J. Plasma Physics (2006), *vol.* 72, *part* 2, *pp.* 143–147.
doi:10.1017/S0022377805004204 First published online 2 December 2005 Printed in the United Kingdom

Letter to the Editor

The three-wave coupling coefficients for a cold magnetized plasma

L. STENFLO and G. BRODIN

Department of Physics, Umeå University, SE-901 87 Umeå, Sweden
(lennart.stenflo@physics.umu.se)
(gert.brodin@physics.umu.se)

(Received 29 August 2005 and accepted 19 October 2005)

Abstract. The resonant interaction between three waves in a uniform magnetized plasma is considered. Using the somewhat inaccessible result of the general theory, we deduce the explicit expressions for the coupling coefficients of a cold magnetized two-component plasma.

1. Introduction

The physics of nonlinear waves is a rapidly developing research field that has recently received increased attention (e.g. Azeem and Mirza 2005; Marklund and Shukla 2005; Mendonca et al. 2005; Onishchenko et al. 2004; Shukla 2004; Shukla and Stenflo 2005a, b; Stenflo 2004; Vladimirov and Yu 2004; Wu and Chao 2004). Although there are general formalisms to treat such phenomena, there is also a need to have access to reliable explicit expressions for specific cases. As a particular example, Brodin and Stenflo (1988; 1990) considered the resonant interaction between three magnetohydrodynamic (MHD) waves in a plasma. Starting from the standard MHD theory they thus derived the coupling coefficients. However, the textbook MHD equations are not able to correctly treat the nonlinear interaction between three Alfvén waves (cf. Shukla and Stenflo 2005b). In the present paper, we are therefore going to reconsider the general nonlinear interaction between three waves in a cold, magnetized, two-component plasma, in order to derive the *explicit* expressions for the coupling coefficients. Such expressions have previously been presented for a one-component plasma (Stenflo 1973), but, due to algebraic difficulties, never before for a two-component plasma.

2. Results

Considering the resonant interaction between three waves with frequencies ω_j ($j = 1, 2, 3$) and wavevectors $\mathbf{k}_j$, we assume that the matching conditions

$$\omega_3 = \omega_1 + \omega_2 \tag{1}$$

and

$$\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2 \tag{2}$$

are satisfied. The development of, for example, the z-components (E_{jz}) of the wave electric field amplitudes is then governed by the three coupled bilinear equations

(e.g. Stenflo 1994)

$$\frac{dE_{1z}^*}{dt} = \alpha_1 E_{2z} E_{3z}^* \tag{3a}$$

$$\frac{dE_{2z}^*}{dt} = \alpha_2 E_{1z} E_{3z}^* \tag{3b}$$

and

$$\frac{dE_{3z}}{dt} = \alpha_3 E_{2z} E_{1z} \tag{3c}$$

where the z-axis is along the external magnetic field ($B_0\hat{\mathbf{z}}$), the star denotes complex conjugate, α_j are the coupling coefficients, $d/dt = \partial/\partial t + \mathbf{v}_{gj} \cdot \nabla + \nu_j$, where $\mathbf{v}_{gj}$ is the group velocity of wave j, and ν_j accounts for the linear damping rate. The general formula for α_j for a *hot* magnetized plasma has been derived previously (e.g. Stenflo 1994; Stenflo and Larsson 1977) and it is therefore only presented in Appendix A here. Although, in principle, it covers all interaction mechanisms in uniform plasmas, it is not easy to apply it directly to, for example, Alfvén waves in a *cold* two-component plasma. However, after some straightforward, but rather lengthy evaluation of the formula in Appendix A, we can finally write α_j in the comparatively simple form

$$\alpha_{1,2} = \frac{M_{1,2}}{\partial D(\omega_{1,2}, \mathbf{k}_{1,2})/\partial \omega_{1,2}} C \tag{4a, b}$$

and

$$\alpha_3 = -\frac{M_3}{\partial D(\omega_3, \mathbf{k}_3)/\partial \omega_3} C \tag{4c}$$

