \sum_{j=1}^2 \partial_j^2$, and $\operatorname{div} u = \sum_{j=1}^2 \partial_j u_j$.

System (1.1) is a fundamental linearized problem when one considers the flow around a rigid body translating with a constant speed. In this paper, we are interested in a global-in-time estimate for solutions to (1.1) when α is small.

To state the results, let us first introduce the basic function spaces used in this paper. We denote by $L^q(\Omega)$, $1 \leqslant q \leqslant \infty$, the usual Lebesgue space of all measurable functions whose L^q norm, $\|f\|_{L^q} = (\int_\Omega |f|^q\,dx)^{\frac{1}{q}}$ for $q < \infty$ and $\|f\|_{L^\infty} = \mathrm{ess.sup}_{x\in\Omega}|f(x)|$ for $q = \infty$, is finite. The class of smooth and compactly supported functions in Ω is denoted by $C_0^\infty(\Omega)$ and the class of test functions for solenoidal vector fields in Ω is defined by $C_{0,\sigma}^\infty(\Omega) = \{f \in C_0^\infty(\Omega)^2 \mid \operatorname{div} f = 0 \text{ in } \Omega\}$. The space of all L^q solenoidal vector fields in Ω is denoted by $L^q_\sigma(\Omega)$, which is characterized for $1 < q < \infty$ as $L^q_\sigma(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{\|f\|_{L^q}} = \{f \in L^q(\Omega)^2 \mid \operatorname{div} f = 0 \text{ in } \Omega,\ f\cdot n = 0 \text{ on } \partial\Omega\}$. Here, $n = n(x)$ is the exterior unit normal vector at $x \in \partial\Omega$. The L^q Sobolev space of order k in Ω is denoted by $W^{k,q}(\Omega)$, and we also introduce the space $W_0^{1,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|f\|_{W^{1,q}}}$. As is well known, for $q \in (1,\infty)$, the space $L^q(\Omega)^2$ is written as the direct sum $L^q(\Omega) = L^q_\sigma(\Omega) \oplus G^q(\Omega)$, where $G^q(\Omega) = \{\nabla p \in L^q(\Omega)^2 \mid p \in L^q_{loc}(\overline{\Omega})\}$. Then the Helmholtz projection $\mathbb{P}_\Omega : L^q(\Omega)^2 \to L^q_\sigma(\Omega)$ is well defined, which is an orthogonal projection when $q = 2$; see [17] or [18] for details. For simplicity, the Helmholtz projection in $\mathbb{R}^2$ is denoted by $\mathbb{P}$ instead of $\mathbb{P}_{\mathbb{R}^2}$. For $q \in (1,\infty)$, the second order elliptic operators $A_{\alpha,\Omega}$ in $L^q(\Omega)$ is defined by

$$D_{L^q}(A_{\alpha,\Omega}) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), \quad A_{\alpha,\Omega} f = -\Delta f + \alpha\partial_1 f, \quad f \in D_{L^q}(A_{\alpha,\Omega}).$$

Then the Oseen operators $\mathbb{A}_{\alpha,\Omega}$ in $L^q_\sigma(\Omega)$ is defined by

$$\begin{aligned} &D_{L^q}(\mathbb{A}_{\alpha,\Omega}) = W^{2,q}(\Omega)^2 \cap W_0^{1,q}(\Omega)^2 \cap L^q_\sigma(\Omega),\\ &\mathbb{A}_{\alpha,\Omega} f = \mathbb{P}_\Omega A_{\alpha,\Omega} f. \end{aligned}$$

To simplify the notations, the counterparts of these operators in $L^q(\mathbb{R}^2)^2$ or $L^q_\sigma(\mathbb{R}^2)$ are written as A_α or $\mathbb{A}_\alpha$ instead of $A_{\alpha,\mathbb{R}^2}$ or $\mathbb{A}_{\alpha,\mathbb{R}^2}$.

When $\alpha = 0$, the operator $\mathbb{A}_\Omega = \mathbb{A}_{0,\Omega}$ is called the Stokes operator, and it is well known that, for $1 < q < \infty$, $-\mathbb{A}_\Omega$ generates a bounded C_0-analytic semigroup in $L^q_\sigma(\Omega)$; cf. [5]. The reader is also referred to [2], [3], and [1] for a recent progress in the L^∞ theory of the Stokes semigroup in an exterior domain. The global L^q-L^r estimate of the Stokes semigroup $\{e^{-t\mathbb{A}_\Omega}\}_{t\geqslant 0}$ has been a fundamental tool in this research field, and it is known that

$$\|e^{-t\mathbb{A}_\Omega} f\|_{L^q(\Omega)} \leqslant \frac{C_{q,r}}{t^{\frac{1}{r}-\frac{1}{q}}}\|f\|_{L^r(\Omega)}, \quad t > 0, \quad f \in L^r_\sigma(\Omega) \tag{1.2}$$

holds for $1 < r \leqslant q < \infty$ or $1 < r < q = \infty$. Indeed, the case $1 < r \leqslant q < \infty$ is proved by P. Maremonti and V. Solonnikov [16] and W. Dan and Y. Shibata [6], while the case $1 < r < q = \infty$ is proved by W. Dan and Y. Shibata [7].

When $\alpha \neq 0$, the difficulty arises in obtaining the global estimate due to the parabolic distribution of the spectrum $\sigma(-\mathbb{A}_{\alpha,\Omega}) \subset \{\lambda \in \mathbb{C} \mid |\Im(\lambda)|^2 \leqslant -\alpha^2\Re(\lambda)\}$ and also due to a two-dimensionality. Since the term $\alpha\mathbb{P}_\Omega\partial_1 f$ is of lower order, it is not difficult to show that the perturbed operator $-\mathbb{A}_{\alpha,\Omega}$ also generates a C_0-analytic semigroup in $L^q_\sigma(\Omega)$. So the difficulty lies in the estimate for large time. In fact, it is very recent that the following L^q-L^r estimates are established for $e^{-t\mathbb{A}_{\alpha,\Omega}}$ in a pioneering work by T. Hishida [12].

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Theorem 1.1 [12]. *Let $1 < r \leqslant q < \infty$. Fix $M > 0$ and let $\alpha \in [0, M]$. Then we have for any $f \in L^r_\sigma(\Omega)$,*

$$\|e^{-t\mathbb{A}_{\alpha,\Omega}} f\|_{L^q(\Omega)} \leqslant \frac{C_{q,r}}{\alpha^\kappa t^{\frac{1}{r}-\frac{1}{q}}} \|f\|_{L^r(\Omega)}, \quad t > 0. \tag{1.3}$$

Here the constant C depends only on M, q, r, and Ω, while $\kappa > 1$ depends only on q and r.

Our interest here is the singularity $O(\alpha^{-\kappa})$ in (1.3) for small $\alpha > 0$. This singularity does not appear for a higher-dimensional case. Indeed, for n-dimensional exterior problem with $n \geqslant 3$, it is known that

$$\|e^{-t\mathbb{A}_\Omega} f\|_{L^q(\Omega)} \leqslant \frac{C_{q,r}}{t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}} \|f\|_{L^r(\Omega)}, \quad t > 0,\ f \in L^r_\sigma(\Omega) \tag{1.4}$$

holds for $1 < r \leqslant q < \infty$. Estimate (1.4) is proved by T. Kobayashi and Y. Shibata [14] when $n = 3$ and by Y. Enomoto and Y. Shibata [8, 9] when $n \geqslant 4$. The key difference between the cases $n = 2$ and $n \geqslant 3$ is that when $n = 2$ the Oseen term $\alpha\partial_1$ leads to a drastic change of the decay structure for the kernel of the resolvent operator. For example, this is directly seen for the kernel of $\mathbb{A}_\alpha^{-1}$ and $\mathbb{A}^{-1}$; the kernel of $\mathbb{A}^{-1}$ contains a logarithmic growth term $\log|x|$, while the kernel of $\mathbb{A}_\alpha^{-1}$ decays at spatial infinity in the slow variable αx that is a key in [10, 11] for the existence of physically reasonable solutions in a two-dimensional exterior domain. This drastic difference of the structure in small resolvent parameters λ shows that the analysis of (1.1) for $t \gg 1$ exhibits a nature of the singular perturbation when $\alpha \to 0$. Note that the limit $\lambda \to 0$ is also considered as a singular limit from the resolvent to the endpoint of the continuous spectrum and, thus, the real difficulty lies in the joint limit $\alpha, \lambda \to 0$ (in particular, $\lambda = i\mu$ with $\mu \in \mathbb{R} \to 0$, as explained below). As a summary, the difficulties in analyzing (1.1) for large time are listed as follows.

(0) (Exterior domain) the presence of the nontrivial compact boundary,
(i) (Two-dimensionality) the possible logarithmic singularity of the resolvent near the origin, which is also related to the estimate $\|e^{-t\mathbb{A}_\alpha} f\|_{L^\infty(\mathbb{R}^2)} \leqslant Ct^{-1}\|f\|_{L^1(\mathbb{R}^2)}$ that is not integrable over $(1, \infty)$,
(ii) (Time dependence with large time) the parabolic distribution of the spectrum $\sigma(-\mathbb{A}_{\alpha,\Omega}) \subset \{\lambda \in \mathbb{C} \mid |\Im(\lambda)|^2 \leqslant -\alpha^2\Re(\lambda)\}$,
(iii) (Singular perturbation) the nature of the singular perturbation in the joint limit $\alpha \to 0$ and $\lambda = i\mu \to 0$ for the resolvent problem.

As for **(0)**, we note that, in the whole space case, it is straightforward from the explicit formula that $\|e^{-t\mathbb{A}_\alpha} f\|_{L^q(\mathbb{R}^2)} \leqslant Ct^{-\frac{1}{r}+\frac{1}{q}}\|f\|_{L^r(\mathbb{R}^2)}$ holds for $t > 0$, $f \in L^r_\sigma(\mathbb{R}^2)$, and $1 \leqslant r \leqslant q \leqslant \infty$. The difficulties **(0)** and **(i)** are common with the analysis of the Stokes semigroup ($\alpha = 0$), while **(ii)** and **(iii)** are specific to the case $\alpha \neq 0$. It should be emphasized that, as stated in **(ii)**, the difficulty is related to the time dependence and the problem is more delicate than the steady case. Indeed, what is needed for the unsteady case is the analysis of the behavior of $(i\mu + \mathbb{A}_{\alpha,\Omega})^{-1}$ when $\mu \in \mathbb{R} \to 0$, rather than the

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analysis of the fixed operator $\mathbb{A}_{\alpha,\Omega}^{-1}$ (or the limit of $(\mu+\mathbb{A}_{\alpha,\Omega})^{-1}$ as $\mu \in \mathbb{R}_+ \to 0$) as in the steady problem. Note that we should distinguish the behaviors of $(\mu+\mathbb{A}_{\alpha,\Omega})^{-1}$ and $(i\mu+\mathbb{A}_{\alpha,\Omega})^{-1}$ when $\mu \in \mathbb{R}_+ \to 0$; for the former one, the distance from the continuous spectrum is μ, while for the latter one, the distance from the continuous spectrum is $O(\mu^2)$ when $\alpha \neq 0$ due to the parabolic distribution. Therefore, one is potentially faced with a stronger singularity in the analysis of the unsteady problem for $\alpha \neq 0$. Roughly speaking, the difficulties **(0)**, **(i)**, and **(ii)** are overcome in [12], and the contribution of this paper is to resolve **(iii)**.

To obtain the L^q-L^r estimates of $\{e^{-t\mathbb{A}_{\alpha,\Omega}}\}_{t\geqslant 0}$, the key step is to establish the local energy decay estimate of $e^{-t\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega f$, which is a “local–local” estimate in the sense that we aim the estimate of $e^{-t\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega f$ near the boundary $\partial\Omega$ when f is compactly supported. The local energy decay estimate is a common tool in the exterior problem; for the Stokes semigroup, it was established by H. Iwashita [13] for the 3D case and by W. Dan and Y. Shibata [6] for the 2D case, while for the Oseen semigroup, it was obtained by T. Kobayashi and Y. Shibata [14] for the 3D case, by Y. Enomoto and Y. Shibata [8] for the higher-dimensional case, and by T. Hishida [12] for the 2D case (but with a singularity in small α in the 2D case). The first result of this paper is stated as follows, which is a significant improvement of [12, Theorem 2.1].

Theorem 1.2 (Local energy decay estimate). *Set $\Omega_4 = \Omega \cap \{|x| \leqslant 4\}$ and let $1 < q < \infty$. Then there exists a number $\delta_q \in (0, \frac{1}{2}]$ such that the following statements hold for all $\alpha \in (0, \delta_q]$. Assume that $f \in L^q(\Omega)^2$ and* supp $f \subset \{|x| \leqslant 5\}$. *Then for $j = 0, 1, 2$,*

$$\|\nabla^j e^{-t\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega f\|_{L^q(\Omega_4)} \leqslant \begin{cases} \dfrac{C}{t^{\frac{j}{2}}}\|f\|_{L^q(\Omega)}, & 0 < t \leqslant 3, \\ \left(\dfrac{C}{t|\log t|^2} + \dfrac{C\alpha^2}{|\log t|}\right)\|f\|_{L^q(\Omega)}, & 2 \leqslant t \leqslant \alpha^{-2}, \\ \dfrac{C}{t^2\alpha^2|\log \alpha|}\|f\|_{L^q(\Omega)}, & t \geqslant \alpha^{-2}, \end{cases} \tag{1.5}$$

and the associated pressure field $p[\mathbb{P}_\Omega f](t) = p_u(t)$, $\int_{\Omega_4} p[\mathbb{P}_\Omega f](t)\,dx = 0$, satisfies

$$\|p[\mathbb{P}_\Omega f](t)\|_{L^q(\Omega_4)} \leqslant \begin{cases} \dfrac{C}{t^{\frac{1}{2}(1+\frac{1}{q})}}\|f\|_{L^q(\Omega)}, & 0 < t \leqslant 3, \\ \left(\dfrac{C}{t|\log t|^2} + \dfrac{C\alpha^2}{|\log t|}\right)\|f\|_{L^q(\Omega)}, & 2 \leqslant t \leqslant \alpha^{-2}, \\ \dfrac{C}{t^2\alpha^2|\log \alpha|}\|f\|_{L^q(\Omega)}, & t \geqslant \alpha^{-2}, \end{cases} \tag{1.6}$$

Here, C depends only on q and Ω.

Remark 1.3. (i) The crucial point of Theorem 1.2 is that $\int_1^\infty \|\nabla^j e^{-t\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega f\|_{L^q(\Omega_4)}$ can be bounded uniformly in small α. This L^1 integrability in time is essential to obtain the global L^q-L^r estimate for the exterior problem through a standard cut-off argument connecting with the estimate of the Oseen semigroup in the whole space.

(ii) The order $O(\alpha^{-2})$ is a natural time scale in this problem. Heuristically, the time before $O(\alpha^{-2})$ is regarded as the Stokes scale, i.e., the transport term $\alpha\partial_1$ is a perturbation, while the time after $O(\alpha^{-2})$ is the Oseen scale, where the transport term plays a dominant role. By taking the formal limit $\alpha \to 0$ in (1.5), we obtain the estimate $\|\nabla^j e^{-t\mathbb{A}_\Omega}\mathbb{P}_\Omega f\|_{L^q(\Omega_4)} \leqslant \frac{C}{t|\log t|^2}\|f\|_{L^q(\Omega)}$ for $t \geqslant 2$, which is exactly the local energy decay estimate obtained by W. Dan and Y. Shibata [6] for the Stokes semigroup.

As an important application of Theorem 1.2, we obtain the L^q-L^r estimate of the Oseen semigroup $e^{-t\mathbb{A}_{\alpha,\Omega}}$ with a uniform bound for small $\alpha > 0$ as follows.

Theorem 1.4. *Let $1 < r \leqslant q < \infty$ and let $\alpha \in (0, \min\{\delta_r, \delta_q\}]$. Then it follows that for any $f \in L^r_\sigma(\Omega)$,*

$$\|e^{-t\mathbb{A}_{\alpha,\Omega}} f\|_{L^q(\Omega)} \leqslant C_{q,r} t^{-\frac{1}{r}+\frac{1}{q}} \|f\|_{L^r(\Omega)}, \quad t > 0. \tag{1.7}$$

If $1 < r \leqslant q < 2$, then

$$\|\nabla e^{-t\mathbb{A}_{\alpha,\Omega}} f\|_{L^q(\Omega)} \leqslant C_{q,r} t^{-\frac{1}{2}-\frac{1}{r}+\frac{1}{q}} \|f\|_{L^r(\Omega)}, \quad t > 0. \tag{1.8}$$

Here $C_{q,r}$ is a constant depending only on q, r, and Ω.

