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On Generalised Single-Heading Navigation

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Introducing the notion of a pseudoloxodrome, we generalise a single-heading navigation to conformally flat Riemannian manifolds, under the action of a perturbing vector field (wind, current) of arbitrary force. The findings are applied to time-optimal navigation with the use of the Euler–Lagrange equations. We refer to the Zermelo navigation problem admitting space and time dependence of both a perturbation and a ship's speed. The necessary conditions for single-heading time-optimal navigation are obtained and the pseudoloxodromes of minimum and maximum time are discussed. Furthermore, we describe winds which yield the pseudoloxodromic and loxodromic time extremals. Our research is also illustrated with the examples in dimension two emphasising the single-heading solutions among the time-optimal trajectories in the presence of some space-dependent winds.

KEYWORDS

1. Single-Heading Navigation. 2. Pseudoloxodrome. 3. Zermelo Navigation Problem. 4. Time-Optimal Path.

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1. INTRODUCTION AND MOTIVATION. Roughly speaking, a single-heading (also known as constant-heading) navigation is a strategy applied in dimension two with only one angular control of a ship, i.e., *a heading* (a steering angle). This is the angle subtended by the ship's self-velocity and an axis of the orthogonal coordinate system, which is constant with respect to a flowing medium (air, water) during a travel (flight, sea passage) between two waypoints of a route and under the action of a perturbing wind or current. In the absence of perturbation, a single-heading navigation is equivalent to a loxodromic (rhumb line) navigation. The latter is widely applied and essential, in particular, in marine passage planning including traditional and electronic navigational charts and systems, e.g., Electronic Chart Display and Information System (ECDIS) (Kos et al., 1999; Norris, 2010; Tseng et al., 2012; Weintrit and Kopacz, 2012). In navigation based on a loxodrome, a ship keeps a constant course in the presence of wind or/and current with respect to the fixed ground, while the corresponding heading angle varies in general. Thus, on the rotational surface model, a course line (a track) crosses the meridians or parallels of latitude at the

same angle. In both cases the distance travelled is not the shortest, since this is represented by the orthodromic passage (great circle, great ellipse, geodesic navigation) on the spherical or spheroidal model of the Earth, where both courses, i.e., the heading and the direction of a tangent vector to the trajectory (course over ground), are variable during the passage (see Earle, 2006; Weintrit and Kopacz, 2011; Pallikaris and Latsas, 2012). The strategy is convenient from the navigational point of view; however, it does not guarantee an optimal travel in the sense of time. One can say that the ship remains parallel to itself and is carried freely by the flow, but with self-speed (airspeed) non-zeroed, so the ship's engine is not stopped. By analogy, we will call such a travelled single-heading path (in arbitrary dimension) *a pseudoloxodrome*.

Many of the properties of the ship's motion investigated in a two-dimensional stationary field of flow are of great practical interest for navigation both at sea and in the air (see, e.g., De Jong, 1974). The single-heading path passing through the starting and destination points is unambiguously defined for a given heading. It is, however, possible that several single-heading paths pass through both positions, but for different values of the constant heading. Apart from its simple steering principle, such a strategy had in practice also other benefits. Thus, it was in many cases preferred to other methods of navigation. For example, in aviation the lateral drift may often become so large that areas of bad weather are avoided. According to De Jong (1956) the time of navigation along a single-heading trajectory is generally shorter than along the geometrically shortest route.

The systems or techniques of navigation can refer to different skills that involve the determination of position and direction, as well as covering positioning and orienteering by comparison to known locations or patterns. An interesting aspect of the problem in nature concerns the navigation strategies in the presence of a perturbing medium and the corresponding skills of migrating animals, that is, swimmers and flyers. Recent bioscience works have introduced a new perspective in the analysis of wildlife tracking datasets, with different animal groups potentially exhibiting different levels of complexity in goal attainment during migration (Hays et al., 2014). There has been a lot of published work looking at the ability of flying and swimming animals to deal with cross-flows to reach their goal (e.g., Krupczynski and Schuster, 2008; Chapman et al., 2011). Recently some focus has been given to navigational strategies including (sub)optimal tracks and single-heading routes in animal ecology or avian biology (see McLaren et al., 2014; 2016; Chapman et al., 2015). The animals' route strategies between two locations may affect short daily passages and long-distance travel that vary spatially and seasonally, e.g., from the place where they rest to another place where they feed. There is an a priori expectation that it may be difficult for animals migrating at sea to assess current flows due to the general absence of fixed reference points, and hence achieving the time-minimal route may be very challenging (Chapman et al., 2011). For example, in Hays et al. (2014) two simple planar strategies tested two rules for swimming direction: heading adjusted to the goal (i.e., goal-oriented strategy) and constant heading to the goal (i.e., single-heading strategy).

We remark that, before modern technology including navigational equipment and positioning systems was available, a single-heading flight had obvious advantages from the point of view of the pilot. Having assumed in practice that the airplane was flying on an isobaric surface, namely the wind was geostrophic and unchanging in time, and that the Coriolis parameter was the same everywhere, an additional advantage for navigation was that one coordinate of the airplane's position could always be determined easily if the airplane was flying a constant-heading course. It has been shown that the single-heading flight is neither necessarily faster nor necessarily slower than the straight-line flight (Arrow, 1949).

In our study we aim to extend the single-heading strategy to non-Euclidean backgrounds and higher dimensions, considered on conformally flat Riemannian manifolds in the presence of a perturbing vector field. We admit both a perturbation and a ship's self-speed as functions of position and time. The paper is organised as follows. Section 2 is dedicated to preliminaries, including the background of the problem under consideration and time-extremal navigation in an arbitrary wind. We recall the condition (6) for optimal navigation, which stands for the generalisation of the classical navigation formula of Zermelo. This allows us to distinguish locally time-minimal and time-maximal paths (Proposition 2.1). We also describe generalised loxodromic navigation, emphasising time-extremal loxodromes (Proposition 2.2). In Section 3 we introduce the notion of a pseudoloxodrome and study the generalisation of single-heading navigation (Propositions 3.2 and 3.3). Next, we obtain the necessary conditions for pseudoloxodromic time-extremal navigation (Theorem 3.4) and discuss pseudoloxodromes of minimum and maximum time (Theorem 3.5) referring to the Zermelo navigation problem. Furthermore, we focus on perturbations that generate time-extremal pseudoloxodromes (Proposition 3.6) and time-extremal loxodromes (Proposition 3.7). Finally, Section 4 presents some examples in dimension two, where pseudoloxodromes are emphasised among the time-optimal paths in the presence of space-dependent winds.

2. PRELIMINARIES.

2.1. *Background of the navigation problem under consideration.* We begin with some preliminaries concerning the description of the background for the study of the single-heading navigation problem. Let (M, h) be a Riemannian manifold of dimension n that is locally conformal to n-dimensional Euclidean space (*conformally flat Riemannian manifold*), i.e., the positive definite fundamental metric tensor h_{ij} of the Riemannian metric h satisfies

$$h_{ij}(x) = \frac{1}{S^2(x)}\delta_{ij}, \quad i,j = 1, \ldots, n \tag{1}$$

for any $x \in M$, where the conformal factor is represented by a positive smooth function S defined on M and δ_{ij} is the Kronecker delta; see also Catino (2016). Let $(x^1, \ldots, x^n)$ be the local coordinates of $x \in M$. The tangent space T_xM is spanned by $\partial/\partial x^1, \ldots, \partial/\partial x^n$ and so any $v \in T_xM$ can be expressed as $v = \sum_{i=1}^{n} v^i(\partial/\partial x^i)$, where $v^1, \ldots, v^n$ are the coordinates of v. The length of v with respect to h is

$$|v|_h = \sqrt{\sum_{i,j=1}^{n} h_{ij}\, v^i v^j} = \frac{1}{S}\sqrt{\sum_{i=1}^{n}(v^i)^2}$$

or, briefly, $|v|_h = (1/S)\sqrt{(v^i)^2}$ is a sum with respect to $i = 1, \ldots, n$, since $(v^i)^2 = v^i v^i$. From this point on, in each product where the same index appears twice, it should be read as a sum with respect to this index without writing the explicit symbol Σ for the sum.

