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BIRKHOFF ORTHOGONALITY IN CLASSICAL M-IDEALS

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Abstract

The Birkhoff orthogonality has been recently intensively studied in connection with the geometry of Banach spaces and operator theory. The main aim of this paper is to characterize the Birkhoff orthogonality in $\mathcal{L}(X; Y)$ under the assumption that $\mathcal{K}(X; Y)$ is an M-ideal in $\mathcal{L}(X; Y)$. Moreover, we survey the known results, as well as giving some new and more general ones. Furthermore, we characterize an approximate Birkhoff orthogonality in $\mathcal{K}(X; Y)$.

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1. Introduction

Throughout this paper $\mathbf{K}$ will denote either the real field, $\mathbf{R}$, or the complex field, $\mathbf{C}$. Let X be a real or complex Banach space. If the norm is generated by an inner product $\langle\cdot|\cdot\rangle$, we consider the standard orthogonality relation: $x \perp y :\Leftrightarrow \langle y|x\rangle = 0$. In the general case, we may consider the definition introduced by Birkhoff [6]:

$$x \perp_{\mathrm{B}} y \quad :\Leftrightarrow \quad \forall_{\lambda\in\mathbf{K}}\, \|x\| \leq \|x + \lambda y\|.$$

Let $\mathcal{H}$ be a finite-dimensional Hilbert space. Let $T, A \in \mathcal{L}(\mathcal{H})$. Bhatia and Šemrl [4] and Bhattacharyya and Grover [5] independently proved that $T \perp_{\mathrm{B}} A$ if and only if there exists $x \in \mathcal{H}$ with $\|x\| = 1$ such that $\|Tx\| = \|T\|$ and $Tx \perp Ax$. Similar investigations have been carried out by Grover for the space $M_n(\mathbf{C})$ in [12].

Li and Schneider [15] gave examples of finite-dimensional normed linear spaces X in which there exist operators $T, A \in \mathcal{L}(X)$ such that $T \perp_{\mathrm{B}} A$ but there exists no $x \in S(X)$ such that $\|Tx\| = \|T\|$ and $Tx \perp_{\mathrm{B}} Ax$. Benítez *et al.* [3] proved that X is an inner product space if and only if every $T \in \mathcal{L}(X)$ satisfies the Bhatia–Šemrl (B–Š) property (that is, we say that T satisfies the *(B–Š) property* if for any $A \in \mathcal{L}(X)$, $T \perp_{\mathrm{B}} A$ implies that there exists $x \in S(X)$ such that $\|Tx\| = \|T\|$ and $Tx \perp_{\mathrm{B}} Ax$).

Remark 1.1. Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. Bhatia and Šemrl [4, Theorem 1.1 and Remark 3.1] also obtained the following characterization of the Birkhoff orthogonality for Hilbert space operators: if $A, B \in \mathcal{L}(\mathcal{H})$, then $A \perp_B B$ if and only if there exists a sequence (x_n) of unit vectors of $\mathcal{H}$ such that $\lim_{n\to\infty} \|Ax_n\| = \|A\|$ and $\lim_{n\to\infty} \langle Ax_n | Bx_n \rangle = 0$.

Similar investigations have been carried out by Arambašić and Rajić for the C^*-modules in [2].

In [20], it was proved that if T is a linear operator on a finite-dimensional real normed linear space X such that T attains norm only on $\pm D$, where D is a connected closed subset of $S(X)$, then T satisfies the Bhatia–Šemrl property. Let us quote a result from [20].

Theorem 1.2 [20]. *Let X be a finite-dimensional real normed linear space. Let $T \in \mathcal{L}(X)$ be such that T attains its norm at only $\pm D$, where D is a connected subset of $S(X)$. Then for $A \in \mathcal{L}(X)$ with $T \perp_B A$ there exists $x \in D$ such that $Tx \perp_B Ax$.*

The above results motivate this paper. We will extend (in some sense) some of the above results in the subsequent section. Our results generalize and complement those in [4, 20] to some extent.

2. Preliminaries

Let X be a real or complex Banach space. The closed unit ball of X is denoted by $B(X)$. The unit sphere of X is denoted by $S(X)$. Fix $x \in X \setminus \{0\}$. We consider the set $J(x)$ defined as follows:

$$J(x) := \{x^* \in X^* : \|x^*\| = 1,\ x^*(x) = \|x\|\}.$$

It is easy to check that the set $J(x)$ is convex and closed, and $J(x) \subset S(X^*)$. By the Hahn–Banach theorem, we get $J(x) \neq \emptyset$ for all $x \in X \setminus \{0\}$.

