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# The response of a damped pendulum to a large driving force

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Analysing the motion of a driven, damped pendulum as a function of the amplitude of the driving force, we show, first, that for moderate values and larger of the amplitude, deviations from a simple motion with the period of the driving force are bounded by a constant times the inverse square root of the amplitude, for late times. For amplitudes above a larger threshold we are able to show that, for late times, the motion becomes a periodic motion with the period of the driving force. The manner in which this periodic motion is achieved with the passage of time is analysed.

## 1 Introduction

A damped pendulum is suspended from a pivot at one end and constrained to move in the plane perpendicular to the pivot. Its displacement is described by the angle $\theta$ giving the deviation of the pendulum from its stable rest position. When the pendulum is subjected to a driving force which is sinusoidal in time, its motion is described by the equation

$$\ddot{\theta}+\lambda\dot{\theta}+\omega_0^2\sin\theta = G\sin(\omega t). \tag{1.1}$$

Here $\lambda > 0$ is the damping, $\omega_0 > 0$ is the characteristic frequency of the pendulum (depending on gravity, mass of the pendulum, and moment of inertia of the pendulum), and $G > 0$ and $\omega > 0$ are the amplitude and frequency, respectively, of the driving (generalized) force.

Solutions of (1.1) have been studied extensively. Indeed, the family of such solutions for different values of the parameters $\omega$, $G$ and $\lambda$ has served as a paradigm for the discussion of different regimes of periodic motion, and of phenomena like period doubling, bifurcation, and chaos [1]. It is the late-time behaviour of solutions of (1.1) which is of primary interest. This is the behaviour after the effect of initial conditions has disappeared (as always happens eventually when $\lambda > 0$). For small $G$, this late-time motion is periodic with period $T = 2\pi/\omega$. As $G$ is increased, there is a point of bifurcation, after which the period doubles to $2T$. Further doublings of the period occur, until one arrives at a state of apparently chaotic motion.

Some systems of differential equations having chaotic solutions may be studied experimentally by observing the motion of actual physical devices. There are some electromechanical systems of this type [2]. In the case of the damped driven pendulum, a series of papers by Blackburn, Smith and coworkers [4–6] have laid the foundation for the experimental study of solutions of (1.1). Their efforts have resulted in the design of a pendulum which is commercially available, and which can be used to make very precise measurements [3]. Such measurements of the late-time motion of the damped driven pendulum, for different values $G$ of the amplitude of the driving force, suggest that, as $G$ increases, the pendulum slips into and out of apparently chaotic motions [7].

One can explore the nature of the late-time motion as $G$ is varied with greater versatility on the computer. Numerical solution of (1.1) by the fourth-order Runge–Kutta method is quite appropriate for obtaining such solutions, and these solutions are the basis for much of the analysis presented by Baker & Gollub [1]. The computer results bear out the experimental observations of Hammel [7], with regard to the alternating intervals in $G$ of periodicity and apparent chaos.

Nevertheless, the range of $G$ values for which the motion was studied by Baker and Gollub computationally, and by Hammel experimentally, was limited. A larger range of $G$ values was used in the computer simulations of Hinczewski [8]. His results suggested that at much higher $G$ values than those for which apparent chaotic behaviour first manifested itself, the apparent chaos disappeared permanently, and the late-time behaviour was periodic with period $T$. The mathematical analysis to support such a conclusion is the main purpose of this paper.

Standard definitions of chaos [2, 9] describe it as the phenomenon associated with a map $F$ from a metric space $M$ (often a Cantor-like subset of Euclidean space, with its own metric) into itself, with the following properties: (i) $F$ is unstable in $M$, with respect to unlimited repetitions of $F$, and in fact some repetition of $F$ on any open set in $M$ will intersect any other open set in $M$; (ii) points which are mapped into themselves after a finite number of applications of $F$ are dense in $M$. One should check the references above for further details.

In the context of the pendulum motion, $M$ will be a subset of $\mathbb{R}^2$, consisting of pairs $(\theta, T\dot{\theta})$ at a given time. $F$ will transform a pair $(\theta, T\dot{\theta})$ into the corresponding quantities a time $T$ later. Although the standard definition of chaos given above does not ascribe a 'size' to the chaos, we can assign such a size as the diameter of $M$ in the Euclidean norm in $\mathbb{R}^2$.

The central observation of this paper is that, for $G$ large, the nonlinear term in (1.1) has a relatively small effect on the motion. In particular, in the next section (Theorem 2.2) we will see that, writing $\theta = \theta_1 + \tilde{\theta}$, where $\theta_1$ satisfies (1.1) without the nonlinear term, for

$$\omega G \geqslant \lambda^2 \sqrt{\omega^2+\lambda^2} \tag{1.2}$$

and $t \geqslant$ a threshold value dependent upon the parameters of the problem and the initial conditions; we have

$$|\dot{\tilde{\theta}}(t)| \leqslant \frac{A_1}{\sqrt{G}} \tag{1.3}$$

for a constant $A_1 = A_1(\lambda, \omega, \omega_0, G)$. In the third section (equation (3.11)) we will find that, for almost all $G$ sufficiently large and $t \geqslant$ another threshold value, we have

$$|\tilde{\theta}(t) - A_2| \leqslant \frac{A_3}{\sqrt{G}}, \tag{1.4}$$

where $A_2$ and $A_3$ depend upon the parameters of the problem, and in addition, $A_2$ depends upon the initial conditions.

In conformity with the discussion of chaos above, we can say that the magnitude of any chaotic component of the motion is no more than $O(G^{-1/2})$ for $G$ sufficiently large. The question naturally arises as to whether Hinczewski's results [8] actually showed a vanishing of chaotic motion for $G$ large, or whether they were incapable of discerning a motion with such a small amplitude. This question is answered in §§4 and 5 of this paper. There we show (Theorem 5.3) that there is a constant $A_4$ such that, when $G \geqslant A_4$,

$$\theta(t+T)-\theta(t) \to 0 \quad \text{as } t \to \infty. \tag{1.5}$$

In general, the constant $A_4$ is much larger than the minimum value of $G$ given by (1.2). (Some numbers are given in the final section of the paper.) It follows from (1.5) that any components of the motion which are doubly periodic, multiply periodic, or chaotic, are precisely zero, and not just small. We will see, however, that the larger $G$ is, the longer it takes for this motion of period $T$ to manifest itself.

At this point we not that, to our knowledge, there is no proof that there is such a thing as a 'chaotic pendulum' satisfying (1.1) for *any* value of $G > 0$. For sufficiently small $G$ we know that the motion becomes periodic with period $T$ as $t \to \infty$, and we have shown the same thing in this paper for large $G$. For intermediate values of $G$ the results of computations are suggestive of chaos, but we cannot be sure that we do not have a motion of period $2^{20}\,T$, for example.

The methods of analysis we use in this paper are largely standard methods of classical asymptotic analysis, using for the most part little more than integration by parts. The asymptotic régime we study is considerably different from that studied by Cartwright [10]. In her case, $\lambda$ and $G$ are both large. Cartwright considers the late-time behaviour of solutions of a broad class of equations, but our equation (1.1) does not fall into that class. For example, she requires that the nonlinear term represented here by $\sin\theta$ – call it $g(\theta)$ – satisfies $g(\theta)\,\text{sgn}(\theta) > 0$ for $|\theta| \geqslant 1$, and also $g'(\theta) > 0$. Cartwright's starting point is to multiply the analogue of (1.1) by $\dot{\theta}$ and integrate over time, and to get an energy-type integral. Our starting point is to use the large amplitude of the oscillations of $\theta_1$ defined above, and to see how this brings about rapid oscillations in $\sin\theta$ in (1.1), with consequent small effect when integrated over time.

## 2 Domination of pendulum motion by driving force

We decompose the pendulum angle $\theta$ in (1.1) into two parts,

$$\theta = \theta_1 + \tilde{\theta}, \tag{2.1}$$

where

$$\ddot{\theta}_1 + \lambda\dot{\theta}_1 = G\sin(\omega t) \tag{2.2a}$$

and

$$\ddot{\tilde{\theta}} + \lambda\dot{\tilde{\theta}} = -\omega_0^2 \sin(\theta_1 + \tilde{\theta}). \tag{2.2b}$$

The solution of (2.2a) is

$$\theta_1 = c_1\,\mathrm{e}^{-\lambda t} + c_2 - \tilde{G}\cos(\omega t - \phi), \tag{2.3a}$$

where

$$\tilde{G} = \frac{G}{\omega\sqrt{\omega^2+\lambda^2}} \tag{2.3b}$$

and

$$\cos\phi = \frac{\lambda}{\sqrt{\omega^2+\lambda^2}}, \quad \sin\phi = \frac{\omega}{\sqrt{\omega^2+\lambda^2}}. \tag{2.3c}$$

We choose $c_1$ and $c_2$ so that

$$\theta_1(0) = \theta(0) \quad \text{and} \quad \dot{\theta}_1(0) = \dot{\theta}(0), \tag{2.4a}$$

or

$$c_1 = \frac{-G\omega}{\lambda(\omega^2+\lambda^2)} - \frac{\dot{\theta}(0)}{\lambda} \quad \text{and} \quad c_2 = \frac{G}{\lambda\omega} + \frac{\dot{\theta}(0)}{\lambda} + \theta(0). \tag{2.4b}$$

We shall refer to the solution $\theta_1$ of (2.2a) as a 'simple harmonic motion', although strictly speaking it becomes periodic only as $t \to \infty$, and one usually thinks of simple harmonic motions as involving only small values of the displacement $\theta$, for example, $|\theta| \ll \pi/2$. That is not the case here. The angle $\tilde{\theta}$ will be called simply the 'correction' to the simple harmonic motion. To the extent that the motion described by (1.1) is 'chaotic', the 'chaos' will be contained in the correction $\tilde{\theta}$ to the simple harmonic motion $\theta_1$. In this section we will obtain some bounds for the correction as a function of the driving amplitude $\tilde{G}$.

