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A CASE STUDY IN EARLY MATHEMATICAL ECONOMICS: PIETRO VERRI AND PAOLO FRISI, 1772

BY

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I. INTRODUCTION

In eighteenth-century Italy a number of interesting efforts were made to apply mathematical methods to economic analysis. Scholars living in the Milan area were especially keen on mathematizing economic issues. R. D. Theocharis identified a common current of thought, which he called the Milanese School, in his pioneering research on early mathematical economics (Theocharis 1983, p. 2).

The following study sets out to examine one of these attempts at a mathematized economic theory: the sixth edition of Pietro Verri's *Reflections on Political Economy*, published in 1772. Verri (1728–1797), an economist and philosopher, served in the Austrian administration in Milan. He first published his book in 1771; its success was considerable, with five different editions produced in just one year. The main body of the text contained no mathematics at all, but the sixth edition included some anonymous footnotes, in which an effort was made to translate Verri's arguments into mathematical equations. These notes aroused a lively controversy in Italy about the application of mathematical tools to economic reasoning. They have been attributed to a well-known Milanese mathematician of the time, Paolo Frisi (1728–1784), who also wrote a review of Welsh economist Henry Lloyd's *An Essay on the Theory of Money* (1771), included as an appendix to the sixth edition.

Historically, the attempt to mathematize the *Reflections on Political Economy* has been rather unsuccessful, in that Verri suppressed Frisi's notes in later editions of the book. Today, few historians of economic thought know about them.

This paper addresses the question of whether the application of mathematical methods by Frisi succeeds in improving Verri's economic theory. Today, mathematized theories are expected to achieve a number of significant gains: logical

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rigor, clarity of expression, generality, simplicity, and conciseness. Does the formalization of Verri and Frisi achieve these aims? If the answer is affirmative, then sheer historical reasons are enough to explain the suppression of Frisi's notes by Verri. If it is negative, then the elimination of mathematical symbols and equations in subsequent editions of the book may be tied to theoretical reasons as well. This paper will seek to highlight both the achievements and the shortcomings of the formalization.

At the core of Verri's economic analysis is price theory, and it was Frisi's mathematical treatment of it that created the greatest interest. For this reason, this paper will focus on it. Both its economic content and its mathematical form will be discussed. Since Verri and Frisi worked independently, their contributions will be examined separately, and their common results will be assessed at the end.

Quotations are taken from the American edition (1993) of the English translation of Verri's book, first edited by Peter Groenewegen in Sydney in 1986. However, the text of this translation is not based on the 1772 edition, which is to be studied here, but on the seventh version, published in Italy in 1781. Thus, whenever the text of the sixth edition differs from the 1781, translations will be made from the 1772 Italian original.

II. THE ECONOMIST'S CONTRIBUTION

The Law of Supply and Demand: Preliminary Statements

Verri's point of departure is the idea that "The price of things is determined by two mutual causes, *want* and *scarcity*. The stronger these two causes are when taken together, the higher the price of things rises; and conversely, the more a commodity increases in abundance or the need for it declines, the more its price goes down and the cheaper it becomes" (Verri 1993, p. 15). This statement can be interpreted as an intuitive formulation of the law of supply and demand: the price of a good rises in the case of excess demand, and diminishes when supply exceeds demand.

Verri recognized that want and scarcity are not objective but psychological factors. First, he explained that although some goods are unquestionably useful, nobody would pay a positive price for them, due to their abundance. "Water, air and sunlight have no price, and yet nothing is more *useful*, indeed essential to us, than these" (Verri 1993, p. 15). Then he illustrated scarcity with the example of those goods that have no price, even though they are rare, because nobody is willing to buy them. "A medallion, an antique broach, a natural curiosity and suchlike objects, although they may be extremely rare and of great value to a few interested people or collectors, would nevertheless generally have a low price, or none at all, in the marketplace" (Verri 1993, p. 15). Not only is scarcity insufficient to determine the price, but scarcity itself depends on subjective motives. "Every quantity of goods withheld from trading has no influence on the price, and is as if non-existent. Potential offers will only produce a potential abundance" (Verri 1993, p. 15). Accordingly, it is not the absolute quantity of an existing good that determines its price, but rather its "apparent plenty," that is, the amount actually offered for sale. Contrary to absolute quantity, apparent abundance does not hinge on any physical or natural circumstances, but only on a psychological element, i.e., eagerness to sell the good.

In other words, *individual willingness to buy or sell influences prices*. Value and price are the results of *subjective* evaluations. Verri's emphasis on supply and demand, combined with references to utility and scarcity in the context of market price determination, explains why he has been considered a precursor of marginalist and neoclassical economics.

Yet Verri's very terminology shows that his economic imagery is still, to a certain extent, pre-modern. Some critics have considered his expression "apparent plenty" a synonym for the more modern "effective supply" (Porta and Scazzieri 1999, p. 825), but it should be seen as a remnant of medieval economic thought. In the Middle Ages, emphasis was placed on imperfect and asymmetrical information in economic transactions. Indeed, merchants were often able to gain extra profits due to consumers' ignorance and weakness. The market was not yet a pure concept, but still tied to a physical space where people gathered for the purchase and sale of commodities. It was meant to ensure plain visibility of merchants' actions, like a form of social control imposed upon them. Accordingly, policymakers were expected to guarantee that all transactions would take place in a market context, and that commodities would be physically abundant in the marketplace (Kaplan 1976, pp. 62–63). The term "appearance" denotes all this. Thus, Necker, for example, condemned the behavior of those sellers who had enough information to foresee poor harvests, and who then speculated, thus causing the price of corn to rise through a reduction of its appearance (Necker 1986, p. 31).

