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J-stability of expanding maps in non-Archimedean dynamics

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Abstract. The aim of this paper is to show J-stability of expanding rational maps over an algebraically closed, complete and non-Archimedean field of characteristic zero. More precisely, we will show that for any expanding rational map, there exists a neighborhood of it such that the dynamics on the Julia set of any rational map in the neighborhood is the same as the dynamics of the expanding rational map as a non-Archimedean analogue of a corollary of Mañé, Sad and Sullivan's result [On the dynamics of rational maps. *Ann. Sci. Éc. Norm. Supér. (4)* **16** (1983), 193–217] in complex dynamics.

1. *Introduction*

Let K be an algebraically closed field of characteristic zero with a complete, non-trivial and non-Archimedean norm $|\cdot| : K \to \mathbb{R}$. For instance, one can consider the field $\mathbb{C}_p$ of complex p-adic numbers or a smallest algebraically closed and complete extension of the field of formal Laurent series $\mathbb{C}((T))$ equipped with a non-Archimedean norm. We refer to [**Rob00**] for the details on non-Archimedean norms. In this paper, we study stability on the chaotic locus of iterations of expanding rational maps with coefficients in K. We will give a rigorous definition of expanding rational maps later in this section (Definition 1.1).

This paper contributed to the development of the study of iterations of rational maps acting on the projective line over non-Archimedean fields. We refer to [**Hs00**, **Ben01**, **Bez01**, **Ben02**, **Bez04**, **Riv03**, **Kiw06**, **Silv07**, **BH12**] for interesting developments of the theory of dynamical systems in this non-Archimedean settings. The main result of this paper is motivated by Mañé, Sad and Sullivan's result [**MSS83**] in complex dynamics. They introduced the notion of J*-stability* of rational maps, which roughly means that the dynamics on the Julia sets of two given rational maps are dynamically equivalent if those two rational maps are close enough. They also showed that a rational map is J-stable if it has a neighborhood in the set of rational maps on which the number of attracting cycles is constant. As a corollary of their theorem, we can obtain the J-stability theorem for expanding rational maps in complex dynamics. We refer the reader to [**McMull94**] for more details on J-stability in complex dynamics. The main purpose of this paper is to give a non-Archimedean analogue of the J-stability theorem for expanding rational maps. Here we prepare some notation to introduce our main result (Theorem 1.2).

The *projective line* $\mathbb{P}^1_K$ over K is defined as the quotient space

$$\mathbb{P}^1_K := (K^2 \backslash \{\mathbf{0}\})/\sim,$$

where $\sim$ is the equivalence relation defined by $(z_0, z_1) \sim (w_0, w_1)$ if there exists a $c \in K\backslash\{0\}$ such that $(z_0, z_1) = (c \cdot w_0, c \cdot w_1)$. The *chordal metric* ρ on $\mathbb{P}^1_K$ is defined by

$$\rho((z_0 : z_1), (w_0 : w_1)) := \frac{|z_0 \cdot w_1 - w_0 \cdot z_1|}{\max\{|z_0|, |z_1|\} \cdot \max\{|w_0|, |w_1|\}}$$

for any $(z_0 : z_1), (w_0 : w_1) \in \mathbb{P}^1_K$. Note that this metric is complete and invariant under the action of $\phi \in \mathrm{GL}(2; \mathcal{O})$, where $\mathcal{O} := \{z \in K \mid |z| \leq 1\}$ is the ring of integers of K. We refer to [**Silv07**, §2.1] for more details on the chordal metric. A *rational map* $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ of degree d over K is a map given by

$$f(z_0 : z_1) = (F_0(z_0, z_1) : F_1(z_0, z_1)),$$

where F_0 and F_1 are two-variable homogeneous polynomials of degree d over K with no common factors. Note that we can identify $(z : 1) \in \mathbb{P}^1_K$ with $z \in K$ and $(1 : 0) \in \mathbb{P}^1_K$ with ∞. Hence a rational map naturally induces an action on $K \cup \{\infty\}$. In the rest of this paper, we will regard the projective line $\mathbb{P}^1_K$ as the union $K \cup \{\infty\}$ and a rational map as an action on $K \cup \{\infty\}$. For a given rational map $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$, let us denote by f^k the *kth iteration* of f. The *Fatou set* $\mathcal{F}_f$ of f is defined as the largest open set on which $\{f^k\}_{k=0}^{\infty}$ is equicontinuous with respect to ρ and the *Julia set* $\mathcal{J}_f$ of f is defined as $\mathbb{P}^1_K \backslash \mathcal{F}_f$. We will review some standard facts on rational maps and the Julia sets in §2. Throughout this paper, we denote by $\mathbb{N}$ the set $\{1, 2, \ldots\}$ of all natural numbers.

Definition 1.1. Let $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ be a rational map of degree $d \geq 2$. We say that f is *expanding* if it satisfies the following properties.

(1) The Julia set $\mathcal{J}_f$ of f is non-empty and it is contained in K.

(2) There exist a $c > 0$ and a $\lambda > 1$ such that, for any $z \in \mathcal{J}_f$ and $n \in \mathbb{N}$,

$$|(f^n)'(z)| \geq c \cdot \lambda^n.$$

Note that the non-emptiness of the Julia set is necessary, unlike in complex dynamics. We refer to [**Silv07**, §2.5] and [**Ben14**] for more details on rational maps whose Julia sets are empty.

Remark 1. One can define expanding rational maps in terms of spherical derivatives as complex dynamics. More precisely, we define the *spherical derivative* $f^{\#} : \mathbb{P}^1_K \to \mathbb{R}$ of a rational map $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ by

$$f^{\#}(z) := \lim_{w \to z} \frac{\rho(f(w), f(z))}{|w - z|}$$

and say that f is *expanding with respect to the spherical derivative* if it has non-empty Julia set (possibly containing infinity) and there exists a $C > 0$ and a $\lambda > 1$ such that, for any $z \in \mathcal{J}_f$ and $n \in \mathbb{N}$,

$$(f^n)^{\#}(z) \geq C \cdot \lambda^n.$$

Note that $f^{\#}:\mathbb{P}^1_K \to \mathbb{R}$ and $(\phi \circ f)^{\#} = f^{\#}$ for any $\phi \in \mathrm{GL}(2; \mathcal{O})$ because the spherical metric ρ is invariant under the action of $\mathrm{GL}(2; \mathcal{O})$. This implies that if f is expanding with respect to the spherical derivative, then $\phi^{-1} \circ f \circ \phi$ is also expanding, although the constant C may be different. Moreover, it is not difficult to check that if f is expanding with respect to the spherical derivative, then f is also expanding in the sense of Definition 1.1. Indeed, since $\mathcal{F}_f$ is non-empty (see, for example, [**Silv07**, Corollary 5.19] for more details), there exists a $\phi \in \mathrm{GL}(2; \mathcal{O})$ such that $\infty \in \phi(\mathcal{F}_f)$. Thus replacing f by $\phi^{-1} \circ f \circ \phi$, we can assume that $\infty \notin \mathcal{J}_f$. Hence f is expanding in the sense of Definition 1.1 because $\mathcal{J}_f$ is bounded and

$$f^{\#}(z) = \frac{|f'(z)|}{\max\{1, |f(z)|^2\}}$$

for any $z \in K$. We refer to [**Ben08**] for more details on this spherical derivative.