First, we note that (2.2b) is a first-order linear differential equation for $\dot{\tilde{\theta}}$, with solution

$$\dot{\tilde{\theta}}(t) = -\omega_0^2 \int_0^t e^{-\lambda(t-\xi)} \sin(\theta_1(\xi) + \tilde{\theta}(\xi))\, d\xi, \tag{2.5}$$

since $\dot{\tilde{\theta}}(0) = 0$, by (2.4). Equation (2.5) will be used to bound $\dot{\tilde{\theta}}$. The first bound we obtain is simply

$$|\dot{\tilde{\theta}}(t)| \leqslant \omega_0^2 \int_0^t e^{-\lambda(t-\xi)}\, d\xi \leqslant \frac{\omega_0^2}{\lambda}. \tag{2.6}$$

A sharper bound is obtained by breaking the interval of integration $[0, t]$ in (2.5) into two equal parts:

$$|\dot{\tilde{\theta}}(t)| \leqslant \frac{\omega_0^2\, e^{-\lambda t/2}}{\lambda} + \omega_0^2 \left| \int_{t/2}^t e^{-\lambda(t-\xi)} \sin(\theta_1(\xi) + \tilde{\theta}(\xi))\, d\xi \right|. \tag{2.7}$$

The first term on the right of (2.7) is bounded by $\omega_0^2/(\omega\sqrt{\tilde{G}})$, provided that $t$ is large enough that

$$e^{-\lambda t/2} \leqslant \frac{\lambda}{\omega\sqrt{\tilde{G}}}. \tag{2.8}$$

A bound for the second term on the right is provided by the following lemma.

**Lemma 2.1** *Let $\tilde{G}$ and $t$ satisfy the inequalities*

$$\tilde{G} \geqslant \lambda^2/\omega^2, \tag{2.9a}$$

$$\left|c_1 \frac{\lambda}{\omega} e^{-\lambda t/2}\right| \leqslant \sqrt{\tilde{G}}, \tag{2.9b}$$

*where $c_1$ is given by* (2.4b). *Then*

$$\left| \int_{t/2}^t e^{-\lambda(t-\xi)} \sin(\theta_1(\xi) + \tilde{\theta}(\xi))\, d\xi \right| \leqslant \left| \int_{t/2}^t e^{-\lambda(t-\xi)} \exp(i(\theta_1(\xi) + \tilde{\theta}(\xi)))\, d\xi \right| \leqslant \frac{2\pi}{\omega\sqrt{\tilde{G}}} \frac{2.7 + \omega_0/\lambda}{1 - e^{-\pi\lambda/\omega}}. \tag{2.10}$$

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**Proof** For a constant $\alpha \geqslant 2$, let

$$S_\alpha = \{\xi \in [t/2, t] \,|\, |\sin(\omega\xi - \phi)| \geqslant \alpha \tilde{G}^{-1/2}\}. \tag{2.11}$$

$S_\alpha$ consists of a collection of intervals $[a, b]$ in $[t/2, t]$ over which $|\sin(\omega\xi - \phi)| \geqslant \alpha \tilde{G}^{-1/2}$. Let us bound the contribution to the second integral of (2.10) of the form

$$I = \int_a^b e^{-\lambda(t-\xi)} \exp(i(\theta_1(\xi) + \tilde{\theta}(\xi)))\, d\xi. \tag{2.12}$$

We note that, over $[a, b]$,

$$\|\dot{\theta}_1(\xi)| \geqslant \tilde{G}\omega |\sin(\omega\xi - \phi)| - \sqrt{\tilde{G}}\,\omega \geqslant \tilde{G}\omega(1 - 1/\alpha)|\sin(\omega\xi - \phi)|$$
$$\geqslant \frac{\tilde{G}\omega}{2}|\sin(\omega\xi - \phi)| \geqslant \frac{\alpha}{2}\omega\sqrt{\tilde{G}}, \tag{2.13}$$

on account of (2.3a), (2.9b), and the restriction $\alpha \geqslant 2$. We integrate (2.12) by parts:

$$\begin{aligned} I = & -i e^{-\lambda(t-\xi)} \frac{e^{i\theta_1(\xi)} e^{i\tilde{\theta}(\xi)}}{\dot{\theta}_1(\xi)}\bigg|_a^b \\ & + i\lambda \int_a^b e^{-\lambda(t-\xi)} \frac{e^{i\theta_1(\xi)} e^{i\tilde{\theta}(\xi)}}{\dot{\theta}_1(\xi)}\, d\xi \\ & - \int_a^b e^{-\lambda(t-\xi)} \frac{e^{i\theta_1(\xi)} e^{i\tilde{\theta}(\xi)}}{\dot{\theta}_1(\xi)} \dot{\tilde{\theta}}(\xi)\, d\xi \\ & - i \int_a^b e^{-\lambda(t-\xi)} \frac{e^{i\theta_1(\xi)} e^{i\tilde{\theta}(\xi)}}{(\dot{\theta}_1(\xi))^2} \ddot{\theta}_1(\xi)\, d\xi. \end{aligned} \tag{2.14}$$

The integrated term in (2.14) and the first integral on the right-hand side of (2.14) are bounded by

$$\frac{4 e^{-\lambda(t-b)}}{\alpha\omega\sqrt{\tilde{G}}} \quad \text{and} \quad \frac{2 e^{-\lambda(t-b)}}{\alpha\omega\sqrt{\tilde{G}}}, \quad \text{respectively,} \tag{2.15}$$

on account of (2.13). With regard to the second integral on the right-hand side of (2.14), (2.6) and (2.13) give the bound

$$\frac{2\omega_0^2}{\lambda^2 \alpha\omega\sqrt{\tilde{G}}} e^{-\lambda(t-b)}. \tag{2.16}$$

Finally, in the last term of (2.14), we have

$$|\ddot{\theta}_1(\xi)| \leqslant 2\tilde{G}\omega^2,$$

on account of (2.3a) and (2.9). Thus, in view of (2.13), a bound for the last integral is

$$e^{-\lambda(t-b)} \frac{8}{\tilde{G}} \int_a^b \frac{d\xi}{\sin^2(\omega\xi - \phi)} \leqslant \frac{16}{\omega\alpha\sqrt{\tilde{G}}} e^{-\lambda(t-b)}. \tag{2.17}$$

The bounds in (2.15), (2.16) and (2.17) add up to

$$\frac{1}{\omega\alpha\sqrt{\tilde{G}}}\left(22 + \frac{2\omega_0^2}{\lambda^2}\right) e^{-\lambda(t-b)}. \tag{2.18}$$

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Adding up terms like (2.18) for all the intervals $[a, b]$ contained in $S_\alpha$ results in the bound

$$\left|\int_{S_\alpha} \mathrm{e}^{-\lambda(t-\xi)} \exp\left(i(\theta_1(\xi)+\tilde{\theta}(\xi))\right) \mathrm{d}\xi\right| \leqslant \frac{2}{\omega\,\alpha\sqrt{\tilde{G}}}\frac{11+\omega_0^2/\lambda^2}{1-\mathrm{e}^{-\pi\lambda/\omega}}. \tag{2.19}$$

We have to bound the contributions to the second integral on the left-hand side of (2.10) for which

$$|\sin(\omega\xi-\phi)| < \alpha\,\tilde{G}^{-1/2}. \tag{2.20}$$

Equation (2.20) implies that

$$|\omega\xi-\phi-n\pi| \leqslant \frac{\pi}{2}\frac{\alpha}{\sqrt{\tilde{G}}} \quad \text{for some integer } n. \tag{2.21}$$

Thus, the contributions to the integral for $\xi\in[t/2, t]-S_\alpha$ are bounded by

$$\frac{\pi\alpha}{\omega\sqrt{\tilde{G}}}\quad\frac{1}{1-\mathrm{e}^{-\pi\lambda/\omega}}. \tag{2.22}$$

Putting all our results together, we get

$$\left|\int_{t/2}^{t} \mathrm{e}^{-\lambda(t-\xi)} \sin\left(\theta_1(\xi)+\tilde{\theta}(\xi)\right) \mathrm{d}\xi\right| \leqslant \frac{2}{\omega\,\alpha\sqrt{\tilde{G}}}\frac{11+\omega_0^2/\lambda^2}{1-\mathrm{e}^{-\pi\lambda/\omega}} + \frac{\pi\alpha}{\omega\sqrt{\tilde{G}}}\frac{1}{1-\mathrm{e}^{-\pi\lambda/\omega}}. \tag{2.23}$$

Letting

$$\alpha = \left(\frac{2}{\pi}(11+\omega_0^2/\lambda^2)\right)^{1/2} < \sqrt{\frac{2}{\pi}}\left(\sqrt{11}+\frac{\omega_0}{\lambda}\right) < 2{\cdot}7+\frac{\omega_0}{\lambda} \tag{2.24}$$

gives us the result (2.10). □

Equations (2.10) and (2.7) immediately yield a bound for $|\dot{\tilde{\theta}}|$, formalized in the following theorem.