Supply

What has been said above raises the question of how Verri was able to assess individual willingness to sell, and by what means he evaluated its influence on price determination. In his opinion, the number of sellers of a given good is the concrete, visible market expression of its owners' willingness to sell it: "*apparent plenty is measured by the number of sellers*" (Verri 1993, p. 18). Let V (from the Italian *venditori*, i.e., sellers) be this number. Verri tried to give an explanation of this statement, in that he distinguished three different cases:

1. If $V = 1$:

> If a city had sufficient food to feed its people for one year, but this food were under the control of a single person, that single seller would bring to the market, on any one day, only a quantity sufficient for that day's sales; thus the quantity offered would be reduced to a minimum level, the *apparent plenty* would be the minimum possible, and consequently the *price* would be the maximum possible since it would depend solely on the discretion of that one despotic seller (Verri 1993, p. 18).

This is a monopoly.

2. If $V = 2$: "Now, suppose these provisions to be divided between two sellers" (Verri 1993, p. 18). Two different outcomes are possible:

> If they were to come to an agreement, we would be in the same position as before. If not, then the beginnings of competition between them would emerge, because even though there might be a quite substantial profit in selling the food to half the city, it is human nature to want more. Hence, speculation develops between the two sellers to estimate what advantage there is in lowering the price, and whether the benefit from the portion [of the market] taken from the competitor outweighs the general fall in price (Verri 1993, p. 18).

The same happens when $V = 3, 4, 5$, i.e., for any finite number, provided it is sufficiently small. "If a third, fourth, fifth, sixth seller, and so on, should appear in the market offering the same specific commodity, the portion each can sell becomes smaller and smaller, and likewise the loss from reducing the price, being easily recoverable through extended sales, diminishes" (Verri 1993, p. 18). This case is nowadays known as imperfect competition, or oligopoly, though it was not called like this in the eighteenth century.

3. $V = n$, where n is a sufficiently large number:

> When the number of sellers is increased in this manner, naturally enough the more there are, the more difficult it is for them to agree among themselves, and the more the increased sales make up for the decrease in price, with the result that rivalry and competition become livelier; so it follows that *apparent* plenty will increase and the price of the commodity will fall proportionately (Verri 1993, p. 18).

This case is today called perfect competition, but once more this phrase was not in use at the time.

It was Luigi Einaudi who in 1938 first noticed that Verri's explanation outlines a theory of monopoly, imperfect competition, and perfect competition. He observed that Verri perceived some of the most important features of Cournot's theory of market structures long before the publication of the latter's *Recherches* in 1838. Indeed, Cournot reproduced Verri's scheme: starting from one single seller, he first added a second, then a third, and so on, until their number became very high. Verri's conclusions—in passing from situation 1 to 2 and then to 3, an increasing augmentation of V and of the offered quantity can be observed, together with a progressive drop in price—are also confirmed by Cournot's analysis (Einaudi 1938, pp. 66–70). More recently, Theocharis has acknowledged Verri's theory of oligopoly as a forerunner of some modern ideas in this field (Theocharis 1983, pp. 151–54). There are only a few other contributions that are earlier than Cournot's. In addition to Verri, Theocharis mentions two Italian authors: Giammaria Ortes and Melchiorre Gioja.

However, it should be remarked that Verri's analysis is still far from Cournot's.

1. $V = 1$. According to Verri, in a monopoly the equilibrium quantity will be smaller, and the price higher than in the perfect competition case, but he offered nothing to support this statement. The solution to this problem can be attributed unmistakably to Cournot, who seventy years later succeeded in determining equilibrium quantity and price by introducing optimization assumptions. Given

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the market demand function, the monopolist maximizes his profit, subject to his own cost function. The optimization principle is meant to formally represent his economic behavior, that is, the pursuit of self-interest. The whole process can be expressed in mathematical form, and a unique solution can be found. Evidently, no such assumptions can be found in earlier works. Although Verri recognized that market supply and demand ultimately depend on individual decisions, he confined his economic analysis to the aggregate level without pushing it further to the study of the rules of individual behavior. *A fortiori*, it can be said that his theory of monopoly lacks solid microeconomic foundations.

In a certain definite sense Verri seems to be closer to medieval and pre-modern thought than to modern theories. In the eighteenth century, monopoly was often considered as a legal and political notion, rather than an economic one. In d'Alembert and Diderot's *Encyclopédie*, it was explicitly stated that it belonged to the field of jurisprudence. Accordingly, monopoly was sometimes seen as a *privilege* awarded by the sovereign. Adam Smith often employed the term to refer to industries enjoying statutory protection, whatever the number of firms might be (West 1987, p. 538). Sometimes, monopoly was treated as a *crime*. The entry "Monopoly" in the *Encyclopédie* defined it as an "illicit and hateful traffic" (Diderot and d'Alembert 1966–67, vol. 10, p. 668), punished according to the laws in force. Both medieval scholastic authors and sixteenth-seventeenth century mercantilist writers seem to have had a similar view. "Monopoly is a kind of commerce, in buying, selling, changing or bartering, usurped by a few and sometimes but by one person, and forestalled from all others, to the gaine of the monopolist and the detriment of other men." Its consequence is "the setting of the price at the pleasure of the monopolian to his private benefit and the prejudice of the public" (Misselden 1622, quoted in de Roover 1951, pp. 510–11). In fact, the number of sellers does not matter: the commerce is "usurped by *a few* and sometimes but by *one* person." Verri himself used the term monopoly in this sense in another passage of the book:

> When a nation's wealth is concentrated in the hands of a few, it is from those few that the people must receive their nourishment, and with their arbitrary price these sellers will force the common people into poverty and dependence. A few magnates, whose wealth permits them to devour every kind of commodity, will frequently occasion monopolies and artificial shortages in that nation (Verri 1993, p. 23).