Let us denote by $\mathcal{R}_d$ the *set of rational maps* of degree d. The set $\mathcal{R}_d$ is equipped with the topology induced from the $(2d+1)$-dimensional projective space over K.We refer to [**Silv07**, Proposition 2.13 and §4.3] for more details.

THEOREM 1.2. *Let $f:\mathbb{P}^1_K \to \mathbb{P}^1_K$ be a rational map of degree $d \geq 2$. If f is expanding, then it is J-stable in $\mathcal{R}_d$, that is, there exists a neighborhood $U \subset \mathcal{R}_d$ of f such that, for all $g \in U$, there exists a homeomorphism $h: \mathcal{J}_f \to \mathcal{J}_g$ such that*

$$h \circ f(z) = g \circ h(z)$$

for any $z \in \mathcal{J}_f$. Moreover, the homeomorphism $h: \mathcal{J}_f \to \mathcal{J}_g$ is isometric if g is close enough to f.

Note that it is an non-Archimedean analogue of the J-stability theorem for expanding rational maps in complex dynamics. However, there is a difference in the property of the conjugation. More precisely, Mañé, Sad and Sulilvan's theorem states that the conjugation is quasi-conformal although the conjugation in Theorem 1.2 is an isometry. It is also natural to consider that there is a rational map which is not expanding but J-stable. Such a problem is still open in complex dynamics and it is also known that this problem is equivalent to the *Fatou conjecture*, which is one of the most famous conjectures in complex dynamics. We refer the reader to [**GS98**] for more details on the Fatou conjecture.

This paper is organized as follows. In §2, we will recall some basic notions of non-Archimedean dynamical systems. In §3, we will prove Theorem 1.2 by using some key lemmas. The key lemmas used will be proved in §4.

2. *Preliminaries*

In this section, we give a brief exposition of rational maps and Julia sets. Let us denote by $B(a; r)$ a *closed ball*

$$B(a; r) := \{z \in K \mid |z - a| \leq r\}$$

centered at $a \in K$ with radius $r > 0$.

PROPOSITION 2.1. *Let $f:\mathbb{P}^1_K \to \mathbb{P}^1_K$ be a non-constant rational map. If f has no zeros or poles in $B(a; r)$, then, for any $z \in B(a; r)$,*

$$|f(z)| = |f(a)|.$$

Proof of Proposition 2.1. If $\alpha \in K$ satisfies $|a - \alpha| > r$, then

$$|z - \alpha| = \max\{|z - a|, |a - \alpha|\} = |a - \alpha|$$

for any $z \in B(a; r)$. Since K is algebraically closed, we can write f as a quotient of products of polynomials of degree one and thus we complete our proof. □

Next let us briefly summarize the basic properties of Julia sets that we will need in this paper.

PROPOSITION 2.2. *Let $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ be a non-constant rational map. Then the following properties hold.*

(1) *The Julia set $\mathcal{J}_f$ is completely invariant under f, that is,*

$$f^{-1}(\mathcal{J}_f) = \mathcal{J}_f.$$

(2) *For any $z \in \mathcal{J}_f$,*

$$\mathcal{J}_f = \overline{\bigcup_{k=0}^{\infty} f^{-k}(\{z\})}.$$

(3) *For a closed subset $E \subset \mathbb{P}^1_K$ satisfying $f^{-1}(E) \subset E$ and $\#(E) \geq 2$, $\mathcal{J}_f \subset E$.*

See [**Silv07**, Proposition 5.30 and Corollary 5.32] for the proof.

The following proposition deals with the relationship between repelling periodic points and the Julia set. Recall that a periodic point x of f with period m is said to be *repelling* if $|(f^m)'(x)| > 1$. We refer the reader to [**Silv07**, §2.2] for more details.

PROPOSITION 2.3. *Let $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ be a non-constant rational map. Then the Julia set $\mathcal{J}_f$ is contained in the closure of the set of periodic points of f.*

See [**Hs00**, Theorem 3.1] for the proof. Hsia [**Hs00**] conjectured that the Julia set is the closure of the repelling periodic points as a non-Archimedean analogue of Fatou and Julia's results in complex dynamics. His conjecture is still open. However, we remark that Bézivin [**Bez01**] proved that it is true if the rational map has at least one repelling periodic point. See also [**Kiw06**, **Okuy11**, **BH12**, **DJR15**] for more details on the recent progress on this topic.

The following proposition yields information about how many zeros of a rational map are in a closed ball not containing any poles of the rational map.

PROPOSITION 2.4. *Let $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ be a non-constant rational map. Suppose that there exists a sequence $\{a_k\}_{k=0}^{\infty} \subset K$ such that*

$$f(z) = a_0 + a_1 \cdot (z - a) + a_2 \cdot (z - a)^2 + \cdots$$

for any $z \in B(a; r)$. Setting a non-negative integer

$$l := \max\{m \in \{0, 1, \ldots\} \mid \forall n \in \{0, 1, \ldots\}, |a_m| \cdot r^m \geq |a_n| \cdot r^n\},$$

we have exactly l zeros of f in $B(a; r)$, counted with multiplicity. In particular, if

$$|a_0| > |a_n| \cdot r^n$$

for any $n \in \mathbb{N}$, then f has no zeros in $B(a; r)$.

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See [**BR10**, Proposition A.16] for the proof. As an application of Proposition 2.4, we obtain a criterion for bijective maps.

COROLLARY 2.5. *Under the hypotheses of Proposition 2.4, if*

$$|a_1| \cdot r > |a_n| \cdot r^n$$

for any $n \geq 2$, then

$$f|B(a; r) \to B(f(a); |a_1| \cdot r)$$

is bijective.

See [**BR10**, Corollary A.17] for the proof. We end this section with an important constant in this paper.