The proof of the local energy estimate in Theorem 1.2 proceeds along the line in [12]. The main object is the resolvent problem

$$\begin{cases} \lambda v - \Delta v + \alpha\partial_1 v + \nabla q = f, \quad \operatorname{div} v = 0, \quad x \in \Omega, \\ \qquad\qquad v|_{\partial\Omega} = 0, \end{cases} \tag{1.9}$$

where $f \in L^q(\Omega)^2$ with $\operatorname{supp} f \subset \{|x| \leqslant 5\}$ and $\lambda \in \mathbb{C}$ is a resolvent parameter. The force f is not needed to be solenoidal. It is known that (1.9) is uniquely solvable in L^q for $\lambda \in \{z \in \mathbb{C} \mid |\Im(z)|^2 > -\alpha^2\Re(z)\}$ (cf. [12]). We set $\Sigma_{\frac{3\pi}{4}} = \{z \in \mathbb{C}\setminus\{0\} \mid |\arg(z)| \leqslant \frac{3\pi}{4}\}$, and let us denote by χ_B the characteristic function of the set B. The solution of (1.9) is constructed by gluing the solution to the whole space problem and the solution in the bounded domain. To this end, we also introduce the bounded domain $D = \Omega \cap \{|x| < 5\}$, and let $\mathbb{A}_{\alpha,D}$ and $\mathbb{P}_D$, respectively, be the Oseen operator and the Helmholtz projection in the L^q space over the domain D. Note that since D is bounded, $\mathbb{A}_{\alpha,D}$ has a spectral gap uniformly in small α. Roughly speaking, we will show that there exists $\delta_q > 0$ such that if $\lambda \in \Sigma_{\frac{3\pi}{4}}$ with $|\lambda| \leqslant \delta_q$ and $|\Im(\lambda)|^2 > -\alpha^2\Re(\lambda)$ and if $0 < \alpha \leqslant \delta_q$, then $\chi_{\Omega_4}(\lambda + \mathbb{A}_{\alpha,\Omega})^{-1}\mathbb{P}_\Omega f$ has the expansion of the form

$$\begin{aligned} &\chi_{\Omega_4}(\lambda + \mathbb{A}_{\alpha,\Omega})^{-1}\mathbb{P}_\Omega f \\ &\quad = \chi_{\Omega_4}\mathbb{A}_{\alpha,D}^{-1}\mathbb{P}_D\Big(V_0 f + \frac{1}{\log(4\lambda + \alpha^2)}\big(W_{1,1} f + d(\alpha,\lambda) W_{1,2} f\big)\Big) + \mathcal{Q}_\alpha(\lambda) f, \end{aligned} \tag{1.10}$$

where V_0, $W_{1,1}$, $W_{1,2}$ are bounded operators on $L^q_{[5]}(\Omega)^2 = \{f \in L^q(\Omega)^2 \mid \operatorname{supp} f \subset \{|x| \leqslant 5\}\}$ and are independent of α and λ, while $\mathcal{Q}_\alpha(\lambda) f$ is a remainder satisfying the estimate

$$\|\mathcal{Q}_\alpha(\lambda) f\|_{L^q(\Omega_4)} \leqslant \frac{C}{|\log(4\lambda + \alpha^2)|^2} \|f\|_{L^q_{[5]}(\Omega)},$$

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and $d(\alpha, \lambda)$ is a complex number defined as

$$d(\alpha, \lambda) = \int_0^1 \frac{\alpha^2 s}{4\lambda + \alpha^2 s}\, ds.$$

Moreover, the right-hand side of (1.10) is shown to provide the analytic extension about λ to the sector $\Sigma_{\frac{3\pi}{4}}$ near the origin. Expansion (1.10) is a new and key achievement of this paper and describes the behavior of the localized resolvent when λ and α are small. It should be emphasized that, in the level of "local–local" estimate, expansion (1.10) solves not only the difficulty **(iii)** but also the difficulty **(ii)** through the analytic extension to the sector. The semigroup estimate in Theorem 1.2 is obtained by using the Dunford formula $\chi_{\Omega_4} e^{-t\mathbb{A}_{\alpha,\Omega}} \mathbb{P}_\Omega f = \frac{1}{2\pi t} \int_\Gamma e^{t\lambda} \chi_{\Omega_4} (\lambda + \mathbb{A}_{\alpha,\Omega})^{-1} \mathbb{P}_\Omega f \, d\lambda$ with a suitably chosen curve Γ and by applying the Cauchy theorem in the complex analysis.

The argument to show Theorem 1.4 from Theorem 1.2 is rather standard, which is based on a cut-off argument using the Bogovskii operator to recover the divergence-free condition. Another important application of Theorem 1.2 is given in the paper [15], where the asymptotic stability of the physically reasonable solution is proved when α is small enough. This is the first stability result of the stationary solutions constructed by R. Finn and D. R. Smith [11] in two dimensions, which has remained open for a long time since their famous work in the 1960s.

This paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.2 and is the core of this paper. In § 2.1, we study the resolvent kernel for the Oseen operator in $\mathbb{R}^2$ and give the expansion for small λ and α. In § 2.2, we state the estimate of the Oseen operator in the bounded domain D, which is more or less standard. In §§ 2.3 and 2.4, we establish the expansion (1.10) by using the argument of [12] and [6], and Theorem 1.2 is proved. Theorem 1.4 is shown in § 3. Some estimates used in the proof are collected in the appendix for the reader's convenience.

2. Local energy estimate

In this section, we prove Theorem 1.2.

2.1. Expansion of resolvent kernel in $\mathbb{R}^2$

We denote by $E^\alpha_\lambda(x)$ the function given as

$$E^\alpha_\lambda(x) = \int_0^\infty e^{-\lambda t} \Phi(t, x - \alpha t \mathbf{e}_1)\, dt, \quad \Re(\lambda) > 0. \tag{2.1}$$

Here, $\Phi(t, x)$ is the kernel of the Stokes semigroup in $\mathbb{R}^2$, and it is given in terms of the Fourier transform:

$$\Phi(t, x) = \mathcal{F}^{-1}\left[e^{-t|\xi|^2} \left(\mathbb{I} - \frac{\xi \otimes \xi}{|\xi|^2} \right) \right](x).$$

Here, $\mathbb{I}$ is the 2×2 identity matrix and $\xi \otimes \xi = (\xi_i \xi_j)_{1 \leqslant i,j \leqslant 2}$. By [12, equation (4.6)], the following representation of E^α_λ is obtained in terms of the modified Bessel functions: with

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the notation $z(s,x)=\sqrt{s(\lambda+(\frac{\alpha}{2})^2 s)}\,|x|$, we have

$$\begin{aligned}E^{\alpha}_{\lambda}(x)=\sum_{j=1}^{5}E^{\alpha}_{\lambda,j}(x) &= \frac{\mathbb{I}}{2\pi}e^{\frac{\alpha}{2}x_1}K_0\Bigg(\sqrt{\lambda+\Big(\frac{\alpha}{2}\Big)^2}\,|x|\Bigg)\\ &\quad-\frac{\mathbb{I}}{4\pi}\int_0^1 e^{\frac{\alpha}{2}x_1 s}K_0(z(s,x))\,ds\\ &\quad+\frac{x\otimes x}{4\pi|x|^2}\int_0^1 e^{\frac{\alpha}{2}x_1 s}z(s,x)K_1(z(s,x))\,ds\\ &\quad+\frac{(-\frac{\alpha}{2})\big(x\otimes\mathbf{e}_1+\mathbf{e}_1\otimes x\big)}{4\pi}\int_0^1 se^{\frac{\alpha}{2}x_1 s}K_0(z(s,x))\,ds\\ &\quad+\frac{(\frac{\alpha}{2})^2|x|^2\mathbf{e}_1\otimes\mathbf{e}_1}{4\pi}\int_0^1\frac{s^2e^{\frac{\alpha}{2}x_1 s}}{z(s,x)}K_1(z(s,x))\,ds. \qquad (2.2)\end{aligned}$$

Here, $K_0(z)$ and $K_1(z)$ are modified Bessel functions of the second kind of orders 0 and 1, respectively (cf. [12, equation (4.7)]), which have series representations as follows. Let $\psi(k)$, $k=1,2,\dots,$ be such that

$$\psi(k)=-\gamma-\frac{1}{k}+k\sum_{l=1}^{\infty}\frac{1}{l(l+k)},\qquad \gamma:\text{Euler's constant.}$$

In particular, $\psi(1)=-\gamma$. Then,

$$\begin{aligned}K_0(z) &= -\Big(\log\frac{z}{2}\Big)\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\Big(\frac{z}{2}\Big)^{2k}+\sum_{k=0}^{\infty}\frac{\psi(k+1)}{(k!)^2}\Big(\frac{z}{2}\Big)^{2k}\\ &= -\log\frac{z}{2}-\gamma+O\Bigg(\Big(\frac{z}{2}\Big)^2\log\frac{z}{2}\Bigg)\quad\text{for small }|z|, \qquad (2.3)\\ K_1(z) &= \frac{1}{z}+\sum_{k=0}^{\infty}\frac{1}{k!(k+1)!}\Big(\frac{z}{2}\Big)^{2k+1}\Big(\log\frac{z}{2}-\frac{1}{2}\psi(k+1)-\frac{1}{2}\psi(k+2)\Big)\\ &= \frac{1}{z}+O\Big(\frac{z}{2}\log\frac{z}{2}\Big)\quad\text{for small }|z|. \qquad (2.4)\end{aligned}$$

In particular, the right-hand side of (2.2) makes sense for $x\neq 0$ and either $\Im(\lambda)\neq 0$ or $\Re(\lambda)\geqslant 0$, and for each fixed $x\neq 0$, it is an analytic extension of (2.1) about λ to $\mathbb{C}\setminus\{\Re(\lambda)\leqslant 0\}$. We are interested in the case $|x|\leqslant 10$ and $|\lambda+\frac{\alpha^2}{4}|$ is small enough, which is the regime studied in the latter section. For $\theta\in(\pi/2,\pi)$, let Σ_θ be the sector in $\mathbb{C}$ with angle θ defined as

$$\Sigma_\theta=\{z\in\mathbb{C}\setminus\{0\}\mid|\arg z|\leqslant\theta\}. \qquad (2.5)$$

Then, by using the expansion $e^{\frac{\alpha}{2}x_1 s}=\sum_{k=0}^{\infty}\frac{1}{k!}(\frac{\alpha}{2}x_1 s)^k$ if necessary, we have the following expansion for each $E^{\alpha}_{\lambda,j}$:

$$E^{\alpha}_{\lambda,1}(x)=\frac{1}{2\pi}\Bigg(-\frac{1}{2}\log\Big(\lambda+\frac{\alpha^2}{4}\Big)-\log|x|+\log 2-\gamma\Bigg)\mathbb{I}+\tilde{E}^{\alpha}_{\lambda,1}(x),$$

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$$\begin{aligned}
E^\alpha_{\lambda,2}(x) &= \frac{1}{4\pi}\left(\int_0^1 \log(z(s,x))\,ds - \log 2 + \gamma\right)\mathbb{I} + \tilde E^\alpha_{\lambda,2}(x),\\
E^\alpha_{\lambda,3}(x) &= \frac{x\otimes x}{4\pi|x|^2} + \tilde E^\alpha_{\lambda,3}(x),\\
E^\alpha_{\lambda,4}(x) &= \frac{\left(-\frac{\alpha}{2}\right)(x\otimes \mathbf{e}_1 + \mathbf{e}_1\otimes x)}{4\pi}\int_0^1 s(-\log(z(s,x)) + \log 2 - \gamma)\,ds + \tilde E^\alpha_{\lambda,4}(x),\\
E^\alpha_{\lambda,5}(x) &= \frac{\left(\frac{\alpha}{2}\right)^2 \mathbf{e}_1\otimes\mathbf{e}_1}{4\pi}\int_0^1 \frac{s}{\lambda+\frac{\alpha^2}{4}s}\,ds + \tilde E^\alpha_{\lambda,5}(x).
\end{aligned}$$

Here, each $\tilde E^\alpha_{\lambda,j}(x)$ is a remainder whose leading part (worst term) is given by the sum of "a product of $e^{\frac{\alpha}{2}x_1}-1$ (or $e^{\frac{\alpha}{2}x_1 s}-1$) and the leading term of the modified Bessel function K_j" and "a constant multiple of the second leading term of K_j". In particular, they satisfy when $|x|\leqslant 10$ and $|\lambda+\frac{\alpha^2}{4}|\leqslant\frac{1}{2}$ with $\lambda\in\Sigma_{3\pi/4}$,

$$\begin{aligned}
|\tilde E^\alpha_{\lambda,j}(x)| + |\nabla \tilde E^\alpha_{\lambda,j}(x)| &\leqslant C(|\lambda|+\alpha^2)^{\frac{1}{2}}|\log(4\lambda+\alpha^2)|(|\log|x||+1),\\
|\partial_\lambda \tilde E^\alpha_{\lambda,j}(x)| + |\nabla\partial_\lambda \tilde E^\alpha_{\lambda,j}(x)| &\leqslant \frac{C}{(|\lambda|+\alpha^2)^{\frac{1}{2}}}(|\log|x||+1),\\
|\partial^2_\lambda \tilde E^\alpha_{\lambda,j}(x)| + |\nabla\partial^2_\lambda \tilde E^\alpha_{\lambda,j}(x)| &\leqslant \frac{C}{(|\lambda|+\alpha^2)^{\frac{3}{2}}}(|\log|x||+1).
\end{aligned} \tag{2.6}$$

Indeed, (2.6) follows from the bound

$$\frac{1}{C}(|\lambda|+\alpha^2 s)\leqslant|4\lambda+\alpha^2 s|\leqslant C(|\lambda|+\alpha^2 s)\quad\text{for }\lambda\in\Sigma_{\frac{3\pi}{4}},\ s\in(0,1], \tag{2.7}$$

with C independent of α, λ, and s. Estimate (2.7) is proved as follows: if $\Re(\lambda)\geqslant 0$, then $|4\lambda+\alpha^2|\geqslant 4|\lambda|$, while if $\Re(\lambda)<0$ and $\lambda\in\Sigma_{\frac{3\pi}{4}}$, then $|4\lambda+\alpha^2|\geqslant 4|\Im(\lambda)|\geqslant|\lambda|$, which implies $|4\lambda+\alpha^2|\geqslant|\lambda|$ for $\lambda\in\Sigma_{\frac{3\pi}{4}}$. On the other hand, if $|\lambda|\geqslant 8^{-1}\alpha^2 s$ and $\lambda\in\Sigma_{\frac{3\pi}{4}}$, then we have $|4\lambda+\alpha^2|\geqslant|\lambda|\geqslant 8^{-1}\alpha^2 s$, while if $|\lambda|\leqslant 8^{-1}\alpha^2 s$, then $|4\lambda+\alpha^2 s|\geqslant 2^{-1}\alpha^2 s$, which gives $|4\lambda+\alpha^2|\geqslant 8^{-1}\alpha^2 s$. Combining these, we obtain the lower bound in (2.7), and the upper bound in (2.7) is trivial from the triangle inequality. We note that the leading term of $E^\alpha_{\lambda,4}(x)$ also satisfies the same estimate as (2.6), and, thus, $E^\alpha_{\lambda,4}(x)$ is also regarded as a remainder. As for the first term in the leading term of $E^\alpha_{\lambda,2}$, we have

$$\begin{aligned}
\frac{1}{4\pi}\int_0^1 \log(z(s,x))\,ds &= \frac{1}{8\pi}\int_0^1 \log s\,ds + \frac{1}{8\pi}\int_0^1 \log\left(\lambda+\frac{\alpha^2}{4}s\right)ds + \frac{1}{4\pi}\log|x|\\
&= -\frac{1}{8\pi} + \frac{1}{8\pi}\log\left(\lambda+\frac{\alpha^2}{4}\right) - \frac{1}{8\pi}\int_0^1 \frac{\alpha^2 s}{4\lambda+\alpha^2 s}\,ds + \frac{1}{4\pi}\log|x|.
\end{aligned}$$

By setting

$$\tilde E^\alpha_\lambda = \tilde E^\alpha_{\lambda,1} + \tilde E^\alpha_{\lambda,2} + \tilde E^\alpha_{\lambda,3} + E^\alpha_{\lambda,4} + \tilde E^\alpha_{\lambda,1},$$

we finally obtain the following expansion of E^α_λ:

$$E^\alpha_\lambda(x) = E^0_0(x) + \left(-\frac{1}{8\pi}\log\left(\lambda+\frac{\alpha^2}{4}\right)\right)\mathbb{I} + \mathbb{J}(\alpha,\lambda) + \tilde E^\alpha_\lambda(x), \tag{2.8}$$

where

$$
\begin{aligned}
E_0^0(x) &= \left(-\frac{1}{4\pi}\log|x|\right)\mathbb{I} + \frac{x\otimes x}{4\pi|x|^2},\\
\mathbb{J}(\alpha,\lambda) &= \frac{1}{4\pi}\left(\log 2-\gamma-\frac{1}{2}-\frac{d(\alpha,\lambda)}{2}\right)\mathbb{I} + \frac{d(\alpha,\lambda)}{4\pi}\mathbf{e}_1\otimes\mathbf{e}_1,
\end{aligned}
\tag{2.9}
$$

with

$$
d(\alpha,\lambda) = \int_0^1 \frac{\alpha^2 s}{4\lambda+\alpha^2 s}\,ds. \tag{2.10}
$$

Note that in virtue of (2.7), we have for $\lambda\in\Sigma_{\frac{3\pi}{4}}$ and $\alpha>0$,

$$
\begin{aligned}
|d(\alpha,\lambda)| &\leqslant C\min\left\{1,\frac{\alpha^2}{|\lambda|}\right\},\\
|\partial_\lambda d(\alpha,\lambda)| &\leqslant \frac{C}{|\lambda|^{\frac{1}{2}}(|\lambda|+\alpha^2)^{\frac{1}{2}}},\\
|\partial_\lambda^2 d(\alpha,\lambda)| &\leqslant \frac{C}{|\lambda|(|\lambda|+\alpha^2)}.
\end{aligned}
\tag{2.11}
$$

Here, C is independent of λ and α; see Appendix A. The matrix $E_0^0(x)$ is nothing but the Stokes fundamental solution, and the remainder $\tilde{E}_\lambda^\alpha$ satisfies from (2.6),

$$
\begin{aligned}
|\tilde{E}_\lambda^\alpha(x)|+|\nabla\tilde{E}_\lambda^\alpha(x)| &\leqslant C(|\lambda|+\alpha^2)^{\frac{1}{2}}|\log(4\lambda+\alpha^2)|\,(\log|x|+1),\\
|\partial_\lambda\tilde{E}_\lambda^\alpha(x)|+|\nabla\partial_\lambda\tilde{E}_\lambda^\alpha(x)| &\leqslant \frac{C}{(|\lambda|+\alpha^2)^{\frac{1}{2}}}(\log|x|+1),\\
|\partial_\lambda^2\tilde{E}_\lambda^\alpha(x)|+|\nabla\partial_\lambda^2\tilde{E}_\lambda^\alpha(x)| &\leqslant \frac{C}{(|\lambda|+\alpha^2)^{\frac{3}{2}}}(\log|x|+1),
\end{aligned}
\tag{2.12}
$$

if $|x|\leqslant 10$ and $|4\lambda+\alpha^2|\leqslant\frac{1}{2}$ with $\lambda\in\Sigma_{3\pi/4}$. The expansion (2.8) is an improvement of [12, equation (4.19)], and the key difference is that we allow a dependence on λ in the leading term, i.e., the term $\log(\lambda+\frac{\alpha^2}{4})$ and $d(\alpha,\lambda)$. The advantage of this additional dependence is that the key estimates are derived for the wider regime of λ than in [12, Theorem 6.1] and is taken uniformly in small α, by virtue of the estimate for the remainder $\tilde{E}_\lambda^\alpha$ as in (2.12). It is important that $d(\alpha,\lambda)$ is uniformly bounded as stated in (2.11). The next proposition gives the estimate for $(\lambda+\mathbb{A}_\alpha)^{-1}\mathbb{P}f$ when f is compactly supported.