The aim of this paper is to present some results concerning the geometry of Banach spaces in the case of the space of all linear, continuous mappings from a Banach space X into a Banach space Y equipped with the operator norm (we will denote it by $\mathcal{L}(X; Y)$, and $\mathcal{L}(X)$ if $X = Y$). The set of compact operators from X into Y is denoted by $\mathcal{K}(X; Y)$, and $\mathcal{K}(X) := \mathcal{K}(X; X)$.

2.1. *M*-ideals. Let X be a Banach space. Suppose that V is a closed subspace of X. The subspace V is said to be an *M-ideal in X* if $X^* = V^* \oplus_1 V^\perp$, where $V^\perp := \{x^* \in X^* : V \subset \ker x^*\}$, and, if $x^* = x_1^* + x_2^*$ is the unique decomposition of x^* in X^*, then $\|x^*\| = \|x_1^*\| + \|x_2^*\|$.

The aim of this subsection is to recall when $\mathcal{K}(X; Y)$ is an M-ideal in $\mathcal{L}(X; Y)$. Hennefeld [14] and Saatkamp [19] have proved that $\mathcal{K}(l^p; l^q)$ are M-ideals when $1 < p \leq q < \infty$. Note that if $1 \leq q < p < \infty$, then $\mathcal{K}(l^p; l^q) = \mathcal{L}(l^p; l^q)$ [18]. Several authors have observed that $\mathcal{K}(X, c_0)$ is an M-ideal for all Banach spaces X [16, 19, 21]. It is known that $\mathcal{K}(l^1; l^1)$ and $\mathcal{K}(l^\infty; l^\infty)$ are not M-ideals [21]. Many of these topics can be found in [13].

2.2. Extreme points. The main tool is a theorem due to Collins and Ruess [9], which characterizes the extremal points of the unit ball in $\mathcal{K}(X;Y)^*$ in terms of extremal points of the closed unit balls in Y^* and X^* (see also [17]). By $\operatorname{Ext} V$ we will denote the set of all extremal points of a given set V. By the Krein–Milman theorem, the closed unit ball $B(Y^*)$ has many extreme points. In particular, $\operatorname{Ext} B(\mathcal{K}(X;Y)^*) \neq \emptyset$.

Theorem 2.1 [9, 17]. *If X and Y are Banach spaces, then*

$$\operatorname{Ext} B(\mathcal{K}(X;Y)^*) = \{x^{**} \otimes y^* \in \mathcal{K}(X;Y)^* : x^{**} \in \operatorname{Ext} B(X^{**}), y^* \in \operatorname{Ext} B(Y^*)\},$$

where $x^{**} \otimes y^* : \mathcal{K}(X;Y) \to \mathbf{K}$, $(x^{**} \otimes y^*)(T) := x^{**}(T^* y^*)$ for every $T \in \mathcal{K}(X;Y)$.

In particular, if X is a reflexive Banach space, then $\operatorname{Ext} B(X) \neq \emptyset$. From Theorem 2.1, we obtain the following result.

Corollary 2.2. *If X is a reflexive Banach space, then*

$$\operatorname{Ext} B(\mathcal{K}(X;Y)^*) = \{y^* \otimes x \in \mathcal{K}(X;Y)^* : x \in \operatorname{Ext} B(X), y^* \in \operatorname{Ext} B(Y^*)\},$$

where $y^* \otimes x \colon \mathcal{K}(X;Y) \to \mathbf{K}$, $(y^* \otimes x)(T) := y^*(Tx)$ for every $T \in \mathcal{K}(X;Y)$.