PROPOSITION 2.6. *Let $p \geq 0$ be the residue characteristic of K. Define*

$$\kappa := \begin{cases} |p|^{1/(p-1)} & \text{if } p > 0, \\ 1 & \text{if } p = 0. \end{cases}$$

Then $\kappa = \min\{|k|^{1/(k-1)} \mid k \in \{2, 3, \ldots\}\} > 0$.

Proof of Proposition 2.6. If $p = 0$, then $|n| = 1$ for any $n \in \mathbb{N}$ and thus

$$\min\{|k|^{1/(k-1)} \mid k \in \{2, 3 \ldots\}\} = 1 = \kappa > 0.$$

Otherwise, since $p = \min\{n \in \mathbb{N} \mid |n| < 1\}$,

$$\min\{|k|^{1/(k-1)} \mid k \in \{2, 3, \ldots\}\} = \min\{|p|^{n/(p^n-1)} \mid n \in \mathbb{N}\} = |p|^{1/(p-1)} = \kappa > 0. \qquad \square$$

3. *Proof of Theorem 1.2*

In this section, we prove Theorem 1.2. We divide the proof into a sequence of lemmas. The proofs of these lemmas can be found in §4.

Proof of Theorem 1.2. Let us first construct a nested sequence of neighborhoods of $\mathcal{J}_f$. Put

$$N := \min\{n \in \mathbb{N} \mid c \cdot \lambda^n > 1\}, \tag{3.1}$$

where $c > 0$ and $\lambda > 1$ are the constants as in Definition 1.1.

LEMMA 3.1. *There exists a nested sequence $\{\Omega_{n,f}\}_{n=0}^{\infty}$ of closed sets such that $\Omega_{0,f}$ is open (as well as closed) and*

$$\Omega_{n,f} \subset f(\Omega_{n,f}) = \Omega_{n-1,f}$$

for any $n \in \mathbb{N}$. Moreover, $\Omega_{0,f}$ does not contain any critical points or poles of $\{f^k\}_{k=1}^N$. Furthermore,

$$\mathcal{J}_f = \Omega_{\infty,f} := \bigcap_{n=0}^{\infty} \Omega_{n,f}.$$

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We will explicitly construct a sequence satisfying the conditions of Lemma 3.1 in the proof. We then fix the constructed sequence $\{\Omega_{n,f}\}_{n=0}^{\infty}$. Let us next find a neighborhood $U \subset \mathcal{R}_d$ of f.

LEMMA 3.2. *For a small enough $r > 0$, there exists a neighborhood $U \subset \mathcal{R}_d$ of f such that, for any $g \in U$, $z \in \Omega_{0,f}$ and $n \in \{1, 2, \ldots, N\}$, $|f(z) - g(z)| < r$ and $|(f^n)'(z)| = |(g^n)'(z)|$. In particular, g^n has no critical points or poles in $\Omega_{0,f}$.*

We fix a neighborhood $U \subset \mathcal{R}_d$ of f for a small enough r. We will give an explicit upper bound for r in the proof of Lemma 3.2. Let us fix a $g \in U$. Our next goal is to construct a nested sequence of neighborhoods of $\mathcal{J}_g$.

LEMMA 3.3. *There exists a nested sequence $\{\Omega_{n,g}\}_{n=0}^{\infty}$ of closed sets such that $\Omega_{0,g} = \Omega_{0,f}$ and*

$$\Omega_{n,g} \subset g(\Omega_{n,g}) = \Omega_{n-1,g}$$

for any $n \in \mathbb{N}$.

As in the case of Lemma 3.1, we will explicitly construct a sequence satisfying the conditions of Lemma 3.3 in the proof. We also fix the constructed sequence $\{\Omega_{n,g}\}_{n=0}^{\infty}$. Let us set

$$\Omega_{\infty,g} := \bigcap_{n=0}^{\infty} \Omega_{n,g}.$$

Although the existence of $\Omega_{\infty,g}$ is not guaranteed at this step, it will turn out that

$$\Omega_{\infty,g} = \mathcal{J}_g \neq \emptyset$$

at the end of this proof. Now let us construct a map from $\Omega_{\infty,f}$ to $\Omega_{\infty,g}$.

LEMMA 3.4. *There exists a uniformly Cauchy sequence $\{h_n : \Omega_{n,f} \to \Omega_{n,g}\}_{n=0}^{\infty}$ of maps such that*

$$h_{n-1} \circ f(z) = g \circ h_n(z)$$

for any $n \in \mathbb{N}$ and $z \in \Omega_{n,f}$.

By Lemma 3.4, we can define a map $h : \Omega_{\infty,f} \to \Omega_{\infty,g}$ by

$$h(z) := \lim_{n\to\infty} h_n(z) \tag{3.2}$$

for any $z \in \Omega_{\infty,f}$. Moreover,

$$h \circ f(z) = \lim_{n\to\infty} h_{n-1} \circ f(z) = \lim_{n\to\infty} g \circ h_n(z) = g \circ \lim_{n\to\infty} h_n(z) = g \circ h(z).$$

We next show that h is isometric.

LEMMA 3.5. *For any $z, w \in \mathcal{J}_f$,*

$$|h(z) - h(w)| = |z - w|.$$

We end this proof with a lemma which yields that $h(\Omega_{\infty,f}) = \mathcal{J}_g$. In particular, this implies that $h : \mathcal{J}_f \to \mathcal{J}_g$ is a conjugacy of the dynamics on $\mathcal{J}_f$ and $\mathcal{J}_g$.

LEMMA 3.6. *The Julia set $\mathcal{J}_g$ is non-empty. Moreover,*

$$\mathcal{J}_g = h(\mathcal{J}_f) = \Omega_{\infty,g}. \qquad \square$$

4. *Proof of key lemmas*

In this section, we prove lemmas used in §3. Let f be an expanding rational map of degree $d \geq 2$ defined in Definition 1.1. We put

$$M := \sup\{|(f^n)'(z)| \mid z \in \mathcal{J}_f, n \in \{0, 1, \ldots, N\}\},$$

where N is the natural number defined as (3.1). Note that $M \geq 1$ is well defined because $\mathcal{J}_f$ is bounded and it is bounded away from all poles of f^n for each $n \in \{1, 2, \ldots, N\}$. We denote by $\mathcal{C}_f$ the *set of critical points or poles* of $\{f^n\}_{n=1}^N$. Fix a $\delta > 0$ satisfying

$$\mathcal{C}_f \cap \bigcup_{z \in \mathcal{J}_f} B(z; M^N \cdot \delta) = \emptyset.$$

Let us set

$$\Omega := \bigcup_{z \in \mathcal{J}_f} B(z; M^N \cdot \delta).$$

We define functions $\nu, \mu : \Omega \to \mathbb{R}$ by

$$\nu(z) := \frac{1}{\sum_{n=0}^{N-1} |(f^n)'(z)|}$$

and

$$\mu(z) := \frac{\sum_{n=0}^{N-1} |(f^n)'(z)|}{\sum_{n=1}^{N} |(f^n)'(z)|}.$$

Remark 2. The functions ν and μ are constant on $B(z; M^N \cdot \delta)$ for any $z \in \mathcal{J}_f$. Indeed, it follows from Proposition 2.1 that

$$|(f^n)'(z)| = |(f^n)'(w)|$$

for any $n \in \{1, 2, \ldots, N\}$ and $w \in B(z; M^N \cdot \delta)$.

