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THE APPLICATIONS OF CRITICAL-POINT THEORY TO DISCONTINUOUS FRACTIONAL-ORDER DIFFERENTIAL EQUATIONS

YU TIAN[1]* AND JUAN J. NIETO[2,3]

[1]*School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, People's Republic of China*
(tianyu2992@163.com)
[2]*Departamento de Análise Matemática, Facultade de Matemáticas, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain*
[3]*Faculty of Science, King Abdulaziz University, PO Box 80203, 21589, Jeddah, Saudi Arabia*

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Abstract We consider a fractional equation involving the left and right Riemann–Liouville fractional integrals and with Sturm–Liouville boundary-value conditions. We establish the variational structure of the problem and, by using critical-point theory, the existence of an unbounded sequence of solutions is obtained.

Keywords: fractional differential equation; discontinuous; Sturm–Liouville boundary conditions; critical-point theory

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1. Introduction

In this paper we will study the existence of solutions for the Sturm–Liouville problem of discontinuous fractional-order differential equation

$$\left.\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \tfrac{1}{2}{}_tD_T^{-\beta}(u'(t))) = \lambda f(u(t)) \quad \text{almost every (a.e.) } t \in [0,T],\\ au(0) - b(\tfrac{1}{2}{}_0D_t^{-\beta}u'(0) + \tfrac{1}{2}{}_tD_T^{-\beta}u'(0)) = 0,\\ cu(T) + d(\tfrac{1}{2}{}_0D_t^{-\beta}u'(T) + \tfrac{1}{2}{}_tD_T^{-\beta}u'(T)) = 0, \end{aligned}\right\} \tag{1.1}$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann–Liouville fractional integrals of order $0 \leqslant \beta < 1$, respectively, $a, c > 0$, $b, d \geqslant 0$, $f\colon \mathbb{R} \to \mathbb{R}$ is an almost everywhere continuous function and λ is a positive parameter. If $\beta = 0$, boundary-value problem (BVP)

* Corresponding author.

(1.1) is the standard second-order differential equation with Sturm–Liouville boundary condition

$$-u''(t) = \lambda f(u(t)) \quad \text{a.e. } t \in [0,T],$$
$$au(0) - bu'(0) = 0, \quad cu(T) + du'(T) = 0.$$

In recent years, fractional differential equations have attracted much attention since they have proved to be very valuable tools in the modelling of many phenomena in various fields of science and engineering such as viscoelasticity, neurons, electrochemistry, control, porous media, and electromagnetism (see [**11**, **14**, **16**, **17**]). For background and applications of the theory of fractional differential equations, we refer the reader to the monographs [**13**, **18**, **21**, **22**] and the papers [**1**–**3**, **9**, **10**, **12**, **15**, **23**, **24**].

Recently, many results were obtained that deal with the existence of solutions for fractional differential equations. Some classical tools have been used to study fractional differential equations in the literature. These classical tools include fixed-point theory, topological degree theory and comparison method (see [**7**, **20**, **25**]).

Critical-point theory is an effective tool for dealing with some boundary-value problems for fractional differential equations. Some new and interesting results can be obtained by using critical-point theory. In [**12**], Jiao and Zhou studied the fractional BVP

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \tfrac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\right) + \nabla F(t,u(t)) = 0 \quad \text{a.e. } t \in [0,T],$$
$$u(0) = u(T) = 0.$$

The variational structure was established and various criteria on the existence of solutions were obtained. In [**23**], Teng *et al.* studied

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \tfrac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\right) \in \partial F(t,u(t)) \quad \text{a.e. } t \in [0,T],$$
$$u(0) = u(T) = 0.$$

By using a variational method based on non-smooth critical-point theory, the existence and multiplicity of solutions were proved.

As far as we know, there are no results for discontinuous fractional differential equations with Sturm–Liouville boundary conditions. As a result, the goal of this paper is to fill the gap in this area. We shall apply the critical-point theory for non-differentiable functions to establish the existence results of infinitely many solutions to problem (1.1). With the Sturm–Liouville boundary condition, we establish the new variational structure and prove the equivalence between the usual norm and new norm in the space $E^{\alpha,2}$ (see Lemma 4.7). With the discontinuity, we prove that the generalized critical point of the functional is the generalized solution of problem (1.1) (see Lemma 5.9). In addition, we point out that problem (1.1) cannot be studied by fixed-point theory because it cannot be expressed as an integral equation. The results hold for the problem with continuous nonlinearity and with Dirichlet boundary condition.

This paper is organized as follows: in § 2 we recall some basic knowledge of non-smooth analysis and abstract results that we are going to apply; in § 3 we introduce fractional integrals and derivatives; in § 4 we introduce fractional derivative space and prove some lemmas; in § 5 we establish the variational structure; in § 6 we prove the main results for $bd \neq 0$; in § 7 we prove the main results for $bd = 0$.

2. Non-smooth analysis

We collect some basic notions and results of non-smooth analysis, namely, the calculus for locally Lipschitz functionals developed by Clarke [**8**] and the monograph of Motreanu and Panagiotopoulos [**19**].

Let $(X, \|\cdot\|_X)$ be a real Banach space, let $(X^*, \|\cdot\|_{X^*})$ be its topological dual, and let $\varphi\colon X \to \mathbb{R}$ be a functional. We recall that φ is locally Lipschitz (l.L.) if, for all $u \in X$, there exists a neighbourhood U_u of u and a real number $L_u \geqslant 0$ such that

$$|\varphi(v) - \varphi(w)| \leqslant L_u \|v - w\|_X \quad \text{for all } v, w \in U_u.$$

Definition 2.1 (Motreanu and Panagiotopoulos [19, Definition 2.161]). Let $\varphi\colon X \to \mathbb{R}$ be l.L. and fix two points $u, v \in X$. The generalized directional derivative of φ at u in the direction $v \in X$ is defined as

$$\varphi^0(u; v) = \limsup_{w \to u,\, \tau \to 0^+} \frac{\varphi(w + \tau v) - \varphi(w)}{\tau}.$$

Definition 2.2 (Motreanu and Panagiotopoulos [19, Definition 2.166]). The generalized gradient of an l.L. functional $\varphi\colon X \to \mathbb{R}$ at a point $u \in X$ is the subset of X^* defined by

$$\partial\varphi(u) = \{u^* \in X^*\colon \langle u^*, v\rangle \leqslant \varphi^0(u; v) \text{ for all } v \in X\}.$$

So $\partial\varphi\colon X \to 2^{X^*}$ is a multifunction. We say that φ has compact gradient if $\partial\varphi$ maps bounded subsets of X into relatively compact subsets of X^*.

The following proposition gives the relationship between the generalized directional derivative $\varphi^0(u; v)$ and the usual directional derivative $\varphi'(u; v)$, the generalized gradient $\partial\varphi(u)$, and the Fréchet differential $\varphi'(u)$.

Proposition 2.3 (Motreanu and Panagiotopoulos [19, Proposition 1.1]). *Let $\varphi \in C^1(X)$ be a functional. Then φ is l.L. and*

$$\varphi^0(u; v) = \varphi'(u; v) \quad \textit{for all } u, v \in X, \tag{2.1}$$

$$\partial\varphi(u) = \{\varphi'(u)\} \quad \textit{for all } u \in X. \tag{2.2}$$

We say that $u \in X$ is a (generalized) critical point of φ when

$$\varphi^0(u; v) \geqslant 0 \quad \forall v \in X$$

clearly signifies that $0 \in \partial\varphi(x)$.

In the proof of our main results, we will use Theorem 2.4. For this, we assume that X is a reflexive real Banach space, that Φ is a sequentially weakly lower semi-continuous functional, that $\Upsilon\colon X \to \mathbb{R}$ is a sequentially weakly upper semi-continuous functional, and that λ is a real positive parameter. Write $\Psi := \Upsilon$, $I_\lambda = \Phi - \lambda\Psi = \Phi - \lambda\Upsilon$.

Provided that $r > \inf_X \Phi$, we can define

$$\varphi(r) := \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{(\sup_{u\in\Phi^{-1}((-\infty,r))}\Psi(u)) - \Psi(u)}{r-\Phi(u)},$$
$$\gamma := \liminf_{r\to+\infty}\varphi(r), \qquad \delta := \liminf_{r\to(\inf_X\Phi)^+}\varphi(r).$$

Assuming that Φ, Ψ are locally Lipschitz functionals, we will use the following result.

Theorem 2.4 (Bonanno and Bisci [4]). *Under the above assumptions on X, Φ and Ψ, one has the following.*

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}((-\infty, r))$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(b) *If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$ the following holds: either*

  (b_1) *I_λ possesses a global minimum, or*

  (b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n\to+\infty}\Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$, then for each $\lambda \in (0, 1/\delta)$ the following holds: either*

  (c_1) *there is a global minimum of Φ that is a local minimum of I_λ, or*

  (c_2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ, with $\lim_{n\to+\infty}\Phi(u_n) = \inf_X \Phi$, which weakly converges to a global minimum of Φ.*

3. Fractional calculus

Definition 3.1 (left and right Riemann–Liouville fractional integrals [13,22]). Let f be a function defined on $[a, b]$. The left and right Riemann–Liouville fractional integrals of order γ for the function f, denoted by ${}_aD_t^{-\gamma}f(t)$ and ${}_tD_b^{-\gamma}f(t)$, respectively, are defined by

$${}_aD_t^{-\gamma}f(t) = \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}f(s)\,\mathrm{d}s, \quad t\in[a,b],\ \gamma>0,$$

and

$${}_tD_b^{-\gamma}f(t) = \frac{1}{\Gamma(\gamma)}\int_t^b (s-t)^{\gamma-1}f(s)\,\mathrm{d}s, \quad t\in[a,b],\ \gamma>0,$$

provided that the right-hand sides are pointwise defined on $[a, b]$, where $\Gamma > 0$ is the classical gamma function.

Remark 3.2. For $n \in \mathbb{N}$, if $\gamma = n$, Definition 3.1 coincides with the nth integrals of the form [**13**, **22**]

$$ {}_aD_t^{-n} f(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)\,\mathrm{d}s, \quad t \in [a,b],\ n \in \mathbb{N}, $$

and

$$ {}_tD_b^{-n} f(t) = \frac{1}{(n-1)!} \int_t^b (s-t)^{n-1} f(s)\,\mathrm{d}s, \quad t \in [a,b],\ n \in \mathbb{N}. $$

Definition 3.3 (left and right Riemann–Liouville fractional derivatives [13, 22]). Let f be a function defined on $[a,b]$. The left and right Riemann–Liouville fractional derivatives of order $\gamma > 0$ for the function f, denoted by ${}_aD_t^{\gamma} f(t)$ and ${}_tD_b^{\gamma} f(t)$, respectively, are defined by

$$ {}_aD_t^{\gamma} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} {}_aD_t^{\gamma-n} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(\int_a^t (t-s)^{n-\gamma-1} f(s)\,\mathrm{d}s \right) $$

and

$$ {}_tD_b^{\gamma} f(t) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} {}_tD_b^{\gamma-n} f(t) = \frac{1}{\Gamma(n-\gamma)} (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(\int_t^b (s-t)^{n-\gamma-1} f(s)\,\mathrm{d}s \right), $$

where $t \in [a,b]$, $n-1 \leqslant \gamma < n$ and $n \in \mathbb{N}$. In particular, if $0 \leqslant \gamma < 1$, then

$$ {}_aD_t^{\gamma} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} {}_aD_t^{\gamma-1} f(t) = \frac{1}{\Gamma(1-\gamma)} \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_a^t (t-s)^{-\gamma} f(s)\,\mathrm{d}s \right), \qquad t \in [a,b], $$

and

$$ {}_tD_b^{\gamma} f(t) = -\frac{\mathrm{d}}{\mathrm{d}t} {}_tD_b^{\gamma-1} f(t) = -\frac{1}{\Gamma(1-\gamma)} \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_t^b (s-t)^{-\gamma} f(s)\,\mathrm{d}s \right), \quad t \in [a,b]. $$

Remark 3.4. For $n \in \mathbb{N}$, if γ becomes an integer $n-1$, according to Definition 3.3, the left and right Riemann–Liouville fractional derivatives become the usual definitions, namely,

$$ {}_aD_t^{n-1} f(t) = f^{(n-1)}(t) \quad \text{and} \quad {}_tD_b^{n-1} f(t) = (-1)^{n-1} f^{(n-1)}(t), \quad t \in [a,b], $$

where $f^{(n-1)}(t)$ is the usual derivative of order $n-1$.

Remark 3.5. If $f \in C([a,b],\mathbb{R}^N)$, it is obvious that the Riemann–Liouville fractional integral of order $\gamma > 0$ exists on $[a,b]$. On the other hand, following [**13**, Lemma 2.2, p. 73], we know that the Riemann–Liouville fractional derivative of order $\gamma \in [n-1,n)$ exists almost everywhere on $[a,b]$ if $f \in AC^n([a,b],\mathbb{R}^N)$, where $C^k([a,b],\mathbb{R}^N)$ $(k = 0,1,\dots)$ denotes the set of mappings that are k-times continuously differentiable on $[a,b]$, $AC([a,b],\mathbb{R}^N)$ is the space of functions that are absolutely continuous on $[a,b]$, and $AC^{(k)}([a,b],\mathbb{R}^N)$ $(k = 1,2,\dots)$ is the space of functions f such that $f \in C^{k-1}([a,b],\mathbb{R}^N)$ and $f^{(k-1)} \in AC([a,b],\mathbb{R}^N)$. In particular, $AC([a,b],\mathbb{R}^N) = AC^1([a,b],\mathbb{R}^N)$. If $f \in L^1([a,b],\mathbb{R}^N)$, the Riemann–Liouville fractional integral is also in $L^1([a,b],\mathbb{R}^N)$.