Proposition 2.1. *Let* $|4\lambda+\alpha^2|\leqslant\frac{1}{2}$ *and* $\lambda\in\Sigma_{\frac{3\pi}{4}}$. *Assume that* $f\in L^q(\mathbb{R}^2)^2$, $1<q<\infty$, *and* $\operatorname{supp} f\subset\{|x|\leqslant 5\}$. *Then for* $|x|\leqslant 5$, *we have*

$$
\begin{aligned}
(\lambda+\mathbb{A}_\alpha)^{-1}\mathbb{P}f(x) &= E_\lambda^\alpha * f(x)\\
&= E_0^0 * f(x) + \left(-\frac{1}{8\pi}\log\left(\lambda+\frac{\alpha^2}{4}\right)+\mathbb{J}(\alpha,\lambda)\right)\int_{\mathbb{R}^2} f\,dx\\
&\quad + \tilde{E}_\lambda^\alpha * f(x)
\end{aligned}
\tag{2.13}
$$

with $\mathbb{J}(\alpha, \lambda)$ *defined as* (2.9)*, and*

$$\begin{aligned}
\|\tilde{E}^{\alpha}_{\lambda} * f\|_{W^{1,\infty}(\{|x|\leqslant 5\})} &\leqslant C(|\lambda| + \alpha^2)^{\frac{1}{2}} \, |\log(4\lambda + \alpha^2)| \|f\|_{L^q}, \\
\|\partial_\lambda \tilde{E}^{\alpha}_{\lambda} * f\|_{W^{1,\infty}(\{|x|\leqslant 5\})} &\leqslant \frac{C}{(|\lambda| + \alpha^2)^{\frac{1}{2}}} \|f\|_{L^q}, \\
\|\partial^2_\lambda \tilde{E}^{\alpha}_{\lambda} * f\|_{W^{1,\infty}(\{|x|\leqslant 5\})} &\leqslant \frac{C}{(|\lambda| + \alpha^2)^{\frac{3}{2}}} \|f\|_{L^q}.
\end{aligned} \tag{2.14}$$

Here, $C > 0$ *depends only on* q*.*

Proof. The expansion (2.13) follows from (2.8), while (2.14) is a consequence of (2.12) and the Young inequality for convolution. The proof is complete. □

2.2. Resolvent estimate for interior problem

Set $D = \Omega \cap B_5(0)$, which is a bounded domain in $\mathbb{R}^2$ with a smooth boundary. In this section, we consider the interior problem

$$\begin{cases}
\lambda u_D - \Delta u_D + \alpha \partial_1 u_D + \nabla p_D = f, & x \in D, \\
\operatorname{div} u_D = 0, & x \in D, \\
u_D = 0, & x \in \partial D.
\end{cases} \tag{2.15}$$

Let us denote by $\mathbb{A}_{\alpha,D}$ the operator $\mathbb{A}_{\alpha,D} u = \mathbb{P}_D(-\Delta u + \alpha \partial_1 u)$, which is realized in $L^q_\sigma(D)$ and $\mathbb{P}_D : L^q(D)^2 \to L^q_\sigma(D)$, $1 < q < \infty$, is the Helmholtz projection.

Proposition 2.2. *Let* $1 < q < \infty$*. There exists* $\alpha_q > 0$ *such that the following statement holds for all* $0 \leqslant \alpha \leqslant \alpha_q$*. The set* $\overline{\Sigma_{\frac{3\pi}{4}}}$ *belongs to the resolvent set of* $-\mathbb{A}_{\alpha,D}$ *in* $L^q_\sigma(D)$*, and*

$$\|\partial^j_\lambda (\lambda + \mathbb{A}_{\alpha,D})^{-1} \mathbb{P}_D f\|_{W^{2,q}(D)} \leqslant C \|f\|_{L^q(D)}, \quad f \in L^q(D)^2, \quad \lambda \in \overline{\Sigma_{\frac{3\pi}{4}}}, \quad j = 0, 1. \tag{2.16}$$

Moreover, the solution $u_D[\lambda] f = (\lambda + \mathbb{A}_{\alpha,D})^{-1} \mathbb{P}_D f$ *satisfies*

$$\|u_D[\lambda] f - \mathbb{A}_D^{-1} \mathbb{P}_D f\|_{W^{2,q}(D)} \leqslant C(|\lambda| + \alpha) \|f\|_{L^q(D)}. \tag{2.17}$$

Here, $\mathbb{A}_D = -\mathbb{P}_D \Delta$ *is the Stokes operator in* $L^q_\sigma(D)$*. The associated pressure* $p_D[\lambda] f$ *with* $\int_D p_D[\lambda] f \, dx = 0$ *also satisfies the decomposition as in* (2.17)*, i.e.,*

$$\|\nabla p_D[\lambda] f - \mathbb{Q}_D(f + \Delta \mathbb{A}_D^{-1} \mathbb{P}_D f)\|_{L^q(D)} \leqslant C(|\lambda| + \alpha) \|f\|_{L^q(D)}. \tag{2.18}$$

Here, $\mathbb{Q}_D = I - \mathbb{P}_D$*. Moreover,* $p_D[\lambda] f$ *is analytic with respect to* λ *in the topology of* $L^q(D)$ *and*

$$\|\partial^j_\lambda p_D[\lambda] f\|_{L^q(D)} \leqslant C \|f\|_{L^q(D)}, \quad j = 0, 1, 2. \tag{2.19}$$

In the above estimates, the constant C *is independent of* $\alpha \in [0, \alpha_q]$ *and* $\lambda \in \overline{\Sigma_{\frac{3\pi}{4}}}$*.*

Proof. It is well known that, since D is bounded, the resolvent set of $-\mathbb{A}_D$ in $L^q_\sigma(D)$ contains the set $\{\lambda \in \mathbb{C} \mid \Re(\lambda) \geqslant -\lambda_0\} \cup \overline{\Sigma_{\frac{3\pi}{4}}}$ for some $\lambda_0 > 0$, and

$$\|(\lambda + \mathbb{A}_D)^{-1} f\|_{L^q(D)} \leqslant \frac{C}{|\lambda| + 1}\|f\|_{L^q(D)},$$
$$\|(\lambda + \mathbb{A}_D)^{-1} f\|_{W^{2,q}(D)} \leqslant C\|f\|_{L^q(D)},$$

for all $f \in L^q_\sigma(D)$ and $\lambda \in \{\lambda \in \mathbb{C} \mid \Re(\lambda) \geqslant -\lambda_0\} \cup \overline{\Sigma_{\frac{3\pi}{4}}}$. Then we have

$$\|(\lambda + \mathbb{A}_D)^{-1}\mathbb{P}_D\partial_1 f\|_{W^{2,q}(D)} \leqslant C\|\partial_1 f\|_{L^q(D)}$$

for $f \in D_{L^q}(\mathbb{A}_D)$ and $\lambda \in \{\lambda \in \mathbb{C} \mid \Re(\lambda) \geqslant -\lambda_0\} \cup \overline{\Sigma_{\frac{3\pi}{4}}}$. These estimates imply that, for a sufficiently small $\alpha > 0$, the resolvent operators of $-\mathbb{A}_{\alpha,D}$ in $L^q_\sigma(D)$ are constructed around those of $-\mathbb{A}_D$. In particular, if $\alpha > 0$ is sufficiently small, then the set $\{\lambda \in \mathbb{C} \mid \Re(\lambda) \geqslant -\lambda_0\} \cup \overline{\Sigma_{\frac{3\pi}{4}}}$ is contained in the resolvent set of $-\mathbb{A}_{\alpha,D}$ in $L^q_\sigma(D)$, and we have for such λ and $f \in L^q(D)^2$,

$$\|(\lambda + \mathbb{A}_{\alpha,D})^{-1}\mathbb{P}_D f\|_{L^q(D)} \leqslant \frac{C}{|\lambda| + 1}\|f\|_{L^q(D)},$$
$$\|(\lambda + \mathbb{A}_{\alpha,D})^{-1}\mathbb{P}_D f\|_{W^{2,q}(D)} \leqslant C\|f\|_{L^q(D)}.$$

Hence, (2.16) holds. Estimate (2.17) follows from the formula

$$\begin{aligned}(\lambda + \mathbb{A}_{\alpha,D})^{-1}\mathbb{P}_D f &= \mathbb{A}_D^{-1}\mathbb{P}_D f - \lambda\mathbb{A}_D^{-1}(\lambda + \mathbb{A}_{\alpha,D})^{-1}\mathbb{P}_D f \\ &\quad - \alpha\mathbb{A}_D^{-1}\mathbb{P}_D\partial_1(\lambda + \mathbb{A}_{\alpha,D})^{-1}\mathbb{P}_D f.\end{aligned}$$

These perturbation arguments are quite standard, and the details are omitted here. Finally, the estimate and the decomposition of the pressure terms are the consequence of those of the velocity field. The proof is complete. □

2.3. Resolvent analysis and local energy decay

The aim of this subsection is to improve Hishida's result in [12] and to establish the local energy estimate for $\{e^{-t\mathbb{A}_{\alpha,\Omega}}\}_{t\geqslant 0}$ in two dimensions for small $\alpha > 0$. To this end, we consider the resolvent problem

$$\begin{cases} \lambda u + A_\alpha u + \nabla p_u = f, & x \in \Omega, \\ \operatorname{div} u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.20}$$

We assume that $f \in L^q(\Omega)^2$ satisfies $\operatorname{supp} f \subset \{x \in \mathbb{R}^2 \mid |x| \leqslant 5\}$. Let $\chi \in C_0^\infty(\mathbb{R}^2)$ be a cut-off function such that $\chi(x) = 1$ for $|x| \leqslant 4$ and $\chi(x) = 0$ for $|x| \geqslant 5$. Set

$$\mathcal{U}_\alpha[\lambda]f = (1-\chi)u_{\mathbb{R}^2}[\lambda]f + \mathbb{B}[\nabla\chi \cdot u_{\mathbb{R}^2}[\lambda]f] + \chi u_D[\lambda]f - \mathbb{B}[\nabla\chi \cdot u_D[\lambda]f], \tag{2.21}$$

$$\mathcal{P}_\alpha[\lambda]f = (1-\chi)p_{\mathbb{R}^2}(f) + \chi\left(p_D[\lambda]f + \frac{1}{|D|}\int_D p_{\mathbb{R}^2}(f)\,dx\right). \tag{2.22}$$

https://doi.org/10.1017/S1474748019000355 Published online by Cambridge University Press

Here, $(u_{\mathbb{R}^2}[\lambda]f, p_{\mathbb{R}^2}(f)) = ((\lambda + \mathbb{A}_\alpha)^{-1}\mathbb{P}f, -(-\Delta_{\mathbb{R}^2})^{-1}\nabla\cdot f)$ is the solution in the whole space in Proposition 2.1 with f extended to $\mathbb{R}^2$ by zero, $(u_D[\lambda]f, p_D[\lambda]f)$ is the solution to (2.15) with $\int_D p_D[\lambda]f\,dx = 0$ obtained in Proposition 2.2 with f restricted on D, and $\mathbb{B} = \mathbb{B}_4$ is the Bogovskii operator in the annulus $D_4 = \{x \in \mathbb{R}^2 \mid 4 < |x| < 5\}$, i.e., $\mathbb{B}[g]$ satisfies

$$\operatorname{div}\mathbb{B}[g] = g \quad \text{in } D_4, \qquad \mathbb{B}[g] = 0 \quad \text{on } \partial D_4,$$

for a given function $g \in C_0^\infty(D_4)$ with $\int_{D_4} g\,dx = 0$. As is well known (see, e.g., [4]), the Bogovskii operator $\mathbb{B}$ is extended to a bounded operator from $W_0^{k,q}(D_4)$ to $W_0^{k+1,q}(D_4)^2$ for any $1 < q < \infty$ and $k = 0, 1, \ldots$, together with the estimate

$$\|\nabla^{k+1}\mathbb{B}[g]\|_{L^q(D_4)} \leqslant C_{q,k}\|\nabla^k g\|_{L^q(D_4)}, \quad 1 < q < \infty,\ k = 0, 1, \ldots. \tag{2.23}$$

By its definition, the couple $(\mathcal{U}_\alpha[\lambda]f, \nabla\mathcal{P}_\alpha[\lambda]f)$ satisfies

$$(\lambda + A_\alpha)\mathcal{U}_\alpha[\lambda]f + \nabla\mathcal{P}_\alpha[\lambda]f = (\mathbb{I} + T_\alpha[\lambda])f, \quad \operatorname{div}\mathcal{U}_\alpha[\lambda]f = 0 \tag{2.24}$$

in Ω, and $\mathcal{U}_\alpha[\lambda]f = 0$ on $\partial\Omega$, where $T_\alpha[\lambda]f$ is given by

$$\begin{aligned} T_\alpha[\lambda]f \;=\; & (\Delta\chi)u_{\mathbb{R}^2} + 2\nabla\chi\cdot\nabla u_{\mathbb{R}^2} - \alpha(\partial_1\chi)u_{\mathbb{R}^2} - (\nabla\chi)p_{\mathbb{R}^2} \\ & + (\lambda + A_\alpha)\mathbb{B}[\nabla\chi\cdot u_{\mathbb{R}^2}] \\ & - (\Delta\chi)u_D - 2\nabla\chi\cdot\nabla u_D + \alpha(\partial_1\chi)u_D + (\nabla\chi)\Big(p_D + \frac{1}{|D|}\int_D p_{\mathbb{R}^2}(f)\,dx\Big) \\ & - (\lambda + A_\alpha)\mathbb{B}[\nabla\chi\cdot u_D]. \end{aligned} \tag{2.25}$$

Here, we have used the abbreviated notations such as $u_{\mathbb{R}^2}$ and u_D instead of $u_{\mathbb{R}^2}[\lambda]f$ and $u_D[\lambda]f$. Note that $T_\alpha[\lambda]f$ is supported in $\{x \in \mathbb{R}^2 \mid 4 \leqslant |x| \leqslant 5\}$. It is already shown by [12, Lemma 6.2] that for all $\alpha > 0$ and $\lambda \in \{z \in \mathbb{C} \mid (\Im(z))^2 > -\alpha^2\Re(z)\}$, operator $\mathbb{I} + T_\alpha[\lambda]$ is invertible in the Banach space

$$L^q_{[5]}(\Omega)^2 = \{f \in L^q(\Omega)^2 \mid f = 0 \ a.e.\ |x| \geqslant 5\}, \quad 1 < q < \infty. \tag{2.26}$$

Hence, the couple

$$(u, \nabla p) = (\mathcal{U}_\alpha[\lambda](\mathbb{I} + T_\alpha[\lambda])^{-1}f, \nabla\mathcal{P}_\alpha[\lambda](\mathbb{I} + T_\alpha[\lambda])^{-1}f)$$

gives the solution to (2.20) when $f \in L^q_{[5]}(\Omega)^2$, and $\mathcal{U}_\alpha[\lambda](\mathbb{I} + T_\alpha[\lambda])^{-1}f$ is analytic in λ from its construction. What is important is then to know the concrete behavior of $(\mathbb{I} + T_\alpha[\lambda])^{-1}f$ when $|\lambda| + \alpha$ is small so that the sharp dependence of its operator norm on λ and α is derived. This is discussed in [12, Theorem 6.1], but the argument in [12] works only for $\lambda \in \{z \in \mathbb{C} \mid |\Im(z)|^2 > -\alpha^2\Re(z)\}$ with $|\lambda| \leqslant \delta\alpha^2$ for some small δ independent of α, resulting in the appearance of the singularity in α^{-1} in the final estimate that we have to remove in this paper. The key is to use the expansion of the resolvent kernel in $\mathbb{R}^2$ shown in Proposition 2.1. A careful analysis below enables us to obtain the expansion of $(\mathbb{I} + T_\alpha[\lambda])^{-1}$ in the regime $\{\lambda \in \Sigma_{\frac{3\pi}{4}} \mid |\lambda| \leqslant c_q\}$ for some small $c_q > 0$ but independent of small α.

The next proposition is the core of this subsection. Let $\alpha_q > 0$ be the number in Proposition 2.2.

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Proposition 2.3. *Let $1 < q < \infty$. There exists $c_q \in (0, \alpha_q]$ such that the following statement holds. Set $\mathcal{O}_{c_q} = \{\lambda \in \Sigma_{\frac{3\pi}{4}} \mid |\lambda| \leqslant c_q\}$, where $\Sigma_{\frac{3\pi}{4}}$ is the sector with angle $\frac{3\pi}{4}$ defined as (2.5). If $\lambda \in \mathcal{O}_{c_q}$ and $0 < \alpha \leqslant c_q$, then the map $T_\alpha[\lambda] : L^q_{[5]}(\Omega)^2 \to L^q_{[5]}(\Omega)^2$ is well defined and bounded, and $\mathbb{I} + T_\alpha[\lambda]$ is invertible and satisfies*

$$\sup_{\lambda \in \mathcal{O}_{c_q}, 0 < \alpha \leqslant c_q} \|(\mathbb{I} + T_\alpha[\lambda])^{-1} f\|_{L^q_{[5]}(\Omega)} \leqslant C\|f\|_{L^q_{[5]}(\Omega)}, \tag{2.27}$$

and

$$\left| \int_\Omega (\mathbb{I} + T_\alpha[\lambda])^{-1} f \, dx \right| \leqslant \frac{C}{|\log(4\lambda + \alpha^2)|} \|f\|_{L^q_{[5]}(\Omega)} \tag{2.28}$$

for all $\lambda \in \mathcal{O}_{c_q}$ and $0 < \alpha \leqslant c_q$. Moreover, $T_\alpha[\lambda]$ and $(\mathbb{I} + T_\alpha[\lambda])^{-1}$ are analytic with respect to $\lambda \in \mathcal{O}_{c_q}$ in the topology of $\mathcal{L}(L^q_{[5]}(\Omega)^2)$, and the following expansion holds:

$$(\mathbb{I} + T_\alpha[\lambda])^{-1} = \Theta_0^{-1} + W_0 + \frac{1}{\log(4\lambda + \alpha^2)} \big(W_{1,1} + d(\alpha, \lambda) W_{1,2}\big) + W_2(\alpha, \lambda), \tag{2.29}$$

where Θ_0 is an invertible operator in $L^q_{[5]}(\Omega)^2$, W_0, $W_{1,1}$, and $W_{1,2}$ are bounded and finite rank operators in $L^q_{[5]}(\Omega)^2$, and

(i) *Θ_0, W_0, $W_{1,1}$, $W_{1,2}$ are independent of α and λ,*

(ii) *$W_2(\alpha, \lambda)$ satisfies*

$$\begin{aligned}
\|W_2(\alpha, \lambda)\|_{\mathcal{L}(L^q_{[5]}(\Omega))} &\leqslant \frac{C}{|\log(4\lambda + \alpha^2)|^2}, \\
\|\partial_\lambda W_2(\alpha, \lambda)\|_{\mathcal{L}(L^q_{[5]}(\Omega))} &\leqslant \frac{C}{|\log(4\lambda + \alpha^2)|^2 |\lambda|^{\frac{1}{2}} (|\lambda| + \alpha^2)^{\frac{1}{2}}}, \\
\|\partial_\lambda^2 W_2(\alpha, \lambda)\|_{\mathcal{L}(L^q_{[5]}(\Omega))} &\leqslant \frac{C}{|\log(4\lambda + \alpha^2)|^2 |\lambda| (|\lambda + \alpha^2)}.
\end{aligned} \tag{2.30}$$

Here, the above constants C depend only on q and Ω.