Remark 3. If $z \in \Omega$ satisfies $f(z) \in \Omega$, then

$$\frac{\nu(f(z))}{|f'(z)|} = \mu(z) \cdot \nu(z) = \frac{1}{\sum_{n=1}^{N} |(f^n)'(z)|} < 1.$$

Indeed, by the chain rule,

$$\frac{\nu(f(z))}{|f'(z)|} = \frac{1}{\sum_{n=0}^{N-1} |(f^n)'(f(z))| \cdot |f'(z)|} = \frac{1}{\sum_{n=1}^{N} |(f^n)'(z)|} = \mu(z) \cdot \nu(z).$$

We set

$$\tilde{\nu}(z) := \nu(z) \cdot \delta \cdot \kappa,$$

where κ is the constant defined in Proposition 2.6.

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LEMMA 4.1. *For any $z \in \mathcal{J}_f$, let us denote by $\{z_k\}_{k=1}^d$ the pre-image of z by f. Let μ_k denote $\mu(z_k)$ for each $k \in \{1, 2, \ldots, d\}$. For any $0 \le r \le 1$,*

$$f|B(z_k;\, \mu_k \cdot r \cdot \tilde{\nu}(z_k)) \to B(z;\, r \cdot \tilde{\nu}(z))$$

is bijective.

Proof of Lemma 4.1. Since f has no poles in $B(z_k;\, M^N \cdot \delta)$, there exists a sequence $\{a_n\}_{n=1}^{\infty} \subset K$ such that

$$f(w) = f(z_k) + a_1 \cdot (w - z_k) + \cdots$$

and

$$f'(w) = a_1 + 2 \cdot a_2 \cdot (w - z_k) + \cdots$$

for any $w \in B(z_k;\, M^N \cdot \delta)$. Note that $a_1 = f'(z_k)$. By Remark 3,

$$|a_1| \cdot \mu_k \cdot r \cdot \tilde{\nu}(z_k) = |f'(z_k)| \cdot \mu_k \cdot r \cdot \tilde{\nu}(z_k) = r \cdot \tilde{\nu}(z).$$

Moreover, since $B(z_k;\, \delta)$ has no critical points of f, it follows from Proposition 2.4 that

$$|a_1| > |n| \cdot |a_n| \cdot \delta^{n-1}$$

for any $n \ge 2$. Since $\mu_k \cdot r \cdot \nu(z_k) < 1$, this implies that

$$\begin{aligned} |a_1| \cdot \mu_k \cdot r \cdot \tilde{\nu}(z_k) &> |n| \cdot |a_n| \cdot \delta^{n-1} \cdot \mu_k \cdot r \cdot \tilde{\nu}(z_k) \\ &\ge |a_n| \cdot (\delta \cdot \kappa)^n \cdot \mu_k \cdot r \cdot \nu(z_k) \\ &\ge |a_n| \cdot (\delta \cdot \kappa \cdot \mu_k \cdot r \cdot \nu(z_k))^n = |a_n| \cdot (\mu_k \cdot r \cdot \tilde{\nu}(z_k))^n. \end{aligned}$$

Thus, by Corollary 2.5, $f|B(z_k;\, \mu_k \cdot r \cdot \tilde{\nu}(z_k)) \to B(z;\, r \cdot \tilde{\nu}(z))$ is bijective. □

We put

$$\Lambda := 1 - \frac{c \cdot \lambda^N - 1}{N \cdot M}.$$

Remark 4. The function μ is uniformly bounded by Λ. Indeed, we calculate

$$\mu(z) = \frac{\sum_{n=0}^{N-1} |(f^n)'(z)|}{\sum_{n=1}^{N} |(f^n)'(z)|} = 1 - \frac{(f^N)'(z) - 1}{\sum_{n=1}^{N} |(f^n)'(z)|} \le 1 - \frac{c \cdot \lambda^N - 1}{N \cdot M} = \Lambda$$

for any $z \in \Omega$. In particular, Λ is positive, because μ is positive on Ω.

Proof of Lemma 3.1. Let us set

$$\Omega_{n,f} := f^{-n}\Bigg(\bigcup_{z \in \mathcal{J}_f} B(z;\, \tilde{\nu}(z))\Bigg)$$

for each $n \in \{0, 1, \ldots\}$. It is obvious that $\Omega_{0,f}$ is closed, and thus so is $\Omega_{n,f}$.

Let us show that

$$\Omega_{n,f} = \bigcup_{z \in \mathcal{J}_f} B\Bigg(z;\, \prod_{k=0}^{n-1} \mu(f^k(z)) \cdot \tilde{\nu}(z)\Bigg)$$

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by induction on $n \in \mathbb{N}$. In particular, this implies that $\Omega_{n,f} \subset \Omega_{n-1,f}$ because μ is uniformly bounded by $\Lambda < 1$.