Definition 3.6 (left and right Caputo fractional derivatives [13]). Let $\gamma \geqslant 0$ and $n \in \mathbb{N}$.

(i) If $\gamma \in (n-1, n)$ and $f \in AC^n([a,b], \mathbb{R}^N)$, then the left and right Caputo fractional derivatives of order γ for the function f, denoted by ${}_a^cD_t^\gamma f(t)$ and ${}_t^cD_b^\gamma f(t)$, respectively, exist almost everywhere on $[a,b]$. ${}_a^cD_t^\gamma f(t)$ and ${}_t^cD_b^\gamma f(t)$ are represented by

$$ {}_a^cD_t^\gamma f(t) = {}_aD_t^{\gamma-n} f^{(n)}(t) = \frac{1}{\Gamma(n-\gamma)}\left(\int_a^t (t-s)^{n-\gamma-1} f^{(n)}(s)\,\mathrm{d}s\right) $$

and

$$ {}_t^cD_b^\gamma f(t) = (-1)^n_t D_b^{\gamma-n} f^{(n)}(t) = \frac{(-1)^n}{\Gamma(n-\gamma)}\left(\int_t^b (s-t)^{n-\gamma-1} f^{(n)}(s)\,\mathrm{d}s\right), $$

respectively, where $t \in [a,b]$. In particular, if $0 < \gamma < 1$, then

$$ {}_a^cD_t^\gamma f(t) = {}_aD_t^{\gamma-1} f'(t) = \frac{1}{\Gamma(1-\gamma)}\left(\int_a^t (t-s)^{-\gamma} f'(s)\,\mathrm{d}s\right), \qquad t \in [a,b], $$

and

$$ {}_t^cD_b^\gamma f(t) = -{}_tD_b^{\gamma-1} f'(t) = -\frac{1}{\Gamma(1-\gamma)}\left(\int_t^b (s-t)^{-\gamma} f'(s)\,\mathrm{d}s\right), \quad t \in [a,b]. $$

(ii) If $\gamma = n-1$ and $f \in AC^{n-1}([a,b], \mathbb{R}^N)$, then ${}_a^cD_t^{n-1} f(t)$ and ${}_t^cD_b^{n-1} f(t)$ are represented by

$$ {}_a^cD_t^{n-1} f(t) = f^{(n-1)}(t) \quad \text{and} \quad {}_t^cD_b^{n-1} f(t) = (-1)^{n-1} f^{(n-1)}(t), \quad t \in [a,b]. $$

In particular, ${}_a^cD_t^0 f(t) = {}_t^cD_b^0 f(t) = f(t)$, $t \in [a,b]$.

Property 3.1 (Kilbas *et al.* [13]). The left and right Riemann–Liouville fractional integral operators have the property of a semigroup, i.e.

$$ {}_aD_t^{-\gamma_1}({}_aD_t^{-\gamma_2} f(t)) = {}_aD_t^{-\gamma_1-\gamma_2} f(t) \quad \text{and} \quad {}_tD_b^{-\gamma_1}({}_tD_b^{-\gamma_2} f(t)) = {}_tD_b^{-\gamma_1-\gamma_2} f(t) \quad \forall \gamma_1, \gamma_2 > 0 $$

in any point $t \in [a,b]$ for the continuous function f and for almost every point in $[a,b]$ if the function $f \in L^1([a,b], \mathbb{R}^N)$.

Property 3.2 (Kilbas *et al.* [13], Samko *et al.* [22]). We have the following property of fractional integration:

$$ \int_a^b [{}_aD_t^{-\gamma} f(t)] g(t)\,\mathrm{d}t = \int_a^b [{}_tD_b^{-\gamma} g(t)] f(t)\,\mathrm{d}t, \quad \gamma > 0, $$

provided that $f \in L^p([a,b], \mathbb{R}^N)$, $g \in L^q([a,b], \mathbb{R}^N)$ and $p \geqslant 1$, $q \geqslant 1$, $1/p + 1/q \leqslant 1 + \gamma$ or $p \neq 1$, $q \neq 1$, $1/p + 1/q = 1 + \gamma$.

Property 3.3 (Kilbas *et al.* [13]). Let $n \in \mathbb{N}$ and $n-1 < \gamma \leqslant n$. If $f \in AC^n([a,b],\mathbb{R}^N)$ or $f \in C^n([a,b],\mathbb{R}^N)$, then

$$ {}_aD_t^{-\gamma}({}_a^cD_t^{\gamma}f(t)) = f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)}{j!}(t-a)^j $$

and

$$ {}_tD_b^{-\gamma}({}_t^cD_b^{\gamma}f(t)) = f(t) - \sum_{j=0}^{n-1}\frac{(-1)^j f^{(j)}(b)}{j!}(b-t)^j $$

for $t \in [a,b]$. In particular, if $0 < \gamma \leqslant 1$ and $f \in AC([a,b],\mathbb{R}^N)$ or $f \in C^1([a,b],\mathbb{R}^N)$, then

$$ {}_aD_t^{-\gamma}({}_a^cD_t^{\gamma}f(t)) = f(t) - f(a) \quad \text{and} \quad {}_tD_b^{-\gamma}({}_t^cD_b^{\gamma}f(t)) = f(t) - f(b). $$

Definition 3.7 (left and right Riemann–Liouville fractional integrals and fractional derivatives on the real line [13, 21]). Let f be a function defined on R. The left and right Riemann–Liouville fractional integrals of order $\gamma > 0$ on the real line for the function f, denoted by ${}_{-\infty}D_t^{-\gamma}f(t)$ and ${}_tD_{\infty}^{-\gamma}f(t)$, respectively, are defined by

$$ {}_{-\infty}D_t^{-\gamma}f(t) = \frac{1}{\Gamma(\gamma)}\int_{-\infty}^{t}(t-s)^{\gamma-1}f(s)\,\mathrm{d}s $$

and

$$ {}_tD_{\infty}^{-\gamma}f(t) = \frac{1}{\Gamma(\gamma)}\int_{t}^{\infty}(s-t)^{\gamma-1}f(s)\,\mathrm{d}s, $$

where $t \in \mathbb{R}$ and $\gamma > 0$.

The left and right Riemann–Liouville fractional derivatives of order $\gamma > 0$ on the real line for the function f, denoted by ${}_{-\infty}D_t^{\gamma}f(t)$ and ${}_tD_{\infty}^{\gamma}f(t)$, respectively, are defined by

$$ {}_{-\infty}D_t^{\gamma}f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}{}_{-\infty}D_t^{\gamma-n}f(t) = \frac{1}{\Gamma(n-\gamma)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\biggl(\int_{-\infty}^{t}(t-s)^{n-\gamma-1}f(s)\,\mathrm{d}s\biggr) $$

and

$$ {}_tD_{\infty}^{\gamma}f(t) = (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}{}_tD_{\infty}^{\gamma-n}f(t) = \frac{1}{\Gamma(n-\gamma)}(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}\biggl(\int_{t}^{\infty}(s-t)^{n-\gamma-1}f(s)\,\mathrm{d}s\biggr), $$

where $t \in \mathbb{R}$, $n-1 \leqslant \gamma < n$ and $n \in \mathbb{N}$. In particular, if γ becomes an integer $n-1$, then

$$ {}_{-\infty}D_t^{n-1}f(t) = f^{(n-1)}(t) \quad \text{and} \quad {}_tD_{\infty}^{n-1}f(t) = (-1)^{n-1}f^{(n-1)}(t), \quad t \in \mathbb{R},\ n \in \mathbb{N}, $$

where $f^{(n-1)}(t)$ is the usual derivative of order $n-1$.

If $0 \leqslant \gamma < 1$, then

$$ {}_{-\infty}D_t^{\gamma}f(t) = \frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_{-\infty}^{t}(t-s)^{-\gamma}f(s)\,\mathrm{d}s\biggr), \quad t \in \mathbb{R}, $$

and

$$ {}_tD^{\gamma}_{\infty}f(t) = -\frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_t^{\infty}(s-t)^{-\gamma}f(s)\,\mathrm{d}s\biggr),\quad t\in\mathbb{R}. $$

Property 3.4 (Fourier transform property [10]). Let $\sigma>0$, $u\in L^p(R)$, $p\geqslant 1$. The Fourier transform of the left and right Riemann–Liouville fractional integrals satisfy

$$ \mathcal{F}({}_{-\infty}D_x^{-\sigma}u(x)) = (\mathrm{i}\omega)^{-\sigma}\hat{u}(\omega), $$
$$ \mathcal{F}({}_xD_{\infty}^{-\sigma}u(x)) = (-\mathrm{i}\omega)^{-\sigma}\hat{u}(\omega), $$

where $\hat{u}(\omega)$ denotes the Fourier transform of u, $\hat{u}(\omega)=\int_{\mathbb{R}}\mathrm{e}^{-\mathrm{i}\omega x}u(x)\,\mathrm{d}x$.

Property 3.5 (Fourier transform property [10]). Let $\mu>0$ and let $u\in C_0^{\infty}(\Omega_0)$, where Ω_0 is a subset of R. The Fourier transform of the left and right Riemann–Liouville fractional derivatives satisfy

$$ \mathcal{F}({}_{-\infty}D_x^{\mu}u(x)) = (\mathrm{i}\omega)^{\mu}\hat{u}(\omega), $$
$$ \mathcal{F}({}_xD_{\infty}^{\mu}u(x)) = (-\mathrm{i}\omega)^{\mu}\hat{u}(\omega). $$

4. Fractional derivative space

Definition 4.1. Let $\alpha\in(\frac{1}{2},1]$, $p\in[1,+\infty)$. The fractional derivative space

$$ E^{\alpha,p} = \{u\colon[0,T]\to\mathbb{R}^N\colon u \text{ is absolutely continous and } {}_0^cD_t^{\alpha}u\in L^p([0,T],\mathbb{R}^N)\} $$

is defined by the closure of $C^{\infty}([0,T],\mathbb{R}^N)$ with the norm

$$ \|u\|_{\alpha,p} = \biggl(\int_0^T|u(t)|^p + |{}_0^cD_t^{\alpha}u(t)|^p\,\mathrm{d}t\biggr)^{1/p}. \tag{4.1} $$

When $p=2$, we write $E^{\alpha,2}=E^{\alpha}$. It is obvious that the fractional derivative space $E^{\alpha,p}$ is the space of functions $u\in L^p([0,T],\mathbb{R}^N)$ having an α-order Caputo fractional derivative ${}_0^cD_t^{\alpha}u\in L^p([0,T],\mathbb{R}^N)$.

Lemma 4.2. *Let $\alpha\in(0,1]$, $p\in(1,+\infty)$; the space $E^{\alpha,p}$ is a reflexive and separable Banach space.*

Proof. The proof is similar to that in [**12**, Proposition 3.1]. We state it as follows. $L^p([0,T],\mathbb{R}^N)\times L^p([0,T],\mathbb{R}^N)$ is a reflexive and separable Banach space with the norm

$$ \|v\|_{L^p([0,T],\mathbb{R}^N)\times L^p([0,T],\mathbb{R}^N)} = (\|v_1\|_{L^p}^p + \|v_2\|_{L^p}^p)^{1/p}, $$

where $v=(v_1,v_2)\in L^p\times L^p$, since $L^p([0,T],\mathbb{R}^N)$ is a reflexive and separable Banach space. Now we consider the space $\Omega=\{(u,{}_0^cD_t^{\alpha}u)\colon u\in E^{\alpha,p}\}\subseteq L^p([0,T],\mathbb{R}^N)\times L^p([0,T],\mathbb{R}^N)$.

We claim that Ω is a closed subset of $L^p([0,T],\mathbb{R}^N)\times L^p([0,T],\mathbb{R}^N)$.