If monopoly is a legal and political notion, price and quantity will depend on the power of the seller or sellers and not on any general economic laws. This explains both Misselden's idea that the price could be set "at the pleasure of the monopolian," and Verri's passage, according to which price and quantity "depend solely on the *discretion* of that one *despotic* seller."

In conclusion it is important to appreciate that, contrary to most eighteenth-century writers, Verri anticipated the modern concept of monopoly, understood as a single uncontested firm. However, there is in his book a good deal of confusion about what this notion involves. Some of Verri's ideas can be traced back to medieval definitions of monopoly as any kind of political restrictions. He was not able to elaborate any general economic law of monopoly price determination.

2. $V = 2, 3, \ldots$ Clearly, there is no unique solution in the case of imperfect competition, since it is impossible to determine whether there will be collusion among sellers or not. Once more, major advances did not come until 1838, when Cournot explicitly excluded coalitions from consideration, in order to avoid indeterminacy: "This restriction is very important, as it will soon appear; for if they should come to an agreement so as to obtain for each the greatest possible income, the results would be entirely different, and would not differ, so far as consumers are concerned, from those obtained in treating of a monopoly" (Cournot 1929, p. 80).

In all fairness to Verri, it should be recognized that while still being behind Cournot, he was ahead of his time. It has been mentioned that most eighteenth century authors considered monopoly as an advantage—whether legal or not—concerning *one or a few* privileged firms: in other words, oligopolies could in their opinion only give rise to coalitions, thus always yielding the same results as monopoly. At least Verri explored the possibility of some sort of competition in oligopolistic situations.

He went even further and explained how sellers are driven to competition in his *Riflessioni sulle leggi vincolanti principalmente nel commercio de' grani* (1769). Even though oligopolists come to an agreement, he said, the more they are, the more difficult it will be for them to enforce it:

> A single seller, seeking a quicker profit than his associates, immediately breaks the conspiracy, as no real obstacle prevents him to lower his price. He will thus encourage all buyers to deal with him rather than with the others. Consequently, all of them will be compelled to lower their own prices, and to compete against one another, in order to be able to sell their goods (Verri 1804, p. 214).

Although Verri did not explicitly say whether a position of equilibrium would be reached, Theocharis has argued that "presumably as long as it pays each seller to undercut his rival, he will do so and equilibrium will be reached when price settles at the free competition level." His conclusion is that "Verri's approach through price adjustments is thus similar to what is described in current oligopoly theory as the Bertrand Case" (Theocharis 1983, p. 153).

3. $V = n$. In the case of perfect competition, Verri based his reasoning entirely upon a *quantitative* characteristic of the market, namely the number of sellers. A *qualitative* definition of perfect competition was first given by Cournot, who characterized it as a situation in which each individual firm believes that it is too small to have any influence on prices, so that its decisions will concern only quantities. Contrary to the case of imperfect competition, sellers cannot choose their strategic variable, price, or quantity. No similar definition can be found in Verri's *Reflections*. As stated earlier, he was not able to include the rules of individual behavior in his economic analysis, and as a result he did not grasp the strategic difference between pure and imperfect competition.

Verri's sole qualitative criterion for distinguishing case 3 from case 2 is that no agreement among sellers is likely to succeed in perfect competition. For him this is a direct result of the number of sellers: the more they are, the more difficult it is for them to come to a binding agreement. His quantitative criterion is a key factor in his analysis. However, Verri failed to specify a line V^* separating case 2 from case 3. The notion of *many* individuals in case 3 is not made precise.

It is important to notice that Verri was one of the first economists to associate perfect competition with a large number of people participating in the exchange. Subsequent work by nineteenth and twentieth century economists has refined this idea, which is still widespread in economic thought, despite some counter-examples, such as the Bertrand oligopoly model in which the equilibrium price curiously equals the marginal cost of the firm, as in the case of perfect competition.

What has been said above leads to the conclusion that Verri's argument is weak. Since the dividing line between imperfect and perfect competition is indefinite, the model is indeterminate for any $V > 1$. Both the monopolistic outcome and the competitive one are virtually possible, and the final equilibrium situation cannot be foreseen with exactitude. All that can be said is that as far as the number V increases, the competitive result is more likely to be observed. Of course, this is not enough to prove the existence of a relation between the number V and the market price of a good.

Demand

As far as the demand side is concerned, Verri maintained that the number of buyers (C, from the Italian *compratori*) represents aggregate demand. In order to demonstrate this, he tried to develop an argument whose features are more or less similar to his explanation of supply:

> Let us say that there is one person with an exclusive monopoly of a given commodity. We have seen that in this case the apparent plenty will be minimal, but if there is only one buyer, then the want too will be minimal, consequently price will depend upon an equal clash between two single opinions. However, if the monopolist has two buyers instead of one, he can raise his asking price, and then, as the number of buyers increases so does the want component of price (all other things being equal) (Verri 1993, pp. 18–19).