(1) For any $w \in \Omega_{1,f}$, there exists a $z \in \mathcal{J}_f$ such that $f(w) \in B(z; \tilde{\nu}(z))$. By Lemma 4.1, there exists a $z' \in f^{-1}(\{z\}) \subset \mathcal{J}_f$ such that

$$w \in B(z'; \mu(z') \cdot \tilde{\nu}(z')).$$

On the other hand, for any

$$w \in \bigcup_{z \in \mathcal{J}_f} B(z; \mu(z) \cdot \tilde{\nu}(z)),$$

there exists a $z \in \mathcal{J}_f$ such that $w \in B(z; \mu(z) \cdot \tilde{\nu}(z))$. Thus

$$f(w) \in B(f(z); \tilde{\nu}(f(z))) \subset \Omega_{0,f}.$$

(2) Assume that

$$\Omega_{n,f} = \bigcup_{z \in \mathcal{J}_f} B\left(z; \prod_{k=0}^{n-1} \mu(f^k(z)) \cdot \tilde{\nu}(z)\right)$$

for $n \in \mathbb{N}$. For any $w \in \Omega_{n+1,f}$, there exists a $z \in \mathcal{J}_f$ such that

$$f(w) \in B\left(z; \prod_{k=0}^{n-1} \mu(f^k(z)) \cdot \tilde{\nu}(z)\right).$$

By Lemma 4.1, there exists a $z' \in f^{-1}(\{z\}) \subset \mathcal{J}_f$ such that

$$w \in B\left(z'; \prod_{k=0}^{n} \mu(f^k(z')) \cdot \tilde{\nu}(z')\right).$$

On the other hand, for any

$$w \in \bigcup_{z \in \mathcal{J}_f} B\left(z; \prod_{k=0}^{n} \mu(f^k(z)) \cdot \tilde{\nu}(z)\right),$$

there exists a $z \in \mathcal{J}_f$ such that

$$w \in B\left(z; \prod_{k=0}^{n} \mu(f^k(z)) \cdot \tilde{\nu}(z)\right)$$

and thus

$$f(w) \in B\left(f(z); \prod_{k=0}^{n-1} \mu(f^k(f(z))) \cdot \tilde{\nu}(f(z))\right) \subset \Omega_{n,f}.$$

By Proposition 2.2(3), $\mathcal{J}_f \subset \Omega_{\infty,f}$. On the other hand, any $w \in \Omega_{\infty,f}$ satisfies

$$w \in \Omega_{n,f} = \bigcup_{z \in \mathcal{J}_f} B\left(z; \prod_{k=0}^{n-1} \mu(f^k(z)) \cdot \tilde{\nu}(z)\right) \subset \bigcup_{z \in \mathcal{J}_f} B(z; \Lambda^{n+1} \cdot \delta \cdot \kappa)$$

for any $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$, there exists a $z_n \in \mathcal{J}_f$ such that

$$|w - z_n| \leq \Lambda^{n+1} \cdot \delta \cdot \kappa.$$

Since $\mathcal{J}_f$ is closed, this implies that $w \in \mathcal{J}_f$. □

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Remark 5. As an immediately corollary of Lemma 3.1, one can find a repelling periodic point of f. Indeed, it follows from Proposition 2.3 that there exists a periodic point $x \in \Omega_{0,f}$ of f^N with period m. On the other hand, by Lemma 3.1,

$$f^{-N}(\Omega_{0,f}) = \Omega_{N,f} \subset \Omega_{0,f}.$$

This implies that

$$\{x, f^N(x), f^{2N}(x), \ldots, f^{(m-1)N}(x)\} \subset \bigcup_{k=0}^{\infty} f^{-kN}(\{x\}) \subset \Omega_{0,f}.$$

Thus,

$$|(f^{mN})'(x)| = \left|\prod_{k=0}^{m-1} (f^N)'(f^{kN}(x))\right| \geq (c \cdot \lambda^N)^m > 1.$$

For the proof of Lemma 3.2, we write

$$\Xi_k := \bigcup_{z \in \mathcal{J}_f} B(z; M^k \cdot \delta \cdot \kappa) \subset \Omega$$

for each $k \in \{0, 1, \ldots, N\}$.

Proof of Lemma 3.2. It is not difficult to find a neighborhood U of f such that

$$|f(z) - g(z)| < \delta \cdot \kappa$$

and

$$|f'(z) - g'(z)| < c \cdot \lambda$$

for any $g \in U$ and $z \in \Xi_N$ because of the continuity of the coefficients of a rational map. In particular, this implies that

$$|f'(z)| = |g'(z)|$$

for any $z \in \Xi_N$. Let us fix a $g \in U$.

We first show that the following properties are satisfied for any $k \in \{1, 2, \ldots, N\}$.

(1) $g(\Xi_{k-1}) \subset \Xi_k$.

(2) For any $z \in \Xi_{k-1}$ and $n \in \{1, 2, \ldots, N\}$,

$$|(f^n)'(f(z))| = |(f^n)'(g(z))|.$$

Since $z \in \Xi_{k-1}$, there exists a $z' \in \mathcal{J}_f$ such that $z \in B(z'; M^{k-1} \cdot \delta \cdot \kappa)$. Since f has no poles in $B(z'; M^N \cdot \delta)$, there exists a sequence $\{a_m\}_{m=1}^{\infty} \subset K$ such that

$$f(w) = f(z') + a_1 \cdot (w - z') + \cdots$$

and

$$f'(w) = a_1 + 2 \cdot a_2 \cdot (w - z') + \cdots$$

for any $w \in B(z'; M^N \cdot \delta)$. Note that $a_1 = f'(z')$. Since $B(z'; M^{k-1} \cdot \delta)$ has no critical points of f, it follows from Proposition 2.4 that

$$|a_1| > |m| \cdot |a_m| \cdot (M^{k-1} \cdot \delta)^{m-1}$$

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for any $m \geq 2$. Thus

$$M \cdot M^{k-1} \cdot \delta \cdot \kappa \geq |a_1| \cdot M^{k-1} \cdot \delta \cdot \kappa$$
$$> |m| \cdot |a_m| \cdot (M^{k-1} \cdot \delta)^m \cdot \kappa \geq |a_m| \cdot (M^{k-1} \cdot \delta \cdot \kappa)^m.$$

It follows from Corollary 2.5 that

$$f(z) \in f(B(z;\, M^{k-1} \cdot \delta \cdot \kappa)) = f(B(z';\, M^{k-1} \cdot \delta \cdot \kappa)) \subset B(f(z');\, M^k \cdot \delta \cdot \kappa)$$

and thus

$$|f(z) - f(z')| \leq M^k \cdot \delta \cdot \kappa.$$

In particular, this implies that

$$g(z) \in B(f(z);\, \delta \cdot \kappa) \subset B(f(z);\, M^k \cdot \delta \cdot \kappa) = B(f(z');\, M^k \cdot \delta \cdot \kappa) \subset \Xi_k.$$

Moreover, since f^n has no critical points in $B(f(z');\, M^k \cdot \delta \cdot \kappa)$,

$$|(f^n)'(f(z))| = |(f^n)'(g(z))|$$

for any $n \in \{1, 2, \ldots, N\}$. This completes the proof of (1) and (2).