In fact, let

$$\bar{u}_n = (u_n, {}_0^cD_t^\alpha u_n) \to (u_0, v_0) = \bar{u}_0 \quad \text{in } L^p([0,T],\mathbb{R}^N) \times L^p([0,T],\mathbb{R}^N);$$

then $u_n \to u_0$ and ${}_0^cD_t^\alpha u_n \to v_0$ in $L^p([0,T],\mathbb{R}^N)$. Since $E^{\alpha,p}$ is closed and $u_n \in E^{\alpha,p}$, one has $u_n \to u^0$ in $E^{\alpha,p}$. So $u_n \to u^0$ and ${}_0^cD_t^\alpha u_n \to {}_0^cD_t^\alpha u^0$ in $L^p([0,T],\mathbb{R}^N)$. We claim that $u_0 = u^0$, $v_0 = {}_0^cD_t^\alpha u^0$ almost everywhere in $[0,T]$. In fact,

$$\|u_0 - u^0\|_{L^p} = \|u_0 - u_n + u_n - u^0\|_{L^p} \leqslant \|u_n - u_0\|_{L^p} + \|u_n - u^0\|_{L^p} \to 0$$

as $n \to \infty$. So $u_0 = u^0$ almost everywhere in $[0,T]$. Then

$$\begin{aligned}\|v_0 - {}_0^cD_t^\alpha u^0\|_{L^p} &= \|v_0 - {}_0^cD_t^\alpha u_n + {}_0^cD_t^\alpha u_n - {}_0^cD_t^\alpha u^0\|_{L^p} \\ &\leqslant \|{}_0^cD_t^\alpha u_n - v_0\|_{L^p} + \|{}_0^cD_t^\alpha u_n - {}_0^cD_t^\alpha u^0\|_{L^p} \to 0\end{aligned}$$

as $n \to \infty$. So $v_0 = {}_0^cD_t^\alpha u^0$ almost everywhere in $[0,T]$, $\bar{u}_0 \in \Omega$. Therefore, Ω is also a reflexive and separable Banach space with respect to the norm $\|\cdot\|_{L^p \times L^p}$. Define an operator $A\colon E^{\alpha,p} \to \Omega$ as

$$A\colon u \to (u, {}_0^cD_t^\alpha u) \quad \forall u \in E^{\alpha,p}.$$

It is obvious that the operator $A\colon u \to (u, {}_0^cD_t^\alpha u)$ is an isometric mapping and $E^{\alpha,p}$ is isometric isomorphic to the space Ω. Thus, $E^{\alpha,p}$ is a reflexive and separable Banach space. □

Lemma 4.3 (Jiao and Zhou [12, Lemma 3.1]). *Let $0 < \alpha \leqslant 1$ and $1 \leqslant p < \infty$. For any $f \in L^p([0,T],\mathbb{R}^N)$, we have*

$$\|{}_0D_\xi^{-\alpha} f\|_{L^p([0,t])} \leqslant \frac{t^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^p([0,t])} \quad \text{for } \xi \in [0,t],\ t \in [0,T].$$

Lemma 4.4. *Let $0 < \alpha \leqslant 1$ and $1 \leqslant p < \infty$. For any $f \in L^p([0,T],\mathbb{R}^N)$ we have*

$$\|{}_\xi D_T^{-\alpha} f\|_{L^p([t,T])} \leqslant \frac{(T-t)^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^p([t,T])} \quad \text{for } \xi \in [t,T],\ t \in [0,T].$$

Proof. If $p = 1$, we have

$$\begin{aligned}\|{}_\xi D_T^{-\alpha} f\|_{L^1([t,T])} &= \frac{1}{\Gamma(\alpha)}\int_t^T \left|\int_\xi^T (s-\xi)^{\alpha-1} f(s)\,\mathrm{d}s\right| \mathrm{d}\xi \\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_t^T \int_\xi^T (s-\xi)^{\alpha-1}|f(s)|\,\mathrm{d}s\,\mathrm{d}\xi \\ &= \frac{1}{\Gamma(\alpha)}\int_t^T \int_t^s (s-\xi)^{\alpha-1}\,\mathrm{d}\xi|f(s)|\,\mathrm{d}s \\ &= \frac{1}{\Gamma(\alpha)}\int_t^T \frac{(s-t)^\alpha}{\alpha}|f(s)|\,\mathrm{d}s \\ &\leqslant \frac{(T-t)^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^1([t,T])}.\end{aligned}$$

If $1 < p < +\infty$, then let $g \in L^q([0,T],\mathbb{R}^N)$, where $1/p + 1/q = 1$. Define

$$H_{\xi * f}\colon L^q([0,T],\mathbb{R}^N) \to \mathbb{R}$$

by

$$H_{\xi * f}(g) = \int_t^T g(\xi)\bigg(\int_\xi^T (s-\xi)^{\alpha-1} f(s)\,\mathrm{d}s\bigg)\,\mathrm{d}\xi. \tag{4.2}$$

We compute

$$\begin{aligned}\bigg|\int_t^T g(\xi)\bigg(\int_\xi^T (s-\xi)^{\alpha-1} f(s)\,\mathrm{d}s\bigg)\,\mathrm{d}\xi\bigg| &= \bigg|\int_t^T g(\xi)\bigg(\int_\xi^T (T-\tau)^{\alpha-1} f(T-\tau+\xi)\,\mathrm{d}\tau\bigg)\,\mathrm{d}\xi\bigg| \\ &= \bigg|\int_t^T (T-\tau)^{\alpha-1}\bigg(\int_t^\tau g(\xi) f(T-\tau+\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}\tau\bigg| \\ &\leqslant \frac{(T-t)^\alpha}{\alpha}\|f\|_{L^p([t,T])}\|g\|_{L^q([t,T])} \end{aligned} \tag{4.3}$$

for $t \in [0,T]$. So $H_{\xi * f} \in (L^q([0,T],\mathbb{R}^N))^*$, where $(L^q([0,T],\mathbb{R}^N))^*$ denotes the dual space of $L^q([0,T],\mathbb{R}^N)$.

Therefore, by (4.2), (4.3) and the Riesz representation theorem, there exists $h \in L^p([0,T],\mathbb{R}^N)$ such that

$$\int_t^T g(\xi)h(\xi)\,\mathrm{d}\xi = H_{\xi * f}(g) = \int_t^T g(\xi)\bigg(\int_\xi^T (s-\xi)^{\alpha-1} f(s)\,\mathrm{d}s\bigg)\,\mathrm{d}\xi \tag{4.4}$$

for all $g \in L^q([0,T],\mathbb{R}^N)$. By (4.3), $\|h\|_{L^p([t,T])} \leqslant ((T-t)^\alpha/\alpha)\|f\|_{L^p([t,T])}$.

By (4.4),

$$\frac{h(\xi)}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)}\int_\xi^T (s-\xi)^{\alpha-1} f(s)\,\mathrm{d}s = {}_\xi D_T^{-\alpha} f(\xi) \quad \text{for } \xi \in [t,T],$$

which means that

$$\|{}_\xi D_T^{-\alpha} f\|_{L^p([t,T])} \leqslant \frac{\|h\|_{L^p([t,T])}}{\Gamma(\alpha)} \leqslant \frac{(T-t)^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^p[t,T]}.$$ □

Lemma 4.5. *For $u(x)$ a real-valued function, one has*

$$\begin{aligned}\int_{-\infty}^{+\infty} ({}_{-\infty}D_t^{\alpha-1}u'(t), {}_tD_\infty^{\alpha-1}u'(t))\,\mathrm{d}t &= -\cos\pi\alpha \int_{-\infty}^{+\infty} |{}_{-\infty}D_t^{\alpha-1}u'(t)|^2\,\mathrm{d}t \\ &= -\cos\pi\alpha \int_{-\infty}^{+\infty} |{}_tD_\infty^{\alpha-1}u'(t)|^2\,\mathrm{d}t. \end{aligned} \tag{4.5}$$

Proof. The idea of the proof comes from [**10**, Lemma 2.4]. By the Fourier transform properties,

$$\int_{\mathbb{R}} u\bar{v}\,\mathrm{d}x = \int_{\mathbb{R}} \hat{u}\bar{\hat{v}}\,\mathrm{d}w, \tag{4.6}$$

where $\hat{u}(w) = \mathcal{F}(u)$, and, with the observation that

$$\overline{(\mathrm{i}w)^{\alpha-1}} = \begin{cases} \exp(-\mathrm{i}\pi(\alpha-1))\overline{(-\mathrm{i}w)^{\alpha-1}} & \text{if } w \geqslant 0, \\ \exp(\mathrm{i}\pi(\alpha-1))\overline{(-\mathrm{i}w)^{\alpha-1}} & \text{if } w < 0, \end{cases} \tag{4.7}$$

we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} ({}_{-\infty}D_t^{\alpha-1}u'(t), {}_tD_{\infty}^{\alpha-1}u'(t))\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \mathcal{F}({}_{-\infty}D_t^{\alpha-1}u'(t))\overline{\mathcal{F}({}_tD_{\infty}^{\alpha-1}u'(t))}\,\mathrm{d}w \quad \text{(we have used (4.6))} \\
&= \int_{\mathbb{R}} (\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)\overline{(-\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)}\,\mathrm{d}w \quad \text{(we have used Property (3.4))} \\
&= \int_{-\infty}^{0} (\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)\exp(-\mathrm{i}\pi(\alpha-1))\overline{(\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)}\,\mathrm{d}w \\
&\quad + \int_{0}^{+\infty} (\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)\exp(\mathrm{i}\pi(\alpha-1))\overline{(\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)}\,\mathrm{d}w \quad \text{(we have used (4.7))} \\
&= \int_{-\infty}^{0} (\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)\overline{(\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)}\,\mathrm{d}w\cos\pi(\alpha-1) \\
&\quad - \mathrm{i}\int_{-\infty}^{0} (\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)\overline{(\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)}\,\mathrm{d}w\sin\pi(\alpha-1) \\
&\quad + \int_{0}^{+\infty} (\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)\overline{(\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)}\,\mathrm{d}w\cos\pi(\alpha-1) \\
&\quad + \mathrm{i}\int_{0}^{+\infty} (\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)\overline{(\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)}\,\mathrm{d}w\sin\pi(\alpha-1) \\
&= \int_{-\infty}^{+\infty} (\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)\overline{(\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)}\,\mathrm{d}w\cos\pi(\alpha-1) \\
&\quad - \mathrm{i}\sin\pi(\alpha-1)\biggl(\int_{-\infty}^{0} (\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)\overline{(\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)}\,\mathrm{d}w \\
&\qquad - \int_{0}^{+\infty} (\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)\overline{(\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)}\,\mathrm{d}w\biggr).
\end{aligned} \tag{4.8}$$

We compute

$$\begin{aligned}
&\int_{-\infty}^{0} (\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)\overline{(\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)}\,\mathrm{d}w - \int_{0}^{+\infty} (\mathrm{i}\omega)^{\alpha-1}\widehat{u'}(\omega)\overline{(\mathrm{i}\omega)^{\alpha-1}\widehat{u'}(\omega)}\,\mathrm{d}\omega \\
&= \int_{-\infty}^{0} (\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)\overline{(\mathrm{i}w)^{\alpha-1}}\widehat{u'}(-w)\,\mathrm{d}w - \int_{0}^{+\infty} (\mathrm{i}\omega)^{\alpha-1}\widehat{u'}(\omega)\overline{(\mathrm{i}\omega)^{\alpha-1}}\widehat{u'}(-\omega)\,\mathrm{d}\omega \\
&= \int_{0}^{+\infty} (-\mathrm{i}t)^{\alpha-1}\widehat{u'}(-t)\overline{(-\mathrm{i}t)^{\alpha-1}}\widehat{u'}(t)\,\mathrm{d}t - \int_{0}^{+\infty} (\mathrm{i}\omega)^{\alpha-1}\widehat{u'}(\omega)\overline{(\mathrm{i}\omega)^{\alpha-1}}\widehat{u'}(-\omega)\,\mathrm{d}\omega
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{+\infty} \exp\left(-\frac{\pi}{2}(\alpha-1)\mathrm{i}\right)t^{\alpha-1}\overline{\exp\left(-\frac{\pi}{2}(\alpha-1)\mathrm{i}\right)t^{\alpha-1}\widehat{u'}(-t)}\widehat{u'}(t)\,\mathrm{d}t \\
&\quad - \int_0^{+\infty} \exp\left(\frac{\pi}{2}(\alpha-1)\mathrm{i}\right)\omega^{\alpha-1}\overline{\exp\left(\frac{\pi}{2}(\alpha-1)\mathrm{i}\right)\omega^{\alpha-1}\widehat{u'}(\omega)}\widehat{u'}(-\omega)\,\mathrm{d}\omega \\
&= 0. \qquad (4.9)
\end{aligned}
$$

Combining (4.8) and (4.9), one has

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} ({}_{-\infty}D_t^{\alpha-1}u'(t), {}_tD_{\infty}^{\alpha-1}u'(t))\,\mathrm{d}t \\
&\qquad = \int_{-\infty}^{+\infty} (\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)\overline{(\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)}\,\mathrm{d}w\cos\pi(\alpha-1) \\
&\qquad = \int_{-\infty}^{+\infty} \mathcal{F}({}_{-\infty}D_t^{\alpha-1}u'(t))\overline{\mathcal{F}({}_{-\infty}D_t^{\alpha-1}u'(t))}\,\mathrm{d}\omega\cos\pi(\alpha-1) \\
&\qquad = \int_{-\infty}^{+\infty} |{}_{-\infty}D_t^{\alpha-1}u'(t)|^2\,\mathrm{d}t\cos\pi(\alpha-1) \quad \text{(we have used (4.6))} \\
&\qquad = -\cos\pi\alpha\int_{-\infty}^{+\infty} |{}_{-\infty}D_t^{\alpha-1}u'(t)|^2\,\mathrm{d}t.
\end{aligned}
$$

Since

$$
\int_{-\infty}^{+\infty} (\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)\overline{(\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)}\,\mathrm{d}w = \int_{-\infty}^{+\infty} (-\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)\overline{(-\mathrm{i}w)^{\alpha-1}\widehat{u'}(w)}\,\mathrm{d}w,
$$

one has

$$
\int_{-\infty}^{+\infty} |{}_{-\infty}D_t^{\alpha-1}u'(t)|^2\,\mathrm{d}t = \int_{-\infty}^{+\infty} |{}_tD_{\infty}^{\alpha-1}u'(t)|^2\,\mathrm{d}t.
$$

The result follows. □

Lemma 4.6. *If $\alpha \in (\frac{1}{2}, 1]$, then for any $u \in E^\alpha$ we have*

$$
-\cos\pi\alpha\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,\mathrm{d}t \leqslant -\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))\,\mathrm{d}t \leqslant -\frac{1}{\cos\pi\alpha}\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,\mathrm{d}t. \qquad (4.10)
$$

Proof. Let $u \in E^\alpha$ and define

$$
\tilde{u}(t) = \begin{cases} u(t), & t \in [0, T], \\ u(0), & t \leqslant 0, \\ u(T), & t \geqslant T. \end{cases} \qquad (4.11)
$$