**Theorem 2.2** *Let $\tilde{G}$ and $t$ satisfy* (2.9) *and* (2.8). *Then*

$$|\dot{\tilde{\theta}}(t)| \leqslant \frac{\omega_0^2(2\pi)(3+\omega_0/\lambda)}{\omega\sqrt{\tilde{G}}(1-\mathrm{e}^{-\pi\lambda/\omega})}. \tag{2.25}$$

The bound (2.25) holds as long as the inequality (2.9 a) is satisfied, provided that one waits long enough, as required in (2.9 b) and (2.8). Theorem 2.2 then states that the deviation of the pendulum angular velocity, chaotic or not, from the angular velocity of a simple harmonic motion is no more than $O(1/\sqrt{\tilde{G}})$ for $\tilde{G}$ large. Thus, even if there is a chaotic part of the angular velocity, it may not be discernible for large $\tilde{G}$.

No matter how small a bound we have on the size of $\tilde{\theta}$, so long as the bound is positive there is the possibility of large excursions in the angle $\tilde{\theta}$ as the time increases without limit. The behaviour of $\tilde{\theta}$ will be studied in more detail in the sequel.

## 3 Intervals of attraction for pendulum angle at late times

In this section we shall indicate the mechanism whereby, for almost all values of $\tilde{G}$, the pendulum correction angle $\tilde{\theta}$ is trapped in a certain interval for late times. The width of the interval, when it exists, is bounded by a multiple of $1/\sqrt{\tilde{G}}$. We will not prove our assertion,

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but our argument can serve as an outline of a proof, which we leave for completion to the sufficiently interested reader.

Our starting point is the integral equation (2.5), which we rewrite by integration by parts:

$$\dot{\tilde{\theta}}(t) = -\omega_0^2 e^{-\lambda t} H_1(t) \cos\tilde{\theta}(t) - \omega_0^2 e^{-\lambda t} H_2(t) \sin\tilde{\theta}(t) + \omega_0^2 \int_0^t e^{-\lambda(t-\xi)} [-e^{-\lambda\xi} H_1(\xi) \sin\tilde{\theta}(\xi) + \epsilon^{-\lambda\xi} H_2(\xi) \cos\tilde{\theta}(\xi)] \dot{\tilde{\theta}}(\xi)\, d\xi, \quad (3.1a)$$

where

$$e^{-\lambda t} H_1(t) = \int_0^t e^{-\lambda(t-\xi)} \sin(\theta_1(\xi))\, d\xi \quad (3.1b)$$

and

$$e^{-\lambda t} H_2(t) = \int_0^t e^{-\lambda(t-\xi)} \cos(\theta_1(\xi))\, d\xi. \quad (3.1c)$$

It is easy to bound $e^{-\lambda t} H_1(t)$ and $e^{-\lambda t} H_2(t)$. Clearly,

$$|e^{-\lambda t} H_i(t)| \leqslant \frac{1}{\lambda}, \quad i = 1, 2. \quad (3.2a)$$

Bounds when the conditions (2.9) and (2.8) are satisfied are obtained in the same way we bounded the right-hand side of (2.5), except that to bound $e^{-\lambda t} H_1(t)$ we replace $\tilde{\theta}(\xi)$ by 0, and to bound $e^{-\lambda t} H_2(t)$ we replace $\tilde{\theta}(\xi)$ by $\pi/2$. The result is

$$|e^{-\lambda t} H_i(t)| \leqslant \frac{6\pi}{\omega\sqrt{\tilde{G}}(1-e^{-\pi\lambda/\omega})}, \quad i = 1, 2. \quad (3.2b)$$

Thus, in (3.1a) it is clear that the integrated terms are the dominant ones, and that the integrals left over are no more than $O(1/\tilde{G})$ in magnitude, on account of (2.25) and (3.2).

We can integrate (3.1a) from $t$ to $t+2\pi/\omega$, to get

$$\begin{aligned}\tilde{\theta}\left(t+\frac{2\pi}{\omega}\right) - \tilde{\theta}(t) = &-\omega_0^2 \left(\int_t^{t+2\pi/\omega} e^{-\lambda\xi} H_1(\xi)\, d\xi\right) \cos\tilde{\theta}(t) \\ &-\omega_0^2 \left(\int_t^{t+2\pi/\omega} e^{-\lambda\xi} H_2(\xi)\, d\xi\right) \sin\tilde{\theta}(t) \\ &-\omega_0^2 \int_t^{t+2\pi/\omega} F_1(\xi) \sin\tilde{\theta}(\xi)\, \dot{\tilde{\theta}}(\xi)\, d\xi + \omega_0^2 \int_t^{t+2\pi/\omega} F_2(\xi) \cos\tilde{\theta}(\xi)\, \dot{\tilde{\theta}}(\xi)\, d\xi \\ &+\frac{\omega_0^2}{\lambda} e^{-\lambda t}(1-e^{-2\pi\lambda/\omega}) \int_0^t [-H_1(\xi)\sin\tilde{\theta}(\xi) + H_2(\xi)\cos\tilde{\theta}(\xi)]\, \dot{\tilde{\theta}}(\xi)\, d\xi \\ &+\frac{\omega_0^2}{\lambda} \int_t^{t+2\pi/\omega} (e^{-\lambda\xi} - e^{-\lambda(t+2\pi/\omega)}) [-H_1(\xi)\sin\tilde{\theta}(\xi) + H_2(\xi)\cos\tilde{\theta}(\xi)]\, \dot{\tilde{\theta}}(\xi)\, d\xi,\end{aligned} \quad (3.3a)$$

where

$$F_i(\xi) = \int_{t+2\pi/\omega}^{\xi} e^{-\lambda\eta} H_i(\eta)\, d\eta, \quad i = 1, 2. \quad (3.3b)$$

Now we will study an approximate version of (3.3a) which holds for $\lambda t$ and $\tilde{G}$ large. From (3.1b) and (2.3a) we have

$$e^{-\lambda t} H_1(t) = \int_0^t e^{-\lambda\xi} \sin(c_1 e^{-\lambda(t-\xi)} + c_2 - \tilde{G}\cos(\omega(t-\xi)-\phi))\, d\xi$$

and, with $K_1(t)$ defined by

$$K_1(t) = \int_0^\infty e^{-\lambda\xi} \sin(c_2 - \tilde{G}\cos(\omega(t-\xi)-\phi))\, d\xi, \tag{3.4a}$$

one easily derives

$$|e^{-\lambda t} H_1(t) - K_1(t)| \leqslant \left(c_1 t + \frac{1}{\lambda}\right) e^{-\lambda t}. \tag{3.4b}$$

Similarly, for

$$K_2(t) = \int_0^\infty e^{-\lambda\xi} \cos(c_2 - \tilde{G}\cos(\omega(t-\xi)-\phi))\, d\xi, \tag{3.5a}$$

we have

$$|e^{-\lambda t} H_2(t) - K_2(t)| \leqslant \left(c_1 t + \frac{1}{\lambda}\right) e^{-\lambda t}. \tag{3.5b}$$

Note that $K_1(t)$ and $K_2(t)$ are periodic in $t$ with period $2\pi/\omega$.

Integration of $K_1(t)$ and $K_2(t)$ over a period yields

$$\begin{aligned}\int_t^{t+2\pi/\omega} K_1(\xi)\, d\xi &= \int_0^\infty e^{-\lambda\eta} \int_0^{2\pi/\omega} \sin c_2 \cos(\tilde{G}\cos\omega\xi)\, d\xi\, d\eta \\ &= \frac{2\pi}{\lambda\omega} \sin c_2\, J_0(\tilde{G}),\end{aligned} \tag{3.6a}$$

and

$$\int_t^{t+2\pi/\omega} K_2(\xi)\, d\xi = \frac{2\pi}{\lambda\omega} \cos c_2\, J_0(\tilde{G}), \tag{3.6b}$$

where $J_0(\tilde{G})$ is the Bessel function of order 0 [11].