Proof. We first recall the definition of $T_\alpha[\lambda]$ in (2.25), where $u_{\mathbb{R}^2} = u_{\mathbb{R}^2}[\lambda]f = (\lambda + \mathbb{A}_\alpha)^{-1}\mathbb{P}f$, $p_{\mathbb{R}^2} = -\nabla \cdot (-\Delta_{\mathbb{R}^2})^{-1} f$, $u_D = u_D[\lambda]f = (\lambda + \mathbb{A}_{\alpha,D})^{-1}\mathbb{P}_D f$, and $p_D = p_D[\lambda]f$ is such that $\int_D p_D \, dx = 0$ and $\nabla p_D[\lambda]f = \mathbb{Q}_D(f + \Delta u_D[\lambda]f - \alpha\partial_1 u_D[\lambda]f)$. Set

$$u^{(0)}_{\mathbb{R}^2} = E^0_0 * f, \quad u^{(0)}_D = \mathbb{A}_D^{-1}\mathbb{P}_D f,$$

and let $p^{(0)}_D$ be the pressure field such that $\int_D p^{(0)}_D \, dx = 0$ and

$$\nabla p^{(0)}_D = \mathbb{Q}_D(f + \Delta\mathbb{A}_D^{-1}\mathbb{P}_D f).$$

Then, $u_{\mathbb{R}^2}[\lambda]f$ is decomposed as in Proposition 2.1, and $u_D[\lambda]f$ and $p_D[\lambda]f$ are decomposed as $u_D[\lambda]f = u^{(0)}_D + (u_D[\lambda]f - u^{(0)}_D)$ and $p_D[\lambda]f = p^{(0)}_D + (p_D[\lambda]f - p^{(0)}_D)$. Thus, $T_\alpha[\lambda]f$ is decomposed as

$$T_\alpha[\lambda]f = T_0[0]f + Y_\alpha[\lambda]f + Z_\alpha[\lambda]f, \tag{2.31}$$

where

$$
\begin{aligned}
T_0[0]f &= (\Delta\chi)u^{(0)}_{\mathbb{R}^2} + 2\nabla\chi\cdot\nabla u^{(0)}_{\mathbb{R}^2} - (\nabla\chi)p_{\mathbb{R}^2} + \Delta\mathbb{B}[\nabla\chi\cdot u^{(0)}_{\mathbb{R}^2}] \\
&\quad - (\Delta\chi)u^{(0)}_D - 2\nabla\chi\cdot\nabla u^{(0)}_D + (\nabla\chi)\Big(p^{(0)}_D + \frac{1}{|D|}\int_D p_{\mathbb{R}^2}\,dx\Big) \\
&\quad - \Delta\mathbb{B}[\nabla\chi\cdot u^{(0)}_D],
\end{aligned}
\tag{2.32}
$$

$$
\begin{aligned}
Y_\alpha[\lambda]f &= (\Delta\chi)\Big(-\frac{1}{8\pi}\log\Big(\lambda+\frac{\alpha^2}{4}\Big)\mathbb{I} + \mathbb{J}(\alpha,\lambda)\Big)\int_\Omega f\,dx \\
&\quad - \Delta\mathbb{B}\Big[\nabla\chi\cdot\Big(-\frac{1}{8\pi}\log\Big(\lambda+\frac{\alpha^2}{4}\Big)\mathbb{I} + \mathbb{J}(\alpha,\lambda)\Big)\int_\Omega f\,dx\Big],
\end{aligned}
\tag{2.33}
$$

and $Z_\alpha[\lambda]$ is a bounded linear operator in $L^q_{[5]}(\Omega)^2$ satisfying

$$
\begin{aligned}
\|Z_\alpha[\lambda]f\|_{L^q_{[5]}(\Omega)} &\leqslant C(|\lambda|+\alpha^2)^{\frac12}|\log(4\lambda+\alpha^2)|\|f\|_{L^q_{[5]}(\Omega)}, \\
\|\partial_\lambda Z_\alpha[\lambda]f\|_{L^q_{[5]}(\Omega)} &\leqslant \frac{C}{(|\lambda|+\alpha^2)^{\frac12}}\|f\|_{L^q_{[5]}(\Omega)}, \\
\|\partial_\lambda^2 Z_\alpha[\lambda]f\|_{L^q_{[5]}(\Omega)} &\leqslant \frac{C}{(|\lambda|+\alpha^2)^{\frac32}}\|f\|_{L^q_{[5]}(\Omega)},
\end{aligned}
\tag{2.34}
$$

as long as $\lambda \in \Sigma_{\frac{3\pi}{4}}$ and $|4\lambda+\alpha^2| \leqslant \frac12$. Here, estimate (2.34) follows from (2.14), (2.17), (2.18), and $\|\partial_\lambda u_D[\lambda]f\|_{L^q(D)} + \|\partial_\lambda p_D[\lambda]f\|_{L^q(D)} \leqslant C\|f\|_{L^q(D)}$ with C independent of $\lambda \in \Sigma_{\frac{3\pi}{4}}$ and small α. As in [12, equation (6.26)], we set

$$
w_j = \frac{1}{4\pi}((-\Delta\chi)\mathbf{e}_j + \Delta\mathbb{B}[\partial_j\chi]), \quad j = 1, 2, \tag{2.35}
$$

which are clearly independent of λ and α. Then the finite rank operator $Y_\alpha[\lambda]$ is written as

$$
\begin{aligned}
Y_\alpha[\lambda]f &= \Big(\frac12\log\Big(\lambda+\frac{\alpha^2}{4}\Big) + \gamma + \frac12 + \frac{d(\alpha,\lambda)}{2} - \log 2 - d(\alpha,\lambda)\Big)\langle\mathbf{e}_1, f\rangle_{L^2(\Omega)}w_1 \\
&\quad + \Big(\frac12\log\Big(\lambda+\frac{\alpha^2}{4}\Big) + \gamma + \frac12 + \frac{d(\alpha,\lambda)}{2} - \log 2\Big)\langle\mathbf{e}_2, f\rangle_{L^2(\Omega)}w_2 \\
&= \Big(\frac12\log(4\lambda+\alpha^2) + \gamma + \frac{1-d(\alpha,\lambda)}{2}\Big)\langle\mathbf{e}_1, f\rangle_{L^2(\Omega)}w_1 \\
&\quad + \Big(\frac12\log(4\lambda+\alpha^2) + \gamma + \frac{1+d(\alpha,\lambda)}{2}\Big)\langle\mathbf{e}_2, f\rangle_{L^2(\Omega)}w_2.
\end{aligned}
\tag{2.36}
$$

As stated in [12, pp. 330], it is well known from [6, Lemmas 3.2–3.5] that $T_0[0]$ is a compact operator from $L^q_{[5]}(\Omega)^2$, $1 < q < \infty$, into itself and $\mathbb{I} + T_0[0]$ is injective on the subspace $\{f \in L^q_{[5]}(\Omega)^2 \mid \int_\Omega f\,dx = 0\}$ (for the reader's convenience, the proof of this fact is given in Appendix B), and the dimension of the kernel of $\mathbb{I} + T_0[0]$ is less than or equal

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to 2. Then the Fredholm theory implies that there exist $m_j \in L^q_{[5]}(\Omega)^2$, $j = 1, 2$, such that $L^q_{[5]}(\Omega)^2 = \mathrm{Range}\,(\mathbb{I} + T_0[0]) \oplus \mathrm{Span}\,\{m_1, m_2\}$, and the operator

$$\Theta_0 f = (\mathbb{I} + T_0[0]) f + \langle \mathbf{e}_1, f\rangle_{L^2(\Omega)} m_1 + \langle \mathbf{e}_2, f\rangle_{L^2(\Omega)} m_2 \tag{2.37}$$

is bijective on $L^q_{[5]}(\Omega)^2$. If $\mathrm{Ker}\,(\mathbb{I} + T_0[0]) \leqslant 1$, then m_1 and/or m_2 are taken as zero. Note that m_1 and m_2 are independent of λ and α. From (2.31) and (2.37), we see

$$\begin{aligned}(\mathbb{I} + T_\alpha[\lambda]) f &= (\mathbb{I} + T_0[0]) f + Y_\alpha[\lambda] f + Z_\alpha[\lambda] f \\ &= \Theta_0 f + \tilde{Y}_\alpha[\lambda] f + Z_\alpha[\lambda] f \end{aligned}\tag{2.38}$$

where

$$\tilde{Y}_\alpha[\lambda] f = Y_\alpha[\lambda] f - \langle \mathbf{e}_1, f\rangle_{L^2(\Omega)} m_1 - \langle \mathbf{e}_2, f\rangle_{L^2(\Omega)} m_2.$$

Our aim is to obtain the estimate of the inverse of $\mathbb{I} + T_\alpha[\lambda]$. Since $Z_\alpha[\lambda]$ is a small perturbation when $|\lambda| + \alpha$ is small, we study the inverse of $\Theta_0 + \tilde{Y}_\alpha[\lambda]$. We observe that

$$\tilde{Y}_\alpha[\lambda] f = \sum_{j=1,2} \phi_j \langle \mathbf{e}_j, f\rangle_{L^2(\Omega)} w_j - \sum_{j=1,2} \langle \mathbf{e}_j, f\rangle_{L^2(\Omega)} m_j, \tag{2.39}$$

where

$$\begin{aligned}\phi_1 = \phi_1(\alpha, \lambda) &= \frac{1}{2}\log(4\lambda + \alpha^2) + \gamma + \frac{1 - d(\alpha, \lambda)}{2}, \\ \phi_2 = \phi_2(\alpha, \lambda) &= \frac{1}{2}\log(4\lambda + \alpha^2) + \gamma + \frac{1 + d(\alpha, \lambda)}{2}.\end{aligned}\tag{2.40}$$

Since Θ_0 is invertible and independent of λ and α, we study the invertibility of $\mathbb{I} + \Theta_0^{-1}\tilde{Y}_\alpha[\lambda]$, where

$$\Theta_0^{-1}\tilde{Y}_\alpha[\lambda] h = \sum_{j=1,2} \phi_j \langle \mathbf{e}_j, h\rangle_{L^2(\Omega)} \Theta_0^{-1} w_j - \sum_{j=1,2} \langle \mathbf{e}_j, h\rangle_{L^2(\Omega)} \Theta_0^{-1} m_j. \tag{2.41}$$

The key idea here is to determine, first, the quantity $\int_\Omega h\,dx$ for the solution h to the problem $(\mathbb{I} + \Theta_0^{-1}\tilde{Y}_\alpha[\lambda]) h = f$ with a given $f \in L^q_{[5]}(\Omega)^2$. Once this is done, the solution h is given by $h = f - \Theta_0^{-1}\tilde{Y}_\alpha[\lambda] h$, for $\Theta_0^{-1}\tilde{Y}_\alpha[\lambda] h$ is defined only in terms of $\int_\Omega h\,dx$. To determine $\int_\Omega h\,dx$, we integrate the equation $(\mathbb{I} + \Theta_0^{-1}\tilde{Y}_\alpha[\lambda]) h = f$ over Ω, which yields the linear algebraic equation for $\int_\Omega h\,dx$ of the form

$$\mathbb{M}(\alpha, \lambda) \int_\Omega h\,dx = \int_\Omega f\,dx,$$

where the 2×2 matrix $\mathbb{M}(\alpha, \lambda)$ is given by

$$\begin{aligned}&\mathbb{M}(\alpha, \lambda) \\ &= \begin{pmatrix} 1 + \phi_1 \langle \mathbf{e}_1, \Theta_0^{-1} w_1\rangle_{L^2(\Omega)} - \langle \mathbf{e}_1, m_1\rangle_{L^2(\Omega)} & \phi_2 \langle \mathbf{e}_1, \Theta_0^{-1} w_2\rangle_{L^2(\Omega)} - \langle \mathbf{e}_1, m_2\rangle_{L^2(\Omega)} \\ \phi_1 \langle \mathbf{e}_2, \Theta_0^{-1} w_1\rangle_{L^2(\Omega)} - \langle \mathbf{e}_2, m_1\rangle_{L^2(\Omega)} & 1 + \phi_2 \langle \mathbf{e}_2, \Theta_0^{-1} w_2\rangle_{L^2(\Omega)} - \langle \mathbf{e}_2, m_2\rangle_{L^2(\Omega)} \end{pmatrix}\end{aligned}\tag{2.42}$$

The direct computation shows

$$\det \mathbb{M}(\alpha, \lambda) = \phi_1(\alpha, \lambda)\phi_2(\alpha, \lambda) C^{(0)} + \phi_1(\alpha, \lambda) C_1^{(1)} + \phi_2(\alpha, \lambda) C_2^{(1)} + C^{(2)}, \tag{2.43}$$

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where

$$C^{(0)} := \det \begin{pmatrix} \langle \mathbf{e}_1, \Theta_0^{-1} w_1 \rangle_{L^2(\Omega)} & \langle \mathbf{e}_1, \Theta_0^{-1} w_2 \rangle_{L^2(\Omega)} \\ \langle \mathbf{e}_2, \Theta_0^{-1} w_1 \rangle_{L^2(\Omega)} & \langle \mathbf{e}_2, \Theta_0^{-1} w_2 \rangle_{L^2(\Omega)} \end{pmatrix}, \tag{2.44}$$

and $C_j^{(1)}$ and $C^{(2)}$ are constants independent of λ and α. By [12, equation (6.30)], that is a key observation of [12], we already know that

$$C^{(0)} \neq 0.$$

Hence, recalling that $|d(\alpha, \lambda)| \leqslant C$ by (2.11), we observe that

$$\det \mathbb{M}(\alpha, \lambda) \sim C^{(0)} \left(\frac{1}{2} \log(4\lambda + \alpha^2) \right)^2 \neq 0$$

if $|\lambda| + \alpha$ is small enough, and we have

$$\int_\Omega h \, dx = \mathbb{M}(\alpha, \lambda)^{-1} \int_\Omega f \, dx. \tag{2.45}$$

Then we conclude that $\mathbb{I} + \Theta_0^{-1} \tilde{Y}_\alpha[\lambda]$ is invertible, and by using (2.41), we have

$$\begin{aligned} (\mathbb{I} + \Theta_0^{-1} \tilde{Y}_\alpha[\lambda])^{-1} f &= f - \Theta_0^{-1} \tilde{Y}_\alpha[\lambda] h \\ &= f - \sum_{j,k} b_{jk}(\alpha, \lambda) (\phi_j(\alpha, \lambda) \langle \mathbf{e}_k, f \rangle_{L^2(\Omega)} \Theta_0^{-1} w_j - \langle \mathbf{e}_k, f \rangle_{L^2(\Omega)} m_j). \end{aligned} \tag{2.46}$$

Here, $(b_{jk}(\alpha, \lambda))_{1 \leqslant j,k \leqslant 2} = \mathbb{M}(\alpha, \lambda)^{-1}$. In particular, the exact computation of $\mathbb{M}(\alpha, \lambda)^{-1}$ shows

$$(\mathbb{I} + \Theta_0^{-1} \tilde{Y}_\alpha[\lambda])^{-1} = \mathbb{I} + \tilde{W}_0 + \frac{1}{\log(4\lambda + \alpha^2)} (\tilde{W}_{1,1} + d(\alpha, \lambda) \tilde{W}_{1,2}) + \tilde{W}_2(\alpha, \lambda), \tag{2.47}$$

where each $\tilde{W}_j$ is finite rank and $\tilde{W}_0$, $\tilde{W}_{1,j}$ are independent of λ and α, and for $\lambda \in \mathcal{O}_{c_q}$ and $0 < \alpha \leqslant c_q$,

$$\begin{aligned} &\| \tilde{W}_2(\alpha, \lambda) f \|_{L^q_{[5]}(\Omega)} \leqslant \frac{C}{|\log(4\lambda + \alpha^2)|^2} \left| \int_\Omega f \, dx \right|, \\ &\| \partial_\lambda \tilde{W}_2(\alpha, \lambda) f \|_{L^q_{[5]}(\Omega)} \\ &\quad \leqslant \frac{C}{|\log(4\lambda + \alpha^2)|^2} \left(\frac{1}{|4\lambda + \alpha^2| |\log(4\lambda + \alpha^2)|} + |\partial_\lambda d(\alpha, \lambda)| \right) \left| \int_\Omega f \, dx \right| \\ &\quad \leqslant \frac{C}{|\log(4\lambda + \alpha^2)|^2 |\lambda|^{\frac{1}{2}} (|\lambda| + \alpha^2)^{\frac{1}{2}}} \left| \int_\Omega f \, dx \right|, \\ &\| \partial_\lambda^2 \tilde{W}_2(\alpha, \lambda) f \|_{L^q_{[5]}(\Omega)} \\ &\quad \leqslant \frac{C}{|\log(4\lambda + \alpha^2)|^2} \left(\frac{1}{|4\lambda + \alpha^2|^2 |\log(4\lambda + \alpha^2)|} + |\partial_\lambda d(\alpha, \lambda)|^2 + |\partial_\lambda^2 d(\alpha, \lambda)| \right) \left| \int_\Omega f \, dx \right| \\ &\quad \leqslant \frac{C}{|\log(4\lambda + \alpha^2)|^2 |\lambda| (|\lambda| + \alpha^2)} \left| \int_\Omega f \, dx \right|. \end{aligned} \tag{2.48}$$

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Here, we have used (2.11). We also have from (2.45),

$$\sup_{\lambda\in\mathcal{O}_{c_q},0<\alpha\leqslant c_q}\left|\int_\Omega(\mathbb{I}+\Theta_0^{-1}\tilde{Y}_\alpha[\lambda])^{-1}f\,dx\right|\leqslant\frac{C}{|\log(4\lambda+\alpha^2)|}\left|\int_\Omega f\,dx\right| \tag{2.49}$$

for all $f\in L^q_{[5]}(\Omega)^2$. From (2.47), if we set $W_0=\tilde{W}_0\Theta_0^{-1}$, $W_{1,j}=\tilde{W}_{1,j}\Theta_0^{-1}$, and $W_{2,1}(\alpha,\lambda)=\tilde{W}_2(\alpha,\lambda)\Theta_0^{-1}$, then we have

$$\begin{aligned}\big(\Theta_0+\tilde{Y}_\alpha[\lambda]\big)^{-1}&=(\mathbb{I}+\Theta_0^{-1}\tilde{Y}_\alpha[\lambda])^{-1}\Theta_0^{-1}\\&=\Theta_0^{-1}+W_0+\frac{1}{\log(4\lambda+\alpha^2)}(W_{1,1}+d(\alpha,\lambda)W_{1,2})+W_{2,1}(\alpha,\lambda).\end{aligned} \tag{2.50}$$