It is clear that

$$
{}_{-\infty}D_t^{\alpha-1}\tilde{u}'(t) = 0 \quad \text{for } t < 0
$$

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and

$$ {}_tD_{\infty}^{\alpha-1}\tilde{u}'(t) = 0 \quad \text{for } t > 0. $$

So by Lemma 4.5, we have

$$
\begin{aligned}
&-\int_0^T ({}_0^cD_t^{\alpha}u(t), {}_t^cD_T^{\alpha}u(t))\,\mathrm{d}t \\
&\quad = \int_0^T ({}_0D_t^{\alpha-1}u'(t), {}_tD_T^{\alpha-1}u'(t))\,\mathrm{d}t \quad \text{(we have used the definitions of left} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and right Caputo fractional derivatives)} \\
&\quad = \int_{-\infty}^{+\infty} ({}_0D_t^{\alpha-1}\tilde{u}'(t), {}_tD_T^{\alpha-1}\tilde{u}'(t))\,\mathrm{d}t \quad \text{(we have used (4.11))} \\
&\quad = \int_{-\infty}^{+\infty} ({}_{-\infty}D_t^{\alpha-1}\tilde{u}'(t), {}_tD_{\infty}^{\alpha-1}\tilde{u}'(t))\,\mathrm{d}t \\
&\quad = -\cos\pi\alpha \int_{-\infty}^{+\infty} |{}_{-\infty}D_t^{\alpha-1}\tilde{u}'(t)|^2\,\mathrm{d}t \quad \text{(we have used Lemma 4.5)} \\
&\quad = -\cos\pi\alpha \int_0^{+\infty} |{}_0D_t^{\alpha-1}\tilde{u}'(t)|^2\,\mathrm{d}t \\
&\quad \geqslant -\cos\pi\alpha \int_0^T |{}_0D_t^{\alpha-1}u'(t)|^2\,\mathrm{d}t \\
&\quad = -\cos\pi\alpha \int_0^T |{}_0^cD_t^{\alpha}u(t)|^2\,\mathrm{d}t.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\left|-\int_0^T ({}_0^cD_t^{\alpha}u(t), {}_t^cD_T^{\alpha}u(t))\,\mathrm{d}t\right| \\
&\quad = \left|\int_0^T ({}_0D_t^{\alpha-1}u'(t), {}_tD_T^{\alpha-1}u'(t))\,\mathrm{d}t\right| \\
&\quad \leqslant \int_0^T \frac{1}{\sqrt{2\varepsilon}}|{}_0D_t^{\alpha-1}u'(t)|\sqrt{2\varepsilon}|{}_tD_T^{\alpha-1}u'(t)|\,\mathrm{d}t \\
&\quad \leqslant \int_0^T \frac{1}{4\varepsilon}|{}_0D_t^{\alpha-1}u'(t)|^2 + \varepsilon|{}_tD_T^{\alpha-1}u'(t)|^2\,\mathrm{d}t \\
&\quad = \frac{1}{4\varepsilon}\int_0^T |{}_0^cD_t^{\alpha}u(t)|^2\,\mathrm{d}t + \varepsilon\int_0^{\infty} |{}_tD_{\infty}^{\alpha-1}\tilde{u}'(t)|^2\,\mathrm{d}t \\
&\quad \leqslant \frac{1}{4\varepsilon}\int_0^T |{}_0^cD_t^{\alpha}u(t)|^2\,\mathrm{d}t + \varepsilon\int_{-\infty}^{\infty} |{}_tD_{\infty}^{\alpha-1}\tilde{u}'(t)|^2\,\mathrm{d}t \\
&\quad = \frac{1}{4\varepsilon}\int_0^T |{}_0^cD_t^{\alpha}u(t)|^2\,\mathrm{d}t - \frac{\varepsilon}{\cos\pi\alpha}\left|\int_{-\infty}^{+\infty} ({}_{-\infty}D_t^{\alpha-1}\tilde{u}'(t), {}_tD_{\infty}^{\alpha-1}\tilde{u}'(t))\,\mathrm{d}t\right| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(we have used Lemma 4.5)}
\end{aligned}
$$

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$$= \frac{1}{4\varepsilon}\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,\mathrm{d}t - \frac{\varepsilon}{\cos\pi\alpha}\left|\int_0^T ({}_0D_t^{\alpha-1}u'(t), {}_tD_T^{\alpha-1}u'(t))\,\mathrm{d}t\right|$$
$$= \frac{1}{4\varepsilon}\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,\mathrm{d}t - \frac{\varepsilon}{\cos\pi\alpha}\left|\int_0^T ({}_0^cD_t^{\alpha}u(t), {}_t^cD_T^{\alpha}u(t))\,\mathrm{d}t\right|.$$

So

$$\left(1+\frac{\varepsilon}{\cos\pi\alpha}\right)\left|\int_0^T ({}_0^cD_t^{\alpha}u(t), {}_t^cD_T^{\alpha}u(t))\,\mathrm{d}t\right| \leqslant \frac{1}{4\varepsilon}\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,\mathrm{d}t.$$

By taking $\varepsilon = -\frac{1}{2}\cos\pi\alpha$, we have

$$\left|\int_0^T ({}_0^cD_t^{\alpha}u(t), {}_t^cD_T^{\alpha}u(t))\,\mathrm{d}t\right| \leqslant -\frac{1}{\cos\pi\alpha}\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,\mathrm{d}t.$$

□

Lemma 4.7. *Let $\alpha \in (\frac{1}{2}, 1]$ and $u \in E^\alpha$. The norm $\|u\|_{\alpha,2}$ is equivalent to*

$$\|u\| = \left(-\int_0^T ({}_0^cD_t^{\alpha}u, {}_t^cD_T^{\alpha}u)\,\mathrm{d}t + \frac{c}{d}(u(T))^2 + \frac{a}{b}(u(0))^2\right)^{1/2}.$$

Proof. First we will show that there exists an $M_1 > 0$ satisfying $\|u\|_{\alpha,2} \leqslant M_1\|u\|$. By Property 3.3, ${}_0D_t^{-\alpha}({}_0^cD_t^\alpha u(t)) = u(t) - u(0)$. So

$$\begin{aligned}
\int_0^T |u(t)|^2\,\mathrm{d}t &= \int_0^T |{}_0D_t^{-\alpha}({}_0^cD_t^\alpha u(t)) + u(0)|^2\,\mathrm{d}t \\
&\leqslant 2\int_0^T |{}_0D_t^{-\alpha}({}_0^cD_t^\alpha u(t))|^2 + |u(0)|^2\,\mathrm{d}t \\
&\leqslant 2T|u(0)|^2 + 2\frac{T^{2\alpha}}{(\Gamma(\alpha+1))^2}\|{}_0^cD_t^\alpha u\|_{L^2}^2 \quad \text{(we have used Lemma 4.3)} \\
&\leqslant 2T\frac{b}{a}\frac{a}{b}(u(0))^2 + \frac{2T^{2\alpha}}{(\Gamma(\alpha+1))^2\cos\pi\alpha}\int_0^T ({}_0^cD_t^{\alpha}u, {}_t^cD_T^{\alpha}u)\,\mathrm{d}t \\
&\qquad\qquad \text{(we have used Lemma 4.6)} \\
&\leqslant \max\left\{2T\frac{b}{a}, -\frac{2T^{2\alpha}}{(\Gamma(\alpha+1))^2\cos\pi\alpha}\right\}\|u\|^2.
\end{aligned} \tag{4.12}$$

By (4.12) and Lemma 4.6,

$$\begin{aligned}
\|u\|_{\alpha,2} &= \left(\int_0^T |u(t)|^2 + |{}_0^cD_t^\alpha u|^2\,\mathrm{d}t\right)^{1/2} \\
&\leqslant \left(\max\left\{2T\frac{b}{a}, -\frac{2T^{2\alpha}}{(\Gamma(\alpha+1))^2\cos\pi\alpha}\right\}\|u\|^2 + \frac{1}{\cos\pi\alpha}\int_0^T ({}_0^cD_t^{\alpha}u, {}_t^cD_T^{\alpha}u)\,\mathrm{d}t\right)^{1/2} \\
&\leqslant \left(\max\left\{2T\frac{b}{a}, -\frac{2T^{2\alpha}}{(\Gamma(\alpha+1))^2\cos\pi\alpha}\right\} - \frac{1}{\cos\pi\alpha}\right)^{1/2}\|u\|,
\end{aligned} \tag{4.13}$$

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which means that

$$\|u\|_{\alpha,2} \leqslant M_1 \|u\|, \tag{4.14}$$

where

$$M_1 = \left(\max\left\{ 2T\frac{b}{a}, -\frac{2T^{2\alpha}}{(\Gamma(\alpha+1))^2 \cos \pi\alpha} \right\} - \frac{1}{\cos \pi\alpha} \right)^{1/2}. \tag{4.15}$$

In the following we will show that there exists an $M_2 > 0$ satisfying $\|u\| \leqslant M_2 \|u\|_{\alpha,2}$. By Property 3.3, $u(0) = u(t) - {}_0D_t^{-\alpha}({}_0^cD_t^{\alpha} u(t))$. So

$$\begin{aligned}
u(0) &= \frac{1}{T}\int_0^T u(0)\,\mathrm{d}s \\
&= \frac{1}{T}\int_0^T u(t) - {}_0D_t^{-\alpha}({}_0^cD_t^{\alpha} u(t))\,\mathrm{d}t \\
&\leqslant \frac{1}{T}\left(\int_0^T |u(t)|\,\mathrm{d}t + \int_0^T |{}_0D_t^{-\alpha}({}_0^cD_t^{\alpha} u(t))|\,\mathrm{d}t \right) \\
&\leqslant \frac{1}{T}(T^{1/2}\|u\|_{L^2} + T^{1/2}\|{}_0D_t^{-\alpha}({}_0^cD_t^{\alpha} u(t))\|_{L^2[0,T]}) \\
&\qquad\qquad\qquad \text{(we have used Hölder's inequality)} \\
&\leqslant \frac{1}{T}\left(T^{1/2}\|u\|_{L^2} + \frac{T^{\alpha+1/2}}{\Gamma(\alpha+1)}\|{}_0^cD_t^{\alpha} u\|_{L^2[0,T]} \right) \quad \text{(we have used Lemma 4.3)} \\
&\leqslant \sqrt{2}\max\left\{ T^{-1/2}, \frac{T^{-1/2+\alpha}}{\Gamma(\alpha+1)} \right\}\|u\|_{\alpha,2}.
\end{aligned} \tag{4.16}$$

By Property 3.3, $u(T) = u(t) - {}_tD_T^{-\alpha}({}_t^cD_T^{\alpha} u(t))$. So

$$\begin{aligned}
u(T) &= \frac{1}{T}\int_0^T u(T)\,\mathrm{d}s \\
&= \frac{1}{T}\int_0^T u(t) - {}_tD_T^{-\alpha}({}_t^cD_T^{\alpha} u(t))\,\mathrm{d}t \\
&\leqslant \frac{1}{T}\left(\int_0^T |u(t)|\,\mathrm{d}t + \int_0^T |{}_tD_T^{-\alpha}({}_t^cD_T^{\alpha} u(t))|\,\mathrm{d}t \right) \\
&\leqslant \frac{1}{T}(T^{1/2}\|u\|_{L^2} + T^{1/2}\|{}_tD_T^{-\alpha}({}_t^cD_T^{\alpha} u(t))\|_{L^2[0,T]}) \\
&\qquad\qquad\qquad \text{(we have used Hölder's inequality)} \\
&\leqslant \frac{1}{T}\left(T^{1/2}\|u\|_{L^2[0,T]} + \frac{T^{\alpha+1/2}}{\Gamma(\alpha+1)}\|{}_t^cD_T^{\alpha} u\|_{L^2[0,T]} \right) \quad \text{(we have used Lemma 4.4).}
\end{aligned} \tag{4.17}$$

We compute

$$\begin{aligned}
\|{}_t^cD_T^{\alpha} u\|^2_{L^2[0,T]} &= \|{}_tD_T^{\alpha-1} u'\|_{L^2[0,T]} \\
&\leqslant \int_{\mathbb{R}} ({}_tD_{\infty}^{\alpha-1}\tilde{u}', \overline{{}_tD_{\infty}^{\alpha-1}\tilde{u}'})\,\mathrm{d}t
\end{aligned}$$

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$$
\begin{aligned}
&= \int_{\mathbb{R}} |{}_tD_{\infty}^{\alpha-1}\tilde{u}'|^2\,\mathrm{d}t \\
&= -\frac{1}{\cos\pi\alpha}\int_{\mathbb{R}} ({}_{-\infty}D_t^{\alpha-1}\tilde{u}', {}_tD_{\infty}^{\alpha-1}\tilde{u}')\,\mathrm{d}t \quad \text{(we have used Lemma 4.5)} \\
&= -\frac{1}{\cos\pi\alpha}\int_{\mathbb{R}}\biggl(\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{t}(t-s)^{-\alpha}\tilde{u}'(s)\,\mathrm{d}s, \\
&\qquad\qquad \frac{1}{\Gamma(1-\alpha)}\int_t^{\infty}(s-t)^{-\alpha}\tilde{u}'(s)\,\mathrm{d}s\biggr)\,\mathrm{d}t \\
&= -\frac{1}{\cos\pi\alpha}\int_{\mathbb{R}}\biggl(\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}(t-s)^{-\alpha}\tilde{u}'(s)\,\mathrm{d}s, \\
&\qquad\qquad \frac{1}{\Gamma(1-\alpha)}\int_t^{T}(s-t)^{-\alpha}\tilde{u}'(s)\,\mathrm{d}s\biggr)\,\mathrm{d}t \\
&= -\frac{1}{\cos\pi\alpha}\int_0^T ({}_0D_t^{\alpha-1}u', {}_tD_T^{\alpha-1}u')\,\mathrm{d}t \\
&= -\frac{1}{\cos\pi\alpha}\int_0^T ({}_0^cD_t^{\alpha}u, {}_t^cD_T^{\alpha}u)\,\mathrm{d}t \\
&\leqslant \frac{1}{(\cos\pi\alpha)^2}\int_0^T |{}_0^cD_t^{\alpha}u|^2\,\mathrm{d}t \quad \text{(we have used Lemma 4.6).}
\end{aligned} \tag{4.18}
$$

By (4.17) and (4.18),

$$
\begin{aligned}
|u(T)| &\leqslant \frac{1}{T}\biggl(T^{1/2}\|u\|_{L^2[0,T]} - \frac{T^{\alpha+1/2}}{\Gamma(\alpha+1)\cos\pi\alpha}\|{}_0^cD_t^{\alpha}u\|_{L^2[0,T]}\biggr) \\
&\leqslant \max\biggl\{T^{-1/2}, \frac{-T^{\alpha-1/2}}{\Gamma(\alpha+1)\cos\pi\alpha}\biggr\}(\|u\|_{L^2[0,T]} + \|{}_0^cD_t^{\alpha}u\|_{L^2[0,T]}) \\
&\leqslant \sqrt{2}\max\biggl\{T^{-1/2}, \frac{-T^{\alpha-1/2}}{\Gamma(\alpha+1)\cos\pi\alpha}\biggr\}\|u\|_{\alpha,2}.
\end{aligned} \tag{4.19}
$$

Equations (4.16) and (4.19) and Lemma 4.6 mean that $\|u\| \leqslant M_2\|u\|_{\alpha,2}$ for some $M_2 > 0$. The proof is completed. □

5. Variational structure for $bd \neq 0$

Let $\alpha = 1 - \beta/2$. Then $\alpha \in (\frac{1}{2}, 1]$.