where

$$\begin{aligned} C = \sum_\sigma \frac{q\omega_{\mathrm{p}}^2}{m\omega_1\omega_2\omega_3 k_{1z}k_{2z}k_{3z}} & \\ \times \bigg[\frac{\mathbf{k}_1 \cdot \mathbf{K}_1}{\omega_1} \mathbf{K}_2 \cdot \mathbf{K}_3^* + \frac{\mathbf{k}_2 \cdot \mathbf{K}_2}{\omega_2} \mathbf{K}_1 \cdot \mathbf{K}_3^* + \frac{\mathbf{k}_3 \cdot \mathbf{K}_3^*}{\omega_3} \mathbf{K}_1 \cdot \mathbf{K}_2 & \\ - \frac{i\omega_{\mathrm{c}}}{\omega_3} \left(\frac{k_{2z}}{\omega_2} - \frac{k_{1z}}{\omega_1} \right) \mathbf{K}_3^* \cdot (\mathbf{K}_1 \times \mathbf{K}_2) \bigg] & \end{aligned} \tag{5}$$

$$\begin{aligned} \mathbf{K} = -\Bigg[\mathbf{k}_\perp + i\frac{\omega_{\mathrm{c}}}{\omega}\mathbf{k} \times \hat{\mathbf{z}} + \left(\frac{\sum i(\omega_{\mathrm{c}}/\omega)\left(\omega_{\mathrm{p}}^2/(\omega^2 - \omega_{\mathrm{c}}^2)\right)}{1 - k^2c^2/\omega^2 - \sum\left(\omega_{\mathrm{p}}^2/(\omega^2 - \omega_{c}^2)\right)} \right) \left(\mathbf{k} \times \hat{\mathbf{z}} - i\frac{\omega_{\mathrm{c}}}{\omega}\mathbf{k}_\perp \right) \Bigg] & \\ \times \frac{(1 - k_\perp^2c^2/\omega^2 - \sum(\omega_{\mathrm{p}}^2/\omega^2))\omega^4}{(\omega^2 - \omega_{\mathrm{c}}^2)k_\perp^2c^2} + k_z\hat{\mathbf{z}} & \end{aligned} \tag{6}$$

$$\begin{aligned} D(\omega, \mathbf{k}) = \left(1 - \frac{k^2c^2}{\omega^2} - \sum \frac{\omega_{\mathrm{p}}^2}{\omega^2 - \omega_{\mathrm{c}}^2} \right) & \\ \times \left[\left(1 - \frac{k_z^2c^2}{\omega^2} - \sum \frac{\omega_{\mathrm{p}}^2}{\omega^2 - \omega_{\mathrm{c}}^2} \right) \left(1 - \frac{k_\perp^2c^2}{\omega^2} - \sum \frac{\omega_{\mathrm{p}}^2}{\omega^2} \right) - \frac{k_\perp^2k_z^2c^4}{\omega^4} \right] & \\ - \left(\sum \frac{\omega_{\mathrm{p}}^2\omega_{\mathrm{c}}}{\omega(\omega^2 - \omega_{\mathrm{c}}^2)} \right)^2 \left(1 - \frac{k_\perp^2c^2}{\omega^2} - \sum \frac{\omega_{\mathrm{p}}^2}{\omega^2} \right) & \end{aligned} \tag{7}$$

and

$$M_j = \left(1 - \frac{k_j^2 c^2}{\omega_j^2} - \sum \frac{\omega_p^2}{\omega_j^2 - \omega_c^2}\right)\left(1 - \frac{k_{jz}^2 c^2}{\omega_j^2} - \sum \frac{\omega_\mathrm{p}^2}{\omega_j^2 - \omega_\mathrm{c}^2}\right) - \left(\sum \frac{\omega_p^2 \omega_c}{\omega_j\left(\omega_j^2 - \omega_\mathrm{c}^2\right)}\right)^2 \tag{8}$$

where $k = (k_z^2 + k_\perp^2)^{1/2}$, $\mathbf{k}_\perp$ is the perpendicular (to $\widehat{\mathbf{z}}$) part of the wavevector, ω_p is the plasma frequency (ω_pe for the electrons and ω_pi for the ions), $\omega_\mathrm{c} = qB_0/m$ is the cyclotron frequency, q and m are the particle charge and mass, and c is the speed of light in vacuum. For notational convenience, the subscript σ denoting the various particle species has been left out in the above formulas. We stress that no approximations have to be used to derive the expressions (4)–(8), which are thus quite general for the case of three-wave interactions in a cold magnetized two-component plasma. It can also be verified that (4) agrees with the coupling coefficients for a magnetized one-component (Stenflo 1973; 1994) plasma.