On account of the bounds (3.2) on $H_1$ and $H_2$ (with corresponding bounds obtained from (3.3b) for $F_1(\xi)$ and $F_2(\xi)$ when $\xi \in [t, t+2\pi/\omega]$), the bounds (2.6) and (2.25), the approach of $e^{-\lambda t} H_i(t)$ to $K_i(t)$, $i = 1, 2$, as given by (3.4b) and (3.5b), and formulas (3.6), we are led to consider the recursion relation

$$\hat{\theta}\left(t + \frac{2\pi}{\omega}\right) - \hat{\theta}(t) = \frac{-2\pi\omega_0^2}{\lambda\omega} J_0(\tilde{G}) \sin(c_2 + \hat{\theta}(t)). \tag{3.7}$$

We anticipate that, for large times and large $\tilde{G}$, the solution of (3.3a) will behave like the solution of (3.7). Recall [11] that

$$J_0(\tilde{G}) \to \sqrt{\frac{2}{\pi\tilde{G}}} \cos(\tilde{G} - \pi/4) \quad \text{as } \tilde{G} \to \infty. \tag{3.8}$$

Suppose $\tilde{G}$ is large enough that

$$\frac{2\pi\omega_0^2}{\lambda\omega}|J_0(\tilde{G})| \leqslant |. \tag{3.9}$$

Equation (3.7) is a mapping from $\hat{\theta}(t)$ to $\hat{\theta}(t+2\pi/\omega)$. The fixed points of this mapping occur when $\sin(c_2+\hat{\theta}(t)) = 0$, or $c_2+\hat{\theta}(t) = n\pi$. Suppose that $J_0(\tilde{G}) > 0$ and $\sin(c_2+\hat{\theta}(t)) > 0$. Then $2n\pi < c_2+\hat{\theta}(t) < (2n+1)\pi$ for some integer $n$. From (3.7), $c_2+\hat{\theta}(t+2\pi/\omega) < c_2+\hat{\theta}(t)$ and, if (3.9) holds, $c_2+\hat{\theta}(t+2\pi/\omega) > 2n\pi$. It follows easily that, as $N\to\infty$, $c_2+\hat{\theta}(t+2\pi N/\omega)\to 2n\pi$. If, on the other hand, $J_0(\tilde{G}) > 0$ and $\sin(c_2+\hat{\theta}(t)) < 0$, so that $(2n-1)\pi < c_2+\hat{\theta}(t) < 2n\pi$, $c_2+\hat{\theta}(t) < c_2+\hat{\theta}(t+2\pi/\omega) < 2n\pi$, and $c_2+\hat{\theta}(t+2\pi N\omega)\to 2n\pi$ as $N\to\infty$. Similar arguments apply when $J_0(\tilde{G}) < 0$. We summarize the results as follows:

**Lemma 3.1** *For the mapping* (3.7)

$$c_2+\lim_{N\to\infty}\tilde{\theta}(t+2\pi N/\omega) = \begin{cases} 2n\pi & \text{if } J_0(\tilde{G}) > 0 \\ (2n+1)\pi & \text{if } J_0(\tilde{G}) < 0. \end{cases} \tag{3.10}$$

We have seen that equation (3.3a) for the change in $\tilde{\theta}(t)$ over a period $2\pi/\omega$ is the same as (3.7), with additional terms on the right-hand side which are at most $O(1/\tilde{G})$, for $t$ sufficiently large. We may think of the right-hand side of (3.7) as a term causing $\hat{\theta}+c_2$ to be 'attracted' to $2n\pi$ if $J_0(\tilde{G}) > 0$ and to $(2n+1)\pi$ if $J_0(\tilde{G}) < 0$. As long as $|J_0(\tilde{G})| = 0(1/\sqrt{\tilde{G}})$, i.e.

$$|\cos(\tilde{G}-\pi/4)| = 0(1)), \quad \text{for} \left| c_2+\hat{\theta}-\begin{cases} 2n\pi & J_0(\tilde{G}) > 0 \\ (2n+1)\pi & J_0(\tilde{G}) < 0 \end{cases}\right|$$

more than a sufficiently large multiple of $1/\sqrt{\tilde{G}}$, this attractive term will dominate the additional terms which appear in equation (3.3a) for $\tilde{\theta}(t)$. Hence, we can infer that

$$|\tilde{\theta}(t+2\pi N/\omega)+c_2-2n\pi| = 0(1/\sqrt{\tilde{G}}) \quad \text{for } N \text{ sufficiently large and } J_0(\tilde{G}) > 0; \tag{3.11a}$$

$$|\tilde{\theta}(t+2\pi N/\omega)+c_2-(2n+1)\pi| = 0(1/\sqrt{\tilde{G}}) \quad \text{for } N \text{ sufficiently large and } J_0(\tilde{G}) < 0. \tag{3.11b}$$

The 'late-time' behaviour which has just been noted for repeated applications of the mapping (3.7) is a special case of the late-time behaviour of solutions of the initial-value problem

$$\dot{\phi}(t,\phi_0) = A(t)\cos\phi(t,\phi_0)+B(t)\sin\phi(t,\phi_0), \tag{3.12a}$$

$$\phi(t_0,\phi_0) = \phi_0, \tag{3.12b}$$

where $A(t)$ and $B(t)$ are bounded, measurable, and periodic with period $2\pi/\omega$.

**Lemma 3.2** *If, for some $\phi_0 = \phi^*$ and $t_0 = t^*$, the solution of* (3.12) *satisfies*

$$\phi(t^*+2\pi/\omega,\phi^*) = \phi^*, \tag{3.13}$$

*then for all $\phi_0$ and $t_0$, $\lim_{N\to\infty}\phi(t_0+2\pi N/\omega,\phi_0)$ exists.*

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**Proof** It follows from (3.13) and (3.12a) that there are numbers $\phi_-^*$ and $\phi_+^*$ such that $\phi_-^* \leqslant \phi(t, \phi^*) \leqslant \phi_+^* \,\forall t$. Furthermore, for any $t, \phi(t, \phi_0) - \phi_0$ is periodic in $\phi_0$, with period $2\pi$. Since the right-hand side of (3.12a) is Lipschitz-continuous with respect to $\phi$, $\phi(t), \phi_0)$ has a Lipschitz-continuous dependence on $\phi_0$. Specifically,

$$\mathrm{e}^{-M(t_0, t)} \leqslant \frac{\partial \phi(t, \phi_0)}{\partial \phi_0} \leqslant \mathrm{e}^{M(t_0, t)}, \tag{3.14a}$$

where

$$M(t_0, t) = \left| \int_{t0}^{t} ((A(\xi))^2 + (B(\xi))^2)^{1/2} \, \mathrm{d}\xi \right|. \tag{3.14b}$$

From (3.14a), we conclude that $\phi(t, \phi_0)$ increases monotonically with $\phi_0$. This monotonicity is a direct consequence of the fact that (3.12a) is a first-order equation. In contrast, (2.2b) is a second-order equation which does not generally exhibit such monotonicity. From the periodicity of $A(t)$ and $B(t)$ it follows that $\phi(t + 2\pi/\omega, \phi_0) = \phi(t, \phi(t_0 + 2\pi/\omega, \phi_0))$, so if, for example, $\phi(t_0 + 2\pi/\omega, \phi_0) > \phi_0$, we get $\phi(t + 2\pi/\omega, \phi_0) > \phi(t, \phi_0) \,\forall t$. Choose $m$ as the smallest integer such that $\phi^* + 2m\pi \geqslant \phi_0$. The quantities $\phi(t_0 + 2\pi N/\omega, \phi_0)$ are increasing with increasing $N$, but are all bounded above by $\phi_+^* + 2m\pi$. Thus $\lim_{N \to \infty} \phi(t_0 + 2\pi N/\omega, \phi_0)$ exists. Similarly, if $\phi(t_0 + 2\pi/\omega, \phi_0) < \phi_0$, we get a decreasing sequence $\{\phi(t_0 + 2\pi N/\omega, \phi_0) \mid N = 0, 1, 2, \ldots\}$ which is bounded below by $\phi_-^* + 2(m-1)\pi$, and the desired limit exists. If

$$\phi\left(t_0 + \frac{2\pi}{\omega}, \phi_0\right) = \phi_0, \quad \lim_{N \to \infty} \phi\left(t_0 + \frac{2\pi N}{\omega}, \phi_0\right) = \phi_0.$$

In numerous cases of interest, the condition (3.13) is fulfilled. For example, if $A(t) \equiv 0$, then we obtain (3.13) whenever $\phi^* = n\pi$. A similar situation occurs if $B(t) \equiv 0$. Suppose $A(t) = -A(2t_0 + 2\pi/\omega - t)$ and $B(t) = -B(2t_0 + 2\pi/\omega - t)$ for some $t_0$. Then (3.13) is satisfied for all $\phi^*$. Suppose $A(t) = A(2t_0 + 2\pi/\omega - t)$ and $B(t) = B(2t_0 + 2\pi/\omega - t)$ for some $t_0$. Then from (3.12), starting at 'initial' time, $t_0 + 2\pi/\omega$ and 'angle' $\phi_0 + \pi$, and running backward in time for a period $2\pi/\omega$, we get

$$\phi\left(t_0 + \frac{2\pi}{\omega}, \phi_0\right) - \phi_0 = \phi_1 - (\phi_0 + \pi), \tag{3.15}$$

where $\phi_1$ is the angle for which $\phi(t_0 + 2\pi/\omega, \phi_1) = \phi_0 + \pi$. Thus, if $\phi(t_0 + 2\pi/\omega, \phi_0) - \phi_0 > 0$ ($< 0$, resp.), we will have $\phi(t_0 + 2\pi/\omega, \phi_1) - \phi_1 < 0$ ($> 0$, resp.), and by continuity there is a $\phi^*$ such that $\phi(t_0 + 2\pi/\omega, \phi^*) = \phi^*$, satisfying (3.13).