Defining $D_f := \{z \in \mathbb{R} \mid f \text{ is discontinuous at } z\}$, we recall that f is said to be continuous almost everywhere if D_f is (Lebesgue) measurable and $m(D_f) = 0$. Moreover, if f is locally essentially bounded, we write

$$
f^-(z) = \lim_{\delta\to 0^-}\operatorname*{ess\,inf}_{|t-z|<\delta} f(z), \quad f^+(z) = \lim_{\delta\to 0^+}\operatorname*{ess\,sup}_{|t-z|<\delta} f(z) \quad \text{for each } t \in \mathbb{R}.
$$

We observe that f^-, f^+ are, respectively, lower semi-continuous and upper semi-continuous.

Definition 5.1. A function $u \in E^\alpha$ is said to be a generalized solution of (1.1) if u satisfies the boundary condition of (1.1) and the equation of (1.1) for almost everywhere $t \in [0, T]$.

Remark 5.2. If f is a continuous function, then $({}_0D_t^{-\beta}u')', ({}_tD_T^{-\beta}u')' \in C[0, T]$, and u is a classical solution of problem (1.1).

For each $u \in E^\alpha$, define $\Phi\colon E^\alpha \to \mathbb{R}$ by

$$\Phi(u) = -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u, {}_t^cD_T^\alpha u)\,\mathrm{d}t + \frac{c}{2d}(u(T))^2 + \frac{a}{2b}(u(0))^2 = \tfrac{1}{2}\|u\|^2.$$

$\Upsilon(u) = \int_0^T F(u(t))\,\mathrm{d}t$, where $F(u) := \int_0^u f(s)\,\mathrm{d}s$, $j(u) = 0$, $\Psi(u) = \Upsilon(u) - j(u) = \Upsilon(u)$, $I_\lambda(u) = \Phi(u) - \lambda\Psi(u) = \Phi(u) - \lambda\Upsilon(u)$.

Lemma 5.3. $\Phi\colon E^\alpha \to \mathbb{R}$ *is weakly lower semi-continuous.*

Proof. It is clear that Φ is lower semi-continuous. In order to show that Φ is weakly lower semi-continuous on E^α, it is sufficient to show that Φ is convex on E^α.

Let $\lambda \in (0, 1)$, let $u, v \in E^\alpha$, and let $\tilde{u}$ and $\tilde{v}$ be the extensions of u and v by $[0, T]$ defined in (4.11). Then

$$\begin{aligned}
&\Phi_1((1-\lambda)u + \lambda v)\\
&\quad := -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha((1-\lambda)u + \lambda v), {}_t^cD_T^\alpha((1-\lambda)u + \lambda v))\,\mathrm{d}t\\
&\quad = \frac{1}{2}\int_0^T ({}_0D_t^{\alpha-1}((1-\lambda)u' + \lambda v'), {}_tD_T^{\alpha-1}((1-\lambda)u' + \lambda v'))\,\mathrm{d}t\\
&\quad = \frac{1}{2}\int_{-\infty}^{+\infty} ({}_{-\infty}D_t^{\alpha-1}((1-\lambda)\tilde{u}' + \lambda\tilde{v}'), {}_tD_\infty^{\alpha-1}((1-\lambda)\tilde{u}' + \lambda\tilde{v}'))\,\mathrm{d}t\\
&\quad = -\tfrac{1}{2}\cos\pi\alpha\int_{-\infty}^{+\infty} |{}_{-\infty}D_t^{\alpha-1}((1-\lambda)\tilde{u}' + \lambda\tilde{v}')|^2\,\mathrm{d}t \quad \text{(we have used Lemma 4.5)}\\
&\quad \leqslant -\tfrac{1}{2}\cos\pi\alpha\int_{-\infty}^{+\infty} [(1-\lambda)|{}_{-\infty}D_t^{\alpha-1}\tilde{u}'|^2 + \lambda|{}_{-\infty}D_t^{\alpha-1}\tilde{v}'|^2]\,\mathrm{d}t\\
&\quad = \tfrac{1}{2}(1-\lambda)\int_{-\infty}^{+\infty} ({}_{-\infty}D_t^{\alpha-1}\tilde{u}', {}_tD_\infty^{\alpha-1}\tilde{u}')\,\mathrm{d}t + \tfrac{1}{2}\lambda\int_{-\infty}^{+\infty} ({}_{-\infty}D_t^{\alpha-1}\tilde{v}', {}_tD_\infty^{\alpha-1}\tilde{v}')\,\mathrm{d}t\\
&\quad = \frac{1}{2}\int_{-\infty}^{+\infty} (1-\lambda)({}_0D_t^{\alpha-1}\tilde{u}', {}_tD_T^{\alpha-1}\tilde{u}')\,\mathrm{d}t + \frac{1}{2}\int_{-\infty}^{+\infty} \lambda({}_0D_t^{\alpha-1}\tilde{v}', {}_tD_T^{\alpha-1}\tilde{v}')\,\mathrm{d}t\\
&\quad = \frac{1}{2}\int_0^T (1-\lambda)({}_0D_t^{\alpha-1}u', {}_tD_T^{\alpha-1}u')\,\mathrm{d}t + \frac{1}{2}\int_0^T \lambda({}_0D_t^{\alpha-1}v', {}_tD_T^{\alpha-1}v')\,\mathrm{d}t\\
&\quad = -\frac{1-\lambda}{2}\int_0^T ({}_0^cD_t^\alpha u, {}_t^cD_T^\alpha u)\,\mathrm{d}t - \frac{\lambda}{2}\int_0^T ({}_0^cD_t^\alpha v, {}_t^cD_T^\alpha v)\,\mathrm{d}t\\
&\quad = (1-\lambda)\Phi_1(u) + \lambda\Phi_1(v).
\end{aligned}$$

So $\varPhi_1$ is convex on E^α. Clearly,

$$\varPhi_2(u) := \frac{c}{2d}|u(T)|^2 + \frac{a}{2b}|u(0)|^2$$

is convex on E^α. So $\varPhi(u) := \varPhi_1(u) + \varPhi_2(u)$ is convex on E^α. The proof is complete. □

Lemma 5.4. *For $u \in E^\alpha$, there exists $M_3 > 0$ such that $\|u\|_\infty \leqslant M_3\|u\|$, where*

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)|,$$

$$M_3 = \sqrt{2}M_1 \max\left\{T^{-1/2}, \frac{T^{\alpha-1/2}}{\Gamma(\alpha+1)}\right\} + \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2\alpha-1)^{1/2}\sqrt{|\cos\pi\alpha|}},$$

and M_1 is defined in (4.15).

Proof. For $u \in E^\alpha$,

$$\begin{aligned}
|{}_0D_t^{-\alpha}({}_0^cD_t^\alpha u(t))| &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,{}_0^cD_s^\alpha u(s)\,\mathrm{d}s \\
&\leqslant \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{2(\alpha-1)}\,\mathrm{d}s\right)^{1/2}\left(\int_0^T |{}_0^cD_s^\alpha u(s)|^2\,\mathrm{d}s\right)^{1/2} \\
&\leqslant \frac{1}{\Gamma(\alpha)}\left[\frac{t^{2\alpha-1}}{2\alpha-1}\right]^{1/2}\|{}_0^cD_t^\alpha u\|_{L^2[0,T]} \\
&\leqslant \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2\alpha-1)^{1/2}\sqrt{|\cos\pi\alpha|}}\left[-\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))\,\mathrm{d}t\right]^{1/2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(we have used Lemma 4.6)} \\
&\leqslant \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2\alpha-1)^{1/2}\sqrt{|\cos\pi\alpha|}}\|u\|. \qquad (5.1)
\end{aligned}$$

By (4.14) and (4.16),

$$|u(0)| \leqslant \sqrt{2}\max\left\{T^{-1/2}, \frac{T^{\alpha-1/2}}{\Gamma(\alpha+1)}\right\}\|u\|_{\alpha,2} \leqslant \sqrt{2}M_1\max\left\{T^{-1/2}, \frac{T^{\alpha-1/2}}{\Gamma(\alpha+1)}\right\}\|u\|. \qquad (5.2)$$

By Property 3.3, (5.1) and (5.2),

$$\begin{aligned}
\|u\|_\infty &= |u(0) + {}_0D_t^{-\alpha}({}_0^cD_t^\alpha u(t))| \\
&\leqslant \left[\sqrt{2}M_1\max\left\{T^{-1/2}, \frac{T^{\alpha-1/2}}{\Gamma(\alpha+1)}\right\} + \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2\alpha-1)^{1/2}\sqrt{|\cos\pi\alpha|}}\right]\|u\| \\
&:= M_3\|u\|.
\end{aligned}$$

The result follows. □

Proposition 5.5. *Let $0 < \alpha \leqslant 1$, $1 < p < \infty$. Assume that $\alpha > 1/p$ and the sequence (u_k) converges weakly to u in $E^{\alpha,p}$, i.e. $u_k \rightharpoonup u$. Then $u_k \to u$ in $C([0,T],\mathbb{R}^N)$, i.e. $\|u_k - u\|_\infty \to 0$ as $k \to \infty$.*

Proof. By $\|u\|_\infty \leqslant M_3\|u\|$, the injection of $E^{\alpha,p}$ into $C([0,T],\mathbb{R}^N)$ is continuous, i.e. if $u_k \to u$ in $E^{\alpha,p}$, then $u_k \to u$ in $C([0,T],\mathbb{R}^N)$. Since $u_k \rightharpoonup u$ in $E^{\alpha,p}$, it follows that $u_k \rightharpoonup u$ in $C([0,T],\mathbb{R}^N)$. In fact, for any $h \in (C([0,T],\mathbb{R}^N))^*$, if $u_k \to u$ in $E^{\alpha,p}$, then $u_k \to u$ in $C([0,T],\mathbb{R}^N)$, and thus $h(u_k) \to h(u)$. Therefore, $h \in (E^{\alpha,p})^*$, which means that $(C([0,T],\mathbb{R}^N))^* \subseteq (E^{\alpha,p})^*$. Hence, if $u_k \rightharpoonup u$ in $E^{\alpha,p}$, then for any $h \in (C([0,T],\mathbb{R}^N))^*$ we have $h \in (E^{\alpha,p})^*$, and thus $h(u_k) \to h(u)$, i.e. $u_k \rightharpoonup u$ in $C([0,T],\mathbb{R}^N)$. By the Banach–Steinhaus theorem, (u_k) is bounded in $E^{\alpha,p}$, and hence in $C([0,T],\mathbb{R}^N)$. We are now in a position to prove that the sequence (u_k) is equi-uniformly continuous. For any t_1, t_2 satisfying $0 \leqslant t_1 \leqslant t_2 \leqslant T$, we have, by Property 3.3,

$$|u_k(t_1) - u_k(t_2)| = |{}_0D_{t_1}^{-\alpha}({}_0^cD_{t_1}^{\alpha}u_k(t_1)) - {}_0D_{t_2}^{-\alpha}({}_0^cD_{t_2}^{\alpha}u_k(t_2))|. \tag{5.3}$$