2.3. Norm derivatives. In a real normed linear space $(X, \|\cdot\|)$, the Gateaux derivatives of the norm are given for fixed x and y in X by the two expressions $\lim_{\lambda\to 0^\pm}((\|x+\lambda y\| - \|x\|)/\lambda)$. Note that instead of considering the above norm derivatives, it is more convenient to introduce the functionals

$$\rho'_\pm(x,y) := \lim_{\lambda\to 0^\pm} \frac{\|x+\lambda y\|^2 - \|x\|^2}{2\lambda} = \|x\| \cdot \lim_{\lambda\to 0^\pm} \frac{\|x+\lambda y\| - \|x\|}{\lambda}, \qquad x, y \in X,$$

because when the norm comes from an inner product $\langle\cdot|\cdot\rangle$, we obtain $\rho'_+(x,y) = \langle x|y\rangle = \rho'_-(x,y)$, that is, functionals ρ'_+, ρ'_- are perfect generalizations of inner products. Convexity of the norm yields that the above definition is meaningful. The mappings ρ'_+ and ρ'_- are called the *norm derivatives* and their following properties, which will be useful in the present note, can be found, for example, in [1, 11]:

(ND1) $\forall_{x,y\in X}\ \forall_{\alpha\in\mathbf{R}}\ \rho'_\pm(x, \alpha x + y) = \alpha\|x\|^2 + \rho'_\pm(x,y)$;
(ND2) $\forall_{x,y\in X}\ \forall_{\alpha\geq 0}\ \rho'_\pm(\alpha x, y) = \alpha\rho'_\pm(x,y) = \rho'_\pm(x, \alpha y)$;
(ND3) $\forall_{x,y\in X}\ \forall_{\alpha<0}\ \rho'_\pm(\alpha x, y) = \alpha\rho'_\mp(x,y) = \rho'_\pm(x, \alpha y)$;
(ND4) $\forall_{x,y\in X}\ |\rho'_\pm(x,y)| \leq \|x\| \cdot \|y\|, \rho'_\pm(x,x) = \|x\|^2$;
(ND5) $\rho'_- \leq \rho'_+$.

Moreover, mappings ρ'_+, ρ'_- are continuous with respect to the second variable, but not necessarily with respect to the first one. In a real normed space X, we have (cf. [1, 11])

$$x \perp_B y \quad \Leftrightarrow \quad \rho'_-(x,y) \leq 0 \leq \rho'_+(x,y). \tag{2.1}$$

Now we define orthogonality relations related to $\rho'_\pm$ (cf. [1, 11]):

$$x \perp_{\rho_+} y \quad :\Leftrightarrow \quad \rho'_+(x,y) = 0, \qquad x \perp_{\rho_-} y \quad :\Leftrightarrow \quad \rho'_-(x,y) = 0.$$

Let us recall the following result containing a representation of the norm derivatives ρ'_+, ρ'_- in terms of supporting functionals.

Theorem 2.3 [11]. *Let X be a real normed space. Then one has the representation*

$$\rho'_-(x,y) = \|x\| \cdot \inf\{x^*(y) : x^* \in J(x)\} \quad \textit{and}$$
$$\rho'_+(x,y) = \|x\| \cdot \sup\{x^*(y) : x^* \in J(x)\} \quad \textit{for all } x,y \in X.$$

So, in particular,

$$\forall x^* \in J(x) \quad \rho'_-(x,y) \le \|x\| \cdot x^*(y) \le \rho'_+(x,y). \tag{2.2}$$

Furthermore, the following conditions are equivalent:

$$\text{(a) } X \text{ is smooth;} \quad \text{(b) } \rho'_+(x,\cdot) \text{ is linear} \quad (\text{for all } x \text{ in } X). \tag{2.3}$$

Note that $J(x)$ is a weak*-compact convex set and hence it follows from the Krein–Milman theorem that $J(x)$ is the closed convex hull of its extreme points. As a consequence of the above results and by properties of $J(x)$, one has an even more general result.

Theorem 2.4. *Let X be a real normed space. Then one has the representation*

$$\rho'_-(x,y) = \|x\| \cdot \inf\{x^*(y) : x^* \in \operatorname{Ext} J(x)\} \quad \textit{and}$$
$$\rho'_+(x,y) = \|x\| \cdot \sup\{x^*(y) : x^* \in \operatorname{Ext} J(x)\} \quad \textit{for all } x,y \in X.$$

3. Norm derivatives in $\mathcal{L}(X;Y)$

Norm derivatives are important in approximation theory and in the geometry of Banach spaces. In this section we investigate the norm derivatives in the spaces of bounded operators. Let $\mathcal{M}_T$ denote the set of all unit vectors in $S(X)$ at which T attains norm, that is,

$$\mathcal{M}_T := \{x \in S(X) : \|Tx\| = \|T\|\}.$$

It will be shown that $\mathcal{M}_T \neq \emptyset$ under the additional assumption. This is important in this work.