Let $M \times \mathbb{R}$ be the fibred manifold with a point (x, t) on $M \times \mathbb{R}$, having the local coordinates $(x^1, \ldots, x^n, t)$, $t \in \mathbb{R}$. Instead of a smooth curve on $M \times \mathbb{R}$, we simply consider

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its projection on M, namely, a smooth curve $\gamma : [0, 1] \subset \mathbb{R} \to M$, and the variable t is thought of as a parameter to which is ascribed no geometrical significance (Carathéodory, 1935; Aldea and Kopacz, 2020). The points of the curve γ are denoted by $(x^1(t), \ldots, x^n(t))$, $t \in [0, 1]$, and for the tangent vector to γ we use the notation

$$\dot{x}^i(t) = \frac{dx^i(t)}{dt}, \quad i = 1, \ldots, n.$$

A time-dependent tangent vector to M can be considered as the projection of a tangent vector to $M \times \mathbb{R}$, e.g., $\xi \in T_xM \times \mathbb{R}$ and $\xi = \xi^i(x, t)\partial/\partial x^i + \partial/\partial t$; see also Paláček and Krupková (2012) and Aldea and Kopacz (2020) in this regard.

The ship will be modelled as a particle moving at variable speed with respect to water (air) on (M, h). Under the influence of perturbation W, the ship's resulting velocity will be given by the composed vector $v = W + u$, with the ship's self-velocity $u \neq 0$. Here we consider space- and time-dependent wind W, i.e., $W = W^i(x, t)\partial/\partial x^i \in T_xM$ is the projection of $W^i(x, t)\partial/\partial x^i + \partial/\partial t \in T_xM \times \mathbb{R}$. Also, the ship's self-speed is $|u|_h = f(x, t) \in (0, 1]$ and W^i, $i = 1, \ldots, n$, and f are smooth functions on $M \times \mathbb{R}$. Using the coordinates $(x^1(t), \ldots, x^n(t))$, the resulting velocity v is the tangent vector to the trajectory, and so the global motion $v = W + u$ can be rewritten in the local coordinates as

$$\dot{x}^i(t) = W^i + u^i, \quad i = 1, \ldots, n, \tag{2}$$

which are called the *equations of motion*.

Let $\cos\theta_i(t)$ stand for the directional cosines of the ship's velocity $u(x, t)$. The assumption of local flat conformality allows us to follow an analogous method to that in Arrow (1949) by considering the functions $\alpha_i(x, t) = S(x)\cos\theta_i(t)$, $i = 1, \ldots, n$, with $\sum_i(\alpha_i)^2 = S^2$. It follows that $u^i = f\alpha_i$ and then the equations of motion (2) can be rewritten as

$$\dot{x}^i(t) = W^i + f\alpha_i, \quad i = 1, \ldots, n. \tag{3}$$

The condition $|u|_h^2 = h_{ij}u^iu^j = f^2$ is checked. Further on, under the above setting, the triple (h, W, f) will be called the *navigation data*, which creates the background for the study of the pseudoloxodromic navigation problem.

2.2. *Time-optimal paths.* Now, we recall Zermelo's navigation problem,[1] where the aim is to find the time-minimal trajectories, or rather the corresponding steering angles (optimal controls), of a ship (an aerial vehicle) that sails or flies in a space M, under the influence of a perturbation represented by a vector field W, thought of as a wind or a current. The problem was initially formalised and solved by Ernst Zermelo in 1929–1931 for Euclidean spaces of low dimensions, i.e., $\mathbb{R}^2$ (Zermelo, 1930) and $\mathbb{R}^3$ (Zermelo, 1931). Its new link to Finsler geometry has caused purely geometric investigations recently, with the background model created by a Riemannian manifold (M, h), where h is an arbitrary Riemannian metric (Bao et al., 2004; Chern and Shen, 2005). However, the Finslerian studies were developed under the assumptions that the wind W is time-independent and only weak or critical, i.e., $|W|_h^2 = h(W, W) \leq 1$, as well as a ship's self-speed is constant, $f = 1$; see also Yoshikawa and Sabau (2013), Brody and Meier (2015), Brody et al. (2015), Caponio

[1] 'In an unbounded plane where the wind distribution is given by a vector field as a function of position and time, a blimp or an airplane moves with constant speed relative to the surrounding air mass. How must the ship be steered in order to come from a starting point P_0 to another P_1 in the shortest time?' (Zermelo, 1930).

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et al. (2015), Aldea and Kopacz (2017a;b), Kopacz (2017a;b), Javaloyes and Vitório (2018) and Kopacz (2019) for some recent and generalised investigations in differential geometry and physics.

In the field of optimal control, the Zermelo problem is usually taken with application of Pontryagin's maximum principle (Pontryagin et al., 1962). In what follows we proceed via the Lagrangian and this way is in general equivalent to the discussion on the navigation problem via the Hamiltonian; see Bijlsma (2001; 2009), Hull (2009), Techy and Woolsey (2009), Bijlsma (2010), Techy (2011), Jardin and Bryson (2012), Burns (2013), Li et al. (2013) and Marchidan and Bakolas (2016). However, which of them is more convenient in the sense of solution and computational complexity depends in fact on the specific navigation data. Following the general principles from Levi-Civita (1931), De Mira Fernandes (1932) and Arrow (1949), however, in the more general case due to the extended navigation data (h, W, f), recently we have studied the Lagrangian $L(x(\tau), \dot{x}(\tau), t(\tau))$ in Kopacz (2018b) and Aldea and Kopacz (2019a), which is a positive root of the following equation:

$$\lambda L^2 + 2pL - |v|_h^2 = 0, \tag{4}$$

where $\lambda = f^2 - |W|_h^2$,

$$p = S^{-2} \sum_i W^i \dot{x}^i, \quad |W|_h^2 = S^{-2} \sum_i (W^i)^2, \quad |v|_h^2 = S^{-2} \sum_i (\dot{x}^i)^2$$

and $L(x(t), \dot{x}(t), t) = 1$ along the curves that satisfy Equation (2). Note that the case $\lambda \neq 0$ (namely, Equation (4) is of degree two) refers to a *weak wind* $(\lambda > 0)$ or a *strong wind* $(\lambda < 0)$, while $\lambda = 0$ (namely, Equation (4) is of degree one, under the restriction $p \neq 0$) refers to a *critical wind*. Since $L(x(\tau), \dot{x}(\tau), t(\tau))$ is homogeneous of degree one with respect to $\dot{x}$, it allows us to compute the length, i.e., $T = \int_0^1 L(x(\tau), \dot{x}(\tau), t(\tau))\, d\tau$, which is interpreted here as the physical time necessary to travel along the extremals from a starting point $(x^1(0), \ldots, x^n(0))$ to a destination point$(x^1(1), \ldots, x^n(1))$. The first variation of T, under the condition $dt = L(x(\tau), \dot{x}(\tau), t(\tau))\, d\tau$, leads to the Euler–Lagrange equations (see, e.g., Levi-Civita, 1931), that is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = \frac{\partial L}{\partial t}\frac{\partial L}{\partial \dot{x}^i}, \qquad i = 1, \ldots, n. \tag{5}$$