Equations (3a)–(3c), with (4), significantly improve the (approximate) equations in the previous work by Brodin and Stenflo (1988) for the case when the plasma is cold. Thus, although the main emphasis in that work was on the coupling between Alfvén waves and magnetosonic waves where useful results were derived, it was also mentioned that there is no coupling between Alfvén waves in the MHD limit. The present paper shows, however, that this is not true. Thus, there is a non-zero interaction between, for example, one dispersive Alfvén pump wave (Shukla and Stenflo 2005b) and two inertial Alfvén waves characterized by

$$\omega_{1,2} \simeq \frac{k_{1,2z} V_\mathrm{A}}{1 + k_{1,2\perp}^2 \lambda_\mathrm{e}^2} \tag{9}$$

where V_A is the Alfvén velocity and $\lambda_\mathrm{e} = c/\omega_\mathrm{pe}$. In the particular case when E_z is zero for one of the waves, it is of course straightforward to use other variables, e.g. E_x instead of E_z, to derive expressions similar to those above.

3. Conclusions

In the present paper we have improved the approximate results for three-wave interactions in a MHD plasma (Brodin and Stenflo 1988) and found the *explicit* expressions for the coupling coefficients for wave interactions in a cold magnetized two-component plasma. Our coupling coefficient C can thus be used as a starting point (see, for example, Appendix B) of any estimate of the coupling strength where the interaction between any kind of waves (Alfvén waves, whistler waves, etc.) in a cold plasma has to be considered. It can also be useful in interpretations of stimulated scattering of electromagnetic waves in space plasmas (e.g. Kuo 2001; 2003; Stenflo 1999; Yukhimuk 1998). In the latter case we refer the reader to a short historical account of stimulated electromagnetic emissions in the ionosphere (Stenflo 2004).

Appendix A

When calculating the coupling coefficients, it turns out that they contain a common factor V. It is then possible to write the three coupled equations as

$$\frac{dW_{1,2}}{dt} = -2\omega_{1,2}\,\mathrm{Im}\,V \tag{A 1}$$

https://doi.org/10.1017/S0022377805004204 Published online by Cambridge University Press

and

$$\frac{dW_3}{dt} = 2\omega_3 \,\mathrm{Im}\, V \tag{A 2}$$

where $W = \varepsilon_0 \mathbf{E}^* \cdot (1/\omega)\partial(\omega^2 \boldsymbol{\varepsilon})\mathbf{E}$ is the wave energy, $\boldsymbol{\varepsilon}$ is the usual textbook dielectric tensor, and $\mathrm{Im}\, V$ stands for the imaginary part of V where (Stenflo 1994; Stenflo and Larsson 1977)

$$V = \sum_s m \int d\mathbf{v}\, \mathbf{F}_0(\mathbf{v}) \sum_{\substack{p_1+p_2=p_3 \\ p_j=0,\pm1,\pm2,\ldots}} I_1^{p_1} I_2^{p_2} I_3^{-p_3}$$
$$\times \left[\frac{\mathbf{k}_1 \cdot \mathbf{u}_{1p_1}}{\omega_{1d}} \mathbf{u}_{2p_2} \cdot \mathbf{u}_{3p_3}^* + \frac{\mathbf{k}_2 \cdot \mathbf{u}_{2p_2}}{\omega_{2d}} \mathbf{u}_{1p_1} \cdot \mathbf{u}_{3p_3}^* \right.$$
$$\left. + \frac{\mathbf{k}_3 \cdot \mathbf{u}_{3p_3}^*}{\omega_{3d}} \mathbf{u}_{1p_1} \cdot \mathbf{u}_{2p_2} - \frac{i\omega_{\mathrm{c}}}{\omega_{3d}} \left(\frac{k_{2z}}{\omega_{2d}} - \frac{k_{1z}}{\omega_{1d}} \right) \mathbf{u}_{3p_3}^* \cdot (\mathbf{u}_{1p_1} \times \mathbf{u}_{2p_2}) \right] \tag{A 3}$$

where $\omega_{jd} = \omega_j - k_{jz} v_z - p_j \omega_{\mathrm{c}}$, I_j $(= \exp(i\theta_j)) = (k_{j_x} + ik_{jy})/k_{j\perp}$, and the velocity $\mathbf{u}_{jp_j}$ satisfies