More generally, one can derive from (3.12a) the bound

$$\begin{aligned} &\left| \phi\left(t_0 + \frac{2\pi}{\omega}, \phi_0\right) - \phi_0 - \cos\phi_0 \int_0^{2\pi/\omega} A(\xi)\,\mathrm{d}\xi - \sin\phi_0 \int_0^{2\pi/\omega} B(\xi)\,\mathrm{d}\xi \right. \\ &\quad + \tfrac{1}{2}\cos\phi_0 \sin\phi_0 \left[ \left( \int_0^{2\pi/\omega} A(\xi)\,\mathrm{d}\xi \right)^2 + \left( \int_0^{2\pi/\omega} B(\xi)\,\mathrm{d}\xi \right)^2 \right] - \int_0^{2\pi/\omega} A(\xi) \int_0^{\xi} B(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi \\ &\quad \left. + \cos^2\phi_0 \left( \int_0^{2\pi/\omega} A(\xi)\,\mathrm{d}\xi \right) \left( \int_0^{2\pi/\omega} B(\xi)\,\mathrm{d}\xi \right) \right| \leqslant \frac{1}{6} \left( \int_0^{2\pi/\omega} (|A(\xi)| + |B(\xi)|)\,\mathrm{d}\xi \right)^3. \end{aligned} \tag{3.16}$$

There is no loss of generality in considering the case

$$\int_0^{2\pi/\omega} B(\xi)\,\mathrm{d}\xi = 0, \tag{3.17a}$$

for we can always achieve this by changing the origin of $\phi$. If (3.17a) holds and, in addition, if

$$\frac{1}{2}\left(\int_0^{2\pi/\omega} (|A(\xi)|+|B(\xi)|)\mathrm{d}\xi\right)^2 \leqslant \min\left(1, \left|\int_0^{2\pi/\omega} A(\xi)\,\mathrm{d}\xi\right|\right), \tag{3.17b}$$

we can deduce from (3.16) that there is an angle $\phi_0$ such that $\phi(t_0+2\pi/\omega, \phi_0)-\phi_0 > 0$, there is an angle $\phi_1$ such that $\phi(t_0+2\pi/\omega)\,\phi_1)-\phi_1 < 0$, and hence by continuity there is a $\phi^*$ such that (3.13) holds. For all these cases Lemma 3.2 applies.

## 4 Evolution of pendulum motion after memory of initial conditions has faded

For late times, defined by the condition $t \geqslant t_0$, where $\lambda t_0$ is a large number, the solution $\tilde{\theta}$ of (2.5) and (2.3a) will satisfy approximately

$$\dot{\tilde{\theta}}(t) = -\omega_0^2 \int_0^{\infty} \mathrm{e}^{-\lambda\xi} \sin(c_2 - \tilde{G}\cos(\omega(t-\xi)-\phi)+\tilde{\theta}(t-\xi))\,\mathrm{d}\xi. \tag{4.1}$$

Equation (4.1) suggests that for late times we approximate $\tilde{\theta}(t+t_0)$ by

$$\tilde{\theta}(t+t_0) \cong \tilde{\theta}(t_0)+\psi(t), \tag{4.2}$$

where $\psi(t)$ satisfies

$$\dot{\psi}(t) = -\omega_0^2 \int_0^{\infty} \mathrm{e}^{-\lambda\xi} \sin(\theta_0 - \tilde{G}\cos(\omega(t-\xi)-\phi_0)+\psi(t-\xi))\,\mathrm{d}\xi \tag{4.3a}$$

and

$$\psi(0) = 0. \tag{4.3b}$$

For the sake of simplicity, we have written

$$\theta_0 = c_2+\tilde{\theta}(t_0) \quad \text{and} \quad \phi_0 = \phi - \omega t_0 \tag{4.4}$$

in (4.3a). With the translation (4.2), 'late' time now means $t \geqslant 0$.

In this section we shall show that (4.3) has a unique solution for all time, $-\infty < t < \infty$, if $\tilde{G}$ is large enough. First, for a number $\lambda' \in (0, \lambda/2]$, we will let $V$ denote the space of functions $\psi$ defined for $t \leqslant 0$, satisfying (4.3b), and for which

$$\|\psi\| = \sup_{t\leqslant 0} \frac{1}{\lambda'} |\mathrm{e}^{\lambda' t}\,\dot{\psi}(t)| < \infty. \tag{4.5}$$

$U: V \to V$ is defined as the transformation $\psi \to \chi = U\psi$ given by

$$\dot{\chi}(t) = -\omega_0^2 \int_0^{\infty} \mathrm{e}^{-\lambda\xi} \sin(\theta_0 - \tilde{G}\cos(\omega(t-\xi)-\phi_0)+\psi(t-\xi))\,\mathrm{d}\xi. \tag{4.6a}$$

$$\chi(0) = 0. \tag{4.6b}$$

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Clearly, if $\chi \in V$ is in the range of $U$, then

$$|\dot{\chi}(t)| \leqslant \omega_0^2/\lambda \quad \text{for } t \leqslant 0. \tag{4.7}$$

In terms of $U$, (4.3) takes the form

$$\psi = U\psi. \tag{4.8}$$

In order to prove the existence of a solution of (4.8), we first show that $U$, when restricted to functions $\psi \in V$ for which $|\dot{\psi}| \leqslant \omega_0^2/\lambda$ for $t \leqslant 0$, is a contraction in $V$, for $\tilde{G}$ large enough. Hence, we shall bound $\|\chi_1 - \chi_2\|$ in terms of $\|\psi_1 - \psi_2\|$, where

$$\chi_i(t) = U(\psi_i(t)), \quad i = 1, 2. \tag{4.9}$$

From (4.5), we see that we need to bound, for $t \leqslant 0$,

$$\begin{aligned} \frac{1}{\lambda'} e^{\lambda' t} \omega_0^2 \bigg| \int_{-\infty}^{t} & e^{-\lambda(t-\xi)} [\sin(\theta_0 - \tilde{G}\cos(\omega\xi - \phi_0) + \psi_1(\xi)) \\ & - \sin(\theta_0 - \tilde{G}\cos(\omega\xi - \phi_0) + \psi_2(\xi)] \, d\xi \bigg| \\ \leqslant \frac{1}{\lambda'} e^{\lambda' t} \omega_0^2 & \left| \int_{-\infty}^{t} e^{-\lambda(t-\xi)} \exp(i(\theta_0 - \tilde{G}\cos(\omega\xi - \phi_0)))(e^{i\psi_1} - e^{i\psi_2}) \, d\xi \right|. \end{aligned} \tag{4.10}$$

Now our analysis bears a strong resemblance to the proof of Lemma 2.1. Let

$$\tilde{S}_\beta = \{\xi \in (-\infty, t] \,|\, |\sin(\omega\xi - \phi_0)| \geqslant \beta \tilde{G}^{-1/2}\}. \tag{4.11}$$

$\tilde{S}_\beta$ consists of a countable collection of intervals $[a, b]$ in $(-\infty, t]$ over which $|\sin(\omega\xi - \phi_0)| \geqslant \beta \tilde{G}^{-1/2}$.

We bound

$$\tilde{I} = \int_a^b e^{-\lambda(t-\xi)} e^{-i\tilde{G}\cos(\omega\xi - \phi_0)} (e^{i\psi_1} - e^{i\psi_2}) \, d\xi \tag{4.12}$$

by integration by parts.