Applying Equation (5) to the Lagrangian L given by Equation (4), we have recently obtained the conditions for time-extremal navigation considered on conformally flat Riemannian manifolds. Namely, the general solution to time-extremal navigation problem on (M, h) is represented by Equation (3) and the condition

$$\begin{aligned}\frac{d\alpha_i}{dt} &= -S^2\left(\frac{\partial f}{\partial x^i} + \frac{1}{S^2}\frac{\partial W^j}{\partial x^i}\alpha_j\right) + \frac{1}{S^2}\frac{\partial W^k}{\partial x^j}\alpha_i\alpha_j\alpha_k + \frac{\partial f}{\partial x^j}\alpha_i\alpha_j - Sf\frac{\partial S}{\partial x^i}\\ &\quad + \frac{1}{S}\frac{\partial S}{\partial x^j}(W^j + 2f\alpha_j)\alpha_i, \qquad i = 1, \ldots, n,\end{aligned} \tag{6}$$

under the restriction $\lambda + p \neq 0$, where $\lambda + p = f^2 - |W|_h^2 + p$. Note that the inequality $\lambda + p > 0$ can hold for any wind, but if a wind is weak or critical, then $\lambda + p > 0$. Also, if $\lambda + p \leq 0$, then a wind can only be strong. The curves $x(t)$ that satisfy Equation (3) and $\lambda +$

$p = 0$ are called *anomalous* (or *abnormal*) and Equation (6) may hold; see, e.g., Agrachev and Sachkov (2004) and Carathéodory (1935) in this regard.

Equation (6) is the generalisation of the classical formula of Zermelo, namely, the condition for optimal navigation. We remark that by time-optimal solutions to the navigation problem we understand both minimum and maximum time. So, in this sense, we may consider Zermelo's problem as the particular case of the navigation problem. The solutions of least time are of main interest in practice and theoretical studies, but for a complete exposition concerning optimal navigation in arbitrary winds it is necessary to take into consideration also time-maximal paths. This is due to the fact that the same Equation (6) together with Equation (3) can generate both types of time extremals in strong winds. Thus, they need to be distinguished.

Additional conditions for the time-minimal (time-maximal) solutions are provided by the study of the second variation of T. Namely, some classification results on both types of extremal paths with respect to the navigation data, so in particular the wind force $|W|_h$, are included in the following proposition (Kopacz, 2018b; Aldea and Kopacz, 2019a).

Proposition 2.1. *Let (M, h) be a conformally flat Riemannian manifold of dimension n, where h is given by Equation (1). Let (h, W, f) be the navigation data with $f = f(x)$ and $W = W(x)$. Then*

1. *if $\lambda + p > 0$ for any wind W, then Equations (3) and (6) yield the local solutions of the Zermelo navigation problem on (M, h); and*
2. *if $\lambda + p < 0$, then the wind is strong and Equations (3) and (6) yield the local time-maximal solutions to the navigation problem on (M, h).*

2.3. *Time-optimal generalised loxodromic navigation.* By a *generalised loxodromic navigation* we call a navigation with a constant direction of the resultant velocity v during the entire travel on (M, h) between a starting point and a terminal point, i.e., $d\tilde{\beta}_i/dt = 0$, for all $i = 1, \ldots, n$, where $\tilde{\beta}_i := \cos\tilde{\theta}_i(t)$ are the directional cosines of the ship's resultant velocity $v \neq 0$. The corresponding resulting trajectory on M under the above-mentioned conditions will be called a *generalised loxodrome*. Also, by $(h, W, f, \tilde{\beta}_i = \text{const.})$, $i = 1, \ldots, n$, we mean the *loxodromic (navigation) data*. In this subsection we recall some results from Aldea and Kopacz (2019b) and Kopacz (2018b). Expressing the ship's resultant velocity $v \neq 0$ in the form

$$\dot{x}^i = g(x,t)S\tilde{\beta}_i, \quad i = 1, \ldots, n, \tag{7}$$

where $\dot{x}^i = v^i$ and $g = |v|_h$ (a non-zero smooth function on $M \times \mathbb{R}$) is not known a priori like $f = |u|_h$, and combining Equations (7) and (3), we obtain

$$\frac{d\alpha_i}{dt} = -\frac{1}{f}\left(\frac{\partial W^i}{\partial t} + \frac{\partial f}{\partial t}\alpha_i\right) - \frac{gS}{f}\left(\frac{\partial W^i}{\partial x^j} + \frac{\partial f}{\partial x^j}\alpha_i\right)\tilde{\beta}_j + \frac{S}{f}\frac{dg}{dt}\tilde{\beta}_i + \frac{g^2 S}{f}\frac{\partial S}{\partial x^j}\tilde{\beta}_i\tilde{\beta}_j, \tag{8}$$

with $i = 1, \ldots, n$, where

$$\begin{aligned}\frac{dg}{dt} &= g\rho\frac{\partial W^k}{\partial t}\alpha_k + g\rho S^2\frac{\partial f}{\partial t} + g^2\rho S\frac{\partial W^k}{\partial x^j}\alpha_k\tilde{\beta}_j + g^2\rho S^3\frac{\partial f}{\partial x^j}\tilde{\beta}_j \\ &\quad + g^2\left(f\rho S^2 - 1\right)\frac{\partial S}{\partial x^j}\tilde{\beta}_j,\end{aligned} \tag{9}$$

and $\rho := f/(S^2(\lambda + p))$, under the loxodromic data and $\lambda + p \neq 0$.

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Proposition 2.2. *Let (M, h) be a conformally flat Riemannian manifold of dimension n, where h is given by Equation (1). Let $(h, W, f, \tilde{\beta}_i = \text{const.})$, $i = 1, \ldots, n$, be the* loxodromic navigation data *with $f = f(x)$ and $W = W(x)$, and let g denote a ship's resulting speed $|v|_h \neq 0$. Then*

1. *if $\lambda + p > 0$ for an arbitrary wind W, then the solutions of the system of Equations (3), (6), (7), (8) and (9) yield the time-minimal generalised loxodromes on (M, h); and*
2. *if $\lambda + p < 0$, then the wind is strong and the solutions of the system of Equations (3), (6), (7), (8) and (9) yield the time-maximal generalised loxodromes on (M, h).*

For short, we also write 'a loxodrome' instead of 'a generalised loxodrome' when no confusion can arise.

3. GENERALISED SINGLE-HEADING NAVIGATION.

Let (M, h) be a Riemannian manifold of dimension n with the metric h given by Equation (1) and let (h, W, f) be the navigation data.

3.1. *Generalisation of single-heading navigation.*

Definition 3.1. A *generalised single-heading (or pseudoloxodromic) navigation* is a navigation with a constant direction of the ship's self-velocity u during a travel on (M, h) between a starting point and a terminal point, i.e.,

$$\forall\, i = 1, \ldots, n \quad \theta_i(t) = \text{const.} \tag{10}$$

By analogy to widely applied rhumb line navigation in practice and the occurrence of the well-known notion of a loxodrome in different theoretical studies, a path on M satisfying Equation (10) will be called a *pseudoloxodrome* or a *generalised single-heading trajectory*.

To investigate u with constant direction, we denote by $\beta_i(t)$ the directional cosines of the ship's velocity $u(x, t)$, i.e., $\beta_i(t) := \cos\theta_i(t)$, $i = 1, \ldots, n$. This leads to $\sum_i (\beta_i)^2 = 1$ and

$$\alpha_i = S(x)\beta_i(t). \tag{11}$$

Hence, the equivalent conditions for the pseudoloxodromic navigation are

$$\forall\, i = 1, \ldots, n \quad \frac{d\beta_i}{dt} = 0. \tag{12}$$

By $(h, W, f, \beta_i = \text{const.})$, $i = 1, \ldots, n$, we call this the *pseudoloxodromic navigation data*.

Now, we aim to find the generalised single-heading trajectories. Thus, we have the following.