Lemma 3.1. *Suppose that X is a reflexive Banach space. Suppose that $\mathcal{K}(X;Y)$ is an M-ideal in $\mathcal{L}(X;Y)$. Let $T \in \mathcal{L}(X;Y)$, $\operatorname{dist}(T, \mathcal{K}(X;Y)) < 1$, and $\|T\| = 1$. Then $\mathcal{M}_T \cap \operatorname{Ext} B(X) \neq \emptyset$ and*

$$\operatorname{Ext} J(T) = \{y^* \otimes x \in \mathcal{K}(X;Y)^* : x \in \mathcal{M}_T \cap \operatorname{Ext} B(X), y^* \in \operatorname{Ext} J(Tx)\}. \tag{3.1}$$

Proof. By the Hahn–Banach theorem, $J(T) \neq \emptyset$. Note that $J(T)$ is a weak*-compact convex set and hence it is easy to see that $\operatorname{Ext} J(T) \neq \emptyset$. It is not hard to check that $J(T)$ is an extremal subset of $B(\mathcal{L}(X;Y)^*)$. Hence, $\operatorname{Ext} J(T) \subset \operatorname{Ext} B(\mathcal{L}(X;Y)^*)$. From the assumption,

$$\mathcal{L}(X;Y)^* = \mathcal{K}(X;Y)^* \oplus_1 \mathcal{K}(X;Y)^\perp,$$

whence

$$\operatorname{Ext} J(T) \subset \operatorname{Ext} B(\mathcal{K}(X;Y)^* \oplus_1 \mathcal{K}(X;Y)^\perp).$$

This implies that extreme points of $J(T)$ are either extreme in $B(\mathcal{K}(X;Y)^*)$ or extreme in $B(\mathcal{K}(X;Y)^{\perp})$. We want to show that $\operatorname{Ext} J(T) \subset B(\mathcal{K}(X;Y)^*)$.

Fix $\gamma \in \operatorname{Ext} J(T)$. Now we show that $\gamma \in B(\mathcal{K}(X;Y)^*)$. Assume, contrary to our claim, that $\gamma \in B(\mathcal{K}(X;Y)^{\perp})$. By the inequality $\operatorname{dist}(T, \mathcal{K}(X;Y)) < 1$, there is $K \in \mathcal{K}(X;Y)$ such that $\|T - K\| < 1$. It follows that $1 = \gamma(T) = \gamma(T) - 0 = \gamma(T - K) \leq \|T - K\| < 1$, which is a contradiction. So, we have $\gamma \in B(\mathcal{K}(X;Y)^*)$. We have shown that $\operatorname{Ext} J(T) \subset B(\mathcal{K}(X;Y)^*)$. In particular,

$$\operatorname{Ext} J(T) \subset \operatorname{Ext} B(\mathcal{K}(X;Y)^*). \tag{3.2}$$

Using this fact and Corollary 2.2,

$$\operatorname{Ext} J(T) \subset \{y^* \otimes x \in \mathcal{K}(X;Y)^* : x \in \operatorname{Ext} B(X), y^* \in \operatorname{Ext} B(Y^*)\}. \tag{3.3}$$

To prove (3.1), suppose that $\varphi \in \operatorname{Ext} J(T)$. Then, by (3.3), $\varphi = y^* \otimes x$ for some $x \in \operatorname{Ext} B(X)$, $y^* \in \operatorname{Ext} B(Y^*)$. It follows that $\|T\| = \varphi(T) = y^*(Tx) \leq \|Tx\| \leq \|T\|$, whence $x \in \mathcal{M}_T$. In particular, $\mathcal{M}_T \cap \operatorname{Ext} B(X) \neq \emptyset$. It is easy to see that $J(Tx)$ is an extremal subset of $B(Y^*)$. Hence, $\operatorname{Ext} J(Tx) \subset \operatorname{Ext} B(Y^*)$, so $y^* \in \operatorname{Ext} J(Tx)$.

