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Conflict-Free Colourings of Uniform Hypergraphs With Few Edges

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A colouring of the vertices of a hypergraph $\mathcal{H}$ is called *conflict-free* if each edge e of $\mathcal{H}$ contains a vertex whose colour does not repeat in e. The smallest number of colours required for such a colouring is called the conflict-free chromatic number of $\mathcal{H}$, and is denoted by $\chi_{CF}(\mathcal{H})$. Pach and Tardos proved that for an $(2r-1)$-uniform hypergraph $\mathcal{H}$ with m edges, $\chi_{CF}(\mathcal{H})$ is at most of the order of $rm^{1/r}\log m$, for fixed r and large m. They also raised the question whether a similar upper bound holds for r-uniform hypergraphs. In this paper we show that this is not necessarily the case. Furthermore, we provide lower and upper bounds on the minimum number of edges of an r-uniform simple hypergraph that is not conflict-free k-colourable.

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1. Introduction

Let $\mathcal{H}$ be a hypergraph with vertex set $V(\mathcal{H})$ and edge set $E(\mathcal{H})$. A colouring $c : V(\mathcal{H}) \to \{1, 2, 3, \ldots\}$ of $V(\mathcal{H})$ is a *proper colouring of* $\mathcal{H}$ if no edge of size at least 2 is monochromatic. The minimum number of colours required for such a colouring is called the *chromatic number* of $\mathcal{H}$, and is denoted by $\chi(\mathcal{H})$. A *rainbow colouring* of $\mathcal{H}$ is a proper colouring of $\mathcal{H}$ such that, for every edge e, the colours of all vertices of e are distinct. The minimum

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number of colours required for a rainbow colouring is called the *rainbow chromatic number* of $\mathcal{H}$, and is denoted by $\chi_R(\mathcal{H})$.

In connection with some frequency assignment problems for cellular networks, Even, Lotker, Ron and Smorodinsky [8] introduced (in a geometric setting) the following intermediate colouring. A proper colouring of $\mathcal{H}$ is *conflict-free* if, for each edge e of $\mathcal{H}$, some colour occurs on exactly one vertex of e. The minimum number of colours required for a conflict-free colouring is called the *conflict-free chromatic number* of $\mathcal{H}$, and is denoted by $\chi_{CF}(\mathcal{H})$. Since edges of size one are always conflict-free in any colouring, we may assume in the rest of the paper that all edges have size at least two. Because of applications and some interesting features, this parameter has attracted considerable attention (see, *e.g.*, [2, 3, 4, 6, 8, 9, 13, 12]). In particular, Pach and Tardos [12] discussed this notion for general hypergraphs and proved several interesting results. Clearly, $\chi(\mathcal{H}) \leqslant \chi_{CF}(\mathcal{H}) \leqslant \chi_R(\mathcal{H})$ for every $\mathcal{H}$ with equalities when $\mathcal{H}$ is an ordinary graph. However, for general hypergraphs, the behaviour of χ_{CF} may differ significantly from that of χ and of χ_R. For example, if we truncate an edge of a hypergraph, then χ cannot decrease, χ_R cannot increase, and χ_{CF} may increase, decrease, or stay the same. As yet another example we mention that if $\mathcal{H}$ is a 10^6-uniform hypergraph with 10 edges, then $\chi(\mathcal{H}) = 2$ and $\chi_{CF}(\mathcal{H})$ can be 2, 3, or 4.

Pach and Tardos [12] analysed the conflict-free colourings for graphs and hypergraphs. They proved that $\chi_{CF}(\mathcal{H}) \leqslant 1/2 + \sqrt{2m + 1/4}$ for every hypergraph with m edges, and that this bound is tight. They also showed the following result.

Theorem 1.1 ([12]). *Let r be fixed. Let $\mathcal{H}$ be a hypergraph with m edges such that the size of every edge is at least $2r - 1$. Then $\chi_{CF}(\mathcal{H}) \leqslant C_r m^{1/r} \log m$, where C_r is a positive constant depending only on r.*

In fact, they proved a stronger result. Let us define the *edge degree* of an edge e in a hypergraph $\mathcal{H}$ as the number of other edges intersecting e. The maximum edge degree $D(\mathcal{H})$ is the maximum of the edge degrees over all the edges of $\mathcal{H}$. This parameter has been used by a number of authors to bound the chromatic number (*e.g.*, [1, 5, 7, 11, 14, 15]). As for the conflict-free chromatic number, Pach and Tardos [12] showed that $\chi_{CF}(\mathcal{H}) \leqslant C_r m^{1/r} \log m$ for all hypergraphs $\mathcal{H}$ in which the size of every edge is at least $2r - 1$ and $D(\mathcal{H}) \leqslant m$. They also posed the question whether the same upper bound holds also when every edge has size at least r and intersects at most m others. In this paper we show that this is not the case.