Now we show that

$$|(f^n)'(z)| = |(g^n)'(z)|$$

on Ξ_0 for any $n \in \{1, 2, \ldots, N\}$. It is clear that $|f'(z)| = |g'(z)|$ on Ξ_N. Assume that

$$|(f^n)'(z)| = |(g^n)'(z)|$$

on Ξ_{N-n+1} for each $n \in \{1, 2, \ldots, N-1\}$. For any $z \in \Xi_{N-n}$,

$$g(z) \in \Xi_{N-n+1}$$

and thus

$$|(f^n)'(g(z))| = |(g^n)'(g(z))|.$$

Hence

$$|(f^{n+1})'(z)| = |(f^n)'(f(z))| \cdot |f'(z)|$$
$$= |(f^n)'(g(z))| \cdot |g'(z)|$$
$$= |(g^n)'(g(z))| \cdot |g'(z)| = |(g^{n+1})'(z)|.$$

This implies that

$$|(f^n)'(z)| = |(g^n)'(z)|$$

on Ξ_{N-n} for any $n \in \{1, 2, \ldots, N\}$. Since $\{\Xi_{N-k}\}_{k=0}^{N}$ is nested and $\Omega_{0,f} \subset \Xi_0$, this completes our proof. □

Let U be a neighborhood of f determined in Lemma 3.2 for

$$r = \frac{\delta \cdot \kappa}{M \cdot N}.$$

For a fixed $g \in U$, we define functions $\nu, \mu : \Omega \to \mathbb{R}$ by

$$\nu_g(z) := \frac{1}{\sum_{n=0}^{N-1} |(g^n)'(z)|}$$

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and

$$\mu_g(z) := \frac{\sum_{n=0}^{N-1} |(g^n)'(z)|}{\sum_{n=1}^{N} |(g^n)'(z)|}.$$

Let us set

$$\tilde{\nu}_g(z) := \nu_g(z) \cdot \delta \cdot \kappa.$$

Remark 6. It follows from Lemma 3.2 that $\nu = \nu_g$ and $\mu = \mu_g$ on $\Omega_{0,f}$. Thus, by Remark 2, the functions ν_g and μ_g are constant on $B(z;\, \tilde{\nu}(z))$ for any $z \in \mathcal{J}_f$. In particular, this implies that

$$\frac{\delta \cdot \kappa}{M \cdot N} \leq \tilde{\nu}(z) = \tilde{\nu}_g(z)$$

on $\Omega_{0,f}$. Moreover, by Remark 3, if $z \in \Omega_{0,f}$ satisfies $g(z) \in \Omega_{0,f}$, then

$$\frac{\nu_g(g(z))}{|g'(z)|} = \mu_g(z) \cdot \nu_g(z) = \mu(z) \cdot \nu(z) < 1.$$

LEMMA 4.2. *For any $z \in \Omega_{0,f}$, $g^{-1}(\{z\}) \subset \Omega_{0,f}$. Let us denote by $\{z_k\}_{k=1}^{d}$ the pre-image of z by g. Let μ_k denote $\mu_g(z_k)$ for each $k \in \{1, 2, \ldots, d\}$. For any $0 \leq s \leq 1$,*

$$g|B(z_k;\, \mu_k \cdot s \cdot \tilde{\nu}_g(z_k)) \to B(z;\, s \cdot \tilde{\nu}_g(z))$$

is bijective.

Proof of Lemma 4.2. Let us denote by $\{w_k\}_{k=1}^{d} \subset \Omega_{0,f}$ the pre-image of z by f. We show that $g(w) - z$ has a unique zero $z_k \in B(w_k;\, \mu_k \cdot \tilde{\nu}_g(w_k))$. Set

$$\mu'_k := \mu(w_k).$$

Since g has no poles in $B(w_k;\, M^N \cdot \delta)$, there exists a sequence $\{a_n\}_{n=1}^{\infty} \subset K$ such that

$$g(w) = g(w_k) + a_1 \cdot (w - w_k) + \cdots \tag{4.1}$$

and

$$g'(w) = a_1 + 2 \cdot a_2 \cdot (w - w_k) + \cdots$$

for any $w \in B(w_k;\, M^N \cdot \delta)$. Note that $a_1 = g'(w_k)$. It is easily seen that

$$|a_1| \cdot \mu'_k \cdot s \cdot \tilde{\nu}_g(w_k) = |f'(w_k)| \cdot \mu'_k \cdot s \cdot \tilde{\nu}(w_k) = s \cdot \tilde{\nu}(z) = s \cdot \tilde{\nu}_g(z).$$

On the other hand, since $B(w_k;\, \delta)$ has no critical points of g, it follows from Proposition 2.4 that

$$|a_1| > |n| \cdot |a_n| \cdot \delta^{n-1}$$

for any $n \geq 2$. Thus

$$\begin{aligned} |a_1| \cdot \mu'_k \cdot s \cdot \tilde{\nu}_g(w_k) &> |n| \cdot |a_n| \cdot \delta^{n-1} \cdot \mu'_k \cdot s \cdot \tilde{\nu}(w_k) \\ &\geq |a_n| \cdot (\delta \cdot \kappa)^n \cdot \mu'_k \cdot s \cdot \nu(w_k) \\ &\geq |a_n| \cdot (\delta \cdot \kappa \cdot \mu'_k \cdot s \cdot \nu(w_k))^n = |a_n| \cdot (\mu'_k \cdot s \cdot \tilde{\nu}_g(w_k))^n. \end{aligned} \tag{4.2}$$

Hence it follows from Corollary 2.5 that

$$g|B(w_k;\, \mu'_k \cdot s \cdot \tilde{\nu}_g(w_k)) \to B(g(w_k);\, s \cdot \tilde{\nu}_g(z))$$

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is bijective. Since

$$|g(w_k) - z| = |g(w_k) - f(w_k)| < \frac{\delta \cdot \kappa}{N \cdot M} \leq \tilde{\nu}_g(z),$$

there exists a unique $z_k \in B(w_k;\, \mu'_k \cdot \tilde{\nu}_g(w_k))$ such that $g(z_k) = z$. Furthermore, since g^n has no critical points in $B(w_k;\, \mu'_k \cdot \tilde{\nu}_g(w_k))$,

$$|(g^n)'(w_k)| = |(g^n)'(z_k)|$$

for any $n \in \{1, 2, \ldots, N\}$. This implies that

$$\mu'_k = \mu(w_k) = \mu_g(w_k) = \mu_g(z_k) = \mu_k. \tag{4.3}$$

□

Proof of Lemma 3.3. Set $\Omega_{n,g} := g^{-n}(\Omega_{0,f})$ for each $n \in \{0, 1, \ldots\}$. It follows from Lemma 4.2 that

$$B\bigg(z;\, \prod_{k=0}^{n} \mu_g(g^k(z)) \cdot \tilde{\nu}_g(z)\bigg) \subset \Omega_{n,g}$$

if $z \in \Omega_{n,g}$. Indeed, this can be proved by induction on $n \in \mathbb{N}$, as follows.