We prove the converse inclusion in (3.1). It is easy to show that

$$J(T) \supset \{y^* \otimes x \in \mathcal{K}(X;Y)^* : x \in \mathcal{M}_T \cap \operatorname{Ext} B(X), y^* \in \operatorname{Ext} J(Tx)\}. \tag{3.4}$$

Since $\operatorname{Ext} J(Tx) \subset \operatorname{Ext} B(Y^*)$, it follows from Corollary 2.2 that

$$\operatorname{Ext} B(\mathcal{K}(X;Y)^*) \supset \{y^* \otimes x \in \mathcal{K}(X;Y)^* : x \in \mathcal{M}_T \cap \operatorname{Ext} B(X), y^* \in \operatorname{Ext} J(Tx)\}. \tag{3.5}$$

Combining (3.2), (3.4), and (3.5), we immediately get

$$\operatorname{Ext} J(T) \supset \{y^* \otimes x \in \mathcal{K}(X;Y)^* : x \in \mathcal{M}_T \cap \operatorname{Ext} B(X), y^* \in \operatorname{Ext} J(Tx)\},$$

which concludes the proof. □

From now on we assume that the considered normed spaces are real.

Theorem 3.2. *Suppose that X is a reflexive Banach space. Suppose that $\mathcal{K}(X;Y)$ is an M-ideal in $\mathcal{L}(X;Y)$. Let $T, A \in \mathcal{L}(X;Y)$ and $\|T\| = 1$. Suppose that* $\operatorname{dist}(T, \mathcal{K}(X;Y)) < 1$. *Then the following condition holds:*

$$\rho'_+(T, A) = \sup\{\rho'_+(Tx, Ax) : x \in \mathcal{M}_T \cap \operatorname{Ext} B(X)\}. \tag{3.6}$$

Proof. Fix $t \in (0, 1)$. Fix $x \in \mathcal{M}_T \cap \operatorname{Ext} B(X)$ to obtain

$$\frac{\|Tx + tAx\|^2 - \|Tx\|^2}{2t} = \frac{\|Tx + tAx\|^2 - \|T\|^2}{2t} \leq \frac{\|T + tA\|^2 - \|T\|^2}{2t}. \tag{3.7}$$

Since t was arbitrarily chosen from the interval $(0, 1)$, letting $t \to 0^+$ in (3.7),

$$\rho'_+(Tx, Ax) \leq \rho'_+(T, A).$$

Since x was arbitrarily chosen from the set $\mathcal{M}_T \cap \operatorname{Ext} B(X)$,

$$\sup\{\rho'_+(Tx, Ax) : x \in \mathcal{M}_T \cap \operatorname{Ext} B(X)\} \leq \rho'_+(T, A). \tag{3.8}$$

From Theorem 2.4 and Lemma 3.1,

$$
\begin{aligned}
\rho'_+(T,A) &= \sup\{\varphi(A) : \varphi \in \operatorname{Ext} J(T)\} \overset{(3.1)}{=} \\
&= \sup\{y^*(Ax) : x \in \mathcal{M}_T \cap \operatorname{Ext} B(X), y^* \in \operatorname{Ext} J(Tx)\} \\
&\overset{(2.2)}{\leq} \sup\{\rho'_+(Tx,Ax) : x \in \mathcal{M}_T \cap \operatorname{Ext} B(X)\} \overset{(3.8)}{\leq} \rho'_+(T,A).
\end{aligned}
$$

So, we have $\rho'_+(T,A) = \sup\{\rho'_+(Tx,Ax) : x \in \mathcal{M}_T \cap \operatorname{Ext} B(X)\}$. □

Theorem 3.3. *Suppose that X is a reflexive Banach space. Assume that Y is smooth. Suppose that $\mathcal{K}(X;Y)$ is an M-ideal in $\mathcal{L}(X;Y)$. Let $T, A \in \mathcal{L}(X;Y)$ and $\|T\| = 1$. Suppose that* $\operatorname{dist}(T, \mathcal{K}(X;Y)) < 1$. *Then the following condition holds:*

$$
\rho'_-(T,A) = \inf\{\rho'_+(Tx,Ax) : x \in \mathcal{M}_T \cap \operatorname{Ext} B(X)\}. \tag{3.9}
$$

Proof. Since the space Y is smooth, $\rho'_+(Tx,\cdot) = \rho'_-(Tx,\cdot)$. It follows that

$$
\begin{aligned}
\rho'_-(T,A) &\overset{(\mathrm{ND3})}{=} -\rho'_+(T,-A) \overset{(3.6)}{=} \\
&= -\sup\{\rho'_+(Tx,-Ax) : x \in \mathcal{M}_T \cap \operatorname{Ext} B(X)\} \\
&\overset{(\mathrm{ND3})}{=} -\sup\{-\rho'_-(Tx,Ax) : x \in \mathcal{M}_T \cap \operatorname{Ext} B(X)\} \\
&= -\sup\{-\rho'_+(Tx,Ax) : x \in \mathcal{M}_T \cap \operatorname{Ext} B(X)\} \\
&= \inf\{\rho'_+(Tx,Ax) : x \in \mathcal{M}_T \cap \operatorname{Ext} B(X)\}.
\end{aligned}
$$

The proof is complete. □

We now proceed to the norm derivatives in $\mathcal{K}(X;Y)$.