Proposition 3.2. *Let (M, h) be a conformally flat Riemannian manifold of dimension n, where h is given by Equation (1). Let $(h, W, f, \beta_i = \text{const.})$, $i = 1, \ldots, n$, be the pseudoloxodromic navigation data. If there exist the pseudoloxodromes on (M, h), then they are the solutions of the system of Equations (3) and*

$$\frac{d\alpha_i}{dt} = \frac{1}{S}\frac{\partial S}{\partial x^j}(W^j + f\alpha_j)\alpha_i, \quad i = 1, \ldots, n. \tag{13}$$

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Proof. From Equation (11) it results that

$$\frac{d\alpha_i}{dt} = \frac{\partial S}{\partial x^j}\dot{x}^j \beta_i + S\frac{d\beta_i}{dt}.$$

Next, we require the direction of u to be constant, i.e., Equation (12) holds. This yields

$$\frac{d\alpha_i}{dt} = \frac{\partial S}{\partial x^j}\dot{x}^j \beta_i,$$

which together with Equation (3) leads to Equation (13). ■

Roughly speaking, the existence of the generalised single-heading paths depends on a wind, so in particular its 'force' $|W|_h$.

Note that the spherical coordinates denoted by φ_l, $l = 1, \ldots, n-1$, are more suitable for the navigational description of a ship's motion (steering, controls) than the angles θ_i, $i = 1, \ldots, n$. The latter fit better to the geometrical description of the motion determined by the vectors of the ship's velocities. Therefore, a coordinate system in an n-dimensional Euclidean space can be defined such that the coordinates consist of a radial coordinate, r, and $n-1$ angular coordinates, $\varphi_1, \ldots, \varphi_{n-1}$, where $\varphi_2, \ldots, \varphi_{n-1} \in [-\pi/2, \pi/2]$ and $\varphi_1 \in [0, 2\pi)$. If x^i, $i = 1, \ldots, n$, represent the Cartesian coordinates, then they can be computed from r and $\varphi_1, \ldots, \varphi_{n-1}$. Next, we can find the relation between the directional cosines $\beta_i = \cos\theta_i$ and φ_l. Taking into consideration that β_i is the component of the versor of a given vector, so $r = 1$ and $x^i = \beta_i$, we have the following relation for the ship's velocity u:

$$\beta_k = \prod_{i=k}^{n} \cos\varphi_i \sin\varphi_{k-1}, \tag{14}$$

where $k = 1, \ldots, n$, $\varphi_0 = \pi/2$ and $\varphi_n = 0$. Making use of the spherical coordinates we can prove the following.

Proposition 3.3. *Let (M, h) be a conformally flat Riemannian manifold of dimension n, where h is given by Equation (1). Let $\varphi_m \neq \pm\pi/2$, $m = 2, \ldots, n-1$. The necessary and sufficient conditions for pseudoloxodromic navigation on (M, h) are $\varphi_l = \text{const.}$, $l = 1, \ldots, n-1$.*

Proof. If $\beta_i = \text{const.}$, $i = 1, \ldots, n$, and $\varphi_m \neq \pm\pi/2$, $m = 2, \ldots, n-1$, then by Equation (14) we have $\varphi_l = \text{const.}$, $l = 1, \ldots, n-1$. The converse is obvious. ■

Regarding the above restriction excluding $\pm\pi/2$ see, for instance, dimension three. Namely, take $\theta_1 = \theta_2 = \pi/2 = \text{const.}$, and so $\theta_3 = 0 = \text{const.}$ Then $\varphi_2 = \pi/2 = \text{const.}$, but φ_1 is arbitrary, so not necessarily constant. Thus, in dimension three, by application of the spherical coordinates the three angles of the directional cosines, i.e., θ_1, θ_2 and θ_3, are reduced by Equation (14) to two spherical coordinates. These are the angular controls, i.e., the heading $\varphi_1 := \varphi$ and the elevation $\varphi_2 := \vartheta$. The navigation with $\varphi_1(t) = \text{const.}$ and $\varphi_2(t) \neq \text{const.}$ is not considered here as the generalised single-heading navigation, although the (common) heading is constant in time. This is due to the fact that the (three-dimensional) direction of the ship's velocity u is varying. In higher dimensions, our

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reasoning is analogous. We can also consider a piecewise pseudoloxodromic route (flight), i.e., with different pseudoloxodromic legs between the waypoints.

In dimension two, the heading is $\varphi := \theta_1$. Since $\theta_1 + \theta_2 = \pi/2$, we get $\beta_1 = \cos\varphi$ and $\beta_2 = \sin\varphi$, and so $\alpha_1 = S\cos\varphi$ and $\alpha_2 = S\sin\varphi$. In this case Equation (13) means that $-\dot{\varphi}\sin\varphi = 0$ and $\dot{\varphi}\cos\varphi = 0$, which yield $\dot{\varphi} = 0$ and thus $\varphi = \varphi_0 =$ const. The paths with $\varphi_0 =$ const. are the solutions of the system

$$\dot{x}^1 = W^1 + fS\cos\varphi_0, \quad \dot{x}^2 = W^2 + fS\sin\varphi_0. \tag{15}$$

3.2. *Time-optimal pseudoloxodromes.* When the single-heading navigation (in dimension two) was introduced for the first time, it was taken for granted that the resulting trajectories themselves were extremals and that, consequently, along these trajectories the time of navigation would assume an extreme value. Although, in general, the time of navigation is shorter along a single-heading trajectory than along the geometrically shortest route, it is only in fields of flow with a particular structure that the single-heading trajectories are time-minimising or time-maximising extremals. The single-heading trajectories in $\mathbb{R}^2$, which at the same time are also extremals, were called 'single-heading extremals' in De Jong (1956). We would like to mention that the planar examples of time-minimal trajectories that are single-heading have already been emphasised in Zermelo (1930), Levi-Civita (1931), Arrow (1949) and Carathéodory (1935) in the very first studies on the navigation problem.

We thus arrive at the following theorem.

Theorem 3.4. *Let (M, h) be a conformally flat Riemannian manifold of dimension n, where h is given by Equation (1). Let (h, W, f) be the navigation data and $\lambda + p \neq 0$. The necessary conditions for a pseudoloxodromic time-extremal navigation on (M, h) are*

$$\left(\frac{\partial W^k}{\partial x^j}\beta_k + \frac{\partial (fS)}{\partial x^j}\right)\beta_{ij} = 0, \quad i = 1, \dots, n, \tag{16}$$

where $\beta_{ij} := \delta_{ij} - \beta_i\beta_j$.

Proof. Taking into consideration the assumption for the constant 'headings', i.e., Equation (13) with $\alpha_i = S\beta_i$ and inserting this into Equation (6), we obtain

$$\begin{aligned}\frac{\partial S}{\partial x^j}(W^j + fS\beta_j)\beta_i = -S^2\left(\frac{\partial f}{\partial x^i} + \frac{1}{S}\frac{\partial W^j}{\partial x^i}\beta_j\right) + S\frac{\partial W^k}{\partial x^j}\beta_i\beta_j\beta_k + S^2\frac{\partial f}{\partial x^j}\beta_i\beta_j \\ - Sf\frac{\partial S}{\partial x^i} + \frac{\partial S}{\partial x^j}(W^j + 2fS\beta_j)\beta_i,\end{aligned} \tag{17}$$

which gives Equation (16). ∎

Note that β_{ij}, $i, j = 1, \dots, n$, is not an invertible matrix, but it is symmetric ($\beta_{ij} = \beta_{ji}$, $i, j = 1, \dots, n$) and $\beta_{ij}\beta_i = 0$, $j = 1, \dots, n$. Equations (3) and (16) determine the time-optimal pseudoloxodromes, if they exist.

Now, taking together Proposition 2.1 and Theorem 3.4, we have the next result.