First we compute

$$\begin{aligned}
&|{}_0D_{t_1}^{-\alpha}f(t_1) - {}_0D_{t_2}^{-\alpha}f(t_2)| \\
&\quad = \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}(t_1-s)^{\alpha-1}f(s)\,\mathrm{d}s - \int_0^{t_2}(t_2-s)^{\alpha-1}f(s)\,\mathrm{d}s\right| \\
&\quad \leqslant \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]f(s)\,\mathrm{d}s\right| + \frac{1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s)\,\mathrm{d}s\right| \\
&\quad \leqslant \frac{1}{\Gamma(\alpha)}\left(\int_0^{t_1}|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|^q\,\mathrm{d}s\right)^{1/q}\|f\|_{L^p[0,t_1]} \\
&\qquad + \frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-s)^{(\alpha-1)q}\,\mathrm{d}s\right)^{1/q}\|f\|_{L^p[t_1,t_2]} \\
&\quad \leqslant \frac{1}{\Gamma(\alpha)}\left(\int_0^{t_1}(t_1-s)^{(\alpha-1)q}-(t_2-s)^{(\alpha-1)q}\,\mathrm{d}s\right)^{1/q}\|f\|_{L^p[0,t_1]} \\
&\qquad + \frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-s)^{(\alpha-1)q}\,\mathrm{d}s\right)^{1/q}\|f\|_{L^p[t_1,t_2]} \\
&\quad = \frac{\|f\|_{L^p[0,t_1]}}{\Gamma(\alpha)}\left(\frac{t_1^{(\alpha-1)q+1}}{(\alpha-1)q+1} + \frac{-t_2^{(\alpha-1)q+1}+(t_2-t_1)^{(\alpha-1)q+1}}{(\alpha-1)q+1}\right)^{1/q} \\
&\qquad + \frac{\|f\|_{L^p[t_1,t_2]}}{\Gamma(\alpha)}\frac{(t_2-t_1)^{(\alpha-1)+1/q}}{[(\alpha-1)q+1]^{1/q}} \\
&\quad = \frac{\|f\|_{L^p[0,t_1]}}{\Gamma(\alpha)[(\alpha-1)q+1]^{1/q}}[t_1^{(\alpha-1)q+1} - t_2^{(\alpha-1)q+1} + (t_2-t_1)^{(\alpha-1)q+1}]^{1/q} \\
&\qquad + \frac{\|f\|_{L^p[t_1,t_2]}}{\Gamma(\alpha)[(\alpha-1)q+1]^{1/q}}(t_2-t_1)^{(\alpha-1)+1/q} \\
&\quad \leqslant \frac{2\|f\|_{L^p[0,T]}}{\Gamma(\alpha)[(\alpha-1)q+1]^{1/q}}(t_2-t_1)^{\alpha-1/p}, \quad \text{where } \frac{1}{p}+\frac{1}{q}=1.
\end{aligned} \tag{5.4}$$

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By (5.3) and (5.4),

$$\begin{aligned}|u_k(t_1) - u_k(t_2)| &\leqslant \frac{2(t_2-t_1)^{\alpha-1/p}}{\Gamma(\alpha)[(\alpha-1)q+1]^{1/q}}\|{}_0^cD_t^\alpha u_k\|_{L^p[0,T]}\\ &\leqslant \frac{2\|u_k\|_{E^{\alpha,p}}}{\Gamma(\alpha)[(\alpha-1)q+1]^{1/q}}(t_2-t_1)^{\alpha-1/p}\\ &:= C(t_2-t_1)^{\alpha-1/p}.\end{aligned}$$

By the Arzelà–Ascoli theorem, (u_k) is a relatively compact sequence in $C([0,T],\mathbb{R}^N)$. By the uniqueness of the weak limit in $C([0,T],\mathbb{R}^N)$, every uniformly convergent subsequence of (u_k) converges to u. Thus, (u_k) converges uniformly on $[0,T]$ to u. □

Following this, we shall prove Lemma 5.9; we first present some necessary results.

Lemma 5.6. *Let $u, v \in L^1([0,T],\mathbb{R}^n)$. If, for every $\phi \in C_0^\infty[0,T]$,*

$$\int_0^T (u(t),\phi'(t))\,\mathrm{d}t = -\int_0^T (v(t),\phi(t))\,\mathrm{d}t, \tag{5.5}$$

then u is the primitive of v, that is, $u(t) = \int_0^t v(s)\,\mathrm{d}s + c$ for a.e. $t \in [0,T]$, $c \in \mathbb{R}^n$.

Proof. Let us define $w \in C([0,T],\mathbb{R}^n)$ by $w(t) = \int_0^t v(s)\,\mathrm{d}s$ so that

$$\int_0^T (w(t),\phi'(t))\,\mathrm{d}t = \int_0^T \left(\int_0^t v(s)\,\mathrm{d}s, \phi'(t)\right)\mathrm{d}t.$$

By Fubini's theorem, (5.5), we obtain

$$\begin{aligned}\int_0^T (w(t),\phi'(t))\,\mathrm{d}t &= \int_0^T \left[\int_s^T (v(s),\phi'(t))\,\mathrm{d}t\right]\mathrm{d}s\\ &= -\int_0^T (v(s),\phi(s))\,\mathrm{d}s\\ &= \int_0^T (u(s),\phi'(s))\,\mathrm{d}s.\end{aligned}$$

So

$$\int_0^T (u(s)-w(s),\phi'(s))\,\mathrm{d}s = 0. \tag{5.6}$$

By the fundamental lemma of the calculus of variation, one has $u(t) - w(t) = c$, where $c \in \mathbb{R}$. The proof is complete. □

Lemma 5.7. *Consider the problem*

$$\left.\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \tfrac{1}{2}{}_tD_T^{-\beta}(u'(t))) = h(t) \quad \text{a.e. } t \in [0,T],\\ au(0) - b(\tfrac{1}{2}{}_0D_t^{-\beta}u'(0) + \tfrac{1}{2}{}_tD_T^{-\beta}u'(0)) = 0,\\ cu(T) + d(\tfrac{1}{2}{}_0D_t^{-\beta}u'(T) + \tfrac{1}{2}{}_tD_T^{-\beta}u'(T)) = 0,\end{aligned}\right\} \tag{5.7}$$

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where $h \in L^2([0,T])$. Problem (5.7) has a unique solution $\bar{u} \in E^\alpha$ such that $({}_0D_t^{-\beta}(\bar{u}'))'$, $({}_tD_T^{-\beta}(\bar{u}'))'$ are almost everywhere continuous and $\bar{u}$ satisfies (5.7). Furthermore, $\bar{u}$ is obtained via

$$\min_{v \in E^\alpha} \biggl\{ -\int_0^T ({}_0^cD_t^\alpha v, {}_t^cD_T^\alpha v)\,\mathrm{d}t + \frac{c}{d}v^2(T) + \frac{a}{b}v^2(0) \biggr\}.$$

Proof. If u is a classical solution of (5.7), then, by integration by parts, Property 3.1 and Property 3.2, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^T ({}_0D_t^{-\beta}u' + {}_tD_T^{-\beta}u')'v\,\mathrm{d}t + \int_0^T h(t)v(t)\,\mathrm{d}t \\
&\quad = -\frac{c}{d}u(T)v(T) - \frac{a}{b}u(0)v(0) - \frac{1}{2}\int_0^T ({}_0D_t^{-\beta/2}u', {}_tD_T^{-\beta/2}v') + ({}_tD_T^{-\beta/2}u', {}_0D_t^{-\beta/2}v')\,\mathrm{d}t \\
&\qquad + \int_0^T h(t)v(t)\,\mathrm{d}t \\
&\quad = -\frac{c}{d}u(T)v(T) - \frac{a}{b}u(0)v(0) + \frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u, {}_t^cD_T^\alpha v) + ({}_t^cD_T^\alpha u, {}_0^cD_t^\alpha v)\,\mathrm{d}t \\
&\qquad + \int_0^T h(t)v(t)\,\mathrm{d}t
\end{aligned}$$

for all $v \in E^\alpha$.

Let

$$a(u,v) = -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u, {}_t^cD_T^\alpha v) + ({}_t^cD_T^\alpha u, {}_0^cD_t^\alpha v)\,\mathrm{d}t + \frac{c}{d}u(T)v(T) + \frac{a}{b}u(0)v(0).$$

Clearly, $a(u,v)$ is a continuous coercive bilinear form on E^α. We apply the Lax–Milgram theorem [**5**, Corollary 5.8] with the bilinear form $a(u,v)$ and the linear functional $\varphi\colon v \mapsto \int_0^T h(t)v(t)\,\mathrm{d}t$. We obtain that there exists a unique element $\bar{u} \in E^\alpha$ such that

$$a(\bar{u}, v) = \int_0^T h(t)v(t)\,\mathrm{d}t \tag{5.8}$$

for all $v \in E^\alpha$. Moreover, $\bar{u}$ is obtained by $\min_{v\in E^\alpha}\{\frac{1}{2}a(v,v) - \int_0^T h(t)v(t)\,\mathrm{d}t\}$. So

$$\begin{aligned}
0 &= -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha \bar{u}, {}_t^cD_T^\alpha v) + ({}_t^cD_T^\alpha \bar{u}, {}_0^cD_t^\alpha v)\,\mathrm{d}t + \frac{c}{d}\bar{u}(T)v(T) + \frac{a}{b}\bar{u}(0)v(0) \\
&\qquad - \int_0^T h(t)v(t)\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^T ({}_0D_t^{-\beta/2}\bar{u}', {}_tD_T^{-\beta/2}v') + ({}_tD_T^{-\beta/2}\bar{u}', {}_0D_t^{-\beta/2}v')\,\mathrm{d}t + \frac{c}{d}\bar{u}(T)v(T) \\
&\qquad + \frac{a}{b}\bar{u}(0)v(0) - \int_0^T h(t)v(t)\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^T ({}_0D_t^{-\beta}\bar{u}' + {}_tD_T^{-\beta}\bar{u}', v')\,\mathrm{d}t + \frac{c}{d}\bar{u}(T)v(T) + \frac{a}{b}\bar{u}(0)v(0) - \int_0^T h(t)v(t)\,\mathrm{d}t \qquad (5.9)
\end{aligned}$$

holds for all $v \in E^\alpha$. Without loss of generality, (5.9) holds for all $v \in C_0^\infty \subset E^\alpha$, where $C_0^\infty = \{u \in C^\infty[0,T]\colon u(0) = u(T) = 0\}$. Then (5.9) becomes

$$0 = \frac{1}{2}\int_0^T ({}_0D_t^{-\beta}\bar{u}' + {}_tD_T^{-\beta}\bar{u}', v')\,\mathrm{d}t - \int_0^T h(t)v(t)\,\mathrm{d}t$$

for all $v \in C_0^\infty$. By Lemma 5.6,

$$\tfrac{1}{2}({}_0D_t^{-\beta}\bar{u}' + {}_tD_T^{-\beta}\bar{u}') = -\int_0^t h(s)\,\mathrm{d}s + c \quad \text{a.e. } t \in [0,T],\ c \in \mathbb{R}.$$

So

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}({}_0D_t^{-\beta}\bar{u}' + {}_tD_T^{-\beta}\bar{u}') + h(t) = 0, \quad t \in [0,T]. \tag{5.10}$$

Since h is almost everywhere continuous on $[0,T]$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}({}_0D_t^{-\beta}\bar{u}' + {}_tD_T^{-\beta}\bar{u}')$$

is almost everywhere continuous on $[0,T]$. Substituting (5.10) into (5.9), we have, by integration by parts,

$$\begin{aligned}
0 =& \frac{1}{2}\int_0^T ({}_0D_t^{-\beta}\bar{u}' + {}_tD_T^{-\beta}\bar{u}', v')\,\mathrm{d}t + \frac{c}{d}\bar{u}(T)v(T) + \frac{a}{b}\bar{u}(0)v(0) \\
&+ \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}{}_0D_t^{-\beta}\bar{u}' + \tfrac{1}{2}{}_tD_T^{-\beta}\bar{u}'), v\right)\mathrm{d}t \\
=& \frac{c}{d}\bar{u}(T)v(T) + \frac{a}{b}\bar{u}(0)v(0) + \tfrac{1}{2}({}_0D_t^{-\beta}\bar{u}' + {}_tD_T^{-\beta}\bar{u}', v)|_0^T \\
=& \frac{c}{d}\bar{u}(T)v(T) + \frac{a}{b}\bar{u}(0)v(0) + \tfrac{1}{2}{}_0D_t^{-\beta}\bar{u}'(T)v(T) + \tfrac{1}{2}{}_tD_T^{-\beta}\bar{u}'(T)v(T) \\
&- \tfrac{1}{2}{}_0D_t^{-\beta}\bar{u}'(0)v(0) - \tfrac{1}{2}{}_tD_T^{-\beta}\bar{u}'(0)v(0) \\
=& \left[\frac{c}{d}\bar{u}(T) + \tfrac{1}{2}{}_0D_t^{-\beta}\bar{u}'(T) + \tfrac{1}{2}{}_tD_T^{-\beta}\bar{u}'(T)\right]v(T) \\
&+ \left[\frac{a}{b}\bar{u}(0) - \tfrac{1}{2}{}_0D_t^{-\beta}\bar{u}'(0) - \tfrac{1}{2}{}_tD_T^{-\beta}\bar{u}'(0)\right]v(0)
\end{aligned} \tag{5.11}$$

for all $v \in E^\alpha$. Let $v(0) = 0$, $v(T) \neq 0$. Then

$$\frac{c}{d}\bar{u}(T) + \tfrac{1}{2}{}_0D_t^{-\beta}\bar{u}'(T) + \tfrac{1}{2}{}_tD_T^{-\beta}\bar{u}'(T) = 0.$$

Similarly, we have

$$\frac{a}{b}\bar{u}(0) - \tfrac{1}{2}{}_0D_t^{-\beta}\bar{u}'(0) - \tfrac{1}{2}{}_tD_T^{-\beta}\bar{u}'(0) = 0,$$

i.e. the boundary conditions hold.

It is clear from (5.8) that ${}_0D_t^{-\beta}\bar{u}' \in L^2[0,T]$. Hence, we have ${}_tD_T^{-\beta}\bar{u}' \in L^2[0,T]$. If $h \in C[0,T]$, then ${}_0D_t^{-\beta}u' \in C[0,T]$, $({}_0D_t^{-\beta}u')' \in C[0,T]$. □

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Lemma 5.8. *Let $u \in E^{\alpha}$. If $a(u,v) = 0$ for all $v \in L^2[0,T]$, then $u = 0$ for $t \in [0,T]$.*

Proof. If $a(u,v) = 0$ for all $v \in L^2[0,T]$, then, without loss of generality, $a(u,u) = 0$. Since $a(u,u) = \|u\|^2$, we have $u(t) = 0$ for $t \in [0,T]$. □

Lemma 5.9. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded and almost everywhere continuous function. Assume that*

(i) *for a.e. $t \in [0,T]$, for each $u \in D_f$, $f^-(u) \leqslant 0 \leqslant f^+(u)$ implies that $f(u) = 0$.*

If $u \in E^{\alpha}$ is a generalized critical point of I_{λ}, then u is a generalized solution of BVP (1.1).