This is an attempt at a *symmetrical* theory of supply and demand. However, Verri's analysis of market structures, on the buyers' side, is much weaker than the one he developed on the sellers' side. He failed to recognize the negotiation power of a monopsonist. Even though the term monopsony had not yet been introduced in economics, eighteenth century intellectuals were to some extent familiar with the concept. Some of them—e.g., Verri's colleague Cesare Beccaria—had examined the case of a bilateral monopoly, that is, of a monopolist confronting a monopsonist. Their conclusion was that it would be impossible to establish a general law of price determination, because of the fundamental role of the two parties' personal power. Verri does not seem to have been aware of this difficulty.

Verri seems to have been conscious of his lack of logical rigor. In the seventh edition of his book (1781), he eliminated the phrase "the number of buyers will be the *true* measure of want" (Verri 1772, p. 32, my emphasis), and wrote instead: "the number of buyers constitutes a measure which, *while it may lack the absolute* *precision required by a geometer, is the only one which suffices for practical use as a measure of want*" (Verri 1993, p. 18, my emphasis). He also added,

> These proportions are *approximate*, because strictly speaking, to satisfy mathematical precision all the buyers should purchase equal quantities. The quantities displayed by each seller and sought by each buyer are not always the same, nor does a buyer seeking a single unit have the same capacity to alter price as a buyer seeking ten units [. . . .]. Thus, these proportions are *approximately true, and in practice they will always be found consistent with facts* (Verri 1993, p. 19, my emphasis).

The Law of Supply and Demand: Verri's "Theorem"

Verri drew the following conclusion:

> If the number of *sellers* increases (other things being equal), *plenty* will increase and the *price* will fall; if the number of buyers increases (again, other things being equal), so will the *want* grow and the *price* increase. Thus the *price* is deduced from the *number of sellers* in comparison to the *number of buyers*. The more the former increases or the latter diminishes, the further the price will come down; and the more the former decreases and the latter multiplies, the higher the price will rise (Verri 1993, p. 19).

The idea of symbolizing demand and supply by the numbers of buyers and sellers had already been formulated by John Locke, whose economic writings were translated into Italian in 1751 (*Ragionamenti sopra la moneta, l'interesse del denaro, le finanze e il commercio*). According to Seizo Hotta, "the English Locke was one of the important sources of *Economia politica*, paying necessary attention to Lloyd and Forbonnais" (Hotta 1999, p. 710). Verri mentioned Locke in his *Considerazioni sul commercio nello Stato di Milano* (Verri 1939, p. 64), as well as in a letter to his brother Alessandro, dated November 24, 1770 (Verri 1912–42, IV, p. 71). Locke had written that "The price of any Commodity rises or falls, by the proportion of the number of Buyers and Sellers" (Kelly 1991, pp. 243–44), and "All things that are Bought and Sold, raise and fall their price in proportion, as there are more Buyers and Sellers" (Kelly 1991, p. 253). A similar statement can be found in Cantillon's *Essai sur la nature du commerce en général*, published in 1755: "The price of land, as all other prices, naturally depends on the ratio of buyers to sellers" (Cantillon 1997, p. 123). It is very possible that Verri knew it, given the wide distribution of this book by the time he was writing.

Verri seems to have attempted a mathematical formulation of this idea. In his preface to the sixth edition, he wrote: "Political economy, it seems to me, is close to becoming a science; all that is lacking is the method and organization of theorems to give it form, and it would not be so difficult today to fill the gaps and turn them into a succession of even, comfortable steps" (Verri 1993, p. 3). He maintained that, since he had to deal with quantities, he had to use "the language of the science, which measures them," in order to express himself with exactitude (Verri 1772, p. 33). Hence:

> 'If the number of sellers remains unchanged, prices will be proportionate to the number of buyers; if the number of buyers remains unchanged, prices will

> increase in proportion to the decrease in sellers . . .; *the price of things will be in direct proportion to the number of buyers and in inverse proportion to the number of sellers* (Verri 1993, p. 19).

He often used terms such as "proportion," "increase," "decrease," "ratio," "number," etc. However, it is doubtful whether he actually had in mind a mathematical formulation of the law of supply and demand. As noticed earlier, Verri maintained that there is a relation between the number V and the market price of a good, but nothing resembling a rigorous argument for this statement can be found in his work. In addition to this, it has been shown that Verri's theory of demand is even weaker, and that the relation between the number C and prices is an approximate one, which he assumed to be valid for practical purposes only. In other words, although Verri's statement appears at first glance very specific, the exact form of the relationship between the price, on the one hand, and the numbers of buyers and sellers, on the other, should be considered indefinite.

There is one other important point that requires emphasis. Verri's usage of some terms of mathematical origin is not proof enough that he always used them in their strict mathematical sense. In everyday language, these terms have a more ambiguous, imprecise, and vague meaning. Dmitriev maintains that most eighteenth century authors did not mean a ratio in the mathematical sense by the ratio between supply and demand. Their statements may simply mean that there is some relationship between the change in supply and demand and the change in price. He adds:

> The few exceptions are some of the early Italian economists such as, for example, Valeriani, Genovesi and Verri, of whom the last-named adopts an unusual definition of supply and demand, understanding by demand the number of purchasers, and by supply the number of sellers, so that the whole formula is expressed by him in the form: "the price of things varies directly with the number of buyers and inversely with the number of sellers," *but what other meaning should be attached to this "ratio" remains completely unclarified* (Dmitriev 1974, p. 183, my emphasis).