Theorem 3.5. *Let (M, h) be a conformally flat Riemannian manifold of dimension n, where h is given by Equation (1). Let $(h, W, f, \beta_i = \text{const.})$, $i = 1, \dots, n$, be the pseudoloxodromic navigation data with $f = f(x)$ and $W = W(x)$. Then*

1. *if* $\lambda + p > 0$ *for an arbitrary wind W, then the system of Equations* (3)*,* (6) *and* (16) *yields the time-minimal pseudoloxodromes on* (M, h)*; and*
2. *if* $\lambda + p < 0$*, then the wind is strong and the system of Equations* (3)*,* (6) *and* (16) *yields the time-maximal pseudoloxodromes on* (M, h)*.*

3.3. *Which winds yield the pseudoloxodromic time-optimal navigation?.* Recall that, in the time-extremal navigation problem, one normally aims to find the resulting trajectory or the corresponding optimal control for given navigation data including wind. However, we can also look at the problem from a different point of view. Namely, we can look for the vector fields W in the presence of which the time-extremal solutions have some special properties. For instance, all of them are pseudoloxodromic or loxodromic, or both. We should mention that, under the considered navigation data (with given W) by Proposition 2.2 and Theorem 3.4, one can obtain some pseudoloxodromic and loxodromic paths among the family of time-extremal trajectories, respectively, if they exist. Namely, not all of them are of such special kind.

To extend the study, further on we ask about the types of perturbations in which all existing solutions of extreme time are generalised single-heading or loxodromic, or both at the same time. Thus, we can reformulate Theorem 3.4 in the following way.

Proposition 3.6. *Let* (M, h) *be a conformally flat Riemannian manifold of dimension n, where h is given by Equation* (1)*. Let* $u(x, t)$ *be a ship's self-velocity with* $|u|_h = f(x, t) \in (0, 1]$ *and the directional cosines* $\beta_i = \text{const.}$*,* $i = 1, \ldots, n$*. Then the wind* $W = W^i(x, t)\partial/\partial x^i \in T_xM$*, with* $0 \leq |W|_h$ *and* $\lambda + p \neq 0$*, under which all time-extremal paths yielded by Equations* (3) *and* (6) *are pseudoloxodromic, is the solution of the system of partial differential equations (PDEs)* (16)*.*

Proof. Considering Equations (3) and (6) with unknown wind W as well as combining them with the assumption $\beta_i = \text{const.}$, this leads to Equation (16). ■

In particular, we have the following remarks. Namely, if $f = \text{const.}$, i.e., the standard version of Zermelo's problem (with constant maximum ship's self-speed) or if $S = \text{const.}$, i.e., the background metric is Euclidean, then the PDE system admits simplified forms. Considering $(\mathbb{R}^n, \delta_{ij})$ with $f = 1$, then $W = W(t)$ or $W^i = \text{const.}$, $i = 1, \ldots, n$, are the solutions of Equation (16). Thus, in this last case all existing time-extremal paths are pseudoloxodromic.

In dimension two, since the necessary and sufficient condition for a constant direction of velocity u is $\varphi_0 = \varphi(t) = \text{const.}$, the PDE system (16) is reduced to

$$\begin{aligned} 0 = &-\frac{\partial W^1}{\partial x^2}\cos^2\varphi_0 + \left(\frac{\partial W^1}{\partial x^1} - \frac{\partial W^2}{\partial x^2}\right)\sin\varphi_0\cos\varphi_0 + \frac{\partial W^2}{\partial x^1}\sin^2\varphi_0 \\ &+ \frac{\partial (fS)}{\partial x^1}\sin\varphi_0 - \frac{\partial (fS)}{\partial x^2}\cos\varphi_0, \end{aligned} \tag{18}$$

for any $\varphi_0 = \text{const.}$ and unknown wind components W^1 and W^2. So, the last equation leads to the types of perturbations under which the solutions to the time-extremal problem are single-heading. In particular, if $f = \text{const.}$ and $S = \text{const.}$, then Equation (18) yields the

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condition

$$\frac{\partial W^1}{\partial x^2}\cos^2\varphi_0 - \left(\frac{\partial W^1}{\partial x^1} - \frac{\partial W^2}{\partial x^2}\right)\sin\varphi_0\cos\varphi_0 - \frac{\partial W^2}{\partial x^1}\sin^2\varphi_0 = 0. \tag{19}$$

This is in fact the classical navigation formula of Zermelo (1930; 1931) considered for the constant heading in $\mathbb{R}^2$, $\varphi := \varphi_0$. The condition (19) implies that in $\mathbb{R}^2$ the only stationary winds W in which all the solutions to the Zermelo problem (f = const.) are single-heading are given by the radial, negative radial or constant vector fields. Thus, the above leads to the result from De Jong (1956), that is, 'all extremals are single-heading extremals in convergent or divergent and in uniform rectilinear fields of flow'. Such a theorem was proved by means of Hamilton's PDEs (see De Jong (1956; 1974) for more details and the investigation on the planar vector fields in which the single-heading extremals exist).

Now taking into consideration Proposition 2.2 we have the following.

Proposition 3.7. *Let (M, h) be a conformally flat Riemannian manifold of dimension n, where h is given by Equation (1). Let $u(x, t)$ be a ship's self-velocity with $|u|_h = f(x, t) \in (0, 1]$ and the directional cosines $\tilde{\beta}_i$ = const., $i = 1, \ldots, n$. Then the wind $W = W^i(x, t)\partial/\partial x^i \in T_xM$, with $0 \le |W|_h$ and $\lambda + p \neq 0$, under which all time-extremal paths yielded by Equations (3) and (6) are loxodromes, is the solution of the following PDE system:*

$$\begin{aligned}\left(\frac{\partial W^k}{\partial t} + gS\frac{\partial W^k}{\partial x^j}\tilde{\beta}_j\right)\tilde{\beta}_{ik} - fS\left(\frac{\partial W^k}{\partial x^j}\beta_k + \frac{\partial (fS)}{\partial x^j}\right)\beta_{ij} \\ = g\rho S^3\left(\frac{\partial f}{\partial t} + \frac{\partial (fS)}{\partial x^j}\tilde{\beta}_j\right)\beta_{ik}\tilde{\beta}_k,\end{aligned} \tag{20}$$

$i = 1, \ldots, n$, where $\tilde{\beta}_{ik} := \delta_{ik} - g\rho S^2\tilde{\beta}_i\beta_k$.

Proof. Substituting Equations (8) and (9) into the system of Equations (3) and (6) in which the wind W is unknown, results in Equation (20). ∎

Note that $\tilde{\beta}_{ik}\beta_i = 0$, $\tilde{\beta}_{ik}\tilde{\beta}_k = 0$ and $\beta_{ik}\tilde{\beta}_k = \tilde{\beta}_i - (1/(g\rho S^2))\beta_i$, $k = 1, \ldots, n$. Also $\tilde{\beta}_{ik}$, $i, k = 1, \ldots, n$, is not an invertible matrix. Comparing the PDE systems (16) and (20), under the assumption fS = const., we can emphasise the following fact. If the components of W are arbitrary time-dependent functions, then such wind is the solution of Equation (16). However, it may not solve Equation (20). Therefore, there exist time-extremal trajectories such that they are pseudoloxodromic, but they are not loxodromic.

Finally, by Propositions 3.6 and 3.7 we obtain the following corollary.