The goal of the paper is to give reasonable upper bounds on $\chi_{CF}(\mathcal{H})$ for r-uniform hypergraphs $\mathcal{H}$ with given number of edges or maximum edge degree. It will turn out that for a given m, the nature of the bounds for r-uniform hypergraphs with m edges significantly depends on whether r is small or large with respect to m. We also derive similar bound for *simple* r-uniform hypergraphs, *i.e.*, for hypergraphs in which any two distinct edges share at most one vertex. It turns out that for positive integers r, k with $r \leqslant k/8$, both upper and lower bounds on the minimum number of edges in an r-uniform simple hypergraph that have no conflict-free colourings with k colours are roughly squares of the corresponding bounds for hypergraphs without the restriction of being simple.

For a warm-up, in Section 2 we bound the number of edges in r-uniform hypergraphs with χ_{CF} equal to 3 or 4. In particular, for arbitrarily large even r, there is an r-uniform hypergraph $\mathcal{H}$ with just 7 edges and $\chi_{CF}(\mathcal{H}) = 4$. In Section 3, we find upper bounds on $\chi_{CF}(\mathcal{H})$ in terms of the size/maximum edge degree of $\mathcal{H}$ and present some constructions showing that our bounds are reasonable. In Section 4, we do the same for simple r-uniform hypergraphs.

2. Conflict-free colouring of hypergraphs with very few edges

We define the *s-blow-up* of a graph G to be the hypergraph formed by replacing every vertex v of G with an s element set B_v. The set B_v is called a *blob*. If uv is an edge in G, then $B_u \cup B_v$ is an edge in the blow-up.

Observation 2.1. *For a hypergraph $\mathcal{H}$, if either the degree of every vertex of H is at most* 1*, or if there is a vertex contained in every edge of H, then $\chi_{CF}(\mathcal{H}) = 2$.* □

Observation 2.2. *Let $r \geqslant 2$. If $\mathcal{H}$ is an r-uniform hypergraph which is not conflict-free* 2*-colourable, then it has at least* 3 *edges and the only such graph with* 3 *edges is the* $(r/2)$*-blow-up of K_3.*

Proof. By Observation 2.1, every hypergraph with 2 edges is conflict-free 2-colourable. Moreover, a blow-up of K_3 is not. Now assume that $\mathcal{H}$ is an r-uniform hypergraph with 3 edges e_1, e_2, e_3 which is not conflict-free 2-colourable. If every vertex has degree at most 1 or if there is a vertex of degree 3, then by Observation 2.1 it is conflict-free 2-colourable. So assume that the maximum degree is 2. Without loss of generality assume that $v \in e_1 \cap e_2$. If there exists $u \in e_3 - e_1 - e_2$, then we colour v and u with colour 1 and all the remaining vertices with colour 2. This would give a conflict-free 2-colouring of $\mathcal{H}$, a contradiction. Hence $e_3 \subseteq \{e_1 - e_2\} \cup \{e_2 - e_1\}$. Since $\mathcal{H}$ is r-uniform, we have that $e_3 \nsubseteq e_1$ and $e_3 \nsubseteq e_2$. Thus, $e_1 \cap e_3 \neq \emptyset$ and the above argument holds if v is replaced by a vertex $w \in e_3$. Consequently, $e_1 \subseteq \{e_2 - e_3\} \cup \{e_3 - e_2\}$ and similarly $e_2 \subseteq \{e_1 - e_3\} \cup \{e_3 - e_1\}$. Moreover, since $\mathcal{H}$ is r-uniform, it must be the $(r/2)$-blow-up of K_3. In particular, r is even. □

Lemma 2.3. *Let $r \geqslant 3$. If $\mathcal{H}$ is an r-uniform hypergraph with at most* 6 *edges, then it is always conflict-free* 3*-colourable. Moreover, if $r \geqslant 4$ and r is divisible by* 4*, then there exists an r-uniform hypergraph with* 7 *edges which is not conflict-free* 3*-colourable.*

Proof. We first show that if $\mathcal{H}$ has at most 6 edges then $\chi_{CF}(\mathcal{H}) \leqslant 3$. Let $\Delta(\mathcal{H})$ be the maximum degree of $\mathcal{H}$.

Case 1: $\Delta(\mathcal{H}) \geqslant 4$. Let v be a vertex of degree at least 4. We colour v with colour 1. By Observation 2.2, there is a conflict-free colouring of $\mathcal{H} - v$ with colours 2 and 3. This gives a conflict-free 3-colouring of $\mathcal{H}$.

Case 2: $\Delta(\mathcal{H}) \leqslant 2$. Since $\chi_{CF}(\mathcal{G}) \leqslant \Delta(\mathcal{G}) + 1$ for every hypergraph $\mathcal{G}$, we can conflict-free 3-colour $\mathcal{H}$ (see [12]).