Equation (4.12) looks like (2.12), with $\theta_1$ replaced by $-\tilde{G}\cos(\omega\xi - \phi_0)$ and $e^{i\tilde{\theta}}$ replaced by $e^{i\psi_1} - e^{i\psi_2}$. Thus, the integration by parts yields terms like those on the right-hand side of (2.14). Before proceeding with the bounds, we observe that (4.5) implies that, for almost all $t \leqslant 0$,

$$|\dot{\psi}(t)| \leqslant \lambda' e^{-\lambda' t} \|\psi\|. \tag{4.13}$$

Since

$$\psi(t) = -\int_t^0 \dot{\psi}(\xi) \, d\xi,$$

on account of (4.3b), we get for $t \leqslant 0$

$$|\psi(t)| \leqslant e^{-\lambda' 2t} \|\psi\|. \tag{4.14}$$

Using (4.11), (4.14), the fact that $\lambda > \lambda'$, and the mean-value theorem for $e^{i\psi_1} - e^{i\psi_2}$, we get for the first term (the integrated term) in the integration by parts the bound

$$\frac{2 e^{-\lambda(t-b)}}{\omega\beta\sqrt{\tilde{G}}} e^{-\lambda' b} \|\psi_1 - \psi_2\|. \tag{4.15}$$

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Terms analogous to the second and fourth terms on the right-hand side of (2.14) are bounded in the same way as in §2. The only term that is different is the term corresponding to the '$\dot{\tilde{\theta}}$' term. For this we write

$$\dot{\psi}_1 e^{i\psi_1} - \dot{\psi}_2 e^{i\psi_2} = (\dot{\psi}_1 - \dot{\psi}_2) e^{i\psi_1} + \dot{\psi}_2 (e^{i\psi_1} - e^{i\psi_2}). \tag{4.16}$$

Since we are requiring that $|\dot{\psi}_2| \leqslant \omega_0^2/\lambda$ for $t \leqslant 0$, a suitable bound follows immediately:

$$\frac{e^{-\lambda(t-b)}}{\omega\beta\sqrt{\tilde{G}}} \left( \frac{\lambda'}{\lambda - \lambda'} + \frac{\omega_0^2}{\lambda} \frac{1}{\lambda - \lambda'} \right) e^{-\lambda' b} \, \|\psi_1 - \psi_2\|. \tag{4.17}$$

For $\lambda' \leqslant \lambda/2$, all the contributions to the right-hand side of (4.10) from $\tilde{S}_\beta$ are thus bounded by

$$\frac{\omega_0^2}{\lambda'} \frac{1}{\omega\beta\sqrt{\tilde{G}}} \left( 7 + \frac{2\omega_0^2}{\lambda^2} \right) (2 + e^{-\pi(\lambda-\lambda')/\omega} + e^{-2\pi(\lambda-\lambda')/\omega} + \dots) \, \|\psi_1 - \psi_2\|$$

$$\leqslant \frac{2\omega_0^2}{\lambda'} \frac{1}{\omega\beta\sqrt{\tilde{G}}} \left( 7 + \frac{2\omega_0^2}{\lambda^2} \right) \frac{1}{1 - e^{-\pi\lambda/(2\omega)}} \, \|\psi_1 - \psi_2\|. \tag{4.18}$$

The contributions to the right-hand side of (4.10) from $(-\infty, t] - \tilde{S}_\beta$ are bounded in the same way that similar contributions were bounded in the proof of Lemma 2.1. The bound in this case is

$$\frac{\omega_0^2}{\lambda'} \frac{\pi\beta}{\omega\sqrt{\tilde{G}}} \frac{\|\psi_1 - \psi_2\|}{1 - e^{-\lambda\pi/(2\omega)}}. \tag{4.19}$$

By choosing

$$\beta = \left( \frac{2}{\pi} \left( 7 + \frac{2\omega_0^2}{\lambda^2} \right) \right)^{1/2}, \tag{4.20}$$

we obtain

$$\|U\psi_1 - U\psi_2\| \leqslant \frac{4\omega_0^2}{\lambda'} \frac{\sqrt{\pi/2\,(7 + 2\omega_0^2/\lambda^2)}}{\omega\sqrt{\tilde{G}}\,(1 - e^{-\lambda\pi/(2\omega)})} \|\psi_1 - \psi_2\|. \tag{4.21}$$

We have thus proved the following result:

**Lemma 4.1** *For* $0 < \lambda' \leqslant \lambda/2$ *and*

$$\sqrt{\tilde{G}} > \frac{4\omega_0^2}{\lambda'} \frac{\sqrt{\pi/2(7 + 2\omega_0^2/\lambda^2)}}{\omega(1 - e^{-\lambda\pi/(2\omega)})},$$

the transformation $U$ in (4.6) is a contraction for functions $\psi \in V$ satisfying $|\dot{\psi}| \leqslant \omega_0^2/\lambda$ when $t \leqslant 0$, and the coefficient of contraction is

$$K \leqslant \frac{4\omega_0^2}{\lambda'} \frac{\sqrt{\pi/2\,(7 + 2\omega_0^2/\lambda^2)}}{\omega\sqrt{\tilde{G}}\,(1 - e^{-\lambda\pi/(2\omega)})}. \tag{4.22}$$

We are now in a position to solve (4.3) iteratively when $K < 1$. Let

$$\Psi(t) = \theta_0 + \psi(t). \tag{4.23}$$

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We set

$$\Psi^{(0)}(t) = \theta_0, \quad t \leqslant 0, \tag{4.24a}$$

and for $i \geqslant 0$

$$\Psi^{(i+1)} = \theta_0 + U(\Psi^{(i)} - \theta_0), \tag{4.24b}$$

that is,

$$\dot{\Psi}^{(i+1)}(t) = -\omega_0^2 \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-\xi)} \sin(-\tilde{G}\cos(\omega\xi - \phi_0) + \Psi^{(i)}(\xi))\,\mathrm{d}\xi, \tag{4.24c}$$

with

$$\Psi^{(i+1)}(0) = \theta_0. \tag{4.24d}$$

$\|\Psi^{(0)} - \theta_0\| = 0$. To bound $\|\Psi^{(1)} - \theta_0\|$, we have to bound, for $t \leqslant 0$,

$$\omega_0^2 \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-\xi)} \sin(-\tilde{G}\cos(\omega\xi - \phi_0) + \theta_0)\,\mathrm{d}\xi. \tag{4.25}$$

A bound is obtained in the same way that we bounded the right-hand side of (4.10), except that a number of simplifications occur: we delete terms containing $\psi_2$ or $\dot{\psi}_2$, we replace $\lambda'$ by 0, and we delete terms containing $\dot{\psi}_1$. We decompose the integral in (4.25) into integrals over a set $\tilde{S}_\beta$, as in (4.11), and over $(-\infty, t] - \tilde{S}_\beta$. In this case, we can choose $\beta = \sqrt{10/\pi}$. Thus we find

$$\|\Psi^{(1)} - \theta_0\| \leqslant \frac{2\sqrt{10\pi}\,\omega_0^2}{\lambda'\,\omega\sqrt{\tilde{G}}} \frac{1}{1 - \mathrm{e}^{-\pi\lambda/\omega}}. \tag{4.26}$$

It follows from (4.24) and Lemma 4.1 that

$$\|\Psi^{(i+1)} - \Psi^{(i)}\| \leqslant K^i \|\Psi^{(1)} - \theta_0\|, \tag{4.27a}$$

and

$$\|\Psi - \theta_0\| = \lim_{i\to\infty} \|\Psi^{(i)} - \theta_0\| \leqslant \frac{1}{1-K} \|\Psi^{(1)} - \theta_0\|. \tag{4.27b}$$

**Lemma 4.2** *When*

$$\sqrt{\tilde{G}} \geqslant \frac{8\omega_0^2}{\lambda'} \frac{\sqrt{\pi/2\,(7 + 2\omega_0^2/\lambda^2)}}{\omega(1 - \mathrm{e}^{-\lambda\pi/(2\omega)})}, \tag{4.28a}$$

*equation* (4.3) *has a solution* $\psi$ *with the bound*

$$\|\psi\| \leqslant \frac{4\sqrt{10\pi}\,\omega_0^2}{\lambda'\omega\sqrt{\tilde{G}}} \frac{1}{1 - \mathrm{e}^{-\pi\lambda/\omega}}. \tag{4.28b}$$

The proof of Lemma 4.2 is a trivial consequence of equations (4.27b) and (4.22).

Now that $\psi(t)$ solving (4.3) has been constructed for $t \leqslant 0$, we are in a position to use (4.3a) to find $\psi(t)$ for all $t \in (-\infty, \infty)$. Indeed, once $\psi$ has been determined for $t \leqslant 0$, we can calculate $\dot{\psi}(0)$ from (4.3a) immediately, and that information is sufficient to determine $\psi$ for all time. This is because differentiation of (4.3a) with respect to the time brings us back to the equation

$$\ddot{\psi}(t) + \lambda\,\dot{\psi}(t) = -\omega_0^2 \sin(\theta_0 - \tilde{G}\cos(\omega t - \phi_0) + \psi(t)). \tag{4.29}$$

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For the function $\Psi$ introduced in (4.23), we have

$$\dot{\Psi}(t) = -\omega_0^2 \int_{-\infty}^{t} e^{-\lambda(t-\xi)} \sin(-\tilde{G}\cos(\omega\xi - \phi_0) + \Psi(\xi))\, d\xi, \tag{4.30}$$

and $\Psi(t)$ is determined for all time from its 'initial' value $\Psi(0) = \theta_0$. The only other parameter in (4.30), besides the physical variables of the system, is $\phi_0$ which, through (4.4), is related to the time $t_0$ at which 'initial' conditions are specified for $\tilde{\theta}$. Thus, in a sense the equation for $\tilde{\theta}$ looks like a first-order equation, as opposed to a second-order one, for late times. For any prescription whereby the specification of a single datum $\tilde{\theta}(t_0)$ enables one to compute, no matter in how tedious a fashion, $\dot{\tilde{\theta}}(t_0)$ and the whole trajectory of $\tilde{\theta}(t)$, before and after $t_0$, has to be essentially first-order. This 'first-order' character at late times of the equation for the correction $\tilde{\theta}$ to simple harmonic motion has interesting ramifications with regard to the behaviour of $\tilde{\theta}(t)$ as $t \to \infty$, not unlike the conclusions drawn for equation (3.12), and we shall explore them forthwith. First, we will inquire in more detail how it is that the second-order equation (2.2b) begins looking like a first-order equation at late times.