Theorem 3.4. *Suppose that X is a reflexive Banach space. Assume that Y is smooth. Suppose that $\mathcal{K}(X;Y)$ is an M-ideal in $\mathcal{L}(X;Y)$. Let $T, A \in \mathcal{K}(X;Y)$. Then the following conditions hold:*

$$
\begin{aligned}
\rho'_+(T,A) &= \max\{\rho'_+(Tx,Ax) : x \in \mathcal{M}_T\}, \\
\rho'_-(T,A) &= \min\{\rho'_+(Tx,Ax) : x \in \mathcal{M}_T\}.
\end{aligned} \tag{3.10}
$$

Proof. Since the proofs are similar, we present only one. Without loss of generality, we may assume that $\|T\| = 1$ (see (ND2)). By (3.6), let us choose a sequence $(x_n) \subset \mathcal{M}_T$ such that

$$
\rho'_+(Tx_n, Ax_n) \nearrow \rho'_+(T,A). \tag{3.11}
$$

Recall that in a reflexive Banach space the closed unit ball $B(X)$ is weak-compact. By the Gantmaher–Šmul'yan theorem [10, page 58], $B(X)$ is weakly sequentially compact. Thus, there is a subsequence $(x_{n_k}) \subset B(X)$ and there is a vector x_o in $B(X)$ such that $x_{n_k} \xrightarrow{\text{weak}} x_o$. Since T, A are compact operators, T, A are completely continuous. This means that

$$
Tx_{n_k} \xrightarrow{\|\cdot\|} Tx_o \quad \text{and} \quad Ax_{n_k} \xrightarrow{\|\cdot\|} Ax_o.
$$

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It is helpful to recall that the mappings $\rho'_-, \rho'_+ : Y \times Y \to \mathbf{R}$ are continuous with respect to the second variable. Since the space Y is smooth, the mappings $\rho'_-, \rho'_+ : Y \times Y \to \mathbf{R}$ are also continuous with respect to the first variable. We will show that $\rho'_+(T, A) = \rho'_+(Tx_o, Ax_o)$. By the condition (3.11), it is enough to show that $\rho'_+(Tx_{n_k}, Ax_{n_k}) \longrightarrow \rho'_+(Tx_o, Ax_o)$. Finally, observe that

$$\begin{aligned}
|\rho'_+(Tx_{n_k}, Ax_{n_k}) - \rho'_+(Tx_o, Ax_o)| &\leq |\rho'_+(Tx_{n_k}, Ax_{n_k}) - \rho'_+(Tx_{n_k}, Ax_o)| \\
&\qquad + |\rho'_+(Tx_{n_k}, Ax_o) - \rho'_+(Tx_o, Ax_o)| \\
&\overset{(2.3)}{\leq} |\rho'_+(Tx_{n_k}, Ax_{n_k} - Ax_o)| \\
&\qquad + |\rho'_+(Tx_{n_k}, Ax_o) - \rho'_+(Tx_o, Ax_o)| \\
&\leq \|Tx_{n_k}\| \cdot \|Ax_{n_k} - Ax_o\| \\
&\qquad + |\rho'_+(Tx_{n_k}, Ax_o) - \rho'_+(Tx_o, Ax_o)| \to 0.
\end{aligned}$$

The proof of Theorem 3.4 is complete. □

4. Birkhoff orthogonality in $\mathcal{L}(X; Y)$

Using the notion of norm derivatives, we apply our previous results to characterize orthogonality in the sense of Birkhoff in $\mathcal{L}(X; Y)$. We still assume that the considered normed spaces are real. Our next results generalize and complement those in [4, 20] to some extent.