Moreover, it is worth noting that Verri suppressed the above-mentioned passage of the preface in later editions of the book. He also substituted for his picture of political economy as quantitative science a more ambiguous statement: "A mathematician would put it in this manner."

As a result, one may wonder whether Verri's theory of supply and demand is really open to mathematical treatment. It seems pretty difficult to formalize such an unclear set of ideas.

III. THE MATHEMATICIAN'S CONTRIBUTION

Frisi's Mathematical Reasoning: A Preliminary Presentation

Frisi interpreted Verri's words in their strict mathematical sense: "the price of things will be in direct proportion to the number of buyers and in inverse proportion to the number of sellers" became:

$$P = \frac{C}{V}.$$

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He did this through an elimination process: starting from a more general function, he examined its basic features and deduced that it had to be reduced to the simple ratio of buyers to sellers. He considered two types of general formulas: first,

$$P = M\frac{(C+A)^m}{(V+B)^n},$$

where P, V, C are defined as before and M, A, B, m, n, are constants. According to Frisi, given V, if there are no buyers ($C = 0$), there cannot be any price; therefore, A must be equal to zero. Given C, if there are no sellers ($V = 0$), the price must "rise outside any limits" (Verri 1772, p. 38). This condition requires $B = 0$. The resulting function is

$$P = \frac{C^m}{V^n}.$$

Now Frisi's reasoning is that, given V, if C becomes infinite, P must be an infinite of the same order; therefore, $m = 1$. Likewise, if V approaches infinity, given C, P will be an infinitesimal of the same order; therefore, $n = m = 1$. Hence, the above function is reduced to

$$P = \frac{C}{V}.$$

The second kind of general formula considered by Frisi is

$$P = \frac{MC + aC^{1/2} + bC^{1/3} + \&c.}{MV + aV^{1/2} + bV^{1/3} + \&c.}.$$

where a, b, and M are constants. Given C, an infinite V would make P infinitesimal, and given V, an infinite C would correspond to an infinite P. The additional terms would mean an infinitesimal or infinite price of radical order: this result is, in Frisi's opinion, "not likely" (Verri 1772, p. 38). Thus, this function reduces to a simple ratio of buyers to sellers as well.

Frisi also tried to find the point at which his function would be optimized. Of course, given V, P will be at a minimum ($P = 0$) if $C = 0$; given C, P will approach this minimum ($P \to 0$) when $V \to \infty$. The absolute maximum ($P \to \infty$) will either be obtained when $C \to \infty$, given V, or when $V \to 0$, given C. Besides, Frisi intended to determine a relative maximum or minimum, for simultaneous variations of both C and V. The necessary condition is:

$$dP = d\left(\frac{C}{V}\right) = 0,$$

or

$$\frac{V \cdot dC - C \cdot dV}{V^2} = 0,$$

i.e.

$$\frac{dC}{dV} = \frac{C}{V}.$$

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The Mathematical Validity of Frisi's Argument

Frisi's proof must first be carefully examined from a mathematical point of view. It need not be pointed out that his method is highly questionable. A general functional relation should be written as

$$P = f(C, V),$$

where f is an increasing function of C and a decreasing function of V. Frisi only examined some particular cases, namely fractions, the number of buyers being part of the numerator, and the number of sellers of the denominator. It is small wonder that his final result was itself one of these fractions. He simply selected one of them. Condorcet, one of the earliest commentators on his work, emphasized this lack of generality. He maintained that: "the required conditions are not enough to determine the exact form of the relationship: plenty of other formulae would fit" (Condorcet 1994, 2, pp. 72–73).

However, Frisi's argument cannot be fully understood unless it is set in historical perspective. First, there is some evidence that he was unfamiliar with the modern notion of function. It is true that this had already been defined when he wrote his notes, as indeed Euler had suggested it in 1748. Let x, he said, be a variable quantity. Then any variable y made up from x and from some numbers or constant quantities is called a function of x. He had also introduced the notation $f(x)$. However, this definition seems to have been too abstract to be utilized by all mathematicians at the time. Most of them still followed Descartes, who had banned from geometry all curves whose "analytical expression" could not be clearly identified (Bourbaki 1960, p. 213). This observation explains why Frisi strove to specify the exact form of the relationship between P on the one side, and V and C on the other.

Frisi may have even believed his reasoning to be general enough. According to Pierre Crépel:

> Early eighteenth century mathematicians thought that almost all functions could be approximated by a finite or infinite power series. Afterwards, they became more prudent, but never more precise on this issue. Even Lagrange, who was the king of mathematical analysis, freely used the symbol "&c." … When eighteenth century mathematicians took "a sum of powers + &c." into account, they believed their degree of generality to be much greater, than we think today (Crépel 1998, p. 46).

Finally, present-day readers may wonder why, among all the fractions he took into consideration, Frisi chose the simplest one. Once more, eighteenth century scientific and mathematical culture can help to make sense of it. Newton's law of gravitation suggested the idea that all natural phenomena are governed by simple principles. The mathematical expression of nature's simplicity is linearity. Crépel shows that most eighteenth century mathematical arguments were based on often-implicit linearity assumptions (Crépel 1998). It was Joseph Louis Lagrange who, at the turn of the century, developed a rationale for this approach. His basic idea was that, in a differential equation representing some real process, the contribution of the linear part of the function (the first term in the development of a Taylor series) expressed the fundamental characteristics of

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the process itself. The non-linear part was expected to represent inessential perturbations of the fundamental process; therefore, it could be neglected. Thus, linearization meant a simpler description of a phenomenon in which only the marginal aspects were lost. Throughout the nineteenth century, scientists widely adopted this approach. As far as economics is concerned, Cournot's work is an outstanding example of how linearization, considered as a first approximation of more complex curves, could bring about remarkable results.