Case 3: $\Delta(\mathcal{H}) = 3$. Let v be a vertex of degree 3 contained in the edges e_1, e_2 and e_3. If $\mathcal{H} - \{e_1, e_2, e_3\} = \{e_4, e_5, e_6\}$ is conflict-free 2-colourable, then we colour them conflict-free with colours 2 and 3, colour v with colour 1 and arbitrarily colour the remaining vertices with colours 2 and 3. This gives a conflict-free 3-colouring of $\mathcal{H}$. If not, then by Observation 2.2, $\{e_4, e_5, e_6\}$ forms the $(r/2)$-blow-up of K_3. We may assume that $e_4 \cup e_5 \cup e_6 = B_4 \cup B_5 \cup B_6$, where B_4, B_5 and B_6 are the blobs $e_5 \cap e_6$, $e_4 \cap e_6$ and $e_4 \cap e_5$, respectively. Now, suppose that there is a vertex $u \in (e_4 \cup e_5 \cup e_6) - (e_1 \cup e_2 \cup e_3)$. Without loss of generality assume that $u \in B_6$. Let w be a vertex in B_5. We now colour v and u with colour 1, w with colour 2 and the rest of the vertices with colour 3. This gives a conflict-free 3-colouring of $\mathcal{H}$. Hence $\{e_4 \cup e_5 \cup e_6\} \subseteq \{e_1 \cup e_2 \cup e_3\}$. Thus every vertex in $\{e_4 \cup e_5 \cup e_6\}$ has degree 3.

The above argument holds for each vertex $u \in e_4 \cup e_5 \cup e_6$ by replacing v with u and e_1, e_2, e_3 with the three edges containing u. Hence, by symmetry, the degree of every vertex of $\mathcal{H}$ is 3. We also know that deleting any vertex leaves a copy of the $(r/2)$-blow-up of K_3. Moreover, since $\mathcal{H}$ is r-uniform, $\mathcal{H}$ must be the $(r/2)$-blow-up of K_4. A blow-up of K_4 can be conflict-free 3-coloured as follows. In the first blob we colour a vertex with colour 1 and another with colour 2 and the rest with 3. In the second blob we colour one vertex with 2 and the rest with 3. In the third blob we colour one vertex with 1 and the rest with 3 and in the fourth blob we colour everything with colour 3.

Now, to show that there exists a hypergraph $\mathcal{H}$ with 7 edges which is not 3-conflict-free colourable, we consider the $(r/4)$-blow-up of the Fano plane and take the complement of every edge. The resulting hypergraph $\mathcal{H}$ has seven blobs $B_1, B_2, \ldots, B_7$ and the following edges: $e_1 = B_1 \cup B_2 \cup B_6 \cup B_7$, $e_2 = B_2 \cup B_3 \cup B_4 \cup B_7$, $e_3 = B_4 \cup B_5 \cup B_6 \cup B_7$, $e_4 = B_1 \cup B_2 \cup B_4 \cup B_5$, $e_5 = B_1 \cup B_3 \cup B_4 \cup B_6$, $e_6 = B_2 \cup B_3 \cup B_5 \cup B_6$, and $e_7 = B_1 \cup B_3 \cup B_5 \cup B_7$. Suppose that there is a conflict-free 3-colouring f of $\mathcal{H}$ with colours $1, 2$ and 3.

Claim 1. *No colour can appear in exactly one blob.*

Proof. Assume that a colour, say 1, appears in exactly one blob. Consider the three edges e_2, e_3, e_6 not containing B_1. They must be conflict-free 2-colourable with colours $2, 3$. But they form the $(r/2)$-blow-up of K_3 which is not conflict-free 2-colourable, a contradiction.

Claim 2. *No colour can appear in exactly two blobs.*

Proof. Suppose that colour 1 appears in exactly two blobs. Let B_1, B_2 be the blobs containing vertices of colour 1. Consider the two edges e_1, e_4 containing both B_1 and B_2 and the edge e_3 containing neither B_1 nor B_2. These three edges form the $(r/2)$-blow-up of K_3 with at least two vertices of colour 1 present in a single blob. All other vertices gets colour 2 or 3. With these restrictions there exists no conflict-free 3-colouring of the blow-up of K_3.

Hence by the above claims, every colour appears in at least three blobs.

Since f is a conflict-free 3-colouring of $\mathcal{H}$ which has seven edges, some colour is unique for at least three edges. Assume that this colour is 1.

Claim 3. *A vertex with colour* 1 *cannot be unique for more than one edge.*

Proof. If not, then without loss of generality, assume that a vertex with colour 1 belonging to B_1 is unique for edges e_4 and e_5. Hence the blobs B_2, B_3, B_4, B_5, B_6 do not have any vertices of colour 1. So colour 1 appears only in at most two blobs. This contradicts Claims 1 and 2.