Proof. Let $u_0 \in E^{\alpha}$ be a generalized critical point of I_{λ}, i.e. $I_{\lambda}^0(u_0,v) \geqslant 0$ for all $v \in E^{\alpha}$. From this we obtain

$$\Phi'(u_0)(v) + \lambda(-\Upsilon)^0(u_0;v) \geqslant 0$$

for all $v \in E^{\alpha}$. That is,

$$\frac{1}{2}\int_0^T ({}_0^cD_t^{\alpha}u_0, {}_t^cD_T^{\alpha}v) + ({}_t^cD_T^{\alpha}u_0, {}_0^cD_t^{\alpha}v)\,\mathrm{d}t - \frac{c}{d}u_0(T)v(T) - \frac{a}{b}u_0(0)v(0) \leqslant \lambda(-\Upsilon)^0(u_0;v) \tag{5.12}$$

for all $v \in E^{\alpha}$. Clearly, setting

$$L_{u_0}(v) := \frac{1}{2}\int_0^T ({}_0^cD_t^{\alpha}u_0, {}_t^cD_T^{\alpha}v) + ({}_t^cD_T^{\alpha}u_0, {}_0^cD_t^{\alpha}v)\,\mathrm{d}t - \frac{c}{d}u_0(T)v(T) - \frac{a}{b}u_0(0)v(0)$$

for all $v \in E^{\alpha}$, L_{u_0} is a continuous and linear functional on E^{α}; from which, (5.12) implies that $L_{u_0} \in \lambda\partial(-\Upsilon)|_{E^{\alpha}}(u_0)$. Now, since E^{α} is dense in $L^2[0,T]$, from [**6**, Theorem 2.2] one has $\partial(-\Upsilon)|_{E^{\alpha}}(u_0) \subseteq \partial(-\Upsilon)|_{L^2[0,T]}(u_0)$. So $L_{u_0} \in \lambda\partial(-\Upsilon)|_{L^2[0,T]}(u_0)$, and then L is continuous and linear on $L^2[0,T]$. Therefore, there is an $\bar{h} \in L^2[0,T]$ satisfying $L_{u_0}(v) = \int_0^T \bar{h}(x)v(x)\,\mathrm{d}x$ for all $v \in L^2[0,T]$. From Lemma 5.7, there is a unique $\bar{u} \in E^{\alpha}$ satisfying that $({}_0D_t^{-\beta}(\bar{u}'))'$, $({}_tD_T^{-\beta}(\bar{u}'))'$ are almost everywhere continuous such that (5.7) holds. In particular,

$$\int_0^T \bar{h}(x)v(x)\,\mathrm{d}x = \frac{1}{2}\int_0^T ({}_0^cD_t^{\alpha}\bar{u}, {}_t^cD_T^{\alpha}v) + ({}_t^cD_T^{\alpha}\bar{u}, {}_0^cD_t^{\alpha}v)\,\mathrm{d}x - \frac{c}{d}\bar{u}(T)v(T) - \frac{a}{b}\bar{u}(0)v(0)$$

for all $v \in E^{\alpha}$. Hence,

$$\begin{aligned}\frac{1}{2}\int_0^T ({}_0^cD_t^{\alpha}u_0, {}_t^cD_T^{\alpha}v) + ({}_t^cD_T^{\alpha}u_0, {}_0^cD_t^{\alpha}v)\,\mathrm{d}x - \frac{c}{d}u_0(T)v(T) - \frac{a}{b}u_0(0)v(0)&\\ = L_{u_0}(v) = \int_0^T \bar{h}(x)v(x)\,\mathrm{d}x,&\end{aligned}$$

which means that $a(u_0 - \bar{u}, v) = 0$ for all $v \in L^2[0,T]$. Lemma 5.8 implies that $u_0 = \bar{u}$. So $({}_0D_t^{-\beta}u_0')' \in C[0,T]$, $({}_tD_T^{-\beta}u_0')' \in C[0,T]$ and

$$\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u_0, {}_t^cD_T^\alpha v) + ({}_t^cD_T^\alpha u_0, {}_0^cD_t^\alpha v)\,\mathrm{d}t - \frac{c}{d}u_0(T)v(T) - \frac{a}{b}u_0(0)v(0) = \int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}{}_0D_t^{-\beta}u_0' + \tfrac{1}{2}{}_tD_T^{-\beta}u_0'), v(t)\right)\mathrm{d}t$$

holds for all $v \in E^\alpha$. So

$$\int_0^T \left(\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}{}_0D_t^{-\beta}u_0' + \tfrac{1}{2}{}_tD_T^{-\beta}u_0'), v(t)\right)\mathrm{d}t \leqslant \lambda(-\Upsilon)^0(u_0; v)$$

for all $v \in E^\alpha$. Hence, [**6**, Corollary, p. 111] ensures that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}{}_0D_t^{-\beta}u_0' + \tfrac{1}{2}{}_tD_T^{-\beta}u_0') \in [(-\lambda f)^-(u_0(x)), (-\lambda f)^+(u_0(x))] \tag{5.13}$$

for a.e. $t \in [0,T]$. Since $m(D_f) = 0$, we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}{}_0D_t^{-\beta}u_0' + \tfrac{1}{2}{}_tD_T^{-\beta}u_0') = 0 \quad \text{for a.e. } t \in u_0^{-1}(D_f).$$

From Lemma 5.9 (i), $\lambda f(u_0(t)) = 0$ for a.e. $t \in u_0^{-1}(D_f)$. So

$$-\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}{}_0D_t^{-\beta}u_0' + \tfrac{1}{2}{}_tD_T^{-\beta}u_0') = \lambda f(u_0(t)) \quad \text{for a.e. } t \in u_0^{-1}(D_f).$$

On the other hand, for a.e. $t \in [0,T] \setminus u_0^{-1}(D_f)$, (5.13) reduces to

$$-\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}{}_0D_t^{-\beta}u_0'(t) + \tfrac{1}{2}{}_tD_T^{-\beta}u_0'(t)) = \lambda f(u_0(t)).$$

Hence, our claim is proved and the assertion follows. □

6. Existence results for $bd \neq 0$

Now put

$$A = \liminf_{\xi\to+\infty} \frac{\max_{|t|\leqslant\xi} F(t)}{\xi^2}, \qquad B = \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2},$$

$$M_4 := \frac{1}{(\Gamma(1-\alpha))^2}\frac{T^{1-2\alpha}}{2^{3-4\alpha}(1-\alpha)^2} + \frac{c}{d}, \qquad \lambda_1 = \frac{M_4}{BT}, \qquad \lambda_2 = \frac{1}{2M_3^2TA},$$

where M_3 is defined in Lemma 5.4.

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Theorem 6.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded and almost everywhere continuous function. Put $F(\xi) := \int_0^\xi f(s)\,\mathrm{d}s$ for every $\xi \in \mathbb{R}$, assume that Lemma 5.9 (i) holds and also that*

(ii) $\displaystyle\int_0^\xi F(t)\,\mathrm{d}t \geqslant 0$ *for every* $\xi \geqslant 0$;

(iii) $\displaystyle\liminf_{\xi\to+\infty} \frac{\max_{|t|\leqslant\xi} F(t)}{\xi^2} < \kappa \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2}$, *where* $\kappa = \dfrac{1}{2M_3^2 M_4}$.

Then for each $\lambda \in\,]\lambda_1, \lambda_2[$, the problem (1.1) has a sequence of generalized solutions that is unbounded in E^α.

Proof. We shall apply Theorem 2.4 to prove the theorem. By (iii), there exists a real sequence (c_n) satisfying $\lim_{n\to\infty} c_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\max_{|t|\leqslant c_n} F(t)}{c_n^2} = A < +\infty. \tag{6.1}$$

Put $r_n = \frac{1}{2}(c_n^2/M_3^2)$ for all $n \in \mathbb{N}$. By Lemma 5.4, for all $v \in E^\alpha$ satisfying $\|v\|^2 \leqslant 2r_n$, we have $\|v\|_\infty \leqslant M_3\|v\| \leqslant M_3\sqrt{2r_n} = c_n$. So

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(]-\infty,r_n[)} \frac{(\sup_{u\in\Phi^{-1}(]-\infty,r_n[)} \Psi(u)) - \Psi(u)}{r_n - \Phi(u)} \\
&\leqslant \frac{\sup_{\|u\|^2<2r_n} \Psi(u) - \Psi(0)}{r_n} \\
&= \frac{\sup_{\|v\|^2<2r_n} \int_0^T F(v(x))\,\mathrm{d}x}{r_n} \\
&\leqslant \frac{T\max_{|t|\leqslant c_n} F(t)}{r_n} \\
&= 2\frac{M_3^2 T}{c_n^2} \max_{|t|\leqslant c_n} F(t).
\end{aligned}$$

By (6.1), $\varphi(r_n) \leqslant 2M_3^2TA$. So

$$\gamma := \liminf_{r\to+\infty} \varphi(r) \leqslant \liminf_{n\to\infty} \varphi(r_n) \leqslant 2M_3^2TA < +\infty.$$

Now we claim that I_λ is unbounded from below for

$$\lambda \in (\lambda_1, \lambda_2) = \left(\frac{M_4}{BT}, \frac{1}{2M_3^2TA}\right) \subset \left(0, \frac{1}{\gamma}\right).$$

By (iii), let (d_n) be a real sequence satisfying $\lim_{n\to\infty} d_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{F(d_n)}{d_n^2} = B. \tag{6.2}$$

For all $n \in \mathbb{N}$, define

$$w_n(t) = \begin{cases} \dfrac{2d_n t}{T}, & t \in [0, \frac{1}{2}T], \\ d_n, & t \in [\frac{1}{2}T, T]. \end{cases}$$

Clearly, $w_n \in X$ and

$$\begin{aligned} \|w_n\|^2 &= -\int_0^T ({}_0^c D_t^\alpha w_n, {}_t^c D_T^\alpha w_n)\,\mathrm{d}t + \frac{c}{d}(w_n(T))^2 + \frac{a}{b}(w_n(0))^2 \\ &= \int_0^T \left(\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} w_n'(s)\,\mathrm{d}s, \frac{1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} w_n'(s)\,\mathrm{d}s \right) \mathrm{d}t \\ &\quad + \frac{c}{d} d_n^2 \\ &= \frac{1}{(\Gamma(1-\alpha))^2} \left\{ \int_0^{T/2} \left(\int_0^t (t-s)^{-\alpha} \frac{2d_n}{T}\,\mathrm{d}s, \int_t^{T/2} (s-t)^{-\alpha} \frac{2d_n}{T}\,\mathrm{d}s \right) \mathrm{d}t \right\} + \frac{c}{d} d_n^2 \\ &\leqslant \frac{1}{(\Gamma(1-\alpha))^2} \frac{d_n^2 T^{1-2\alpha}}{2^{3-4\alpha}(1-\alpha)^2} + \frac{c}{d} d_n^2 \\ &:= M_4 d_n^2. \end{aligned}$$

Therefore,

$$\Phi(w_n) - \lambda \Psi(w_n) = \frac{\|w_n\|^2}{2} - \lambda \int_0^T F(w_n(t))\,\mathrm{d}t \leqslant \frac{M_4 d_n^2}{2} - \lambda \int_0^T F(w_n(t))\,\mathrm{d}t.$$

By (ii), we have

$$\int_0^T F(w_n(t))\,\mathrm{d}t \geqslant \int_{T/2}^T F(d_n)\,\mathrm{d}t = F(d_n)(\tfrac{1}{2}T).$$

Therefore,

$$\Phi(w_n) - \lambda \Psi(w_n) \leqslant \frac{M_4}{2} d_n^2 - \frac{\lambda T}{2} F(d_n)$$

for all $n \in \mathbb{N}$.

Now, if $B < +\infty$, by (6.2) for any $\varepsilon \leqslant B - M_4/(T\lambda)$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$F(d_n) > (B - \varepsilon) d_n^2 \quad \text{for all } n > N_\varepsilon.$$

So

$$\Phi(w_n) - \lambda \Psi(w_n) \leqslant \frac{M_4 d_n^2}{2} - \frac{\lambda T}{2}(B - \varepsilon) d_n^2 = \frac{d_n^2}{2}(M_4 - \lambda T(B - \varepsilon)) \to -\infty$$

as $n \to +\infty$.

If $B = +\infty$, fix $M_5 > M_4/(\lambda T)$, and from (6.2) there exists $N_M \in \mathbb{N}$ such that $F(d_n) > M_5 d_n^2$ for all $n > N_M$. Therefore,

$$\Phi(w_n) - \lambda \Psi(w_n) \leqslant \tfrac{1}{2} M_4 d_n^2 - \tfrac{1}{2} \lambda T M_5 d_n^2 = \tfrac{1}{2} d_n^2 (M_4 - \lambda M_5 T) \to -\infty$$

as $n \to +\infty$. Since all the assumptions of Theorem 2.1 (b) are verified, the functional I_λ admits a sequence (u_n) of generalized critical points such that $\lim_{n\to+\infty} \|u_n\| = +\infty$, that is, (u_n) is unbounded in E^α. By Lemma 5.9, (u_n) is a sequence of generalized solutions of BVP (1.1). The proof is complete. □

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Remark 6.2. The constant M_4 in the assumptions is decided by the choice of $w_n(t)$.