These observations may give some insight into the reasons why Frisi strove to eliminate the terms $C^{1/2}$, $C^{1/3}$, etc., from his function,

$$P = \frac{MC + aC^{1/2} + bC^{1/3} + \&c.}{MV + aV^{1/2} + bV^{1/3} + \&c.}.$$

An all-important characteristic of Frisi's contribution to mathematical economics is the introduction of differential calculus. He both utilized the concepts of infinite and infinitesimal, and determined first-order conditions, in order to obtain maxima and minima of his function. When he wrote his notes, calculus was one of the newest and most advanced research fields in mathematics: it had been created independently by Newton and Leibniz in the late seventeenth century. Frisi's application of these tools to economic issues is remarkable. Calculus is founded on the philosophical notion of infinity; it is much more abstract than geometry or arithmetic. This probably explains why most eighteenth century authors did not dare apply it to real life issues. For example, in his *Recherches philosophiques sur l'évidence des vérités géométriques* (1773), François Quesnay explicitly refused "imperceptible mathematics." Although he had always insisted that calculations were of the utmost importance in economic matters, he mistrusted the abstract notion of infinity, originating in metaphysical speculations. In his opinion, true knowledge is derived solely from the senses. Thus, he considered traditional geometry more reliable than calculus, because its development is closely related to the practical necessity of measuring real objects (Quesnay 1773, quoted in Steiner 1998, pp. 20–22). Frisi's application of calculus suggests a vision of economics as an abstract science. Thus his contribution may be considered a step forward in the process of progressive creation of modern economics. In comparison to other eighteenth century formalizations, Frisi's work appears daring and forward-looking.

Furthermore, Frisi's optimization appears to have been the earliest in the history of economics. It is a substantial achievement, since maximization and minimization have had an increasingly relevant role in economic theory, from the so-called marginalist revolution onwards. Frisi should be acknowledged as a forefather of modern applications of mathematics to economic issues.

Nonetheless, his use of calculus is questionable. In particular, his comparison of infinite and infinitesimal quantities of different order is decidedly not rigorous by modern standards. Yet, some answers to this perplexity can be found in the history of mathematics. Nowadays, differential calculus is founded on the rigorously defined notion of limit. This approach is of very recent origin, since it dates from Cauchy and Weierstrass's works in the nineteenth century. When Frisi wrote his notes, mathematicians were on a much less secure ground. Their basic concept was the infinitesimal, which can be intuitively thought of as the infinitesimally small modification of a variable. This is too vague a definition, but no eighteenth century mathematician was able to provide a more rigorous one. Sometimes, an infinitesimal was considered as a finite and non-zero quantity, hence, subject to regular algebraic rules. Sometimes, it was thought of as so small a quantity, that it could be eliminated from the equation (as if it equaled zero). It was not unusual to find both interpretations in the same reasoning: this "meaning shift" (the expression is Berkeley's) made the argument inconsistent. Throughout the eighteenth century, there was a great deal of confusion about the basic principles of calculus, but there was enthusiasm for its results and applications. Mathematicians gradually became more and more dissatisfied with this situation, and their criticisms led toward the so-called Age of Rigor in the nineteenth century (Giorello 1982, pp. 231- 40).

In conclusion it can be said that, although some of Frisi's arguments are unacceptable by modern standards, they cannot be judged without taking the inadequacies of eighteenth century mathematics into account. Since some essential concepts were far from being clearly defined, and rigor standards were less strict than they are now, Frisi's peculiar usage of mathematical tools seems to a certain extent justifiable.

An Evaluation of Frisi's Argument, from an Economic Perspective

As indicated earlier, Frisi's formalization is founded on a literal interpretation of Verri's argument. The idea that the price increases when the number of buyers increases, and diminishes when the number of sellers increases, all other things being equal, is transformed into the formula

$$P = \frac{C}{V}.$$

Is this argument acceptable, from an economic point of view? Some eighteenth century writers first discussed this point. One of the most incisive critiques is an article in the *Nuovo Giornale de' letterati d'Italia* (1773), where an anonymous author identified as the mathematician Giambattista Venturi remarked that infinite variations in the numbers of buyers and sellers are very difficult to conceive. In his opinion, it is absurd to speak of the numbers C and V becoming infinitely small because no human being can be smaller than one unit. It is true that if C and V are interpreted literally as the numbers of buyers and sellers they are integer numbers) and differential calculus cannot be applied. It may be objected, however, that contemporary economic theory does not exclude the possibility of a continuum of agents, like the continuum of points on a line. In his 1964 contribution, Robert J. Aumann maintained that the construction of mathematical models to understand perfect competition should rely on the assumption of a continuum of traders. Although apparently unrealistic, the continuum can be considered an approximation of the actual situation in which there is a finite but large number of agents. The connection between perfect competition and a large number of agents has already been mentioned while examining Verri's argument. A thorough study of such an issue is beyond the scope of this paper, but it is interesting to notice that both Verri's theory and Frisi's mathematization raise this question.