Assume that a vertex of colour 1 in B_1 is unique for e_1. So the blobs B_2, B_6, B_7 do not have vertices of colour 1. Again without loss of generality assume that a vertex of colour 1 in B_3 is unique for the edge e_2. So the blob B_4 does not have any vertex of colour 1. Now there must be a vertex of colour 1 in B_5 which is unique for e_3. We now consider the edges e_4, e_5, e_6. Each of these edges contains exactly two vertices of colour 1. We delete these vertices and consider the new edges e_4', e_5', e_6'. The hypergraph formed by these edges must be conflict-free 2-colourable with colours 2, 3. The edges e_4', e_5', e_6' form the $((r/2)-1)$-blow-up of K_3 which is not conflict-free 2-colourable, a contradiction. □

3. Conflict-free colouring of hypergraphs with few edges

Having dealt with small cases, now we study the bounds for the conflict-free chromatic number of hypergraphs with few edges. We start with a simple probabilistic fact we shall use later on.

Lemma 3.1. *Colour a set T of t points, randomly, with s colours, so that each of s^t colourings is equally likely. Let $p_{t,s}$ be the probability that no colour appears exactly once on T and let $\hat{p}_{t,s}$ be the probability that at most one colour appears exactly once on T. Then*

$$p_{t,s} \leqslant \left(\frac{2t}{s}\right)^{\lceil t/2 \rceil} \tag{3.1}$$

and

$$\hat{p}_{t,s} \leqslant \left(\frac{8t}{s}\right)^{\lceil (t-1)/2 \rceil}. \tag{3.2}$$

Proof. To prove (1), let us randomly colour all elements of T, one by one. Note that if no colour appears exactly once we shall use at most $\lfloor t/2 \rfloor$ of them, and the set T' of the elements that are coloured with a colour which we have already used has at least $\lceil t/2 \rceil$ elements. Thus, since the number of ways to choose T' is at most 2^t, we get

$$p_{t,s} < 2^t \left(\frac{t}{2s}\right)^{\lceil t/2 \rceil} \leqslant \left(\frac{2t}{s}\right)^{\lceil t/2 \rceil}.$$

In order to show (2), we again randomly colour all elements of T one by one. Note that we shall use at most t colours. Furthermore, in this case the set T' of the elements that

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are coloured with a colour which we have already used has at least $\lceil (t-1)/2 \rceil$ elements and the number of ways to choose T' is at most 2^t. Hence

$$\hat{p}_{t,s} < 2^t \left(\frac{t}{s}\right)^{\lceil (t-1)/2 \rceil} \leqslant \left(\frac{8t}{s}\right)^{\lceil (t-1)/2 \rceil}.$$

□

Now we can bound $\chi_{CF}(\mathcal{H})$ for an r-uniform hypergraph $\mathcal{H}$ with m edges.

Theorem 3.2. *Let $\mathcal{H}$ be a r-uniform hypergraph with m edges and maximum edge degree $D(\mathcal{H})$.*

(i) *If $D(\mathcal{H}) \leqslant 2^{r/2}$, and $D(\mathcal{H})$ (and thus r) is large enough, then there exists a vertex colouring of $\mathcal{H}$ with $120 \ln D(\mathcal{H})$ colours such that each edge has at least one colour appearing exactly once. In particular,*

$$\chi_{CF}(\mathcal{H}) \leqslant 120 \ln D(\mathcal{H}) \leqslant 120 \ln m.$$

(ii) *If $m \geqslant 2^{r/2}$, then $\chi_{CF}(\mathcal{H}) \leqslant 4r(16m)^{2/(r+2)}$.*

Proof. **(i)** In order to show (i) we set $p = 1.34 \ln D(\mathcal{H})/r$, choose a subset $\hat{T}$ of vertices of $\mathcal{H}$ independently with probability p, and then colour each vertex of $\hat{T}$ independently with one of $s = 120 \ln D(\mathcal{H})$ colours. Let A_e be the event that no colour appears exactly once in the edge e. Then, by Lemma 3.1,

$$\begin{aligned}
\mathbb{P}(A_e) &\leqslant \sum_{i=0}^{i_0} \binom{r}{i} p^i (1-p)^{r-i} \left(\frac{2i}{s}\right)^{i/2} + \sum_{i=i_0+1}^{r} \binom{r}{i} p^i (1-p)^{r-i} \\
&\leqslant \sum_{i=0}^{i_0} \binom{r}{i} p^i (1-p)^{r-i} \left(\frac{2i_0}{s}\right)^{i/2} + \sum_{i=i_0+1}^{r} \binom{r}{i} p^i (1-p)^{r-i},
\end{aligned}$$

where here and below $i_0 = \lfloor 2.5 \cdot 1.34 \ln D(\mathcal{H}) \rfloor$.