(1) If $z \in \Omega_{1,g}$, then it follows from Lemma 4.2 that

$$\begin{aligned} g(B(z;\, \mu(z) \cdot \tilde{\nu}_g(z))) &= B(g(z);\, \tilde{\nu}_g(g(z))) \\ &= B(g(z);\, \tilde{\nu}(g(z))) \subset \Omega_{0,f} = \Omega_{0,g}. \end{aligned}$$

(2) Assume that

$$B\bigg(z;\, \prod_{k=0}^{n} \mu_g(g^k(z)) \cdot \tilde{\nu}_g(z)\bigg) \subset \Omega_{n,g}$$

for any $z \in \Omega_{n,g}$. If $z \in \Omega_{n+1,g}$, then it follows from Lemma 4.2 that

$$\begin{aligned} g\bigg(B\bigg(z;\, \prod_{k=0}^{n+1} \mu_g(g^k(z)) \cdot \tilde{\nu}_g(z)\bigg)\bigg) &= B\bigg(g(z);\, \prod_{k=0}^{n} \mu_g(g^k(g(z))) \cdot \tilde{\nu}_g(g(z))\bigg) \\ &\subset \Omega_{n,g}. \end{aligned}$$

□

Proof of Lemma 3.4. Let us show this statement by induction on $n \geq 0$.

(1) Let us define $h_0 : \Omega_{0,f} \to \Omega_{0,g}$ by $h_0(z) := z$.

(2) Let us define $h_1 : \Omega_{1,f} \to \Omega_{1,g}$ by

$$h_1 := g^{-1} \circ h_0 \circ f.$$

It is well defined because

$$\begin{aligned} f|B(z;\, \mu(z) \cdot \tilde{\nu}(z)) &\to B(f(z);\, \tilde{\nu}(f(z))), \\ h_0|B(f(z);\, \tilde{\nu}(f(z))) &\to B(f(z);\, \tilde{\nu}_g(f(z))) \end{aligned}$$

and

$$g^{-1}|B(g(z);\, \tilde{\nu}_g(g(z))) \to B(z;\, \mu_g(z) \cdot \tilde{\nu}_g(z))$$

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$$\begin{array}{ccc} \Omega_{0,f} & \xrightarrow{h_0} & \Omega_{0,g} \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ \Omega_{1,f} & \xrightarrow{h_1} & \Omega_{1,g} \end{array}$$

FIGURE 1. The construction of $h_1 : \Omega_{1,f} \to \Omega_{1,g}$.

are bijective for a fixed $z \in \Omega_{1,f}$ (see Figure 1). Note that

$$B(f(z);\ \tilde{\nu}_g(f(z))) = B(g(z);\ \tilde{\nu}_g(g(z)))$$

because

$$|f(z) - g(z)| \leq \frac{\delta \cdot \kappa}{N \cdot M} \leq \tilde{\nu}_g(f(z)) = \tilde{\nu}_g(g(z)).$$

Therefore, we obtain

$$|h_1(z) - h_0(z)| \leq \mu_g(z) \cdot \tilde{\nu}_g(z)$$

and

$$h_0 \circ f(z) = g \circ h_1(z)$$

for any $z \in \Omega_{1,f}$.

(3) Assume that $h_n : \Omega_{n,f} \to \Omega_{n,g}$ is a map satisfying

$$h_{n-1} \circ f(z) = g \circ h_n(z)$$

and

$$|h_n(z) - h_{n-1}(z)| \leq \prod_{k=0}^{n-1} \mu_g(f^k(z)) \cdot \tilde{\nu}_g(z)$$

for any $z \in \Omega_{n,f}$. Then we define $h_{n+1} : \Omega_{n+1,f} \to \Omega_{n+1,g}$ by

$$h_{n+1} := g^{-1} \circ h_n \circ f$$

(see Figure 2). Let us check that h_{n+1} is well defined as follows. Fix a $z \in \Omega_{n+1,f}$. Since

$$B\left(h_n(z);\ \prod_{k=0}^{n} \mu_g(f^k(z)) \cdot \tilde{\nu}_g(z)\right) = B\left(h_n(z);\ \prod_{k=0}^{n} \mu_g(g^k(h_n(z))) \cdot \tilde{\nu}_g(h_n(z))\right),$$

it follows from Lemma 4.2 that

$$g|B\left(h_n(z);\ \prod_{k=0}^{n} \mu_g(f^k(z)) \cdot \tilde{\nu}_g(z)\right) \to B\left(g \circ h_n(z);\ \prod_{k=1}^{n} \mu_g(f^k(z)) \cdot \tilde{\nu}_g(f(z))\right)$$

is bijective. Since

$$\begin{aligned} |h_n \circ f(z) - g \circ h_n(z)| &= |h_n \circ f(z) - h_{n-1} \circ f(z)| \\ &\leq \prod_{k=1}^{n} \mu_g(f^k(z)) \cdot \tilde{\nu}_g(f(z)), \end{aligned}$$

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$$
\begin{array}{ccc}
\Omega_{0,f} & \xrightarrow{h_0} & \Omega_{0,g} \\
{\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\
\Omega_{1,f} & \xrightarrow{h_1} & \Omega_{1,g} \\
{\scriptstyle \cdots}\uparrow & & \uparrow{\scriptstyle \cdots} \\
\Omega_{n,f} & \xrightarrow{h_n} & \Omega_{n,g} \\
{\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\
\Omega_{n+1,f} & \xrightarrow{h_{n+1}} & \Omega_{n+1,g}
\end{array}
$$

FIGURE 2. The construction of $h_{n+1} : \Omega_{n+1,f} \to \Omega_{n+1,g}$.

we conclude that h_{n+1} is well defined. Moreover, it satisfies

$$h_{n+1}(z) \in B\bigg(h_n(z); \prod_{k=0}^{n} \mu_g(f^k(z)) \cdot \tilde{\nu}_g(z)\bigg) \subset \Omega_{n+1,g}$$

and

$$h_n \circ f(z) = g \circ h_{n+1}(z). \qquad \square$$

Let $h : \Omega_{\infty,f} \to \Omega_{\infty,g}$ be a map defined as (3.2). To prove Lemmas 3.5 and 3.6, we first estimate $|h(z) - z|$ for each $z \in \Omega_{\infty,f}$.