In particular, $\|(\Theta_0+\tilde{Y}_\alpha[\lambda])^{-1}\|_{\mathcal{L}(L^q_{[5]}(\Omega))}$ is uniformly bounded in $\lambda\in\Sigma_{\frac{3\pi}{4}}$ when $|4\lambda+\alpha^2|$ is small enough. Next we recall that $Z_\alpha[\lambda]$ is regarded as a small perturbation as in (2.34), and, hence, the Neumann series argument implies

$$\begin{aligned}(\mathbb{I}+T_\alpha[\lambda])^{-1}&=(\mathbb{I}+(\Theta_0+\tilde{Y}_\alpha[\lambda])^{-1}Z_\alpha[\lambda])^{-1}(\Theta_0+\tilde{Y}_\alpha[\lambda])^{-1}\\&=(\Theta_0+\tilde{Y}_\alpha[\lambda])^{-1}+W_{2,2}(\alpha,\lambda),\end{aligned} \tag{2.51}$$

where

$$\begin{aligned}W_{2,2}(\alpha,\lambda)&=\sum_{k=1}^{\infty}(-1)^k((\Theta_0+\tilde{Y}_\alpha[\lambda])^{-1}Z_\alpha[\lambda])^k(\Theta_0+\tilde{Y}_\alpha[\lambda])^{-1}\\&=-(\Theta_0+\tilde{Y}_\alpha[\lambda])^{-1}Z_\alpha[\lambda](\mathbb{I}+T_\alpha[\lambda])^{-1}.\end{aligned}$$

Thus, (2.50) with (2.34) and (2.48) implies

$$\begin{aligned}\|W_{2,2}(\alpha,\lambda)f\|_{L^q_{[5]}(\Omega)}&\leqslant C(|\lambda|+\alpha^2)^{\frac12}\|f\|_{L^q_{[5]}(\Omega)},\\\|\partial_\lambda W_{2,2}(\alpha,\lambda)f\|_{L^q_{[5]}(\Omega)}&\leqslant\frac{C}{|\lambda|^{\frac12}}\|f\|_{L^q_{[5]}(\Omega)},\\\|\partial_\lambda^2 W_{2,2}(\alpha,\lambda)f\|_{L^q_{[5]}(\Omega)}&\leqslant\frac{C}{|\lambda|(|\lambda|+\alpha^2)^{\frac12}}\|f\|_{L^q_{[5]}(\Omega)},\end{aligned} \tag{2.52}$$

and (2.49) combined with $(\Theta_0+\tilde{Y}_\alpha[\lambda])^{-1}=(\mathbb{I}+\Theta_0^{-1}\tilde{Y}_\alpha[\lambda])^{-1}\Theta_0^{-1}$ yields

$$\sup_{\lambda\in\mathcal{O}_{c_q},0<\alpha\leqslant c_q}\left|\int_\Omega(\mathbb{I}+T_\alpha[\lambda])^{-1}f\,dx\right|\leqslant\frac{C}{|\log(4\lambda+\alpha^2)|}\|f\|_{L^q_{[5]}(\Omega)}. \tag{2.53}$$

By setting $W_2(\alpha,\lambda)=W_{2,1}(\alpha,\lambda)+W_{2,2}(\alpha,\lambda)$, we obtain the expansion (2.29) of $(\mathbb{I}+T_\alpha[\lambda])^{-1}$ from (2.51) and (2.50). Estimate (2.30) for $W_2(\alpha,\lambda)$ follows from (2.48) and (2.52). The proof is complete. □

Proposition 2.4. *Set $\Omega_4=\Omega\cap\{|x|\leqslant 4\}$. Let $1<q<\infty$. Let $\lambda\in\mathcal{O}_{c_q}$ and let $\alpha\in(0,c_q]$. Then for $f\in L^q_{[5]}(\Omega)^2$ and $x\in\Omega_4$,*

$$(\mathcal{U}_\alpha[\lambda](\mathbb{I}+T_\alpha[\lambda])^{-1}f)(x)=(\mathbb{A}_{\alpha,D}^{-1}\mathbb{P}_D(\mathbb{I}+T_\alpha[\lambda])^{-1}f)(x)+(R_\alpha^{(u)}[\lambda]f)(x), \tag{2.54}$$

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where $R_\alpha^{(u)}[\lambda]f$ *is analytic in* $\lambda \in \mathcal{O}_{c_q}$ *with value in* $L^q_{[5]}(\Omega)^2$ *and*

$$\|R_\alpha^{(u)}[\lambda]f\|_{W^{2,q}(\Omega_4)} \leqslant C|\lambda|\|f\|_{L^q_{[5]}(\Omega)}. \tag{2.55}$$

Here, C *depends only on* q *and* Ω.

Proof. From the definition of $\mathcal{U}_\alpha[\lambda]$ in (2.21), we see that for any $h \in L^q_{[5]}(\Omega)^2$,

$$\mathcal{U}_\alpha[\lambda]h = u_D[\lambda]h \quad \text{for } x \in \Omega_4,$$

where $u_D[\lambda]h = (\lambda + A_{\alpha,D})^{-1}\mathbb{P}_D h$ is the solution to (2.15) with f replaced by h. Then (2.54) follows by setting $R_\alpha^{(u)}[\lambda]f = ((\lambda + A_{\alpha,D})^{-1} - A_{\alpha,D}^{-1})\mathbb{P}_D(\mathbb{I} + T_\alpha[\lambda])^{-1}f$, and (2.55) is a consequence of (2.16) with the resolvent identity and (2.27). The proof is complete. □

It is well known that the set $\mathbb{C} \setminus \{\lambda \leqslant 0\}$ is the resolvent set of the Stokes operator $-\mathbb{A}_\Omega$ in $L^q_\sigma(\Omega)$. In particular, we have

$$\begin{aligned} |\lambda|\|(\lambda + \mathbb{A}_\Omega)^{-1}f\|_{L^q(\Omega)} + \frac{|\lambda|}{1+|\lambda|}\|(\lambda + \mathbb{A}_\Omega)^{-1}f\|_{W^{2,q}(\Omega)} \leqslant C\|f\|_{L^q(\Omega)}, \\ f \in L^q_\sigma(\Omega), \quad \lambda \in \Sigma_{\frac{4\pi}{5}}. \end{aligned} \tag{2.56}$$

Then the standard perturbation theory of sectorial operators yields the following lemma for the resolvent of $\mathbb{A}_{\alpha,\Omega}$ away from the origin.

Lemma 2.5. *Let* $1 < q < \infty$. *Let* $c_q > 0$ *be the number in Proposition 2.3. Then there exists* $\tilde{\alpha}_q > 0$ *such that if* $0 < \alpha \leqslant \tilde{\alpha}_q$, *then* $\Sigma_{\frac{3}{4}\pi} \cap \{\lambda \in \mathbb{C} \mid |\lambda| \geqslant \frac{c_q}{4}\}$ *is included in the resolvent set of* $-\mathbb{A}_{\alpha,\Omega}$ *in* $L^q_\sigma(\Omega)$ *and satisfies*

$$|\lambda|\|(\lambda + \mathbb{A}_{\alpha,\Omega})^{-1}f\|_{L^q(\Omega)} + \frac{|\lambda|}{1+|\lambda|}\|(\lambda + \mathbb{A}_{\alpha,\Omega})^{-1}f\|_{W^{2,q}(\Omega)} \leqslant C\|f\|_{L^q(\Omega)}, \tag{2.57}$$

for all $f \in L^q_\sigma(\Omega)$ *and* $\lambda \in \Sigma_{\frac{3}{4}\pi} \cap \{\lambda \in \mathbb{C} \mid |\lambda| \geqslant \frac{c_q}{4}\}$.

The proof of this lemma is standard since $\alpha\mathbb{P}_\Omega\partial_1$ is regarded as a small perturbation to $\mathbb{A}_\Omega$ when α is small enough. So we omit the proof of Lemma 2.5. The next proposition is our key local energy estimate in large time.

Proposition 2.6. *Set* $\Omega_4 = \Omega \cap \{|x| \leqslant 4\}$. *Let* $1 < q < \infty$. *If* $0 < \alpha \leqslant \min\{c_q, , \tilde{\alpha}_q\}$, *then the following estimate holds for all* $f \in L^q_{[5]}(\Omega)^2$:

$$\|e^{-t\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega f\|_{W^{2,q}(\Omega_4)} \leqslant \begin{cases} \left(\dfrac{C}{t|\log t|^2} + \dfrac{C\alpha^2}{|\log t|}\right)\|f\|_{L^q_{[5]}(\Omega)}, & 2 \leqslant t \leqslant \alpha^{-2}, \\ \dfrac{C}{t^2\alpha^2|\log \alpha|}\|f\|_{L^q_{[5]}(\Omega)}, & t \geqslant \alpha^{-2}. \end{cases} \tag{2.58}$$

Here, C *depends only on* q *and* Ω.

Proof. It suffices to consider the case $t \geqslant 4\max\{1, c_q^{-1}\}$. Since $-\mathbb{A}_{\alpha,\Omega}$ is sectorial in $L^q_\sigma(\Omega)$ and generates a C_0-analytic semigroup in $L^q_\sigma(\Omega)$, Lemma 2.5 leads to the representation of $e^{-t\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega f$ such as

$$e^{-t\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega f = \frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{t}}} e^{t\lambda}(\lambda + \mathbb{A}_{\alpha,\Omega})^{-1}\mathbb{P}_\Omega f \, d\lambda. \tag{2.59}$$

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Here, for a fixed number $\iota > 0$, the curve $\Gamma_\iota = \Gamma_{\pm,\iota} + \Gamma_{0,\iota}$ in $\mathbb{C}$ is taken as

$$\Gamma_{\pm,\iota} = \left\{\lambda + \iota \mid \arg(\lambda) = \pm\frac{3}{4}\pi, |\lambda| \geqslant \frac{c_q}{2}\right\},$$
$$\Gamma_{0,\iota} = \left\{\lambda + \iota \mid |\arg(\lambda)| \leqslant \frac{3\pi}{4}, |\lambda| = \frac{c_q}{2}\right\},$$

and is oriented counter-clockwise. On the curve $\Gamma_{\pm,\frac{1}{t}}$, we simply apply the uniform bound stated in Lemma 2.5 such as $\|(\lambda + \mathbb{A}_{\alpha,\Omega})^{-1}\mathbb{P}_\Omega f\|_{W^{2,q}(\Omega_4)} \leqslant C\|f\|_{L^q_{[5]}(\Omega)}$ for λ with $\arg(\lambda) = \pm\frac{3\pi}{4}$ and $|\lambda| \geqslant \frac{c_q}{2}$, which yields enough temporal decay such that

$$\left\|\frac{1}{2\pi i}\int_{\Gamma_{\pm,\frac{1}{t}}} e^{t\lambda}(\lambda + \mathbb{A}_{\alpha,\Omega})^{-1}\mathbb{P}_\Omega f\,d\lambda\right\|_{W^{2,q}(\Omega_4)} \leqslant C\int_{\Gamma_{\pm,\frac{1}{t}}} |e^{t\lambda}||d\lambda|\|f\|_{L^q_{[5]}(\Omega)}$$
$$\leqslant \frac{C}{t}e^{-\frac{c_q}{4}t}\|f\|_{L^q_{[5]}(\Omega)}. \tag{2.60}$$

Note that C depends only on q and Ω. The estimate on the curve $\Gamma_{0,\iota}$ needs a detailed computation. Let us recall that

$$(\lambda + \mathbb{A}_{\alpha,\Omega})^{-1}\mathbb{P}_\Omega f = \mathcal{U}_\alpha[\lambda](\mathbb{I} + T_\alpha[\lambda])^{-1}f. \tag{2.61}$$

Hence, we have from Proposition 2.4, for $|x| \leqslant 4$,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma_{0,\frac{1}{t}}} e^{t\lambda}(\lambda + \mathbb{A}_{\alpha,\Omega})^{-1}\mathbb{P}_\Omega f\,d\lambda &= \frac{1}{2\pi i}\int_{\Gamma_{0,\frac{1}{t}}} e^{t\lambda}\mathbb{A}_{\alpha,D}^{-1}\mathbb{P}_D(\mathbb{I} + T_\alpha[\lambda])^{-1}f\,d\lambda\\ &\quad + \frac{1}{2\pi i}\int_{\Gamma_{0,\frac{1}{t}}} e^{t\lambda}R_\alpha^{(u)}[\lambda](\mathbb{I} + T_\alpha[\lambda])^{-1}f\,d\lambda\\ &= I(t)f + II(t)f.\end{aligned} \tag{2.62}$$

(i) Estimate of $I(t)f$: Let us recall that $(\mathbb{I} + T_\alpha[\lambda])^{-1}f$ is analytic in $\lambda \in \mathcal{O}_{c_q}$ with value in $L^q_{[5]}(\Omega)^2$, and, thus, the Cauchy theorem implies

$$I(t)f = \frac{1}{2\pi i}\int_{l_{0,\frac{1}{t}}} e^{t\lambda}\mathbb{A}_{\alpha,D}^{-1}\mathbb{P}_D(\mathbb{I} + T_\alpha[\lambda])^{-1}f\,d\lambda,$$

where $l_{0,\frac{1}{t}} = l_{+,\frac{1}{t}} + l_{-,\frac{1}{t}}$ with $l_{\pm,\frac{1}{t}} = \{\frac{1}{t} + r(-\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}) \mid 0 \leqslant r \leqslant \frac{c_q}{2}\}$ and with the orientation going from $z_- := \frac{1}{t} + \frac{c_q}{2}(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})$ to $z_+ := \frac{1}{t} + \frac{c_q}{2}(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})$. Then the expansion of $(\mathbb{I} + T_\alpha[\lambda])^{-1}$ yields

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$$
\begin{aligned}
I(t)f
&= \mathbb{A}_{\alpha,D}^{-1}\mathbb{P}_D\bigg\{\frac{1}{2\pi i}\int_{l_{0,\frac{1}{t}}} e^{t\lambda}\,d\lambda(\Theta_0^{-1}+W_0)f + \frac{1}{2\pi i}\int_{l_{0,\frac{1}{t}}} e^{t\lambda}\frac{1}{\log(4\lambda+\alpha^2)}\,d\lambda W_{1,1}f\\
&\quad + \frac{1}{2\pi i}\int_{l_{0,\frac{1}{t}}} e^{t\lambda}\frac{d(\alpha,\lambda)}{\log(4\lambda+\alpha^2)}\,d\lambda\, W_{1,2}f + \frac{1}{2\pi i}\int_{l_{0,\frac{1}{t}}} e^{t\lambda}W_2(\alpha,\lambda)f\,d\lambda\bigg\}\\
&= \sum_{j=1}^{4} I_j(t)f.
\end{aligned}
$$

For $I_1(t)f$, the Cauchy theorem gives $|\frac{1}{2\pi i}\int_{l_{0,\frac{1}{t}}} e^{t\lambda}\,d\lambda| \leqslant Ce^{-\frac{cq}{4}t}$, and, thus,

$$
\|I_1(t)f\|_{W^{2,q}(\Omega_4)} \leqslant Ce^{-\frac{cq}{4}t}\|f\|_{L^q_{[5]}(\Omega)}. \tag{2.63}
$$

Here, C depends only on q and Ω. To estimate $I_2(t)f$ and $I_3(t)f$, we consider a given analytic function $h(\alpha,\lambda)$ and then the integration by part yields

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{l_{0,\iota}} e^{t\lambda}h(\alpha,\lambda)\,d\lambda &= \frac{1}{2\pi i t}(e^{tz_+}h(\alpha,z_+) - e^{tz_-}h(\alpha,z_-))\\
&\quad - \frac{1}{2\pi i t^2}(e^{tz_+}\partial_\lambda h(\alpha,z_+) - e^{tz_-}\partial_\lambda h(\alpha,z_-))\\
&\quad + \frac{1}{2\pi i t^2}\int_{l_{0,\frac{1}{t}}} e^{t\lambda}\partial_\lambda^2 h(\alpha,\lambda)\,d\lambda. \qquad (2.64)
\end{aligned}
$$

For $I_2(t)f$, we take $h(\alpha,\lambda) = \frac{1}{\log(4\lambda+\alpha^2)}$. Then, since $z_\pm = \frac{1}{t} + \frac{cq}{2}(-\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}})$, the first four terms in the right-hand side of (2.64) are estimated from above by $Ct^{-1}e^{-\frac{cq}{4}t}$. Set $z_1 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$. As for the last term, we have from (2.7) and by the symmetry of $l_{0,\frac{1}{t}}$ about the real axis,

$$
\begin{aligned}
&\left|\frac{1}{2\pi i t^2}\int_{l_{0,\frac{1}{t}}} e^{t\lambda}\partial_\lambda^2\frac{1}{\log(4\lambda+\alpha^2)}\,d\lambda\right|\\
&\quad\leqslant \frac{C}{t^2}\int_{l_{0,\frac{1}{t}}} \frac{|e^{t\lambda}|}{|4\lambda+\alpha^2|^2|\log(4\lambda+\alpha^2)|^2}|d\lambda|\\
&\quad\leqslant \frac{C}{t^2}\int_0^{\frac{cq}{2}} \frac{1}{(|\frac{1}{t}+rz_1|+\alpha^2)^2(\log(|\frac{1}{t}+rz_1|+\alpha^2))^2}\,dr\\
&\quad\leqslant \frac{C}{t^2}\left(\int_0^{\frac{1}{4t}} + \int_{\frac{1}{4t}}^{\frac{cq}{2}}\right)\\
&\quad\leqslant \frac{C}{t^2}\left(\frac{1}{(\frac{1}{t}+\alpha^2)^2(\log(\frac{1}{t}+\alpha^2))^2}\frac{1}{4t} + \int_{\frac{1}{4t}}^{\frac{cq}{2}}\frac{1}{(r+\alpha^2)^2(\log(r+\alpha^2))^2}\,dr\right)\\
&\quad\leqslant \frac{C}{t^2}\frac{1}{(\frac{1}{t}+\alpha^2)(\log(\frac{1}{t}+\alpha^2))^2}.
\end{aligned}
$$