Theorem 4.1. *Suppose that X is a reflexive Banach space. Assume that Y is smooth. Suppose that $\mathcal{K}(X; Y)$ is an M-ideal in $\mathcal{L}(X; Y)$. Let $T \in \mathcal{L}(X; Y)$ be such that T attains its norm at only $\pm D$ (that is, $\mathcal{M}_T = D \cup -D$), where D is a connected subset of $S(X)$. Suppose that* $\operatorname{dist}(T, \mathcal{K}(X; Y)) < \|T\|$. *Then the following conditions are equivalent:*

(a) $T \perp_{\mathrm{B}} A$;

(b) *there exists $x \in D$ such that $Tx \perp_{\mathrm{B}} Ax$, or $T \perp_{\rho_+} A$, or $T \perp_{\rho_-} A$.*

Proof. The Birkhoff orthogonality $\perp_{\mathrm{B}}$ is homogeneous (that is, if $x \perp_{\mathrm{B}} y$, then $\alpha x \perp_{\mathrm{B}} \beta y$ for all α, β in $\mathbf{R}$). From the assumption, we have that $\operatorname{dist}(T/\|T\|, \mathcal{K}(X; Y)) < 1$. Without loss of generality, we may assume that $\|T\| = 1$, and then $\operatorname{dist}(T, \mathcal{K}(X; Y)) < 1$. For the proof of (a)⇒(b), suppose that $T \perp_{\mathrm{B}} A$. Combining (2.1), (3.6), and (3.9), we immediately get

$$\inf\{\rho'_+(Tx, Ax) : x \in \mathcal{M}_T\} \leq 0 \leq \sup\{\rho'_+(Tx, Ax) : x \in \mathcal{M}_T\}.$$

Let us distinguish three cases.

Case 1. If $0 = \sup\{\rho'_+(Tx, Ax) : x \in \mathcal{M}_T\} = \rho'_+(T, A)$, then $T \perp_{\rho_+} A$.

Case 2. If $0 = \inf\{\rho'_+(Tx, Ax) : x \in \mathcal{M}_T\} = \rho'_-(T, A)$, then $T \perp_{\rho_-} A$.

Case 3. Let us now consider the following chain of inequalities:

$$\inf\{\rho'_+(Tx, Ax) : x \in \mathcal{M}_T\} < 0 < \sup\{\rho'_+(Tx, Ax) : x \in \mathcal{M}_T\}. \tag{4.1}$$

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Then we define a mapping $\varphi : \mathcal{M}_T \to \mathbf{R}$ by $\varphi(\cdot) := \rho'_+(T(\cdot), A(\cdot))$. In a similar way as in the proof of Theorem 3.4, we can prove that the function φ is continuous. By (ND3), it is easy to see that $\varphi(x) = \varphi(-x)$. Moreover, the set D is connected and $\mathcal{M}_T = D \cup -D$. Now the condition (4.1) becomes

$$\inf\{\varphi(x) : x \in D\} < 0 < \sup\{\varphi(x) : x \in D\}.$$

Using the Darboux property, we get $\varphi(x_o) = 0$ for some $x_o \in D$. Thus, for the vector x_o, we have $0 = \varphi(x_o) = \rho'_+(T(x_o), A(x_o))$. Next, from (2.1) and (ND5), we have $T(x_o) \perp_{\mathrm{B}} A(x_o)$.

The converse implication has a trivial verification. □

There is a simple observation that can be made here. It can be shown that if $\mathcal{H}$ is a Hilbert space, then the set $\mathcal{M}_T$ is connected for all T in $\mathcal{L}(\mathcal{H})$ (whenever $\mathcal{M}_T \neq \emptyset$). In fact, the set $\mathcal{M}_T$ is arcwise connected.

In some circumstances, Theorem 4.1 can be strengthened as follows.

Theorem 4.2. *Suppose that X is a reflexive Banach space. Assume that Y is smooth. Suppose that $\mathcal{K}(X; Y)$ is an M-ideal in $\mathcal{L}(X; Y)$. Let $T, A \in \mathcal{K}(X; Y)$, $\mathcal{M}_T = D \cup -D$, where D is a connected subset of $S(X)$. Then the following conditions are equivalent:*

(a) $T \perp_{\mathrm{B}} A$;

(b) *there exists $x \in D$ such that $Tx \perp_{\mathrm{B}} Ax$.*

Proof. This result can be obtained similarly. In the above proof, one should consider Theorem 3.4 instead of (3.6) and (3.9). □

Careful reading of the proof of Theorem 4.1 shows that in fact we have the following. (Compare this with Remark 1.1.)