Be that as it may, it is worth noting that other aspects of Frisi's formalization are problematic. The very idea that the price *equals* the ratio of buyers to sellers is absurd. By definition, the price of a good i is the quantity of another good j that must be given in exchange one unit of i. The quotient C/V is not a price. Yet, it is possible to conceive that in some circumstances the price of things *varies*, when the numbers of buyers and sellers vary. However, neither Verri nor Frisi understood the difference. The economist did not distinguish between equilibrium price determination on the one side and the study of price variations in disequilibrium situations on the other side. The mathematician's shortcoming is even worse. It has been mentioned that one of the reasons for the present-day expansion of mathematical economics is greater clarity of expression. An economic concept is often ambiguous, and mathematical form is expected to substitute for it a mathematical object that is subjected to definite rules of reasoning. In a sense, the mere statement of an economic problem in mathematical form may correct some of its fallacies. However, Frisi was unable to eliminate the inconsistency of Verri's argument. His formula simply reproduces its imperfections.

On the whole, it seems clear that Frisi made no great effort to think carefully about the economic concepts he was expressing in symbols and equations. The formula

$$P = \frac{C}{V}$$

translates only the economist's *conclusion* into mathematical symbols without taking its *premises*—that is, the theory of market structures—into account. Frisi sidestepped some of Verri's essential questions concerning monopoly, oligopoly, and competition, and simply took for granted the relation between the numbers C and V on the one side, and the variable P on the other. In his own words, "without going into further details, it will be enough for the following research to suppose what is very true, that prices always rise when the number of potential buyers increases, and that they always diminish when the number and competition of suppliers increase" (Verri 1772, p. 39). Curiously, what is a fundamental problem for the economist becomes a mere assumption for the mathematician. Moreover, Frisi wrote that the relation between the numbers C and V and the variable P is "very true," whereas Verri had explicitly seen it as an approximation. There is a striking discrepancy between their interpretations.

Frisi's failure to fully understand Verri's economic argument may explain why he does not seem to have realized that the numbers of buyers and sellers stand for demand and supply, respectively. He wrote that his formula was deduced "*assuming*, in this abstract reasoning, *that the quantity of the commodity, as well as the want of buyers do not change*; in other words, that all other things are equal, except the simple number of buyers and sellers" (Verri 1772, p. 233, my emphasis). Similarly, "*assuming as given the quantity of money, the quantity, and the want of the commodity*, and all other circumstances, the Italian author (Verri), who only sought variations arising from the numbers of buyers and sellers, stated that the price varies in direct proportion to the former, and in inverse proportion to the latter" (Verri 1772, p. 244, my emphasis).

This attitude inevitably gives rise to odd results. According to Frisi, given the number of sellers, if the number of buyers approaches infinity, the price must be an infinite of the same order. This argument has been discussed above from a mathematical viewpoint. However, as far as economics is concerned, it must be said that the very idea that a price may approach infinity—whatever the order may be—is hardly acceptable. Generally speaking, even when the number of buyers is very large, scarcity prevents prices from rising outside any limits. Significantly, the economist Verri never mentioned the possibility of an infinite price, thus avoiding such a difficulty. It is to be attributed to the mathematician Frisi, who treated the variable P as a mere mathematical object, without considering its economic meaning.

Frisi's failure to go into economic concepts more closely led to further difficulties when he attempted a generalization of Verri's formula. He did this in his review of Henry Lloyd's book. Lloyd had maintained that the price P of a good is a decreasing function of the total quantity of the good M, and an increasing function of the total quantity of money Q, which is given in exchange for M. He had obtained a price formula, reproduced by Frisi:

$$P = \frac{Q}{M}.$$

This is also an intuition of the law of supply and demand, where Q is a monetary measure of aggregate demand, and M is global supply of a good. Hence, Lloyd's theory of price determination is similar to Verri's. Their statements are but two slightly different versions of the same principle. The similarity of their views is not surprising, since Verri and Lloyd were very close friends and had many discussions on political economy. Both Venturi (1977, 1978) and Hutchison (1988, pp. 305–307) describe relationships between them and examine the influence of Lloyd on Verri. The common source of both formulations may have been Locke: "The rising and falling of the Price of Land, as of other things, depends much on the quantity of Land, set to Sale, compar'd with the quantity of Money design'd for that Traffick, or which amounts to the same thing, upon the number of Buyers and Sellers" (Kelly 1991, 1, p. 270). Another possible source is Cantillon, who wrote that the price of things hinges upon the proportion of the quantity of things offered for sale to the quantity of money offered in exchange, or, what is the same thing, upon the ratio of sellers to buyers (Cantillon 1997, pp. 110–11).

However, Frisi did not fully understand that. So he suggested that the two statements might be combined in a more general formula:

$$P = \frac{C \cdot Q}{V \cdot M}.$$

This would mean that "the price is in compound ratio, consisting of the direct ratio of the number of buyers and of the quantity of money, and the inverse

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ratio of the number of sellers and of the quantity of the good" (Verri 1772, p. 246). It was the first time Frisi was not simply presenting someone else's views, but proposing an idea of his own. This formula should be regarded as his most significant contribution to price theory, but it is impossible to make sense of it: the numerator is the product of two different approximations of the same variable (market demand), and the denominator is the product of one variable (market supply) and an approximation of it! Evidently, this is by no means a generalization of Verri's idea.