Corollary 3.8. *Let (M, h) be a conformally flat Riemannian manifold of dimension n, where h is given by Equation (1). Let $u(x, t)$ be a ship's velocity with $|u|_h = f(x, t) \in (0, 1]$ and the directional cosines $\tilde{\beta}_i$ = const., $i = 1, \ldots, n$. Then the wind $W = W^i(x, t)\partial/\partial x^i \in T_xM$, with $0 \le |W|_h$ and $\lambda + p \neq 0$, under which all time-extremal paths yielded by Equations (3) and (6) are both pseudoloxodromic and loxodromic, is the solution of the PDE system of Equations (16) and*

$$\left(\frac{\partial W^k}{\partial t} + gS\frac{\partial W^k}{\partial x^j}\tilde{\beta}_j\right)\tilde{\beta}_{ik} = g\rho S^3\left(\frac{\partial f}{\partial t} + \frac{\partial (fS)}{\partial x^j}\tilde{\beta}_j\right)\beta_{ik}\tilde{\beta}_k, \quad i = 1, \ldots, n. \tag{21}$$

Proof. Equation (21) results from Equations (16) and (20). ■

In order to emphasise some solutions to Equations (16) and (21), we consider the following cases. First, if $fS = \text{const.}$, then the wind W with $W^i = \text{const.}$, $i = 1, \ldots, n$, is the solution of Equations (16) and (21), and so all time-extremal trajectories are pseudoloxodromic and loxodromic, if they exist. Second, also under the assumption $fS = \text{const.}$, for the wind W with $W^i = ax^i + b_i$, with $a, b_i = \text{const.}$, $i = 1, \ldots, n$, all time-extremal trajectories are pseudoloxodromic and loxodromic. Indeed, since $\partial W^k/\partial t = 0$ and $\partial W^k/\partial x^j = \delta_{jk}$, which imply

$$\frac{\partial W^k}{\partial x^j}\beta_k\beta_{ij} = \beta_j\beta_{ij} = 0 \quad \text{and} \quad \frac{\partial W^k}{\partial x^j}\tilde{\beta}_j\tilde{\beta}_{ik} = \tilde{\beta}_j\tilde{\beta}_{ij} = 0,$$

we have Equations (16) and (21) identically checked. This last case is actually the generalisation of the two-dimensional result from De Jong (1956), i.e., the wind is radial ($a > 0$), negative radial ($a < 0$) or constant ($a = 0$).

4. EXAMPLES IN DIMENSION TWO.

For clarity, we now present two-dimensional examples with the focus on the single-heading time-optimal paths, under the action of some space-dependent winds.

4.1. *Euclidean plane with linear (shear) wind.* First, let (M, h) be $(\mathbb{R}^2, \delta_{ij})$ and let the perturbation be given by the shear river-type vector field, i.e., $W = (x^2, 0)$. Thus, the current of the 'river' increases as a linear function of x^2, reaching its minimum force in the midstream. We mention that Zermelo described the problem in the presence of such perturbation (actually he used $W = (-x^2, 0)$) as 'das einfachste nicht-triviale Beispiel unserer Theorie' ('the simplest nontrivial example of our theory') in his first paper (Zermelo, 1930), where the navigation problem was formalised initially, with $f = 1 = \text{const.}$ An example with the field and the use of the Hamiltonian formalism was also investigated in Carathéodory (1935). In particular, if $|W|_h = |x^2| < 1$, then the ship can arrive at any point of destination on the planar sea, so the solution exists, since the current is weak. Now, however, let $f(t, x^1, x^2) = 0{\cdot}95 \sin^2 x^2 + 0{\cdot}01$, so the ship's speed is varying in space and the restriction $0 < f \leq 1$ is checked. Applying Equation (6) we get the time-optimal trajectories from the system

$$\dot{\varphi} = -\cos^2\varphi - 0{\cdot}95 \sin 2x^2 \cos\varphi, \quad \dot{x}^1 = x^2 + f\cos\varphi, \quad \dot{x}^2 = f\sin\varphi. \tag{22}$$

Having solved the last system, we present the time extremals starting from $(0; -0{\cdot}5)$ for different headings $\varphi(0) = \varphi_0 \in [0; 360^\circ)$, i.e., with the increments $\Delta\varphi_0 = 15^\circ$. They are shown in Figure 1. Since we search the pseudoloxodromic extremals, by Equation (19) we get $\cos^2\varphi_0 + 0{\cdot}95 \sin 2x^2 \cos\varphi_0 = 0$. This implies that the single-heading navigation of minimum time is possible only if $\varphi_0 = 90^\circ$ or $\varphi_0 = 270^\circ$; $\lambda + p = f(f + x_0^2 \cos\varphi_0) > 0$. Thus, the strategy is to sail orthogonally to the flowing current W continuously. The single-heading paths of least time are presented in red and blue, respectively.

For comparison, we solve the problem for different ship's self-speed, which has the form $f(t, x^1, x^2) = (1/\sqrt{k})x^1 - C$, with $C = \text{const.}$ Then the domain is $x^1 \in (C\sqrt{k}; (1 + C)\sqrt{k}]$ due to the restriction on f, and from Equation (19) we get $\cos^2\varphi_0 = (1/\sqrt{k})\sin\varphi_0$. For example, let $k := 16$ and $C := -0{\cdot}33$, so $x^1 \in (-1{\cdot}32; 2{\cdot}68]$. The single-heading time-minimal paths among other time-optimal solutions are shown in Figure 2. Their desired

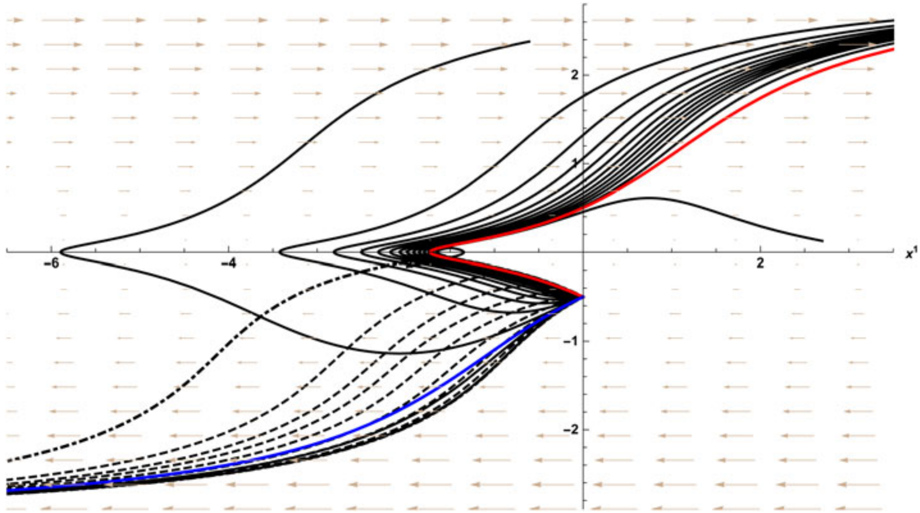

Figure 1. The only single-heading solutions to the time-optimal navigation problem with variable ship's speed $f(t, x^1, x^2) = 0{\cdot}95 \sin^2 x^2 + 0{\cdot}01$ and under the action of the linear vector field $W = (x^2, 0)$ are for the headings $\varphi_0 = 90°$ (time-minimal, red) and $270°$ (time-minimal, blue). In the background are other optimal paths: time-minimal (solid black) for $\varphi_0 \in (62{\cdot}8°, 297{\cdot}2°)$ and time-maximal (dashed black) for $\varphi_0 \in [0, 62{\cdot}8°) \cup (297{\cdot}2°, 360°)$, to an accuracy of $0{\cdot}1°$, obtained with the increments of the initial heading $\Delta\varphi_0 = 15°$. Also, there exist two anomalous paths (dot-dashed black) for $\varphi_0 \in \{62{\cdot}8°, 297{\cdot}2°\}$. The starting point $(0; -0{\cdot}5)$ is within the area of strong current; $t \leq 35$.

headings are: 62° (blue) and 118° (red). We remark that there can also exist single-heading extremals of both types (minimal and maximal), if we shift the initial point slightly, e.g., $x_0 := -0{\cdot}5$. Then the values for both pseudoloxodromic extremals are preserved, but the path with $\varphi_0 = 62°$ becomes time-maximal, since $\lambda + p = f(t_0, x_0^1, x_0^2)(f(t_0, x_0^1, x_0^2) + x_0^2 \cos\varphi_0) < 0$.