Corollary 6.3. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that Theorem 6.1 (ii) and (iii) hold. Then for each $\lambda \in (\lambda_1, \lambda_2)$, problem (1.1) possesses a sequence of pairwise distinct classical solutions.*

Corollary 6.4. *Let f be a locally essentially bounded and almost everywhere continuous function. Assume that (ii) and (iii) of Theorem 6.1 hold. Furthermore, assume that*

(iii$_1$) $\displaystyle\limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} > \frac{M_4}{T}$,

(iii$_2$) $\displaystyle\liminf_{\xi\to+\infty} \frac{\max_{|t|\leqslant\xi} F(t)}{\xi^2} < \frac{1}{2M_3^2 T}$.

Then, for each $\lambda \in (\lambda_1, \lambda_2)$, the problem

$$\left.\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \tfrac{1}{2}{}_tD_T^{-\beta}(u'(t))) &= \lambda f(u(t)) \quad \text{a.e. } t \in [0,T], \\ au(0) - b(\tfrac{1}{2}{}_0D_t^{-\beta}u'(0) + \tfrac{1}{2}{}_tD_T^{-\beta}u'(0)) &= 0, \\ cu(T) + d(\tfrac{1}{2}{}_0D_t^{-\beta}u'(T) + \tfrac{1}{2}{}_tD_T^{-\beta}u'(T)) &= 0 \end{aligned}\right\} \tag{6.3}$$

possesses a sequence of generalized solutions that is unbounded in E^α.

Let

$$A^* = \liminf_{\xi\to0^+} \frac{\max_{|t|\leqslant\xi} F(t)}{\xi^2}, \qquad B^* = \limsup_{\xi\to0^+} \frac{F(\xi)}{\xi^2},$$
$$\lambda_1^* = \frac{M_4}{B^* T}, \qquad \lambda_2^* = \frac{1}{2M_3^2 T A^*}.$$

Theorem 6.5. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded and almost everywhere continuous function. Assume that Theorem 6.1 (ii) holds and that*

(iii$'$) $\displaystyle\liminf_{\xi\to0^+} \frac{\max_{|t|\leqslant\xi} F(t)}{\xi^2} < \kappa \limsup_{\xi\to0^+} \frac{F(\xi)}{\xi^2}$, *where* $\kappa = \dfrac{1}{2M_3^2 M_4}$;

(i$'$) *for a.e. $t \in [0,T]$, for each $u \in D_f$, $f^-(u) \leqslant 0 \leqslant f^+(u)$ implies that $f(u) = 0$.*

Then for each $\lambda \in (\lambda_1^*, \lambda_2^*)$, problem (1.1) has a sequence of generalized solutions that is unbounded in E^α.

Proof. Apply Theorem 2.4 (c), similar to the proof of Theorem 6.1, and we have the result. □

Example 6.6. Put

$$a_n = \frac{2n!(n+2)! - 1}{4(n+1)!}, \qquad b_n = \frac{2n!(n+2)! + 1}{4(n+1)!}$$

for $n \in \mathbb{N}$ and define the non-negative discontinuous function $f\colon \mathbb{R} \to \mathbb{R}$ as

$$f(\xi) = \begin{cases} 2(n+1)![(n+1)!^2 - n!^2], & \xi \in (a_n, b_n), \\ 0, & \xi \notin \bigcup_{n\in\mathbb{N}}(a_n, b_n). \end{cases}$$

Consider the problem

$$\left.\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \tfrac{1}{2}{}_tD_T^{-\beta}(u'(t))) &= \lambda f(u(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) - (\tfrac{1}{2}{}_0D_t^{-\beta}u'(0) + \tfrac{1}{2}{}_tD_T^{-\beta}u'(0)) &= 0, \\ u(T) + (\tfrac{1}{2}{}_0D_t^{-\beta}u'(T) + \tfrac{1}{2}{}_tD_T^{-\beta}u'(T)) &= 0, \end{aligned}\right\} \tag{6.4}$$

where $\beta = \frac{1}{4}$, $T = 1$. Contrast with (1.1) for $\alpha = \beta = \gamma = \sigma = 1$.

We have $\int_{a_n}^{b_n} f(\xi)\,\mathrm{d}\xi = (n+1)!^2 - n!^2$. Then

$$\lim_{n\to+\infty} \frac{F(a_n)}{a_n^2} = 0, \qquad \lim_{n\to+\infty} \frac{F(b_n)}{b_n^2} = 4.$$

So

$$\liminf_{\xi\to+\infty} \frac{\max_{|t|\leqslant\xi} F(t)}{\xi^2} = \liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 0 \quad \text{and} \quad \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 4.$$

Hence, Theorem 6.1 (iii) is satisfied. Clearly, the conditions of Theorem 6.1 are satisfied. By computation, $\lambda_1 > 1/2$, $\lambda_2 = +\infty$. So for $\lambda \in (1/2, +\infty)$, problem (6.4) has a sequence of generalized solutions that is unbounded in E^α.

7. Main results for $bd = 0$

If $b = 0$, $d \neq 0$, then problem (1.1) is reduced to

$$\left.\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \tfrac{1}{2}{}_tD_T^{-\beta}(u'(t))) &= \lambda f(u(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) = 0, \quad cu(T) + d(\tfrac{1}{2}{}_0D_t^{-\beta}u'(T) + \tfrac{1}{2}{}_tD_T^{-\beta}u'(T)) &= 0. \end{aligned}\right\} \tag{7.1}$$

Define the fractional derivative space $X_1 = \{u \in E^\alpha\colon u(0) = 0\}$ with the norm

$$\|u\|_{X_1} = -\int_0^T ({}_0^cD_t^\alpha u, {}_t^cD_T^\alpha u)\,\mathrm{d}t + \frac{c}{d}(u(T))^2.$$

We claim that the norm $\|\cdot\|_{X_1}$ is equivalent to $\|\cdot\|_{\alpha,2}$. Our claim can be proved analogously to the proof of Lemma 4.6 by using the estimates in formulas (4.12)–(4.17). It is clear that X_1 is a reflexive Banach space.

Define $\Phi_1\colon X \to \mathbb{R}$ by

$$\Phi_1(u) = -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u, {}_t^cD_T^\alpha u)\,\mathrm{d}t + \frac{c}{2d}(u(T))^2 = \tfrac{1}{2}\|u\|_{X_1}^2.$$

Let

$$M_6 = \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2\alpha-1)^{1/2}\sqrt{|\cos\pi\alpha|}}.$$

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Remark 7.1. With M_3 replaced by M_6, Theorem 6.1, Corollaries 6.3 and 6.4, and Theorem 6.5 are applicable to problem (7.1).

If $b \neq 0$, $d = 0$, then problem (1.1) is reduced to

$$\left.\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \tfrac{1}{2}{}_tD_T^{-\beta}(u'(t))) = \lambda f(u(t)) \quad \text{a.e. } t \in [0,T], \\ au(0) - b(\tfrac{1}{2}{}_0D_t^{-\beta}u'(0) + \tfrac{1}{2}{}_tD_T^{-\beta}u'(0)) = 0, \quad u(T) = 0. \end{aligned}\right\} \tag{7.2}$$

Define the fractional derivative space $X_2 = \{u \in E^\alpha : u(T) = 0\}$ with the norm

$$\|u\|_{X_2} = \biggl(-\int_0^T ({}_0^cD_t^\alpha u, {}_t^cD_T^\alpha u)\,\mathrm{d}t + \frac{a}{b}(u(0))^2\biggr)^{1/2}.$$

Define $\Phi_2 \colon X \to \mathbb{R}$ by

$$\Phi_2(u) = -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u, {}_t^cD_T^\alpha u)\,\mathrm{d}t + \frac{a}{2b}(u(0))^2 = \tfrac{1}{2}\|u\|_{X_2}^2.$$

Let

$$w_n(t) = \begin{cases} d_n, & t \in [0, \tfrac{1}{2}T], \\ -\dfrac{2d_nt}{T} + 2d_n, & t \in [\tfrac{1}{2}T, T]. \end{cases}$$

Let

$$M_7 = \biggl(\frac{d_n}{T(1-\alpha)\Gamma(1-\alpha)}\biggr)^2 \frac{T^{3-2\alpha}}{2^{3-4\alpha}} + \frac{a}{b}.$$

Remark 7.2. With M_1 replaced by $1/\sqrt{|\cos\pi\alpha|}$, and M_4 replaced by M_7, Theorem 6.1, Corollaries 6.3 and 6.4, and Theorem 6.5 are applicable to problem (7.2).

If $b = 0$, $d = 0$, then problem (1.1) is reduced to the Dirichlet boundary-value problem

$$\left.\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \tfrac{1}{2}{}_tD_T^{-\beta}(u'(t))) = \lambda f(u(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) = 0, \quad u(T) = 0. \end{aligned}\right\} \tag{7.3}$$

Define the fractional derivative space $X_3 = \{u \in E^\alpha : u(0) = 0, u(T) = 0\}$ with the norm

$$\|u\|_{X_3} = -\int_0^T ({}_0^cD_t^\alpha u, {}_t^cD_T^\alpha u)\,\mathrm{d}t.$$

Define $\Phi_3 \colon X \to \mathbb{R}$ by

$$\Phi_3(u) = -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u, {}_t^cD_T^\alpha u)\,\mathrm{d}t = \tfrac{1}{2}\|u\|_{X_3}^2.$$

Let

$$w_n(t) = \begin{cases} \dfrac{2d_nt}{T}, & t \in [0, \tfrac{1}{2}T], \\ -\dfrac{2d_nt}{T} + 2d_n, & t \in [\tfrac{1}{2}T, T]. \end{cases}$$

Let

$$M_8 = \left(\frac{2}{T(1-\alpha)\Gamma(1-\alpha)}\right)^2 \left(\frac{T^{3-2\alpha}}{2^{2-2\alpha}} + \frac{T^{3-2\alpha}}{2^{1-\alpha}}\right).$$

Remark 7.3. With M_4 replaced by M_8, and M_1 replaced by $1/\sqrt{|\cos \pi\alpha|}$, Theorem 6.1, Corollaries 6.3 and 6.4, and Theorem 6.5 are applicable to problem (7.3).

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References

1. I. Area, J. Losada and J. J. Nieto, On fractional derivatives and primitives of periodic functions, *Abstr. Appl. Analysis* **2014** (2014), 392598.
2. M. Belmekki, J. J. Nieto and R. Rodriguez-Lopez, Existence of periodic solution for a nonlinear fractional differential equation, *Bound. Value Probl.* **2009** (2009), 324561.
3. M. Benchohra, S. Hamani and S. K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, *Nonlin. Analysis TMA* **71** (2009), 2391–2396.
4. G. Bonanno and G. M. Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, *Bound. Value Probl.* **2009** (2009), 670675.
5. H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations* (Springer, 2011).
6. K. C. Chang, Variational methods for nondifferentiable functionals and their applications to partial differential equations, *J. Math. Analysis Applic.* **80**(1) (1981), 102–129.
7. Y. K. Chang and J. J. Nieto, Some new existence results for fractional differential inclusions with boundary conditions, *Math. Comput. Modelling* **49** (2009), 605–609.
8. F. H. Clarke, *Optimization and nonsmooth analysis* (Wiley, 1983).
9. J. Cresson, Inverse problem of fractional calculus of variations for partial differential equations, *Commun. Nonlin. Sci. Numer. Simulation* **15** (2010), 987–996.
10. V. J. Ervin and J. P. Roop, Variational formulation for the stationary fractional advection dispersion equation, *Numer. Meth. PDEs* **22** (2006), 58–76.
11. R. Hilfer, *Applications of fractional calculus in physics* (World Scientific, 2000).
12. F. Jiao and Y. Zhou, Existence of solutions for a class of fractional boundary value problems via critical point theory, *Comput. Math. Appl.* **62** (2011), 1181–1199.
13. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Volume 204 (Elsevier, 2006).
14. J. W. Kirchner, X. Feng and C. Neal, Fractal streamchemistry and its implications for contaminant transport in catchments, *Nature* **403** (2000), 524–526.
15. M. Klimek, T. Odzijewicz and A. B. Malinowska, Variational methods for the fractional Sturm–Liouville problem, *J. Math. Analysis Applic.* **416** (2014), 402–426.
16. B. N. Lundstrom, M. H. Higgs, W. J. Spain and A. L. Fairhall, Fractional differentiation by neocortical pyramidal neurons, *Nat. Neurosci.* **11** (2008), 1335–1342.

17. F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, in *Fractals and fractional calculus in continuum mechanics* (ed. A. Carpinteri and F. Mainardi), pp. 291–348 (Springer, 1997).
18. K. S. Miller and B. Ross, *An introduction to the fractional calculus and differential equations* (John Wiley, 1993).
19. D. Motreanu and P. D. Panagiotopoulos, *Minimax theorems and qualitative properties of the solutions of hemivariational inequalities* (Kluwer Academic, Dordrecht, 1999).
20. A. Ouahab, Some results for fractional boundary value problem of differential inclusions, *Nonlin. Analysis TMA* **69** (2008), 3877–3896.
21. I. Podlubny, *Fractional differential equations* (Academic Press, 1999).
22. S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integral and derivatives: theory and applications* (Gordon and Breach, London/New York, 1993).
23. K. M. Teng, H. E. Jia and H. F. Zhang, Existence and multiplicity results for fractional differential inclusions with Dirichlet boundary conditions, *Appl. Math. Computat.* **220** (2013), 792–801.
24. J. R. Wang and Y. Zhou, A class of fractional evolution equations and optimal controls, *Nonlin. Analysis* **12** (2011), 262–272.
25. Y. H. Zhang and Z. B. Bai, Existence of solutions for nonlinear fractional three-point boundary value problem at resonance, *J. Appl. Math. Computat.* **36** (2011), 417–440.