LEMMA 4.3. *For any $n \in \mathbb{N}$ and $z \in \Omega_{n,f}$,*

$$|h_n(z) - z| \le \mu_g(z) \cdot \tilde{\nu}_g(z).$$

In particular, this implies that $\mu_g(h_n(z)) = \mu_g(z)$ and $\tilde{\nu}_g(h_n(z)) = \tilde{\nu}_g(z)$.

Proof of Lemma 4.3. It is easily seen that

$$
\begin{aligned}
|h_n(z) - z| &\le \max_{n\in\mathbb{N}}\{|h_n(z) - h_{n-1}(z)|\} \\
&\le \max_{n\in\mathbb{N}}\bigg\{\prod_{k=0}^{n-1} \mu_g(f^k(z)) \cdot \tilde{\nu}_g(z)\bigg\} \le \mu_g(z) \cdot \tilde{\nu}_g(z). \qquad \square
\end{aligned}
$$

We proceed to show that the map $h : \Omega_{\infty,f} \to \Omega_{\infty,g}$ is isometric.

LEMMA 4.4. *If $z, w \in \Omega_{n,g}$ satisfy $|z - w| \le \mu_g(z) \cdot \tilde{\nu}_g(z)$, then*

$$|g(z) - g(w)| = |g'(z)| \cdot |z - w|.$$

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Proof of Lemma 4.4. It follows from (4.1), (4.2) and (4.3) that

$$\begin{aligned}|g(z)-g(w)| &= \left|a_1\cdot(z-w)+\sum_{k=2}^{\infty}a_k\cdot(z-w)^k\right| \\ &= \left|a_1+\sum_{k=2}^{\infty}a_k\cdot(z-w)^{k-1}\right|\cdot|z-w| \\ &= |a_1|\cdot|z-w| = |g'(z)|\cdot|z-w|.\end{aligned}$$ □

Proof of Lemma 3.5. It is sufficient to show that $\{h_n:\Omega_{n,f}\to\Omega_{n,g}\}_{n=0}^{\infty}$ constructed in the proof of Lemma 3.4 is a sequence of isometries. We now proceed by induction on $n\geq 0$.

(1) It is clear that $h_0:\Omega_{0,f}\to\Omega_{0,g}$ is isometric.

(2) Assume that $h_n:\Omega_{n,f}\to\Omega_{n,g}$ is isometric for $n\geq 0$. Fix $z, w\in\Omega_{n+1,f}$. If $|z-w|>\mu_g(z)\cdot\tilde{\nu}_g(z)$, then $|z-w|>\mu_g(w)\cdot\tilde{\nu}_g(w)$. Indeed, if follows from Remarks 4 and 6 that $|z-w|\leq\mu_g(w)\cdot\tilde{\nu}_g(w)$ implies that $\mu_g(w)=\mu_g(z)$ and $\tilde{\nu}_g(w)=\tilde{\nu}_g(z)$. Hence it follows from Lemma 4.3 that

$$\begin{aligned}|h_{n+1}(z)-h_{n+1}(w)| &= |h_{n+1}(z)-z+z-w+w-h_{n+1}(w)| \\ &= |z-w|.\end{aligned}$$

Otherwise, by Lemma 4.3,

$$|h_{n+1}(z)-h_{n+1}(w)|\leq\mu_g(h_{n+1}(w))\cdot\tilde{\nu}_g(h_{n+1}(w)).$$

It follows from Lemma 4.4 that

$$|f(z)-f(w)| = |f'(z)|\cdot|z-w|$$

and

$$|g(h_{n+1}(z))-g(h_{n+1}(w))| = |g'(h_{n+1}(z))|\cdot|h_{n+1}(z)-h_{n+1}(w)|.$$

Therefore

$$\begin{aligned}|f'(z)|\cdot|z-w| &= |f(z)-f(w)| \\ &= |h_n\circ f(z)-h_n\circ f(w)| \\ &= |g\circ h_{n+1}(z)-g\circ h_{n+1}(w)| \\ &= |g'(h_{n+1}(z))|\cdot|h_{n+1}(z)-h_{n+1}(w)| \\ &= |g'(z)|\cdot|h_{n+1}(z)-h_{n+1}(w)| \\ &= |f'(z)|\cdot|h_{n+1}(z)-h_{n+1}(w)|.\end{aligned}$$ □

Let us end this section with the proof of Lemma 3.6.

Proof of Lemma 3.6. It follows from Remark 5 that there exists a repelling periodic point $x\in\Omega_{0,f}$ of f with period m. Since $y:=h(x)$ is a periodic point of g with period m and $\{g^k(y)\}_{k=1}^{m}\subset\Omega_{\infty,g}$,

$$|(g^m)'(y)|>1.$$

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In particular, this implies that $\mathcal{J}_g \neq \emptyset$. Moreover, it follows from Proposition 2.2(2) that

$$h(\mathcal{J}_f) = h\left(\overline{\bigcup_{k=0}^{\infty} f^{-k}(\{x\})}\right) = \overline{\bigcup_{k=0}^{\infty} h(f^{-k}(\{x\}))} = \overline{\bigcup_{k=0}^{\infty} g^{-k}(\{y\})} = \mathcal{J}_g.$$

We next see that $\Omega_{\infty,g} = \mathcal{J}_g$. By Lemma 4.3,

$$\Omega_{0,f} = \bigcup_{z\in\mathcal{J}_g} B(z;\ \tilde{\nu}_g(z)).$$

Then we can show that

$$\Omega_{n,g} = \bigcup_{z\in\mathcal{J}_g} B\left(z;\ \prod_{k=0}^{n-1} \mu_g(g^k(z)) \cdot \tilde{\nu}_g(z)\right)$$

for any $n \in \mathbb{N}$ and $\Omega_{\infty,g} = \mathcal{J}_g$ in the same manner as in the proof of Lemma 3.1. □

Remark 7. In the proof of Lemma 3.6, we can also prove that y is a repelling periodic point of g with period m by evaluating $|(g^m)'(y)|$, as follows. By Lemma 4.3,

$$|h(z) - z| \leq \mu_g(z) \cdot \tilde{\nu}_g(z)$$

for any $z \in \Omega_{\infty,f}$. Moreover, since f' has no zero in $B(z;\ \mu_g(z) \cdot \tilde{\nu}_g(z))$,

$$|f'(h(z))| = |f'(z)|.$$

Therefore,

$$|(g^m)'(y)| = \prod_{k=0}^{m-1} |f'(h \circ f^k(x))| = \prod_{k=0}^{m-1} |f'(f^k(x))| = |(f^m)'(x)| > 1.$$

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