The last quantity is bounded from above by $\frac{C}{t|\log t|^2}$ if $t \leqslant \alpha^{-2}$ and by $\frac{C}{t^2\alpha^2|\log\alpha|^2}$ if $t \geqslant \alpha^{-2}$. Thus, we conclude that

$$\|I_2(t)f\|_{L^q(\Omega_4)} \leqslant \begin{cases} \dfrac{C}{t|\log t|^2}\|f\|_{L^q_{[5]}(\Omega)}, & 2 \leqslant t \leqslant \alpha^{-2}, \\ \dfrac{C}{t^2\alpha^2|\log\alpha|^2}\|f\|_{L^q_{[5]}(\Omega)}, & t \geqslant \alpha^{-2}. \end{cases} \tag{2.65}$$

For $I_3(t)f$, we take $h(\alpha,\lambda) = \frac{d(\alpha,\lambda)}{\log(4\lambda+\alpha^2)}$. We first consider the case $t \geqslant \alpha^{-2}$. Then, again the first four terms in the right-hand side of (2.64) are bounded from above by $Ct^{-1}e^{-\frac{cq}{4}t}$. To estimate the last term in (2.64), we observe from (2.7) and (2.11) that, for $z_1 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$,

$$\begin{aligned} &\left|\frac{1}{2\pi i t^2}\int_{l_{0,\frac{1}{t}}} e^{t\lambda}\partial_\lambda^2\frac{d(\alpha,\lambda)}{\log(4\lambda+\alpha^2)}\,d\lambda\right| \\ &\quad\leqslant \frac{C}{t^2}\int_0^{\frac{cq}{2}} \frac{e^{-\frac{r}{\sqrt{2}}t}}{|\log(|\frac{1}{t}+rz_1|+\alpha^2)||\frac{1}{t}+rz_1|(|\frac{1}{t}+rz_1|+\alpha^2)}\,dr \\ &\quad\leqslant \frac{C}{t^2}\left(\int_0^{\frac{1}{4t}} + \int_{\frac{1}{4t}}^{\frac{cq}{2}}\right) \\ &\quad\leqslant \frac{C}{t^2}\left(\frac{1}{|\log(\frac{1}{t}+\alpha^2)|(\frac{1}{t}+\alpha^2)} + \int_{\frac{1}{4t}}^{\frac{cq}{2}}\frac{e^{-\frac{r}{\sqrt{2}}t}}{|\log(r+\alpha^2)|r(r+\alpha^2)}\,dr\right), \end{aligned}$$

and since $-\tau\log\tau$ is increasing for $\tau \in (0, e^{-1}]$, the last integral in bounded from above by $\frac{C}{|\log(\frac{1}{t}+\alpha^2)|\frac{1}{t}(\frac{1}{t}+\alpha^2)}\int_{\frac{1}{4t}}^{\frac{cq}{2}} e^{-\frac{r}{\sqrt{2}}t}\,dr$, that gives

$$\begin{aligned} \left|\frac{1}{2\pi i t^2}\int_{l_{0,\frac{1}{t}}} e^{t\lambda}\partial_\lambda^2\frac{d(\alpha,\lambda)}{\log(4\lambda+\alpha^2)}\,d\lambda\right| &\leqslant \frac{C}{t^2}\frac{1}{|\log(\frac{1}{t}+\alpha^2)|(\frac{1}{t}+\alpha^2)} \\ &\leqslant \frac{C}{t^2\alpha^2|\log\alpha|} \quad \text{if } t \geqslant \alpha^{-2}. \end{aligned}$$

When $2 \leqslant t \leqslant \alpha^{-2}$, by using $|d(\alpha,\lambda)| \leqslant C\frac{\alpha^2}{|\lambda|}$ (see (2.11)), we compute directly as

$$\begin{aligned} &\left|\frac{1}{2\pi i}\int_{l_{0,t}} e^{t\lambda}\frac{d(\alpha,\lambda)}{\log(4\lambda+\alpha^2)}\,d\lambda\right| \\ &\quad\leqslant C\alpha^2\int_0^{\frac{cq}{2}}\frac{e^{-\frac{r}{\sqrt{2}}t}}{|\frac{1}{t}+rz_1||\log(|\frac{1}{t}+rz_1|+\alpha^2)|}\,dr \\ &\quad\leqslant C\alpha^2\left(\int_0^{\frac{1}{4t}} + \int_{\frac{1}{4t}}^{\frac{cq}{2}}\right) \\ &\quad\leqslant C\alpha^2\left(\frac{1}{|\log(\frac{1}{t}+\alpha^2)|} + \int_{\frac{1}{4t}}^{\frac{cq}{2}}\frac{e^{-\frac{r}{\sqrt{2}}t}}{(r+\alpha^2)|\log(r+\alpha^2)|}\,dr\right) \quad (\text{since } 2 \leqslant t \leqslant \alpha^{-2}) \end{aligned}$$

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$$\leqslant \frac{C\alpha^2}{|\log(\frac{1}{t}+\alpha^2)|}$$
$$\leqslant \frac{C\alpha^2}{|\log t|}.$$

Thus, we obtain

$$\|I_3(t)f\|_{W^{2,q}(\Omega_4)} \leqslant \begin{cases} \dfrac{C\alpha^2}{|\log t|}\|f\|_{L^q_{[5]}(\Omega)}, & 2\leqslant t\leqslant \alpha^{-2},\\ \dfrac{C}{t^2\alpha^2|\log\alpha|}\|f\|_{L^q_{[5]}(\Omega)}, & t\geqslant \alpha^{-2}. \end{cases} \tag{2.66}$$

The estimate of $I_4(t)f$ is similar to $I_3(t)f$. Indeed, we take $h(\alpha,\lambda)=W_2(\alpha,\lambda)f$ in (2.64), which gives

$$\|I_4(t)f\|_{W^{2,q}(\Omega_4)} \leqslant \frac{C}{t}e^{-\frac{c_q}{4}t}\|f\|_{L^q_{[5]}(\Omega)} + \frac{C}{t^2}\int_{l_{0,\frac{1}{t}}} |e^{t\lambda}|\|\partial_\lambda^2 W_2(\alpha,\lambda)f\|_{W^{2,q}(\Omega_4)}\,|d\lambda|,$$

and the last term is estimated from (2.30) as

$$\begin{aligned}
&\frac{C}{t^2}\int_{l_{0,\frac{1}{t}}} |e^{t\lambda}|\|\partial_\lambda^2 W_2(\alpha,\lambda)f\|_{W^{2,q}(\Omega_4)}\,|d\lambda|\\
&\quad\leqslant \frac{C}{t^2}\int_0^{\frac{c_q}{2}} \frac{e^{-\frac{r}{\sqrt{2}}t}}{|\log(|\frac{1}{t}+rz_1|+\alpha^2)|^2|\frac{1}{t}+rz_1|(|\frac{1}{t}+rz_1|+\alpha^2)}\,dr\,\|f\|_{L^q_{[5]}(\Omega)}\\
&\quad\leqslant \frac{C}{t^2}\left(\int_0^{\frac{1}{4t}}+\int_{\frac{1}{4t}}^{\frac{c_q}{2}}\right)\|f\|_{L^q_{[5]}(\Omega)}\\
&\quad\leqslant \frac{C}{t^2}\left(\frac{1}{|\log(\frac{1}{t}+\alpha^2)|^2(\frac{1}{t}+\alpha^2)}+\int_{\frac{1}{4t}}^{\frac{c_q}{2}}\frac{e^{-\frac{r}{\sqrt{2}}t}}{|\log(r+\alpha^2)|^2 r(r+\alpha^2)}\,dr\right)\|f\|_{L^q_{[5]}(\Omega)}\\
&\quad\leqslant \frac{C}{t^2}\frac{1}{|\log(\frac{1}{t}+\alpha^2)|^2(\frac{1}{t}+\alpha^2)}\|f\|_{L^q_{[5]}(\Omega)}.
\end{aligned}$$

Thus, we see

$$\|I_4(t)f\|_{W^{2,q}(\Omega_4)} \leqslant \begin{cases} \dfrac{C}{t|\log t|^2}\|f\|_{L^q_{[5]}(\Omega)}, & 2\leqslant t\leqslant \alpha^{-2},\\ \dfrac{C}{t^2\alpha^2|\log\alpha|^2}\|f\|_{L^q_{[5]}(\Omega)}, & t\geqslant \alpha^{-2}. \end{cases} \tag{2.67}$$

Collecting (2.63), (2.65), (2.66), and (2.67), we have

$$\|I(t)f\|_{W^{2,q}(\Omega_4)} \leqslant \begin{cases} \dfrac{C}{t|\log t|^2}+\dfrac{C\alpha^2}{|\log t|}, & 2\leqslant t\leqslant \alpha^{-2},\\ \dfrac{C}{t^2\alpha^2|\log\alpha|}, & t\geqslant \alpha^{-2}. \end{cases} \tag{2.68}$$

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(ii) Estimate of $II(t)f$: Again from the Cauchy theorem, we have

$$II(t)f = \frac{1}{2\pi i}\int_{l_{0,\frac{1}{t}}} e^{t\lambda} R_\alpha^{(u)}[\lambda](\mathbb{I}+T_\alpha[\lambda])^{-1} f\, d\lambda.$$

Then, from (2.55), for $R_\alpha^{(u)}[\lambda]$ and (2.27) for $(\mathbb{I}+T_\alpha[\lambda])^{-1}$, we have

$$\begin{aligned}\|II(t)f\|_{W^{2,q}(\Omega_4)} &\leqslant C\int_{l_{0,\frac{1}{t}}} |e^{t\lambda}||\lambda||d\lambda|\|f\|_{L^q_{[5]}(\Omega)} \\ &\leqslant C\int_0^{\frac{c_q}{2}} e^{-\frac{r}{\sqrt{2}}t}\left|\frac{1}{t}+rz_1\right| dr\, \|f\|_{L^q_{[5]}(\Omega)} \\ &\leqslant \frac{C}{t^2}\|f\|_{L^q_{[5]}(\Omega)}. \end{aligned} \tag{2.69}$$

Hence, (2.58) follows from (2.60), (2.68), and (2.69). The proof is complete. □

We have the similar local energy decay estimate for the associated pressure field to $e^{-t\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega f$, which is denoted by $p[\mathbb{P}_\Omega f](t)$ and satisfies $\int_{\Omega\cap\{|x|\leqslant 4\}} p[\mathbb{P}_\Omega f](t)\, dx = 0$ and $\nabla p[\mathbb{P}_\Omega f](t) = -\mathbb{Q}_\Omega A_\alpha e^{-t\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega f$.

Proposition 2.7. *Set $\Omega_4 = \Omega\cap\{|x|\leqslant 4\}$. Let $1<q<\infty$. If $0<\alpha\leqslant \min\{c_q,,\tilde{\alpha}_q\}$, then the following estimate holds for all $f\in L^q_{[5]}(\Omega)^2$:*

$$\|p[\mathbb{P}_\Omega f](t)\|_{L^q(\Omega_4)} \leqslant \begin{cases} \left(\dfrac{C}{t|\log t|^2}+\dfrac{C\alpha^2}{|\log t|}\right)\|f\|_{L^q_{[5]}(\Omega)}, & 2\leqslant t\leqslant \alpha^{-2}, \\ \dfrac{C}{t^2\alpha^2|\log\alpha|}\|f\|_{L^q_{[5]}(\Omega)}, & t\geqslant \alpha^{-2}. \end{cases} \tag{2.70}$$

Here, C depends only on q and Ω.

Proof. The result essentially follows from Proposition 2.6. Indeed, for $u(t) = e^{-t\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega f$, the equality $\partial_t u + A_\alpha u + \nabla p[\mathbb{P}_\Omega f] = 0$ implies that from $\int_{\Omega_4} p[\mathbb{P}_\Omega f](t)\, dx = 0$ and the Poincaré inequality,

$$\begin{aligned}\|p[\mathbb{P}_\Omega f](t)\|_{L^q(\Omega_4)} &\leqslant C\|\nabla p[\mathbb{P}_\Omega f](t)\|_{L^q(\Omega_4)} \\ &\leqslant C(\|\partial_t u(t)\|_{L^q(\Omega_4)} + \|A_\alpha u(t)\|_{L^q(\Omega_4)}) \\ &\leqslant C(\|\partial_t u(t)\|_{L^q(\Omega_4)} + \|u(t)\|_{W^{2,q}(\Omega_4)}).\end{aligned}$$

The term $\|u(t)\|_{W^{2,q}(\Omega_4)}$ is already estimated in Proposition 2.6, and since

$$\partial_t u(t) = \frac{1}{2\pi i}\int_{\Gamma_\iota} e^{t\lambda}\lambda(\lambda+\mathbb{A}_{\alpha,\Omega})^{-1}\mathbb{P}_\Omega f\, d\lambda, \quad \iota = \frac{1}{t}$$

the proof of Proposition 2.6 is directly applied for the estimate of $\partial_t u(t)$ (and provides the better estimate than $u(t)$ itself, as expected). The details are omitted here. The proof is complete. □

Finally, we give the short-time estimate of the pressure $p[\mathbb{P}_\Omega f](t)$ with $\int_{\Omega_4} p[\mathbb{P}_\Omega f](t)\, dx = 0$, where f does not need to be compactly supported.

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Proposition 2.8. *Set* $\Omega_4 = \Omega \cap \{|x| \leqslant 4\}$*. Let* $1 < q < \infty$ *and* $\alpha \in (0, \frac{1}{2}]$*. Then the following estimate holds for all* $f \in L^q(\Omega)^2$*:*

$$\|p[\mathbb{P}_\Omega f](t)\|_{L^q(\Omega_4)} \leqslant \frac{C}{t^{\frac{1}{2}(1+\frac{1}{q})}} \|f\|_{L^q(\Omega)}, \quad 0 < t \leqslant 3. \tag{2.71}$$

Here, C *depends only on* q *and* Ω*.*

Proof. Since the Helmholtz projection is bounded in $L^q(\Omega)^2$, we may assume that $f = \mathbb{P}_\Omega f$, but by assuming that $f \in L^q_\sigma(\Omega)$ instead of $f \in L^q(\Omega)^2$. Let us introduce smooth cut-off functions χ_l, where $\chi_l = 1$ if $|x| \leqslant l$ and $\chi_l = 0$ if $|x| \geqslant l+1$. Let $\mathbb{B}_l$ be the Bogovskii operator in the annulus $D_l = \{l < |x| < l+1\}$, i.e., $\mathbb{B}_l[g]$ satisfies

$$\operatorname{div} \mathbb{B}_l[g] = g \quad \text{in } D_l, \qquad \mathbb{B}_l[g] = 0 \quad \text{on } \partial D_l,$$

for a given function $g \in C_0^\infty(D_l)$ with $\int_{D_l} g\, dx = 0$. The Bogovskii operator $\mathbb{B}_l$ is extended to a bounded operator from $W_0^{k,q}(D_l)$ to $W_0^{k+1,q}(D_l)^2$ for any $1 < q < \infty$ and $k = 0, 1, \ldots,$ together with the estimate

$$\|\nabla^{k+1}\mathbb{B}_l[g]\|_{L^q(D_l)} \leqslant C_{q,k,l}\|\nabla^k g\|_{L^q(D_l)}, \quad 1 < q < \infty,\ k = 0, 1, \ldots. \tag{2.72}$$

We set $f_D = \chi_2 f - \mathbb{B}_2[\nabla\chi_2 \cdot f]$. Note that $\operatorname{supp} f_D \subset \{|x| \leqslant 3\}$ and $\chi_1 f_D = \chi_1 f$. Then we set

$$\begin{aligned} u^{(1)} &= \chi_1 e^{-t\mathbb{A}_{\alpha,D}} f_D - \mathbb{B}_1[\nabla\chi_1 \cdot e^{-t\mathbb{A}_{\alpha,D}} f_D] \\ &\quad + (1-\chi_1)e^{-t\mathbb{A}_\alpha} f + \mathbb{B}_1[\nabla\chi_1 \cdot e^{-t\mathbb{A}_\alpha} f]. \end{aligned} \tag{2.73}$$

Here, f is extended by zero to $\mathbb{R}^2$. Then, since

$$\begin{aligned} &(1-\chi_1)f + \mathbb{B}_1[\nabla\chi_1 \cdot f] + \chi_1 f_D - \mathbb{B}_1[\nabla\chi_1 \cdot f_D] \\ &\quad = (1-\chi_1)f + \mathbb{B}_1[\nabla\chi_1 \cdot f] + \chi_1 f - \mathbb{B}_1[\nabla\chi_1 \cdot f] = f, \end{aligned}$$

we see that $u^{(1)}$ solves

$$\begin{aligned} \partial_t u^{(1)} + A_\alpha u^{(1)} + \nabla(\chi_1 p_D) &= R, \quad t > 0, x \in \Omega, \\ \operatorname{div} u^{(1)} &= 0, \quad t \geqslant 0, x \in \Omega, \\ u^{(1)}|_{\partial\Omega} = 0, \quad u^{(1)}|_{t=0} &= f. \end{aligned}$$

Here, p_D is the associated pressure of $e^{-t\mathbb{A}_{\alpha,D}} f_D$ satisfying $\int_D p_D\, dx = 0$, and R is given by

$$\begin{aligned} R(t) &= -(\Delta\chi_1)u_D(t) - 2\nabla\chi_1 \cdot \nabla u_D(t) + \alpha(\partial_1\chi_1)u_D(t) \\ &\quad + (\nabla\chi_1)p_D(t) - (\partial_t + A_\alpha)\mathbb{B}_1[\nabla\chi_1 \cdot u_D(t)] \\ &\quad + (\Delta\chi_1)u_{\mathbb{R}^2}(t) + 2\nabla\chi_1 \cdot \nabla u_{\mathbb{R}^2}(t) - \alpha(\partial_1\chi_1)u_{\mathbb{R}^2}(t) + (\partial_t + A_\alpha)\mathbb{B}_1[\nabla\chi_1 \cdot u_{\mathbb{R}^2}(t)]. \end{aligned} \tag{2.74}$$

Here, $u_D(t) = e^{-t\mathbb{A}_{\alpha,D}} f_D$ and $u_{\mathbb{R}^2}(t) = e^{-t\mathbb{A}_\alpha} f$, and we note that the pressure associated with $e^{-t\mathbb{A}_\alpha} f$ is taken as zero. Then the original solution $(u, \nabla p_u)$ is constructed in the form $u = u^{(1)} + u^{(2)}$ and $p_u = \chi_1 p_D - \frac{1}{|\Omega_4|}\int_{\Omega_4} \chi_1 p_D\,dx + p_{u^{(2)}}$, and, thus, $(u^{(2)}, p_{u^{(2)}})$ is the solution to

$$\begin{aligned} \partial_t u^{(2)} + A_\alpha u^{(2)} + \nabla p_{u^{(2)}} &= -R, \quad t > 0, x \in \Omega, \\ \operatorname{div} u^{(2)} &= 0, \quad t \geqslant 0, x \in \Omega, \\ u^{(2)}|_{\partial\Omega} = 0, \quad u^{(2)}|_{t=0} = 0, &\quad \int_{\Omega_4} p_{u^{(2)}}\,dx = 0. \end{aligned}$$