$$\omega_{jd} \mathbf{u}_{jp_j} + i\omega_{\mathrm{c}} \widehat{\mathbf{z}} \times \mathbf{u}_{jp_j} = \frac{iq}{m\omega_j} \left\{ \omega_{jd} J_{p_j} \mathbf{E}_j + \left[\left(v_z E_{jz} + \frac{p_j \omega_{\mathrm{c}}}{k_{j\perp}^2} \mathbf{k}_{j\perp} \cdot \mathbf{E}_{j\perp} \right) J_{p_j} \right.\right.$$
$$\left.\left. + \frac{i v_\perp \omega_{\mathrm{c}}}{k_{j\perp}^2} (\widehat{\mathbf{z}} \times \mathbf{k}_j) \cdot \mathbf{E}_j \frac{d}{dv_\perp} J_{p_j} \right] \mathbf{k}_j \right\} \tag{A 4}$$

where $J_{p_j} = J_{p_j}(k_{j\perp} v_\perp / \omega_{\mathrm{c}})$ denotes a Bessel function of order p_j.

Appendix B

The limit when ω is much smaller than ω_{ci} is of special interest. In that case, we approximate (6) by

$$\mathbf{K}_{\mathrm{e}} \simeq -\frac{i\omega}{\omega_{\mathrm{ce}}} \frac{(1 + k_\perp^2 \lambda_{\mathrm{e}}^2)}{k_\perp^2 \lambda_{\mathrm{e}}^2} \mathbf{k} \times \widehat{\mathbf{z}} + k_z \widehat{\mathbf{z}} \tag{B 1}$$

and

$$\mathbf{K}_{\mathrm{i}} \simeq -\frac{i\omega}{\omega_{\mathrm{ci}}} \frac{(1 + k_\perp^2 \lambda_{\mathrm{e}}^2)}{k_\perp^2 \lambda_{\mathrm{e}}^2} \left[\mathbf{k} \times \widehat{\mathbf{z}} - \frac{i\omega}{\omega_{\mathrm{ci}}} \mathbf{k}_\perp \right]. \tag{B 2}$$

We note that the ion contributions dominate the first three terms in (5), whereas the electron contributions are most important for the fourth term in (5). As a result we have a non-zero coupling coefficient C for the particular case of interaction between three Alfvén waves, in contrast to what one obtains from the over-simplified textbook MHD equations (see Brodin and Stenflo (1988), where C_{AAA} was zero).

References

Azeem, M. and Mirza, A. M. 2005 *Phys. Plasmas* **12**, 052306.

Brodin, G. and Stenflo, L. 1988 *J. Plasma Phys.* **39**, 277.

Brodin, G. and Stenflo, L. 1990 *Contr. Plasma Phys.* **30**, 413.

Kuo, S. P. 2001 *J. Plasma Phys.* **66**, 315.

Kuo, S. P. 2003 *J. Plasma Phys.* **69**, 529.

Marklund, M. and Shukla, P. K. 2005 *Rev. Mod. Phys.* (submitted).

Mendonca, J. T., Serbeto, A., Bingham, R. and Shukla, P. K. 2005 *J. Plasma Phys.* **71**, 119.

Onishchenko, O. G., Pokhotelov, O. A., Sagdeev, R. Z., Shukla, P. K. and Stenflo, L. 2004 *Nonlinear Proc. Geophys.* **11**, 241.

Shukla, P. K. (ed.) 2004 Nonlinear physics in action. *Phys. Scripta* **T113**, pp. 1–152.

Shukla, P. K. and Stenflo, L. (eds) 2005a Modern plasma science. *Phys. Scripta* **T116**, pp. 1–135.

Shukla, P. K. and Stenflo, L. 2005b *Phys. Plasmas* **12**, 084502.

Stenflo, L. 1973 *Planet. Space Sci.* **21**, 391.

Stenflo, L. 1994 *Phys. Scripta* **T50**, 15.

Stenflo, L. 1999 *J. Plasma Phys.* **61**, 129.

Stenflo, L. 2004 *Phys. Scripta* **T107**, 262.

Stenflo, L. and Larsson, J. 1977 *Plasma Physics: Nonlinear Theory and Experiments* (ed. H. Wilhelmsson). New York: Plenum, p. 152.

Vladimirov, S. V. and Yu, M. Y. 2004 *Phys. Scripta* **T113**, 32.

Wu, D. J. and Chao, J. K. 2004 *Nonlinear Proc. Geophys.* **11**, 631.

Yukhimuk, V., Voitenko, Yu., Fedun, V. and Yukhimuk, A. 1998 *J. Plasma Phys.* **60**, 485.