One may ask if it is true that $\tilde{\theta}(t+t_0)$ really looks like $\tilde{\theta}(t_0) + \psi(t)$ for $t \geqslant t_0$, even if (4.1) is satisfied approximately at such times. After all, (4.1) is an integral equation which depends on values of $\tilde{\theta}$ at times $t < t_0$, which may not be late, and even on values of $\tilde{\theta}$ at times $t < 0$, before the motion commenced! So, suppose (4.1) applies for $t \geqslant t_0$. How can we justify using it to calculate $\tilde{\theta}$ for $t < t_0$, as was done in the construction which was the essence of the solution of (4.3)? Well, if (4.1) holds for $t \geqslant t_0$, it follows from Theorem 2.2 that, for $\tilde{G}$ large, when calculating $\dot{\tilde{\theta}}(T_0)$ by (4.1), we can replace $\tilde{\theta}(t_0 - \xi)$ on the right-hand side of (4.1) to a good approximation by $\tilde{\theta}(t_0)$. This should be valid up to $\xi = 0(\sqrt{\tilde{G}})$. Suppose $\xi_0$ is such that we can ignore $e^{-\lambda\xi_0}$, but $\xi_0$ is still small compared to $\sqrt{\tilde{G}}$. $\tilde{\theta}(t_0 - \xi)$ will differ from $\tilde{\theta}(t_0)$ by $0(\xi/\sqrt{\tilde{G}})$ for $0 \leqslant \xi \leqslant \xi_0$. The error in using $\tilde{\theta}(t_0)$ instead of $\tilde{\theta}(t-\xi)$ on the right of (4.1) will then give an error $0(1/\tilde{G})$ in the integral, and thus in $\dot{\tilde{\theta}}(t)$ for $t_0 - \xi_0 \leqslant t \leqslant t_0$. This is the essence of the statement of contraction in Lemma 4.1 and given precisely in (4.22). If $\dot{\tilde{\theta}}(t)$ is known with error $0(1/\tilde{G})$ for $t_0 - \xi_0 \leqslant t \leqslant t_0$, then $\tilde{\theta}(t)$ can be determined with the same accuracy by integrating (4.1) with $\tilde{\theta}(t_0)$ instead of $\tilde{\theta}(t-\xi)$ on the right, from $t$ to $t_0$. That the iterative procedure, of which we have just described the first step, converges, and converges rapidly when $\tilde{G}$ is sufficiently large, is what Lemma 4.1 is all about.

Further insight into the significance of Lemma 4.2 may be gained by considering the effective rate of damping of aperiodic terms in $\tilde{\theta}$. Although the coefficient $\lambda$ is referred to as a 'damping' coefficient and gives a decay rate for the effect of the past motion on the present, as in (4.1), a truer measure of the rate of damping is perhaps (in dimensional quantities) $A/\sqrt{\tilde{G}}$, where

$$A = \frac{\omega_0^2}{\omega}(1+\omega_0/\lambda)(1+\omega/\lambda), \tag{4.31}$$

as given in Theorem 2.2. This is on account of the fact that, if the correction $\tilde{\theta}$ is far from a periodic motion at any time, it will take a time $0(\sqrt{\tilde{G}}/A)$ for the motion to adjust in that direction, just because $\dot{\tilde{\theta}}$ is so small and $\tilde{\theta}$ takes so long to adjust. The converse is that, in going backward in time, aperiodic motions tend to grow at the rate $A/\sqrt{\tilde{G}}$, as opposed to the much larger $\lambda$. This enables the integral in (4.1) to converge rapidly. When $K$ is set equal

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to, say, $\frac{1}{2}$, as in (4.28a), it appears that the 'growth' rate $\lambda'$ as time decreases for the correction $\tilde{\theta}$ is $0(A/\sqrt{\tilde{G}})$. When we look at the approach of solutions of the recursion relation (3.7) to limits as $t \to \infty$, we see once again a characteristic decay rate $(\omega_0^2/(\lambda\sqrt{\tilde{G}}))$. (The fact that this rate is less tan $A/\sqrt{\tilde{G}}$ where $A$ is given by (4.31) is a reflection of some crude assumptions we made in obtaining the *a priori* bounds of Theorem 2.2 and Lemma 4.2. For example, the term $\omega_0/\lambda$ in $A$ comes from the rather gross bounds given by (2.6) and (4.7).)

Let us return to the late-time behaviour of $\Psi(t)$ in (4.30). We first show the continuous dependence of $\Psi$ on the initial value $\Psi(0) = \theta_0$. Let $\Psi_1(t)$ satisfy (4.30) with

$$\Psi_1(0) = \theta_{01}. \tag{4.32}$$

$\Psi_1(t)$ is constructed for $t \leqslant 0$ from $\theta_{01}$ the way $\Psi(t)$ is constructed from $\theta_0$ in (4.24). Of course,

$$|\Psi_1^{(0)}(t) - \Psi^{(0)}(t)| = |\theta_{01} - \theta_0|, \quad t \leqslant 0. \tag{4.33}$$

To bound $\|\Psi_1^{(1)} - \Psi^{(1)} - \theta_{01} + \theta_0\|$, we first have to bound

$$|\dot{\Psi}_1^{(1)}(t) - \dot{\Psi}^{(1)}(t)| \leqslant \omega_0^2 \left| \int_{-\infty}^{t} e^{-\lambda(t-\xi)}\, e^{-i\tilde{G}\cos(\omega\xi - \phi_0)}\, (e^{i\theta_{01}} - e^{i\theta_0})\, d\xi \right|, \tag{4.34}$$

and a suitable bound is just $|\theta_{01} - \theta_0|$ times the bound obtained for the integral (4.25). It follows that

$$\|\Psi_1^{(1)} - \theta_{01} - \Psi^{(1)} + \theta_0\| \leqslant \frac{2\sqrt{10\pi}\,\omega_0^2}{\lambda'\,\omega\sqrt{\tilde{G}}} \frac{|\theta_{01} - \theta_0|}{1 - e^{-\pi\lambda/\omega}}. \tag{4.35}$$

Repeated application of Lemma 4.1 and of (4.24b) gives

$$\|\Psi_1 - \theta_{01} - \Psi + \theta_0\| \leqslant \frac{4\sqrt{10\pi}\,\omega_0^2}{\lambda'\,\omega\sqrt{\tilde{G}}} \frac{|\theta_{01} - \theta_0|}{1 - e^{-\pi\lambda/\omega}} \tag{4.36}$$

when the inequality (4.28a) holds.

In particular, (4.36) gives

$$|\dot{\Psi}_1(0) - \dot{\Psi}(0)| \leqslant \frac{4\sqrt{10\pi}\,\omega_0^2}{\omega\sqrt{\tilde{G}}} \frac{|\theta_{01} - \theta_0|}{1 - e^{-\pi/\lambda\omega}}. \tag{4.37}$$

This bound is independent of the angle $\phi_0$ in (4.30). We recall that shifting the time axis by an amount $t_1$ has the effect of shifting $\phi_0$ by $\omega t_1$, as seen already in equation (4.4). Hence at $t_1$ we get a relation like (4.37) bounding $|\dot{\Psi}_1(t_1) - \dot{\Psi}(t_1)|$ in terms of $|\Psi_1(t_1) - \Psi(t_1)|$. Thus, for any $t$, $\dot{\Psi}(t)$ is Lipschitz-continuous in its dependence on $\Psi(t)$, with Lipschitz coefficient

$$C = \frac{4\sqrt{10\pi}\,\omega_0^2}{\omega\sqrt{\tilde{G}}} \frac{1}{1 - e^{-\pi/\lambda\omega}}. \tag{4.38}$$

Then a standard argument from the theory of ordinary differential equations shows that

$$e^{-C|t_1 - t_2|}|\Psi_1(t_2) - \Psi(t_2)| \leqslant |\Psi_1(t_1) - \Psi(t_1)| \leqslant e^{C|t_1 - t_2|}|\Psi_1(t_2) - \Psi(t_2)|. \tag{4.39}$$

The central result of this section is the following theorem.

**Theorem 4.3** *For any value of* $\Psi(0)$, *the solution of* (4.30) *becomes a periodic motion with period* $2\pi/\omega$ *as* $t \to \infty$.

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**Proof** The proof will proceed through the establishment of several easily proved lemmas.

**Lemma 4.4** *The solution of* (4.30) *increases monotonically with* $\Psi(0)$. *That is, if* $\Psi_1(0) > \Psi(0)$, *then* $\Psi_1(t) > \Psi(t)$ *for all* $t$.

**Proof** Suppose the result is not true for some $t_0$. Then by continuity there is a time $t^* \in [\min(0, t_0), \max(0, t_0)]$ for which $\Psi_1(t^*) = \Psi(t^*)$. By (4.39), we get $\Psi_1(0) = \Psi(0)$, giving a contradiction.