Since $p \leqslant 1.34(r/2)(\ln 2)/r \leqslant 0.47$, for $i \geqslant i_0 + 1$, we have

$$\frac{\binom{r}{i+1} p^{i+1} (1-p)^{r-i-1}}{\binom{r}{i} p^i (1-p)^{r-i}} \leqslant \frac{r}{i} \frac{p}{1-p} \leqslant \frac{1}{2.5(1-p)} < \frac{1}{1.325},$$

so the second sum can be bounded from above by a geometric series and consequently

$$\sum_{i=i_0+1}^{r} \binom{r}{i} p^i (1-p)^{r-i} \leqslant 4.08 \binom{r}{i_0+1} p^{i_0+1} (1-p)^{r-i_0-1}.$$

Since $\binom{r}{j} \leqslant (\frac{er}{j})^j$ and $(1-p)^{r-j} \leqslant (1-p)^r \leqslant (e^{-pr/j})^j$, we have

$$\begin{aligned}
\mathbb{P}(A_e) &\leqslant \left(1 + \left(\sqrt{\frac{2i_0}{s}} - 1\right) p\right)^r + 4.08 \left(\frac{erp}{i_0+1} \cdot e^{-pr/(i_0+1)}\right)^{i_0+1} \\
&\leqslant \exp(-0.76pr) + 4.08 \exp\bigl(-0.79 \cdot 1.34 \ln D(\mathcal{H})\bigr) \\
&\leqslant D(\mathcal{H})^{-1.01} + 4.08 D(\mathcal{H})^{-1.05} \leqslant 1/(4D(\mathcal{H}))
\end{aligned}$$

for sufficiently large $D(\mathcal{H})$. Consequently, $D(\mathcal{H})\mathbb{P}(A_e) < 1/4$, and by the Lovász Local Lemma there exists a conflict-free colouring of $\mathcal{H}$ with $s = 120 \ln D(\mathcal{H})$ colours, so (i) follows.

(ii) Now let $s = 2r(16m)^{2/(r+2)}$ and $k = 2s$. We shall show that $\mathcal{H}$ has a conflict-free colouring with at most k colours. Let v be a vertex of maximum degree in $\mathcal{H}$. Reserve a colour c for v and delete v along with all the edges containing it. Repeat this procedure and reserve a different colour every time we delete a vertex of maximum degree in the remaining hypergraph. This procedure is repeated $k/2$ times. Let $\mathcal{H}_1$ denote the hypergraph obtained by $k/2$ repetitions of this procedure. We consider the following two cases.

Case 1: $D(\mathcal{H}_1) < m^{r/(r+2)}$. Colour each vertex of $\mathcal{H}_1$ by a colour chosen randomly among s colours. Let A_e be the event that no colour appears exactly once in the edge e. By Lemma 3.1, $\mathbb{P}(A_e) < (2r/s)^{r/2}$. Thus, for $r \geqslant 2$,

$$4 \cdot D(\mathcal{H}_1) \cdot \mathbb{P}(A_e) < 4 \cdot m^{r/(r+2)} \cdot (2r/s)^{r/2} = 4 \cdot m^{r/(r+2)} \cdot (2r/2r(16m)^{2/(r+2)})^{r/2} \leqslant 1.$$

Hence, by the Lovász Local Lemma, there exists a conflict-free colouring of $\mathcal{H}_1$ with $k/2$ colours. Together with the other $k/2$ colours, we have a conflict-free colouring of $\mathcal{H}$ with $k = 2s = 4r(16m)^{2/(r+2)}$ colours.

Case 2: $D(\mathcal{H}_1) \geqslant m^{r/(r+2)}$. Note that each time we delete a vertex of maximum degree in the remaining hypergraph, we remove at least

$$\Delta(\mathcal{H}_1) \geqslant \frac{D(\mathcal{H}_1)}{r} \geqslant \frac{m^{r/(r+2)}}{r}$$

edges $k/2$ times. Thus, $m \geqslant km^{r/(r+2)}/(2r)$, which implies $k \leqslant 2rm^{2/(r+2)}$, a contradiction. This completes the proof of (ii). □

It is not hard to see that the bound given by Theorem 3.2(i) is tight up to a constant factor. Indeed, the following holds.

Proposition 3.3. *For all $m \geqslant 1$ and for all even $r \geqslant 2$, there exists an r-uniform hypergraph $\mathcal{H}$ with m edges such that $\chi_{CF}(\mathcal{H}) > \frac{1}{2}\log_2 m$.*

Proof. If $1 \leqslant m \leqslant 4$, then $\frac{1}{2}\log_2 m \leqslant 1$, and the statement follows. Let $m \geqslant 5$ and let n be the largest integer such that $\binom{n}{2} \leqslant m$. Let $\mathcal{H}'$ be the $(r/2)$-blow-up of K_n, where the blobs are $B_1, \dots, B_n$. Consider the hypergraph $\mathcal{H}$ obtained from $\mathcal{H}'$ by adding $m - \binom{n}{2}$ isolated edges. By construction, $\mathcal{H}$ has m edges.