Finally, we also show the solutions when the ship proceeds with constant maximum speed, i.e., $f = 1$. Now the same starting point is within the area of weak wind and there are no time-maximal paths. The corresponding single-heading time-minimal trajectories are parabolic (solid red and solid blue in Figure 3).

4.2. *Prolate ellipsoid under weak rotational wind.* Let Σ^2 be an ellipsoid embedded in the Euclidean space $\mathbb{R}^3$, with the Cartesian coordinates (x, y, z), and axes equal to $2r$, $2r$ and $2ar$. The parametrisation of Σ^2 in the spherical coordinate system $(\tilde{\rho}, \phi, \theta)$ leads to the relations $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = ar\cos\theta$, where the azimuth $\phi \in [0, 2\pi)$ and the inclination (colatitude) $\theta \in [0, \pi]$. The parameter $a > 0$ determines the shape of an ellipsoid and as a consequence the flow of the geodesics on a spheroid, which can be oblate $(0 < a < 1)$ or prolate $(a > 1)$. The line element of Σ^2 expressed in terms of the spherical coordinates (ϕ, θ) on Σ^2 is $ds^2 = r^2\sin^2\theta(d\phi)^2 + r^2(\cos^2\theta + a^2\sin^2\theta)(d\theta)^2$. For simplicity, we assume that $r = 1$, so the ellipsoid has the semiaxes $(1, 1, a)$. In particular, travelling between two points on $\mathbb{S}^2$, we can reorient the system of coordinates so that the destination point D is located in one of the poles. This can be done by the appropriate rotation of the sphere and adapting W. In order to solve the navigation problem on the

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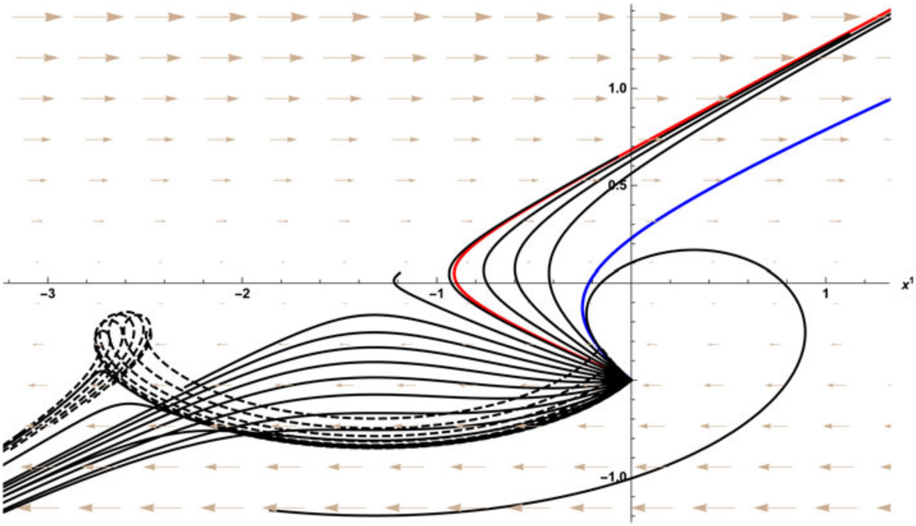

Figure 2. The only single-heading solutions to the time-optimal navigation problem with variable ship's speed $f(t, x^1, x^2) = 0{\cdot}25x^1 + 0{\cdot}33$ and under the action of the linear vector field $W = (x^2, 0)$ are for the headings $\varphi_0 = 62°$ (blue) and $118°$ (red), both time-minimal. In the background are other optimal paths: time-minimal (solid black) for $\varphi_0 \in (48{\cdot}7°, 311{\cdot}3°)$ and time-maximal (dashed black) for $\varphi_0 \in [0, 48{\cdot}7°) \cup (311{\cdot}3°, 360°)$, to an accuracy of $0{\cdot}1°$, obtained with the increments of the initial heading $\Delta\varphi_0 = 15°$. There also exist two anomalous paths for $\varphi_0 \in \{48{\cdot}7°, 311{\cdot}3°\}$. The starting point $(0; -0{\cdot}5)$ is within the area of strong current; $t \leq 8$.

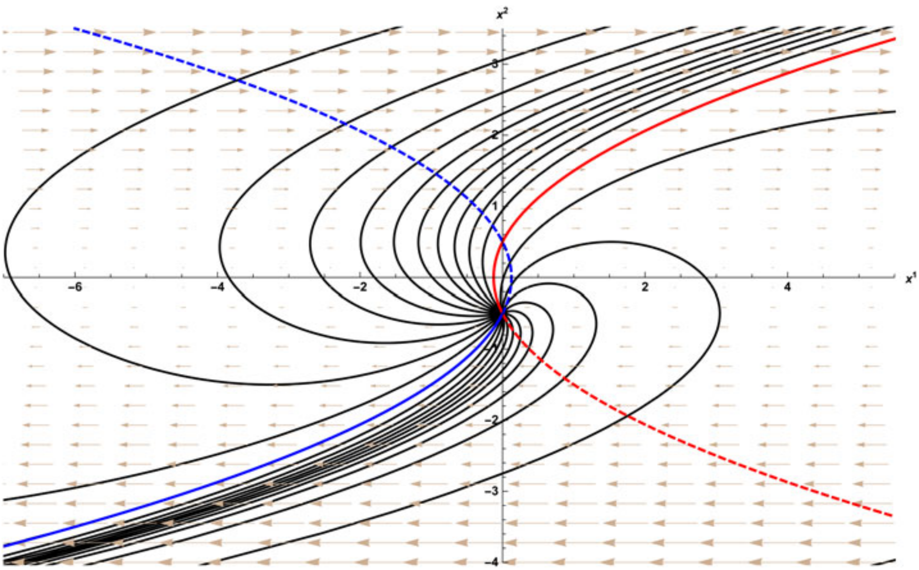

Figure 3. The only single-heading solutions of minimum time (parabolic) to the standard navigation problem ($f = 1 = \text{const.}$), under the action of the linear (shear) vector field $W = (x^2, 0)$ are for the headings $\varphi_0 = 90°$ (solid red) and $270°$ (solid blue). The starting point $(0; -0{\cdot}5)$ is within the area of weak current. In the background are other time-minimal solutions (solid black) generated with the increments of the initial heading $\Delta\varphi_0 = 15°$ (30 paths), $\varphi_0 \in [0, 360°)$. The dashed (red, blue) lines are just the prolongations of the single-heading paths of least time to show their parabolic shape clearly.

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rotational ellipsoid, we consider the transformation

$$x^1 = \phi, \quad x^2 = \int \frac{\sqrt{\varepsilon}}{\sin\theta}\, d\theta, \tag{23}$$

where we have $\varepsilon := \cos^2\theta + a^2\sin^2\theta$, $\theta \in (0, \pi)$. This can be rewritten as $ds^2 = (1/S^2)[(dx^1)^2 + (dx^2)^2]$, with $S = S(x^2) = 1/(r\sin\theta)$, $\theta \in (0, \pi)$.