Corollary 4.3. *Let X, Y, T, A be as in Theorem 4.1. Then $T \perp_{\mathrm{B}} A$ if and only if there exists a sequence (x_n) of unit vectors of X such that $\|Tx_n\| = \|T\|$ (for all n) and* $\lim_{n\to\infty} \rho'_+(Tx_n, Ax_n) = 0$.

5. Approximate Birkhoff orthogonality

In an inner product space an *approximate orthogonality* (ε-orthogonality, with $\varepsilon \in [0, 1)$) of vectors x and y is naturally defined by

$$x \perp^{\varepsilon} y \quad :\Leftrightarrow \quad |\langle x|y\rangle| \le \varepsilon \|x\| \cdot \|y\|.$$

For an *approximate B-orthogonality*, we will follow the definition from [7]:

$$x \perp^{\varepsilon}_{\mathrm{B}} y \quad :\Leftrightarrow \quad \forall_{\lambda \in \mathbf{R}}\ \|x\|^2 \le \|x + \lambda y\|^2 + 2\varepsilon \|x\| \cdot \|\lambda y\|.$$

The next result (cf. [8]) establishes the connection between $\perp^{\varepsilon}_{\mathrm{B}}$ and $\rho'_{\pm}$.

Theorem 5.1 [8, Theorem 3.1]. *Let X be a real normed space and let $\varepsilon \in [0, 1)$. Then*

$$x \perp^{\varepsilon}_{\mathrm{B}} y \quad \Leftrightarrow \quad \rho'_-(x, y) - \varepsilon\|x\| \cdot \|y\| \le 0 \le \rho'_+(x, y) + \varepsilon\|x\| \cdot \|y\|. \tag{5.1}$$

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Now we state and prove the main result of this section.

THEOREM 5.2. *Suppose that X is a reflexive Banach space. Assume that Y is smooth. Suppose that $\mathcal{K}(X;Y)$ is an M-ideal in $\mathcal{L}(X;Y)$. Let $T, A \in \mathcal{K}(X;Y)$. Assume that $\mathcal{M}_T = D \cup -D$, where D is a connected subset of $S(X)$. Suppose that $\mathcal{M}_T \subset \mathcal{M}_A$. Then the following conditions are equivalent:*

(a) $T \perp_B^{\varepsilon} A$;

(b) *there exists $x \in D$ such that $Tx \perp_B^{\varepsilon} Ax$.*

PROOF. We start with proving (a)⇒(b). Suppose that $T \perp_B^{\varepsilon} A$. Combining (5.1) and (3.10), we immediately get

$$\begin{aligned}&\min\{\rho'_+(Tx, Ax) : x \in \mathcal{M}_T\}\\&\quad - \varepsilon\|T\| \cdot \|A\| \le 0 \le \max\{\rho'_+(Tx, Ax) : x \in \mathcal{M}_T\} + \varepsilon\|T\| \cdot \|A\|.\end{aligned}$$

Using the Darboux property again, we get, for some $x_o \in D$,

$$-\varepsilon\|T\| \cdot \|A\| \le \rho'_+(Tx_o, Ax_o) \le \varepsilon\|T\| \cdot \|A\|. \tag{5.2}$$

Since the space Y is smooth, $\rho'_-(Tx_o, Ax_o) = \rho'_+(Tx_o, Ax_o)$. Since $\mathcal{M}_T \subset \mathcal{M}_A$, $\|T\| \cdot \|A\| = \|Tx_o\| \cdot \|Ax_o\|$. Now the condition (5.2) becomes

$$\rho'_-(Tx_o, Ax_o) - \varepsilon\|Tx_o\| \cdot \|Ax_o\| \le 0 \le \rho'_+(Tx_o, Ax_o) + \varepsilon\|Tx_o\| \cdot \|Ax_o\|.$$

Then (5.1) yields $Tx_o \perp_B^{\varepsilon} Ax_o$.

We prove the converse implication. Fix an arbitrary $\lambda \in \mathbf{R}$. From (b),

$$\begin{aligned}\|T\|^2 = \|Tx\|^2 &\le \|Tx + \lambda Ax\|^2 + 2\varepsilon\|Tx\| \cdot \|\lambda Ax\|\\&\le \|T + \lambda A\|^2 + 2\varepsilon\|T\| \cdot \|\lambda A\|\end{aligned}$$

and thus finally we get $T \perp_B^{\varepsilon} A$. □

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