Let $k = \lfloor \log_2 n \rfloor$. Suppose that $\mathcal{H}$ has a conflict-free colouring f with k colours. For $i = 1, \dots, n$, let S_i be the set of colours that appear in the blob B_i. Since there are $2^k - 1$ non-empty distinct subsets of the set $\{1, \dots, k\}$ and $n > 2^k - 1$, there are some $1 \leqslant i < j \leqslant n$ with $S_i = S_j$. Then each colour occurs in the edge $B_i \cup B_j$ an even number of times, a contradiction. So, $\chi_{CF}(\mathcal{H}) \geqslant 1 + k$.

Since

$$m \leqslant \binom{n+1}{2} - 1 = \frac{n^2+n-2}{2} < n^2,$$

we have $\log_2 m < 2\log_2 n < 2(1+k) \leqslant 2\chi_{CF}(\mathcal{H})$. □

Constructing a matching bound for Theorem 3.2(ii), when m is much larger than r, is a harder task. Pach and Tardos [12] showed that if $\mathcal{H}$ is a r-uniform hypergraph with m edges, then $\chi_{CF}(\mathcal{H}) \leqslant rm^{2/(r+1)}\log m$, and they ask whether $\chi_{CF}(\mathcal{H}) \leqslant rm^{1/r}\log m$. We answer their question in the negative. More precisely, we show that if r is much smaller than m, then there exists r-uniform hypergraph $\mathcal{H}$ such that $\chi_{CF}(\mathcal{H}) \geqslant C_r m^{2/(r+2)}/\log m$. Let us start with a simple observation.

Observation 3.4. *Given any colouring f of an n-element set with k colours, we can choose a family $\mathcal{A}_f$ of k disjoint sets such that each set in $\mathcal{A}_f$ has size $\lfloor n/2k \rfloor$ and is monochromatic.*

Proof. Consider the colour classes $A_1, A_2, \ldots, A_k$. For each colour class A_i we partition it into subclasses B_{i_j} of size equal to $\lfloor n/2k \rfloor$ until we can no longer do so. The last subclass, say $B_{i_{j'}}$, for each i will have size less than $\lfloor n/2k \rfloor$. Summing the sizes of these $B_{i_{j'}}$ we get at most $n/2$ vertices. The remaining at least $n/2$ vertices give us a family of k sets such that each set in $\mathcal{A}_f$ has size $\lfloor n/2k \rfloor$ and is monochromatic. □

Theorem 3.5. *For each positive even fixed r, there exists a constant $c_r \leqslant 4(8e^2/r)^{r/2}$ such that for every integer $k \geqslant r/2$ there exists an r-uniform hypergraph $\mathcal{H}$ with less than $1 + c_r k^{(r+2)/2}\log k$ edges such that $\chi_{CF}(\mathcal{H}) > k$.*

Proof. Consider a vertex set V of size n, a multiple of $4k$. Let

$$m = \lceil 4(8e^2/r)^{r/2} k^{(r+2)/2} \log k \rceil. \tag{3.3}$$

We form a random r-uniform hypergraph $\mathcal{H}$ with m edges by choosing m subsets $F_1, F_2, \ldots, F_m$ of V of size r randomly with equal probability and repetitions allowed. We will prove that with a positive probability the conflict-free chromatic number of $\mathcal{H}$ is larger than k.

Let f be any fixed k-colouring of V. By Observation 3.4, there exists a family $\mathcal{A}_f$ of k sets $\{A_1, A_2, \ldots, A_k\}$ such that each of these sets has size $\lfloor n/2k \rfloor$ and is monochromatic. So the probability that edge F_i has a conflict is bounded from below by the probability that it has exactly 2 or 0 vertices from each of the sets in $\mathcal{A}_f$ and no vertices outside $\mathcal{A}_f$, which, in turn, is equal to

$$\binom{k}{r/2}\binom{\lfloor n/(2k) \rfloor}{2}^{r/2}\binom{n}{r}^{-1}.$$

Since

$$\left(\frac{n}{k}\right)^k \leqslant \binom{n}{k} \leqslant \left(\frac{en}{k}\right)^k,$$

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we get

$$\mathbb{P}(\text{edge } F_i \text{ has a conflict}) \geqslant \left(\frac{k}{r/2}\right)^{r/2}\left(\frac{n^2}{16k^2}\right)^{r/2}\left(\frac{r^2}{e^2n^2}\right)^{r/2} = \left(\frac{r}{8e^2k}\right)^{r/2}.$$