Then it is straightforward to see from $\int_{\Omega_4} p_{u^{(2)}}\,dx = 0$ that

$$\begin{aligned} \|p_{u^{(2)}}(t)\|_{L^q(\Omega_4)} &\leqslant C\|\nabla p_{u^{(2)}}(t)\|_{W^{-1,q}(\Omega_4)} \\ &= C\| - \partial_t u^{(2)}(t) - A_\alpha u^{(2)}(t) - R(t)\|_{W^{-1,q}(\Omega_4)} \\ &\leqslant C\big(\|\partial_t u^{(2)}(t)\|_{L^q(\Omega_4)} + \|u^{(2)}(t)\|_{W^{1,q}(\Omega_4)} + \|R(t)\|_{L^q(\Omega_4)}\big). \end{aligned} \tag{2.75}$$

Here, $W^{-1,q}(\Omega_4)$ is the dual space of $W_0^{1,\frac{q}{q-1}}(\Omega_4)$. The norm of $u^{(2)}$ is estimated from the formula

$$u^{(2)}(t) = -\int_0^t e^{-(t-s)\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega R(s)\,ds. \tag{2.76}$$

We have from the local (in time) estimate of the Oseen semigroup,

$$\|u^{(2)}(t)\|_{W^{1,q}(\Omega_4)} \leqslant C\int_0^t (t-s)^{-\frac{1}{2}}\|R(s)\|_{L^q}\,ds, \tag{2.77}$$

while by decomposing $\int_0^t = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t$ and by using

$$\int_{\frac{t}{2}}^t e^{-(t-s)\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega R(s)\,ds = \int_0^{\frac{t}{2}} e^{-s\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega R(t-s)\,ds,$$

we have

$$\partial_t u^{(2)}(t) = \int_0^{\frac{t}{2}} \mathbb{A}_{\alpha,\Omega} e^{-(t-s)\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega R(s)\,ds + \int_0^{\frac{t}{2}} e^{-s\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega \partial_t R(t-s)\,ds. \tag{2.78}$$

Thus, it follows that

$$\|\partial_t u^{(2)}(t)\|_{L^q(\Omega_4)} \leqslant C\int_0^{\frac{t}{2}} (t-s)^{-1}\|R(s)\|_{L^q}\,ds + C\int_0^{\frac{t}{2}} \|\partial_t R(t-s)\|_{L^q}\,ds. \tag{2.79}$$

Therefore, it suffices to estimate $\|R(t)\|_{L^q}$ and $\|\partial_t R(t)\|_{L^q}$. By the definition of $R(t)$, we have from (2.72),

$$\begin{aligned} \|R(t)\|_{L^q} &\leqslant C(\|u_D(t)\|_{W^{1,q}(D_1)} + \|p_D(t)\|_{L^q(D_1)} + \|u_{\mathbb{R}^2}(t)\|_{W^{1,q}(D_1)}), \\ \|\partial_t R(t)\|_{L^q} &\leqslant C(\|\partial_t u_D(t)\|_{W^{1,q}(D_1)} + \|\partial_t p_D(t)\|_{L^q(D_1)} + \|\partial_t u_{\mathbb{R}^2}(t)\|_{W^{1,q}(D_1)}). \end{aligned} \tag{2.80}$$

Here, in the first inequality, we have used $\partial_t \mathbb{B}_1[\nabla\chi \cdot e^{-t\mathbb{A}_\alpha}h] = \mathbb{B}_1[\nabla\chi \cdot (\Delta - \alpha\partial_1)e^{-t\mathbb{A}_\alpha}h]$ (since the pressure is zero) and $\partial_t \mathbb{B}_1[\nabla\chi \cdot e^{-t\mathbb{A}_{\alpha,D}}h] = \mathbb{B}_1[\nabla\chi \cdot ((\Delta - \alpha\partial_1)e^{-t\mathbb{A}_{\alpha,D}}h -$

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$\nabla p_D(t))]$, and since $\nabla\chi = 0$ on the boundary of the annulus $|x| = 1, 2$, one can apply the boundedness of the Bogovskii operator in a negative order space (cf. [4]), i.e.,

$$\|\mathbb{B}_1[\nabla(\nabla\chi\cdot\nabla e^{-t\mathbb{A}_\alpha}h)]\|_{L^q(D_1)} \leqslant C\|\nabla e^{-t\mathbb{A}_\alpha}h\|_{L^q(D_1)},$$
$$\|\mathbb{B}_1[\nabla(\nabla\chi\cdot\nabla e^{-t\mathbb{A}_{\alpha,D}}h)]\|_{L^q(D_1)} \leqslant C\|\nabla e^{-t\mathbb{A}_{\alpha,D}}h\|_{L^q(D_1)},$$
$$\|\mathbb{B}_1[\nabla(p_D(t)\nabla\chi)]\|_{L^q(D_1)} \leqslant C\|p_D(t)\|_{L^q(D_1)}.$$

This argument yields the estimate of $R(t)$ as above. The estimate of $\partial_t R(t)$ is shown in the same manner. Then the standard estimate of the Oseen semigroups $e^{-t\mathbb{A}_{\alpha,D}}$ and $e^{-t\mathbb{A}_\alpha}$ give for $0 < t \leqslant 3$,

$$\|R(t)\|_{L^q} \leqslant C(t^{-\frac{1}{2}}\|f\|_{L^q} + \|p_D(t)\|_{L^q(D_1)}),$$
$$\|\partial_t R(t)\|_{L^q} \leqslant C(t^{-\frac{3}{2}}\|f\|_{L^q} + \|\partial_t p_D(t)\|_{L^q(D_1)}).$$

As for the pressure $p_D(t)$, we have

$$\|\partial_t^j p_D(t)\|_{L^q(D)} \leqslant Ct^{-j-\frac{1}{2}(1+\frac{1}{q})}\|f_D\|_{L^q(D)}, \quad 0 < t \leqslant 3, j = 0, 1. \tag{2.81}$$

The proof of (2.81) is postponed to Appendix C. Then we conclude from $\|f_D\|_{L^q(D)} \leqslant C\|f\|_{L^q}$ that

$$\|\partial_t^j R(t)\|_{L^q} \leqslant Ct^{-j-\frac{1}{2}(1+\frac{1}{q})}\|f\|_{L^q}, \quad 0 < t \leqslant 3, j = 0, 1, \tag{2.82}$$

which gives from (2.77) and (2.79) that, for $0 < t \leqslant 3$,

$$\|u^{(2)}(t)\|_{W^{1,q}(\Omega_4)} \leqslant Ct^{-\frac{1}{2q}}\|f\|_{L^q}, \quad \|\partial_t u^{(2)}(t)\|_{L^q(\Omega_4)} \leqslant Ct^{-\frac{1}{2}(1+\frac{1}{q})}\|f\|_{L^q}. \tag{2.83}$$

Thus, (2.75) yields

$$\|p_{u^{(2)}}(t)\|_{L^q(\Omega_4)} \leqslant Ct^{-\frac{1}{2}(1+\frac{1}{q})}\|f\|_{L^q}, \quad 0 < t \leqslant 3. \tag{2.84}$$

Then, since

$$\|p[\mathbb{P}_\Omega f](t)\|_{L^q(\Omega_4)} = \|p_u(t)\|_{L^q(\Omega_4)} \leqslant C(\|p_D(t)\|_{L^q(\Omega_4)} + \|p_{u^{(2)}}(t)\|_{L^q(\Omega_4)}),$$

estimate (2.71) follows from (2.81) with $\|f_D\|_{L^q(D)} \leqslant C\|f\|_{L^q}$ and (2.84). The proof is complete. □

2.4. Proof of Theorem 1.2

It is standard that $\|\nabla^j e^{-t\mathbb{A}_{\alpha,\Omega}}\mathbb{P}_\Omega f\|_{L^q} \leqslant Ct^{-\frac{j}{2}}\|\mathbb{P}_\Omega f\|_{L^q} \leqslant Ct^{-\frac{j}{2}}\|f\|_{L^q}$ for $0 < t \leqslant 3$. Then estimate (1.5) follows from Proposition 2.6 and estimate (1.6) follows from Propositions 2.7 and 2.8. The proof is complete.

3. L^q-L^r estimate of Oseen semigroup

In this section, we apply the local energy decay estimate of Theorem 1.2 to the L^q-L^r estimate of the Oseen semigroup, which proves Theorem 1.4. Let $\delta_q > 0$ be the number in Theorem 1.2.

Proposition 3.1. *Let $1 < r \leqslant q < \infty$ and let $\alpha \in (0, \delta_q]$. Then for any $f \in L^r_\sigma(\Omega)$,*

$$\|e^{-t\mathbb{A}_{\alpha,\Omega}} f\|_{W^{1,q}(\Omega_4)} \leqslant C(1+t)^{-\frac{1}{r}}\left(\frac{1+t}{t}\right)^{\frac{1}{2}+\frac{1}{r}-\frac{1}{q}} \|f\|_{L^r}, \quad t > 0, \tag{3.1}$$

$$\|p[\mathbb{P}_\Omega f](t)\|_{L^q(\Omega_4)} \leqslant C(1+t)^{-\frac{1}{r}}\left(\frac{1+t}{t}\right)^{\frac{1}{2}(1+\frac{1}{q})+\frac{1}{r}-\frac{1}{q}} \|f\|_{L^r}, \quad t > 0. \tag{3.2}$$

Proof. We consider the case $t \geqslant 2$, and we may take the initial data as $h = e^{-\mathbb{A}_{\alpha,\Omega}} f \in L^\infty(\Omega)^2 \cap L^r_\sigma(\Omega)$ instead of $f \in L^r_\sigma(\Omega)$. Let χ be a cut-off function $\chi \in C_0^\infty(\mathbb{R}^2)$ such that $\chi(x) = 1$ for $|x| \leqslant 4$ and $\chi(x) = 0$ for $|x| \geqslant 5$. Set

$$v(t) = (1-\chi)e^{-t\mathbb{A}_\alpha} h + \mathbb{B}_4[\nabla\chi \cdot e^{-t\mathbb{A}_\alpha} h].$$

Here, h is extended by zero to $\mathbb{R}^2$ and $\mathbb{B}_l$ is the Bogovskii operator in the annulus $D_l = \{x \in \mathbb{R}^2 \mid l < |x| < l+1\}$. Note that the pressure associated with $e^{-t\mathbb{A}_\alpha} h$ is taken as zero, and we have from the Young inequality for convolution,

$$\|e^{-t\mathbb{A}_\alpha} h\|_{W^{1,\infty}} \leqslant C(1+t)^{-\frac{1}{r}} \|h\|_{W^{1,\infty}\cap L^r} \leqslant C(1+t)^{-\frac{1}{r}} \|f\|_{L^r}, \quad t > 0. \tag{3.3}$$

Note that (3.3) is valid for all $\alpha > 0$ with a universal constant C. Then $w = e^{-t\mathbb{A}_{\alpha,\Omega}} h - v(t)$ satisfies

$$\begin{cases} \partial_t w + A_\alpha w + \nabla p_w = -\tilde{R}, & t > 0,\ x \in \Omega, \\ \operatorname{div} w = 0, & t \geqslant 0,\ x \in \Omega, \\ w = 0, & t > 0, x \in \partial\Omega, \\ w|_{t=0} = h_{loc}, & x \in \Omega, \end{cases}$$

with $p_w(t) = p[\mathbb{P}_\Omega h](t)$ (by the construction of v), $h_{loc} = \chi h - \mathbb{B}_4[\nabla\chi \cdot h]$, and

$$\tilde{R}(t) = (\Delta\chi)e^{-t\mathbb{A}_\alpha} h + 2\nabla\chi \cdot \nabla e^{-t\mathbb{A}_\alpha} h - \alpha(\partial_1\chi)e^{-t\mathbb{A}_\alpha} h + (\partial_t + A_\alpha)\mathbb{B}_4[\nabla\chi \cdot e^{-t\mathbb{A}_\alpha} h].$$

Then $\operatorname{supp} R(t) \subset \{4 \leqslant |x| \leqslant 5\}$ and we have from (2.72) and (3.3), and also by arguing as in the derivation of (2.80) for the time derivative,

$$\|\tilde{R}(s)\|_{L^q} \leqslant C\|e^{-s\mathbb{A}_\alpha} h\|_{W^{1,\infty}} \leqslant C(1+s)^{-\frac{1}{r}} \|f\|_{L^r}. \tag{3.4}$$

We write w in the integral form as

$$\begin{aligned} w(t) &= e^{-t\mathbb{A}_{\alpha,\Omega}} h_{loc} - \int_0^t e^{-(t-s)\mathbb{A}_{\alpha,\Omega}} \mathbb{P}_\Omega \tilde{R}(s)\, ds \\ &= w^{(1)}(t) + w^{(2)}(t). \end{aligned} \tag{3.5}$$

Then the associated pressure p_w is written in the form $p_w = p_{w^{(1)}} + p_{w^{(2)}}$, where

$$p_{w^{(1)}}(t) = p[h_{loc}](t), \quad p_{w^{(2)}}(t) = -\int_0^t p[\mathbb{P}_\Omega \tilde{R}(s)](t-s)\, ds \tag{3.6}$$

by following the notation used in Theorem 1.2. Since $e^{-t\mathbb{A}_{\alpha,\Omega}} h = w(t)$ for $|x| \leqslant 4$, it suffices to estimate $w(t)$. From the fact $\operatorname{supp} h_{loc}, \operatorname{supp} \tilde{R}(s) \subset \{|x| \leqslant 5\}$, Theorem 1.2 implies

$$\|e^{-t\mathbb{A}_{\alpha,\Omega}} h_{loc}\|_{W^{1,q}(\Omega_4)} \leqslant Ct^{-1}\|h_{loc}\|_{L^q} \leqslant Ct^{-1}\|f\|_{L^r}, \tag{3.7}$$

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and

$$\begin{aligned}&\left\| \int_0^t e^{-(t-s)\mathbb{A}_{\alpha,\Omega}} \mathbb{P}_\Omega R(s)\, ds \right\|_{W^{1,q}(\Omega_4)} \\ &\quad \leqslant C \int_0^t \left(\frac{\chi_{t-s\leqslant 2}}{(t-s)^{\frac{1}{2}}} + \frac{\chi_{2\leqslant t-s\leqslant \alpha^{-2}}}{(t-s)|\log (t-s)|^2} \right. \\ &\qquad \left. + \frac{\alpha^2 \chi_{2\leqslant t-s\leqslant\alpha^{-2}}}{|\log (t-s)|} + \frac{\chi_{t-s\geqslant \alpha^{-2}}}{(t-s)^2\alpha^2|\log\alpha|} \right) \|\tilde{R}(s)\|_{L^q}\, ds. \qquad (3.8)\end{aligned}$$

Thus, estimates (3.7), (3.8), and (3.4) give

$$\|e^{-t\mathbb{A}_{\alpha,\Omega}} h\|_{W^{1,q}(\Omega_4)} = \|w(t)\|_{W^{1,q}(\Omega_4)} \leqslant C t^{-\frac{1}{r}} \|f\|_{L^r}, \quad t \geqslant 2.$$

This proves (3.1) for $t \geqslant 2$. As for the pressure term, we have from Theorem 1.2,

$$\|p_{w^{(1)}}(t)\|_{L^q(\Omega_4)} \leqslant C t^{-1} \|h_{loc}\|_{L^q} \leqslant C t^{-1} \|f\|_{L^r}, \quad t \geqslant 2,$$

and

$$\begin{aligned}\|p_{w^{(2)}}(t)\|_{L^q(\Omega_4)} \leqslant C \int_0^t &\left(\frac{\chi_{t-s\leqslant 2}}{(t-s)^{\frac{1}{2}(1+\frac{1}{q})}} + \frac{\chi_{2\leqslant t-s\leqslant\alpha^{-2}}}{(t-s)|\log (t-s)|^2} \right. \\ &\left. + \frac{\alpha^2\chi_{2\leqslant t-s\leqslant\alpha^{-2}}}{|\log (t-s)|} + \frac{\chi_{t-s\geqslant\alpha^{-2}}}{(t-s)^2\alpha^2|\log\alpha|} \right) \|\tilde{R}(s)\|_{L^q}\, ds.\end{aligned}$$

Then, as in the proof of the velocity $w(t)$ above, we can derive from (3.4) that

$$\|p_w(t)\|_{L^q(\Omega_4)} \leqslant \|p_{w^{(1)}}(t)\|_{L^q(\Omega_4)} + \|p_{w^{(2)}}(t)\|_{L^q(\Omega_4)} \leqslant C t^{-\frac{1}{r}} \|f\|_{L^r}, \quad t \geqslant 2.$$

This proves (3.2) for $t \geqslant 2$. The estimate for $0 < t \leqslant 2$ is easy to verify. In particular, (3.1) is standard and we omit the details. As for (3.2), from the semigroup property and Proposition 2.8, we have

$$\begin{aligned}\|p[\mathbb{P}_\Omega f](t)\|_{L^q(\Omega_4)} = \left\| p[e^{-\frac{t}{2}\mathbb{A}_{\alpha,\Omega}} f]\left(\frac{t}{2}\right) \right\|_{L^q(\Omega_4)} &\leqslant C t^{-\frac{1}{2}(1+\frac{1}{q})} \|e^{-\frac{t}{2}\mathbb{A}_{\alpha,\Omega}} f\|_{L^q(\Omega_4)} \\ &\leqslant C t^{-\frac{1}{2}(1+\frac{1}{q})-\frac{1}{r}+\frac{1}{q}} \|f\|_{L^r}, \quad 0 < t \leqslant 2.\end{aligned}$$

Thus, (3.2) follows. The proof is complete. □

We are now in a position to state our main estimate for the semigroup $\{e^{-t\mathbb{A}_{\alpha,\Omega}}\}_{t\geqslant 0}$.