Finally, one may wonder whether Frisi's optimization methods fit their economic content. In order to answer this question, the economic argument shall first be examined. In Verri's opinion, abundant commodities and low prices are the conditions required for a social optimum. His vision—like Smith's—can be marked as *cheapness and plenty*, as opposed to Quesnay's *dearness and plenty* perspective (Porta and Scazzieri 1999, p. 824). On the basis of his price theorem, Verri maintained that a State's economic policy should aim at generalized price reductions. He explicitly stated that this must not be achieved through sumptuary laws, because diminishing the number of buyers would inevitably lower production and trade, thus provoking a reduction of the number of sellers as well. In the end, the ratio of buyers to sellers would be more or less the same, whereas industry and trade would lessen. Instead, economic policy must aim at increasing the number of sellers through liberalization of trade and extensive promotion of free competition. In Verri's opinion, an increase in the number of sellers does not lead to the same outcome, in that the number of domestic buyers does not necessarily rise. Since an additional amount of commodities is associated with a lower price, foreign countries may be interested in buying them. So, the number of international purchasers may increase, without affecting the number of domestic buyers. In the end, industry and trade inside the country will be more developed than before, and prices will be reduced to the advantage of the whole nation.

Frisi's optimization is supposed to translate this argument. As indicated earlier, he found that the price would be a maximum or minimum when

$$dP = d\left(\frac{C}{V}\right) = 0.$$

from which he got

$$\frac{dC}{dV} = \frac{C}{V}.$$

However, this is only a necessary condition, not a sufficient one; it only says that the price at this point does not change. The mathematical criticism that can be addressed to Frisi is that he should have added a second-order condition. The economic criticism is far more devastating: Verri was looking for a situation in which the price is at the minimum. He was not interested in the conditions for *either* a minimum or a maximum, let alone for a stationary point only. Knowing under what conditions price would remain unchanged adds nothing to his theory of economic policy. Frisi's result is useless. Likewise, it is not difficult to show that no policy principles can be deduced from the condition

$$\frac{dC}{dV} = \frac{C}{V}.$$

Frisi's mathematical treatment provided no solutions to Verri's questions. It did not advance the discussion of the underlying economic problems.

IV. CONCLUSIONS

The case that has been studied is one of the earliest attempts at a general, quantitative, and abstract price theory. Both the economist and the mathematician contributed to the final result. Verri tried to develop his intuition of the law of supply and demand, and possibly to draw some exact conclusions from it. Frisi endeavored to transform Verri's argument into a mathematical theorem.

There is no doubt that this experiment is, on the whole, unsuccessful. On the one hand, the economist is responsible for this failure. Although he grasped that it was necessary to seek the ultimate causes of price changes in the subjective evaluations of goods by agents, he was not able to formalize them. Hence, he could not give any satisfactory explanation of how market variables originate in individual behaviors. In order to explain how willingness to buy or sell goods affects prices, Verri relied on some more or less arbitrary indexes. In particular, he chose the number of buyers and sellers without providing sufficient evidence that they would be reliable indicators. While Verri prefigured some aspects of marginalist analysis, it has been noticed that his views were to a great extent anchored to medieval and early modern thought.

On the other hand, the mathematical side is also answerable. It has been shown that Frisi's näive utilization of mathematics did not improve Verri's economic theory. One of the causes of his failure is related to the state of pure mathematics at the time. It has been mentioned that, in Frisi's case, the lag from mathematical discovery to economic application was very short. While distinguishing him from other eighteenth-century mathematical economists, this entailed serious difficulties. It has been shown that the status of mathematical analysis was very different from what it is now. The problem of the foundations of the calculus had not yet been solved and was very controversial. Besides, a general definition of functions had just been proposed, but it was not yet widely accepted by mathematicians themselves.

It has been shown that the most important factor explaining Frisi's failure concerns applied mathematics. Frisi's formula appears to be a mere metaphor, not an instrument of scientific inquiry. In fact,

$$P = \frac{C}{V}$$

may be somehow considered as an image of the economist's words. However, as noted above, it can by no means be used as a tool for further discoveries, because it gives rise to inappropriate results. It does not lead to a better understanding

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of economic notions. In particular, Frisi's formalization does not help Verri grasp the difference between the problem of equilibrium price determination and the search for general laws of price variations in disequilibrium situations. As a matter of fact, the economist was not encouraged to refine his analysis of supply and demand, which was reproduced unchanged in subsequent editions of the book.

It has also been shown that Frisi's general formula,

$$P = \frac{C \cdot Q}{V \cdot M},$$

is devoid of economic sense. Hence, his own contribution to price theory is to be rejected. Another key finding of this paper is that Frisi's optimization cannot achieve anything helpful or beneficial, because the condition

$$\frac{dC}{dV} = \frac{C}{V}$$

has no economic meaning. If Frisi's work is interpreted as a translation of Verri's book from the original Italian into mathematical language, it must be noticed that the inverse translation, from mathematical symbols into common language, would be impossible.

Frisi's formalization suggested no new ideas to the economist. The *Reflections on Political Economy* gained nothing from this mathematical treatment. Frisi's work only resulted in his notes being suppressed by Verri!

It can be noted as a last remark that this leads to a more general line of thinking, namely, that two sources of difficulty should be emphasized when studying eighteenth-century mathematical economics. First, primitive formulations of economic concepts are to be considered a major obstacle to mathematical treatment. Secondly, the inadequacies of both pure and applied mathematics at the time are to be taken into account, for they may have prevented a deeper understanding of economic issues.

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