Consequently,

$$\begin{aligned}\mathbb{P}(f \text{ is a conflict-free colouring of } \mathcal{H}) &\leqslant \left(1-\left(\frac{r}{8e^2k}\right)^{r/2}\right)^m \\ &< \exp\left(-m\left(\frac{r}{8e^2k}\right)^{r/2}\right).\end{aligned}$$

There are k^n distinct colourings of $V(\mathcal{H})$, so

$$\begin{aligned}\mathbb{P}(\mathcal{H} \text{ is conflict-free colourable with } k \text{ colours}) &< k^n \exp\left(-m\left(\frac{r}{8e^2k}\right)^{r/2}\right) \\ &\leqslant \exp\left(-m\left(\frac{r}{8e^2k}\right)^{r/2} + n\log k\right).\end{aligned}$$

If $n = 4k$, then by (3.3) the probability that $\mathcal{H}$ is conflict-free colourable is strictly smaller than 1. Hence there exists an r-uniform hypergraph $\mathcal{G}$ with m edges such that $\chi_{CF}(\mathcal{G}) > k$. □

Remark. Solving (3.3) for k, we get $k \sim C_r m^{2/(r+2)}/\log m$, where C_r is a function of r. Thus, Theorem 3.5 shows that for a given m and $r \leqslant C_r m^{2/(r+2)}/\log m$, there exists an r-uniform hypergraph $\mathcal{H}$ with m edges such that $\chi_{CF}(\mathcal{H}) > C_r m^{2/(r+2)}/\log m$.

4. Conflict-free colouring of simple hypergraphs

Although one can show that there exist simple r-uniform hypergraphs $\mathcal{H}$ with $m = C^r$ such that $\chi(\mathcal{H}) = \Theta(r)$, the second part of Theorem 3.2(ii) can be improved in the case of simple hypergraphs. Let us start with the following simple consequence of Lemma 3.1.

Lemma 4.1. *Let $r \leqslant k/8$ and let $\mathcal{H}$ be an r-uniform hypergraph. If*

$$D(\mathcal{H}) < \frac{1}{4}\left(\frac{k}{8r}\right)^{\lceil (r-1)/2\rceil},$$

then there exists a vertex colouring of $\mathcal{H}$ with k colours such that each edge has at least two colours appearing exactly once.

Proof. Consider a random k-colouring of $\mathcal{H}$ and let A_e be the event that the edge e has at most one colour appearing exactly once. By Lemma 3.1, the probability of A_e,

$$\mathbb{P}(A_e) \leqslant \left(\frac{8r}{k}\right)^{\lceil (r-1)/2\rceil}.$$

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Now note that for a given edge e, the event A_e is independent of all but at most $D(\mathcal{H})$ other events $A_{e'}$. Thus, for

$$D(\mathcal{H}) < \frac{1}{4}\left(\frac{k}{8r}\right)^{\lceil (r-1)/2 \rceil},$$

we have $4 \cdot \mathbb{P}(A_e) \cdot D(\mathcal{H}) < 1$, and so by the Lovász Local Lemma there exists a colouring where none of the events A_e occur. Consequently, there exists a colouring of $\mathcal{H}$ with k colours such that every edge has at least two colours appearing exactly once. □

Remark. By Lemma 3.3, for a given m, even if r is arbitrarily large (but even), there is an r-uniform hypergraph $\mathcal{H}$ with m edges and $\chi_{CF}(\mathcal{H}) > 0.5 \log_2 m$. There is no similar statement for simple hypergraphs. Indeed, if the maximum edge degree of a simple r-uniform hypergraph $\mathcal{H}$ is less than r, then we can choose in each edge e a vertex v_e that belongs only to e. Then we colour each v_e with 1, and every other vertex with 2. So, such a hypergraph has a conflict-free colouring with just 2 colours.

Theorem 4.2. *Let $r \leqslant k/8$ and let $\mathcal{H}$ be an r-uniform simple hypergraph with m edges. If*

$$m \leqslant \frac{1}{16r(r-1)^2}\left(\frac{k}{8(r-1)}\right)^{r-2},$$

then $\chi_{CF}(\mathcal{H}) \leqslant k$.

Proof. Assume that $\chi_{CF}(\mathcal{H}) > k$. Let $\mathcal{H}_1$ be the hypergraph obtained from $\mathcal{H}$ by truncating each edge e by a vertex v_e of maximum degree. Observe that $\mathcal{H}_1$ is an $(r-1)$-uniform simple hypergraph, and if f is a k-colouring of $\mathcal{H}_1$, then there exists an edge of $\mathcal{H}_1$ which has at most one colour appearing exactly once, otherwise $\mathcal{H}$ would be conflict-free k-colourable. Now by Lemma 4.1,

$$D(\mathcal{H}_1) \geqslant \frac{1}{4}\left(\frac{k}{8(r-1)}\right)^{\lceil (r-2)/2 \rceil}.$$