Theorem 1.4. It suffices to consider the case $t \geqslant 2$. Let $f \in L^r_\sigma(\Omega)$. We introduce a cut-off function $\chi \in C_0^\infty(\mathbb{R}^2)$ such that $\chi(x) = 1$ for $|x| \leqslant 3$ and $\chi(x) = 0$ for $|x| \geqslant 4$. Set

$$w(t) = (1-\chi) e^{-t\mathbb{A}_{\alpha,\Omega}} f + \mathbb{B}_3[\nabla\chi \cdot e^{-t\mathbb{A}_{\alpha,\Omega}} f].$$

Here, $\mathbb{B}_3$ is the Bogovskii operator in the annulus $\{x \in \mathbb{R}^2 \mid 3 < |x| < 4\}$. Then, w coincides with $e^{-t\mathbb{A}_{\alpha,\Omega}} f$ for $|x| \geqslant 4$ and satisfies the integral equation in $\mathbb{R}^2$:

$$\begin{aligned}w(t) &= e^{-t\mathbb{A}_\alpha}((1-\chi) f + \mathbb{B}_3[\nabla\chi \cdot f]) + \int_0^t e^{-(t-s)\mathbb{A}_\alpha} \mathbb{P} R(s)\, ds \\ &= w^{(1)}(t) + w^{(2)}(t), \qquad (3.9)\end{aligned}$$

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where

$$R(t) = (\Delta\chi)e^{-t\mathbb{A}_{\alpha,\Omega}}f + 2\nabla\chi\cdot\nabla e^{-t\mathbb{A}_{\alpha,\Omega}}f - \alpha(\partial_1\chi)e^{-t\mathbb{A}_{\alpha,\Omega}}f \\ - (\nabla\chi)p[\mathbb{P}_\Omega f](t) + (\partial_t + A_\alpha)\mathbb{B}_3[\nabla\chi\cdot e^{-t\mathbb{A}_{\alpha,\Omega}}f]. \tag{3.10}$$

Here, $p[\mathbb{P}_\Omega f](t)$ is the associated pressure for $e^{-t\mathbb{A}_{\alpha,\Omega}}f$, i.e., the couple $(u(t), p[\mathbb{P}_\Omega f](t)) = (e^{-t\mathbb{A}_{\alpha,\Omega}}f, p[\mathbb{P}_\Omega f](t))$ satisfies (1.1). By Proposition 3.1, with $q = r$, and by also using the similar technique as in (2.80) for the time derivative in (3.10), the term R is estimated as

$$\|R(t)\|_{L^r} \leqslant C(\|e^{-t\mathbb{A}_{\alpha,\Omega}}f\|_{W^{1,r}(\Omega_4)} + \|p[\mathbb{P}_\Omega f](t)\|_{L^r(\Omega_4)}) \\ \leqslant C(1+t)^{-\frac{1}{r}}\left(\frac{1+t}{t}\right)^{\frac{1}{2}(1+\frac{1}{r})}\|f\|_{L^r}, \quad t > 0. \tag{3.11}$$

Set $h = (1-\chi)f + \mathbb{B}_3[\nabla\chi\cdot f]$. It is straightforward to see that

$$\|w^{(1)}(t)\|_{L^q} \leqslant Ct^{-\frac{1}{r}+\frac{1}{q}}\|h\|_{L^r} \leqslant Ct^{-\frac{1}{r}+\frac{1}{q}}\|f\|_{L^r}, \quad t > 0. \tag{3.12}$$

Similarly, we have

$$\|\nabla w^{(1)}(t)\|_{L^q} \leqslant Ct^{-\frac{1}{2}-\frac{1}{r}+\frac{1}{q}}\|f\|_{L^q}, \quad t > 0. \tag{3.13}$$

Next, we estimate $w^{(2)}$. Since $R(s)$ is supported in $\{3 \leqslant |x| \leqslant 4\}$, we have $\|R(s)\|_{L^1} \leqslant C\|R(s)\|_{L^r}$, and, thus, from (3.11),

$$\|w^{(2)}(t)\|_{L^q} \leqslant C\int_0^t (t-s)^{-1+\frac{1}{q}}\|R(s)\|_{L^1}\,ds \\ \leqslant C\int_0^t (t-s)^{-1+\frac{1}{q}}(1+s)^{-\frac{1}{r}}\left(\frac{1+s}{s}\right)^{\frac{1}{2}(1+\frac{1}{r})}ds\,\|f\|_{L^r} \\ \leqslant Ct^{-\frac{1}{r}+\frac{1}{q}}\|f\|_{L^r}, \quad t \geqslant 2. \tag{3.14}$$

Here, we have used $1 < r \leqslant q < \infty$. The derivative estimate of $w^{(2)}$ in L^q with $1 < q < 2$ is shown similarly, for

$$\|\nabla w^{(2)}(t)\|_{L^q} \leqslant C\int_0^t (t-s)^{-\frac{1}{2}-1+\frac{1}{q}}\|R(s)\|_{L^1}\,ds \\ \leqslant C\int_0^t (t-s)^{-\frac{3}{2}+\frac{1}{q}}(1+s)^{-\frac{1}{r}}\left(\frac{1+s}{s}\right)^{\frac{1}{2}(1+\frac{1}{r})}ds\,\|f\|_{L^r} \\ \leqslant Ct^{-\frac{1}{2}-\frac{1}{r}+\frac{1}{q}}\|f\|_{L^r}, \quad t \geqslant 2, 1 < r \leqslant q < 2. \tag{3.15}$$

Hence, (3.12), (3.13), (3.14), and (3.15) together with the fact $w(t) = e^{-t\mathbb{A}_{\alpha,\Omega}}f$ for $|x| \geqslant 4$ show

$$\|e^{-t\mathbb{A}_{\alpha,\Omega}}f\|_{L^q(\Omega\cap\{|x|\geqslant 4\})} \leqslant Ct^{-\frac{1}{r}+\frac{1}{q}}\|f\|_{L^r}, t \geqslant 2, \quad 1 < r \leqslant q < \infty,$$

$$\|\nabla e^{-t\mathbb{A}_{\alpha,\Omega}}f\|_{L^q(\Omega\cap\{|x|\geqslant 4\})} \leqslant Ct^{-\frac{1}{2}-\frac{1}{r}+\frac{1}{q}}\|f\|_{L^r}, t \geqslant 2, \quad 1 < r \leqslant q < 2.$$

The estimate in $\Omega\cap\{|x| \leqslant 4\}$ is already proved in Proposition 3.1. The proof is complete. □

https://doi.org/10.1017/S1474748019000355 Published online by Cambridge University Press

Acknowledgments. The author is partially supported by JSPS grant 17K05320 and 25220702.

Appendix A. Estimate of $d(\alpha, \lambda)$

Let $\alpha > 0$ and $\lambda \in \Sigma_{\frac{3\pi}{4}}$. Let us prove (2.11). In virtue of (2.7), we have

$$|d(\alpha, \lambda)| \leqslant C \int_0^1 \frac{\alpha^2 s}{|\lambda| + \alpha^2 s}\, ds \leqslant C \min\left\{1, \frac{\alpha^2}{|\lambda|}\right\}.$$

Similarly, it is straightforward to see

$$\begin{aligned} |\partial_\lambda d(\alpha, \lambda)| = \left| \int_0^1 \frac{4\alpha^2 s}{(4\lambda + \alpha^2 s)^2}\, ds \right| &\leqslant C \int_0^1 \frac{\alpha^2 s}{(|\lambda| + \alpha^2 s)^2}\, ds \\ &\leqslant \frac{C}{|\lambda|^{\frac{1}{2}}} \int_0^1 \frac{\alpha^2 s}{(|\lambda| + \alpha^2 s)^{\frac{3}{2}}}\, ds \\ &\leqslant \frac{C}{|\lambda|^{\frac{1}{2}} (|\lambda| + \alpha^2)^{\frac{1}{2}}}, \end{aligned}$$

and

$$\begin{aligned} |\partial_\lambda^2 d(\alpha, \lambda)| = \left| \int_0^1 \frac{32\alpha^2 s}{(4\lambda + \alpha^2 s)^3}\, ds \right| &\leqslant C \int_0^1 \frac{\alpha^2 s}{(|\lambda| + \alpha^2 s)^3}\, ds \\ &\leqslant C \int_0^1 \frac{1}{(|\lambda| + \alpha^2 s)^2}\, ds \\ &\leqslant \frac{C}{|\lambda|(|\lambda| + \alpha^2)}. \end{aligned}$$

The proof is complete.

Appendix B. Injectivity of $\mathbb{I} + T_0[0]$ in $\{f \in L^q_{[5]}(\Omega) \mid \int_\Omega f\, dx = 0\}$

In this appendix, we give a brief sketch of the proof for the injectivity of $\mathbb{I} + T_0[0]$ in $\{f \in L^q_{[5]}(\Omega) \mid \int_\Omega f\, dx = 0\}$ for the reader's convenience. Note that this is already proved in [6]. Suppose that $f \in L^q_{[5]}(\Omega)$, $\int_\Omega f\, dx = 0$ satisfies $(\mathbb{I} + T_0[0]) f = 0$. Then the definition of $T_0[0]$ implies

$$-\Delta \mathcal{U}_0[0] f + \nabla \mathcal{P}_0[0] f = (\mathbb{I} + T_0[0]) f = 0, \quad \operatorname{div} \mathcal{U}_0[0] f = 0, \quad x \in \Omega$$

and $\mathcal{U}_0[0] f = 0$ on $\partial\Omega$. Moreover, from the definition of $\mathcal{U}_0[0]$, the vector field $\mathcal{U}_0[0] f$ is equal to $E_0^0 * f$ for $|x| \geqslant 5$, and, thus, the condition $\int_\Omega f\, dx = 0$ yields the decay $\mathcal{U}_0[0] f(x) = O(|x|^{-1})$ as $|x| \to \infty$. On the other hand, we have $\mathcal{P}_0[0] f(x) = -\nabla \cdot (-\Delta_{\mathbb{R}^2})^{-1} f(x) = O(|x|^{-1})$ for $|x| \gg 1$. These decays ensure that

$$\mathcal{U}_0[0] f = 0 \quad \text{and} \quad \mathcal{P}_0[0] f = 0 \quad \text{in } \Omega \tag{B.1}$$

by the injectivity of the Stokes operator in an exterior domain. Since $\mathcal{U}_0[0]f = E_0^0 * f$ for $|x| \geqslant 5$, we obtain $E_0^0 * f = 0$ on $|x| \geqslant 5$. Therefore, $(u_{\mathbb{R}^2}^{(0)}, p_{\mathbb{R}^2}) := (E_0^0 * f, -\nabla \cdot (-\Delta_{\mathbb{R}^2})^{-1} f)$ solves

$$-\Delta u_{\mathbb{R}^2}^{(0)} + \nabla p_{\mathbb{R}^2} = f, \quad \operatorname{div} u_{\mathbb{R}^2}^{(0)} = 0, \quad |x| \leqslant 5, \quad u_{\mathbb{R}^2}^{(0)} = 0, \quad |x| = 5.$$

Next, the fact $\mathcal{U}_0[0]f = u_D^{(0)} := \mathbb{A}_D^{-1}\mathbb{P}_D f$ for $x \in \Omega \cap \{|x| \leqslant 4\}$ and (B. 1) imply $u_D^{(0)} = 0$ for $|x| \leqslant 4$, and, similarly, by recalling (2.22), the associated pressure $p_D^{(0)}$, $\int_D p_D^{(0)}\, dx = 0$, satisfies $p_D^{(0)} + \frac{1}{|D|}\int_D p_{\mathbb{R}^2}\, dx = 0$ for $x \in \Omega \cap \{|x| \leqslant 4\}$. Let $\tilde{u}_D^{(0)}$ be the extension of $u_D^{(0)}$ to $\{|x| < 5\}$ such that $\tilde{u}_D^{(0)} = u_D^{(0)}$ in D and $\tilde{u}_D^{(0)} = 0$ on Ω^c and, similarly, let $\tilde{p}_D^{(0)}$ be the extension of $p_D^{(0)}$ to $\{|x| < 5\}$ such that $\tilde{p}_D^{(0)} = p_D^{(0)}$ in D and $\tilde{p}_D^{(0)} = -\frac{1}{|D|}\int_D p_{\mathbb{R}^2}\, dx$ on Ω^c. Then we verify that $(\tilde{u}_D^{(0)}, \tilde{p}_D^{(0)})$ solves

$$-\Delta \tilde{u}_D^{(0)} + \nabla \tilde{p}_D^{(0)} = f, \quad \operatorname{div} \tilde{u}_D^{(0)} = 0, \quad |x| < 5, \quad \tilde{u}_D^{(0)} = 0, \quad |x| = 5.$$

Here, we have used the fact that $u_D^{(0)} = 0$ and $p_D^{(0)} = -\frac{1}{|D|}\int_D p_{\mathbb{R}^2}\, dx$ in a neighborhood of $\partial\Omega$. By the uniqueness of the solution to the Stokes equations in the bounded domain $\{|x| < 5\}$, we conclude that $u_{\mathbb{R}^2}^{(0)} = \tilde{u}_D^{(0)}$ and $p_{\mathbb{R}^2} - \tilde{p}_D^{(0)} = Const.$ in $\{|x| < 5\}$. Integrating over D, we see $|D|Const. = \int_D p_{\mathbb{R}^2}\, dx - \int_D \tilde{p}_D^{(0)}\, dx = \int_D p_{\mathbb{R}^2}\, dx$ and, thus, $p_{\mathbb{R}^2} = p_D^{(0)} + \frac{1}{|D|}\int_D p_{\mathbb{R}^2}\, dx$ in $\Omega \cap \{|x| \leqslant 5\}$. From the definition of $T_0[0]$, these identities in $\{4 < |x| < 5\}$ yield that $T_0[0]f = 0$. Hence, we have arrived at $f = f + T_0[0]f = (\mathbb{I} + T_0[0])f = 0$. The proof is complete.

Appendix C. Estimate of the pressure in a bounded domain: proof of (2.81)

We give the estimate of p_D satisfying $\int_D p_D\, dx = 0$. Take arbitrary $\phi \in C_0^\infty(D)$ and set $\tilde{\phi} = \phi - c_\phi$, where the constant c_ϕ is taken so that $\int_D \tilde{\phi}\, dx = 0$. Let ω_ϕ with $\int_D \omega_\phi\, dx = 0$ be the unique solution to the Neumann problem: $\Delta\omega_\phi = \tilde{\phi}$ in D and $\partial_n \omega_\phi = 0$ on ∂D. It is classical that $\|\omega_\phi\|_{W^{2,\frac{q}{q-1}}(D)} \leqslant C\|\tilde{\phi}\|_{L^{\frac{q}{q-1}}(D)} \leqslant C\|\phi\|_{L^{\frac{q}{q-1}}(D)}$. Then the condition $\int_D p_D(t)\, dx = 0$ and the integration by parts yield

$$\begin{aligned}\int_D p_D(t)\phi\, dx = \int_D p_D(t)\tilde{\phi}\, dx = \int_D p_D(t)\Delta\omega_\phi\, dx \\ = -\int_D \nabla p_D(t) \cdot \nabla\omega_\phi\, dx.\end{aligned} \tag{C. 1}$$

By the equation of u_D, we have $\nabla p_D = -\partial_t u_D + \Delta u_D - \alpha\partial_1 u_D$, which implies from the integration by parts and $u_D|_{\partial D} = 0$,

$$\begin{aligned}\int_D \nabla p_D \cdot \nabla\omega_\phi\, dx = \int_D (-\partial_t u_D + \Delta u_D - \alpha\partial_1 u_D) \cdot \nabla\omega_\phi\, dx \\ = -\int_D \nabla^\perp \operatorname{rot} u_D \cdot \nabla\omega_\phi\, dx + \alpha\int_D u_D \cdot \partial_1\nabla\omega_\phi\, dx.\end{aligned} \tag{C. 2}$$

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Here, $\nabla^{\perp}=(\partial_2,-\partial_1)$ and $\operatorname{rot} u_D=\partial_1 u_{D,2}-\partial_2 u_{D,1}$. Then the integration by parts and the trace theorem give, for $q'=\frac{q}{q-1}$,

$$\begin{aligned}\left|\int_D \nabla^{\perp}\operatorname{rot} u_D\cdot\nabla\omega_\phi\,dx\right| &= \left|\int_{\partial D}\operatorname{rot} u_D(n_2\partial_1\omega_\phi-n_1\partial_2\omega_\phi)\,dS\right|\\ &\leqslant \|\operatorname{rot} u_D\|_{L^q(\partial D)}\|\nabla\omega_\phi\|_{L^{q'}(\partial D)}\\ &\leqslant C\|\operatorname{rot} u_D\|_{L^q(D)}^{1-\frac1q}\|\nabla\operatorname{rot} u_D\|_{W^{1,q}(D)}^{\frac1q}\|\nabla\omega_\phi\|_{L^{q'}(D)}^{1-\frac{1}{q'}}\|\nabla\omega_\phi\|_{W^{1,q'}(D)}^{\frac{1}{q'}}\\ &\leqslant C\|\operatorname{rot} u_D\|_{L^q(D)}^{1-\frac1q}\|\nabla\operatorname{rot} u_D\|_{W^{1,q}(D)}^{\frac1q}\|\phi\|_{L^{q'}(D)}.\end{aligned}$$

The other term in the right-hand side of (C. 2) is estimated as

$$\begin{aligned}\left|\alpha\int_D u_D\cdot\partial_1\nabla\omega_\phi\,dx\right| &\leqslant C\alpha\|u_D\|_{L^q(D)}\|\nabla^2\omega_\phi\|_{L^{\frac{q}{q-1}}(D)}\\ &\leqslant C\alpha\|u_D\|_{L^q(D)}\|\phi\|_{L^{\frac{q}{q-1}}(D)}.\end{aligned}$$

Collecting these, we have from (C. 1) and the duality argument,

$$\|p_D(t)\|_{L^q(D)}\leqslant C(\|\operatorname{rot} u_D\|_{L^q(D)}^{1-\frac1q}\|\nabla\operatorname{rot} u_D\|_{W^{1,q}(D)}^{\frac1q}+\alpha\|u_D(t)\|_{L^q(D)}). \tag{C. 3}$$

Thus, we have arrived at

$$\|p_D(t)\|_{L^q(D)}\leqslant Ct^{-\frac12(1+\frac1q)}\|f_D\|_{L^q(D)},\quad 0<t\leqslant 3. \tag{C. 4}$$

Arguing exactly in the same manner, we can also show

$$\begin{aligned}\|\partial_t p_D(t)\|_{L^q(D)} &\leqslant C(\|\partial_t\operatorname{rot} u_D\|_{L^q(D)}^{1-\frac1q}\|\nabla\partial_t\operatorname{rot} u_D\|_{W^{1,q}(D)}^{\frac1q}+\alpha\|\partial_t u_D(t)\|_{L^q(D)})\\ &\leqslant Ct^{-1-\frac12(1+\frac1q)}\|f_D\|_{L^q(D)},\quad 0<t\leqslant 3.\end{aligned} \tag{C. 5}$$

The proof of (2.81) is complete.

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