Next, we consider the perturbing (rotational) vector field in the following form:

$$\begin{aligned} W(x,y,z) &= \tilde{W}^1(x,y,z)\frac{\partial}{\partial x} + \tilde{W}^2(x,y,z)\frac{\partial}{\partial y} + \tilde{W}^3(x,y,z)\frac{\partial}{\partial z} \\ &= cy\frac{\partial}{\partial x} - cx\frac{\partial}{\partial y} + 0\frac{\partial}{\partial z}, \quad c \in \mathbb{R}, \end{aligned}$$

which acts on Σ^2 seen as embedded in $\mathbb{R}^3$. After transformation, the expression of W in the base $\{\partial/\partial\phi, \partial/\partial\theta\}$ becomes

$$W(\phi,\theta) = W^1(\phi,\theta)\frac{\partial}{\partial\phi} + W^2(\phi,\theta)\frac{\partial}{\partial\theta} = -c\frac{\partial}{\partial\phi}, \tag{24}$$

and $|W(\phi,\theta)|_h = |c|\sin\theta$, $\theta \in (0,\pi)$, which includes all types of winds, i.e., $|W|_h < f$ (weak), $|W|_h = f$ (critical) and $|W|_h > f$ (strong). Let $f = 1$, $c = \frac{5}{7}$ (weak wind) and the flattening is determined by $a := \frac{3}{2}$. Thus, the two equations of the spheroidal motion are

$$\dot{\phi} = \frac{\cos\varphi}{\sin\theta} - c, \quad \dot{\theta} = -\frac{\sin\varphi}{\sqrt{\varepsilon}}, \tag{25}$$

and by Equation (18) it follows easily that the condition for the generalised single-heading optimal navigation is $\varphi_0 \in \{90^\circ, 270^\circ\}$. Consequently, the strategy is to head the self-velocity u towards the destination point(s), i.e., the pole(s). This means that the ship (u) follows a meridian pointing north or south; however, the spherical track will look variously. Namely, the angle $\tilde{\varphi}(t)$ between the resulting velocity vector v and the meridians varies. This is investigated in Figure 5 for three paths starting from different points on Σ^2, i.e., at three colatitudes: 60° (blue), 90° (red) and 120° (black). If $\tilde{\varphi}(t)$ is taken clockwise from north, then it corresponds to a course over ground in navigation. The corresponding colour-coded trajectories are presented in Figure 4. It is clear that the solutions are not loxodromic, since $\tilde{\varphi} \neq$ const., which Figure 5 shows. However, they are single-heading and time-optimal. We remark that the constant heading φ changes sign, if the point of destination becomes the opposite pole. We also mention that several moons of the Solar System approximate prolate spheroids in shape; however, they are actually triaxial ellipsoids. Also, the algorithms for accurate and global navigational calculations correspond to spheroidal (oblate, $a \in (0; 1)$) geometric models; see, e.g., Earle (2006), Pallikaris and Latsas (2012), Tseng et al. (2012) and Kopacz (2018a) in this regard.

5. CONCLUSION. This study generalises a single-heading navigation to conformally flat Riemannian manifolds under the action of arbitrary winds (currents) admitting the varying ship's self-speed in space and time. Special attention is paid to application of the pseudoloxodromic strategy to time-optimal navigation, i.e., the constant heading solutions

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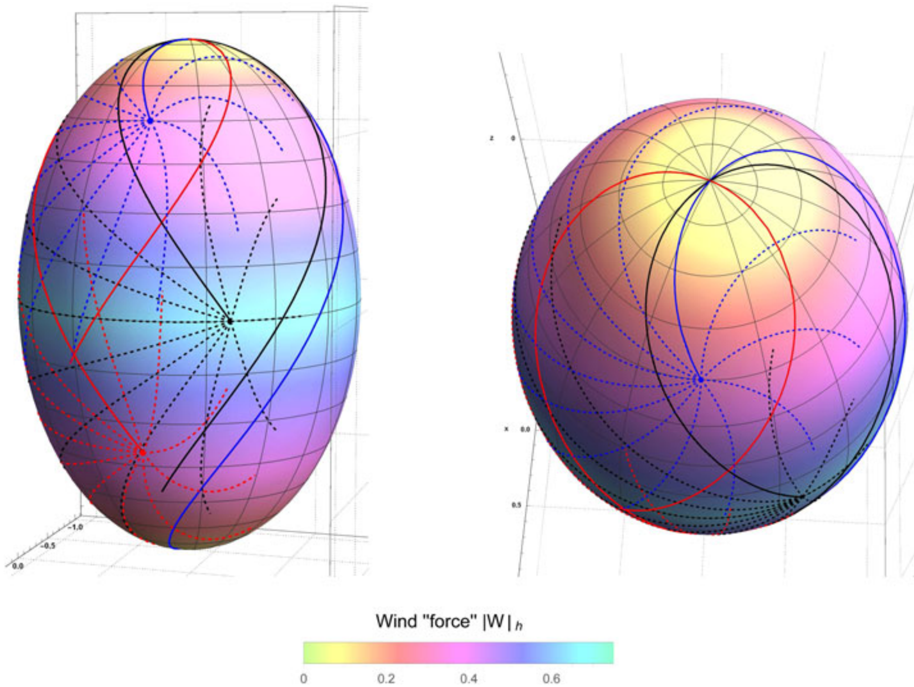

Figure 4. The single-heading (pseudoloxodromic) solutions of minimum time starting from different colatitudes on the prolate ellipsoid ($a := 1{\cdot}5$), i.e., 45° (solid blue), 90° (solid black) and 120° (solid red), among other time-minimal paths (dashed colours, respectively), under weak rotational wind given by Equation (24), with $c := \frac{5}{7}$; $t \leq 1{\cdot}5$. The corresponding time extremals are generated for different initial headings $\varphi(0) = \varphi_0$, i.e., with the steps $\Delta\varphi_0 = 30°$; $t \leq 5$. The colour-coded surface of the ellipsoid shows wind 'force' represented by the norm $|W|_h$ (left, side view; right, top (north) view).

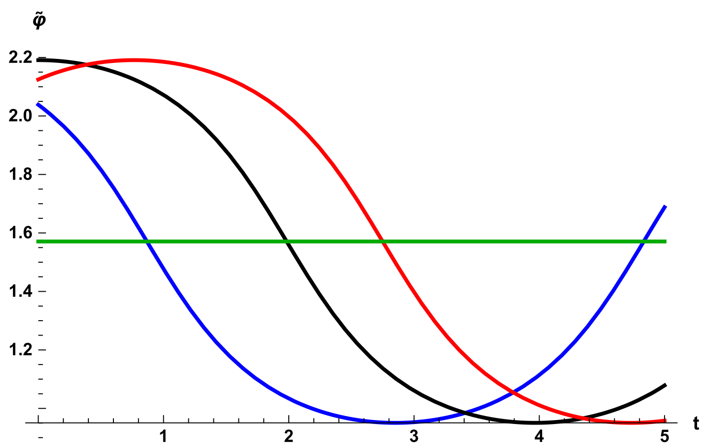

Figure 5. The investigation of varying direction $\tilde{\varphi}(t)$ (course over ground) of the tangent vectors v to the single-heading (pseudoloxodromic) solutions of least time shown in Figure 4, respectively (in colours). The horizontal green line represents the direction of the self-velocity u, i.e., the heading $\varphi(t)$, which points towards a pole and is constant (up to modulus), i.e., $|\varphi(t)| = \varphi_0 = 90° =$ const. for each considered path; $t \leq 5$.

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of the Zermelo navigation problem. Consequently, the corresponding necessary conditions for such type of navigation are obtained. This leads to the types of perturbing vector fields which yield the time-minimal paths of constant direction. By analogy, the investigation shows direct connections to loxodromic (rhumb line) time-efficient navigation, where the direction of a tangent vector to the resulting trajectory is preserved and the corresponding steering angles (optimal controls) vary. In particular, some real-world applications in dimension two combining both single-heading and time-optimal navigation refer to air travel and marine navigation of ships as well as to ecology and avian biology, including mathematical modelling and open problems that concern the navigational strategies of migrating animals in their short daily passages and long-distance travels varying spatially and seasonally. For consistency and comparison, future research will be to extend the study to arbitrary Riemannian manifolds, dropping the assumption on conformal flatness of the background Riemannian sea and focusing on the interplay between pseudoloxodromic and loxodromic navigation, including time-optimal trajectories in arbitrary winds. Also, an interesting area for further investigations concerns the structure of fields of flow in which the time-optimal pseudoloxodromes exist, so when not all of optimal paths are of such special type.

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