Furthermore, $\mathcal{H}_1$ has a vertex of degree at least $D(\mathcal{H}_1)/(r-1)$. If $\mathcal{H}_1$ has a vertex v of degree at least d, then every edge e in $\mathcal{H}_1$ containing v must have a vertex v_e whose degree in $\mathcal{H}$ is at least d. Moreover, since $\mathcal{H}$ is simple, all these d vertices are distinct. Hence $\mathcal{H}$ has at least $D(\mathcal{H}_1)/(r-1)$ vertices of degree at least $D(\mathcal{H}_1)/(r-1)$. So by the degree-sum formula,

$$m \geqslant D(\mathcal{H}_1)^2/r(r-1)^2 > \frac{1}{16r(r-1)^2}\left(\frac{k}{8(r-1)}\right)^{r-2}.$$ □

Note that if we solve the equation

$$m = \frac{1}{16r(r-1)^2}\left(\frac{k}{8(r-1)}\right)^{r-2}$$

with respect to k, then we get $k \sim C_r' m^{1/(r-2)}$. So for large r, the upper bound for the conflict-free chromatic number for simple hypergraphs provided by Theorem 4.2 is roughly

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a square of the bound given by Theorem 3.2 for the general case. The following result shows that, at least for large r, this estimate is not very far from being optimal.

Lemma 4.3. *Let $r \leqslant k$. Then, there exists an r-uniform simple hypergraph $\mathcal{H}$ with*

$$(1+o(1))(4k\ln k)^2\left(\frac{4e^2k}{r}\right)^r$$

edges such that $\chi_{CF}(\mathcal{H}) > k$.

Proof. We first construct an auxiliary $4k$-uniform simple hypergraph $\mathcal{H}_1$ as follows. Let q be a prime which will be chosen later. The vertex set of $\mathcal{H}_1$ is $S = S_1 \cup \cdots \cup S_{4k}$, where all S_i are disjoint copies of $GF(q) = \{0, 1, \ldots, q-1\}$. The edges of $\mathcal{H}_1$ are $4k$-tuples $(x_1, \ldots, x_{4k}) \in S_1 \times \cdots \times S_{4k}$ that are solutions of the system of linear equations

$$\sum_{i=1}^{4k} i^j x_i = 0, \qquad j = 0, 1, \ldots, 4k-3, \tag{4.4}$$

over $GF(q)$.

For any fixed pair of variables in (4.4), we have a $(4k-2)\times(4k-2)$ system of linear equations with Vandermond's determinant, which has a unique solution over $GF(q)$. This means that $\mathcal{H}_1$ is a $4k$-uniform simple hypergraph with $4kq$ vertices, in which each vertex is contained in q edges, so $|E(\mathcal{H}_1)| = q^2$.

Now, from each edge e of $\mathcal{H}_1$ we choose an r-subset A_e randomly and independently. Let $\mathcal{H}$ be the r-uniform simple hypergraph obtained from $\mathcal{H}_1$ by taking the subsets A_e as its edges. Our goal is to show that with positive probability the conflict-free chromatic number of $\mathcal{H}$ is large.

To this end, fix a colouring f. Let B_e denote the event that the edge e has a conflict in the colouring f, and $p = \mathbb{P}(B_e)$. Arguing as in the proof of Theorem 3.5, one can show that

$$p \geqslant \left(\frac{r}{8e^2k}\right)^{r/2}.$$

Since the edges of $\mathcal{H}$ were chosen independently, the probability that f is a conflict-free colouring of $\mathcal{H}$ is $(1-p)^{q^2}$. Moreover, the total number of colourings is k^{4kq}, so the probability that there exists a conflict-free colouring of $\mathcal{H}$ with k colours is at most $k^{4kq}\cdot(1-p)^{q^2}$. This probability is less than 1, provided

$$k^{4kq}\cdot e^{-pq^2} < 1,$$

which holds whenever

$$q > \frac{4k\ln k}{p}.$$

Now if we take the smallest prime q such that

$$q > q_0 = 4k\ln k\left(\frac{8e^2k}{r}\right)^{r/2},$$

then we have an r-uniform simple hypergraph with q^2 edges and $\chi_{CF}(\mathcal{H}) > k$. It is known (see, for instance, [10]) that one can take $q = (1 + o(1))q_0$. Hence

$$|E(\mathcal{H})| = (1 + o(1))(4k \ln k)^2 \left(\frac{8e^2k}{r}\right)^r. \qquad \square$$

Finally, let us remark that if we take $k = r$, then we get a simple r-uniform hypergraph $\mathcal{H}$ with $m = 2^{O(r)}$ edges such that $\chi(\mathcal{H}) > r = \Omega(\ln m)$. So Theorem 3.2(i) cannot be much improved in the case of simple hypergraphs, at least when m grows exponentially with r.

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