**Lemma 4.5** *For any integer* $m$ *and solution* $\Psi(t)$ *of* (4.30), *if* $\Psi_1(0) = \Psi(0) + 2m\pi$, *then* $\Psi_1(t) = \Psi(t) + 2m\pi \,\forall\, t$.

**Proof** This follows by replacing $\Psi(t)$ by $\Psi(t) + 2m\pi$ in (4.30).

**Lemma 4.6** *If* $\Psi(2\pi/\omega) > \Psi(0)$, $\Psi(2\pi/\omega) < \Psi(0)$, *or* $\Psi(2\pi/\omega) = \Psi(0)$, *respectively, then for all* $t$ $\Psi(t + 2\pi/\omega) > \Psi(t)$, $\Psi(t + 2\pi/\omega) < \Psi(t)$, *or* $\Psi(t + 2\pi/\omega) = \Psi(t)$, *respectively*.

**Proof** Let

$$\Phi(t) = \Psi\left(t + \frac{2\pi}{\omega}\right). \tag{4.40}$$

$\Phi(t)$ satisfies the same equation (4.30) as $\Psi(t)$, and $\Phi(0) = \Psi(2\pi/\omega)$. If $\Psi(2\pi/\omega) > \Psi(0)$, then $\Phi(0) > \Psi(0)$ and, from Lemma 4.4, $\Phi(t) = \Psi(t + 2\pi/\omega) > \Psi(t)$ for all $t$. Similar arguments establish the other cases in the lemma.

**Corollary 4.7** *If* $\Psi(2\pi/\omega) > \Psi(0)$ ($\Psi(2\pi/\omega) < \Psi(0)$, *resp.*), *then the sequence* $\{\Psi(t + 2\pi N/\omega) \mid N = 0, 1, 2, \ldots\}$ *is increasing* (*decreasing, resp.*).

**Lemma 4.8** *There is a number* $\Psi^*(0)$ *such that, if* $\Psi^*(t)$ *is a solution of* (4.30) *with initial value* $\Psi^*(0)$, *then*

$$\Psi^*\left(t + \frac{2\pi}{\omega}\right) = \Psi^*(t) \quad \text{for all } t.$$

**Proof** Let $\Psi(t)$ be one solution of (4.30) and let $\tilde{\Psi}(t)$ be another solution, with $\tilde{\Psi}(0)$ chosen so that

$$\tilde{\Psi}\left(\frac{\pi}{\omega}\right) = -\Psi(0). \tag{4.41}$$

We see from (4.30) that

$$\tilde{\Psi}\left(t + \frac{\pi}{\omega}\right) = -\Psi(t) \quad \text{for all } t. \tag{4.42}$$

Suppose $\Psi(2\pi/\omega) > \Psi(0)$. Then $\tilde{\Psi}(3\pi/\omega) < \tilde{\Psi}(\pi/\omega)$ follows from (4.42) and, according to Lemma 4.6, $\tilde{\Psi}(2\pi/\omega) < \tilde{\Psi}(0)$. By continuity, there must be a number $\Psi^*(0)$ between $\Psi(0)$ and $\tilde{\Psi}(0)$ such that the solution $\Psi^*$ of (4.30) with initial value $\Psi^*(0)$ satisfies $\Psi^*(2\pi/\omega) = \Psi^*(0)$, and hence $\Psi^*(t + 2\pi/\omega) = \Psi^*(t) \,\forall\, t$. A similar inference can be made if $\Psi(2\pi/\omega) < \Psi(0)$. The only remaining case if $\Psi(2\pi/\omega) = \Psi(0)$, and then we can use $\Psi(0)$ for $\Psi^*(0)$.

Let $\Psi^*(0)$ be chosen as in Lemma 4.8. $\Psi^*(t)+2m\pi$ is also a solution of (4.30) which is periodic in time with period $2\pi/\omega$. Let $\Psi(t)$ be any solution of (4.30) and let $m$ be chosen so that $\Psi^*(0)+2m\pi \leqslant \Psi(0) < \Psi^*(0)+2(m+1)\pi$. It follows from Lemma 4.4 that $\Psi(2\pi N/\omega)$ lies in the interval $[\Psi^*(0)+2m\pi, \Psi^*(0)+2(m+1)\pi)$ for all $N$. We have already observed in Corollary 4.7 that the sequence $\{\Psi(2\pi N/\omega)\,|\, N=0,1,2,\ldots\}$ is either monotonically increasing, monotonically decreasing, or constant. In any case, $\Psi(2\pi N/\omega)$ approaches a constant as $N\to\infty$, and the motion described by $\Psi(t)$ approaches a periodic motion with period $2\pi/\omega$.

## 5 Passage to periodic motion

In the last section we showed that the solution of the approximate equation (4.1) approaches a periodic motion, with period $2\pi/\omega$, for $\tilde{G}$ large enough. The main result of this section is that, for $\tilde{G}$ large enough, the solution $\tilde{\theta}$ of equation (2.5) also approaches a periodic motion for late times.

In accordance with (4.2), let us define, given $t_0>0$, $\tilde{\psi}(t)$ by

$$\tilde{\psi}(t) = \begin{cases} \tilde{\theta}(t+t_0)-\tilde{\theta}(t_0) & t \geqslant -t_0, \\ \quad -\tilde{\theta}(t_0) & t \leqslant -t_0, \end{cases} \tag{5.1}$$

where we observe, on account of (2.4a) and (2.1), that $\tilde{\psi}(t)$ as defined is continuous in time, and $\dot{\tilde{\psi}}$ is continuous in time.

Of course, $\dot{\tilde{\psi}}(t) = \dot{\tilde{\theta}}(t+t_0)$ for $t \geqslant -t_0$. Equations (5.1) and (2.5) give

$$\dot{\tilde{\psi}}(t) = -\omega_0^2 \int_0^{(t+t_0)_+} \mathrm{e}^{-\lambda\xi} \sin(\theta_0 + c_1\,\mathrm{e}^{-\lambda(t+t_0-\xi)} - \tilde{G}\cos(\omega(t-\xi)-\phi_0) + \tilde{\psi}(t-\xi))\,\mathrm{d}\xi, \tag{5.2a}$$

$$\tilde{\psi}(0) = 0, \tag{5.2b}$$

where $\theta_0$ and $\phi_0$ are given in (4.4). To compare $\tilde{\psi}$ with the solution $\psi$ of (4.3), we shall rewrite (5.2a) in the form

$$\dot{\tilde{\psi}}(t) = -\omega_0^2 \int_0^{\infty} \mathrm{e}^{-\lambda\xi} \sin(\theta_0 - \tilde{G}\cos(\omega(t-\xi)-\phi_0) + \tilde{\psi}(t-\xi))\,\mathrm{d}\xi + g(t), \tag{5.3a}$$

where

$$\begin{aligned} q(t) = {} & \omega_0^2 \int_{(t+t_0)_+}^{\infty} \mathrm{e}^{-\lambda\xi} \sin(\theta_0 - \tilde{G}\cos(\omega(t-\xi)-\phi_0) + \tilde{\psi}(t-\xi))\,\mathrm{d}\xi \\ & + \omega_0^2 \int_0^{(t+t_0)_+} \mathrm{e}^{-\lambda\xi} [\sin(\theta_0 - \tilde{G}\cos(\omega(t-\xi)-\phi_0) + \tilde{\psi}(t-\xi)) \\ & - \sin(\theta_0 + c_1\,\mathrm{e}^{-\lambda(t+t_0-\xi)} - \tilde{G}\cos(\omega(t-\xi)-\phi_0) + \tilde{\psi}(t-\xi))]\,\mathrm{d}\xi. \end{aligned} \tag{5.3b}$$

Equations (5.2b) and (5.3a) are equivalent to

$$\tilde{\psi} = U\tilde{\psi} + Q, \tag{5.4a}$$

where the operator $U$ is given in (4.6) and

$$Q(t) = -\int_t^0 q(\xi)\,\mathrm{d}\xi. \tag{5.4b}$$

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**Lemma 5.1** *Let* $\|\cdot\|$ *be the norm given by* (4.5). *For* $0 < \lambda' \leqslant \lambda/2$,

$$\|Q| \leqslant \frac{\omega_0^2}{\lambda\lambda'}(1+|c_1|)\,e^{-\lambda' t_0}. \tag{5.5}$$

**Proof** The first integral on the right-hand side of (5.3b) is bounded by $\omega_0^2/\lambda\, e^{-\lambda(t+t_0)_+}$. We have to find

$$\sup_{t\leqslant 0}\frac{1}{\lambda'}\frac{\omega_0^2}{\lambda}e^{-\lambda(t+t_0)_+}e^{\lambda' t} = \frac{\omega_0^2}{\lambda\lambda'}e^{-\lambda' t_0}. \tag{5.6}$$

The second integral on the right-hand side of (5.3b) is bounded by

$$\omega_0^2\int_0^{(t+t_0)_+}|c_1|\,e^{-\lambda(t+t_0)}\,d\xi = |c_1|\,\omega_0^2\,e^{-\lambda(t+t_0)}\,(t+t_0)_+, \tag{5.7}$$

on account of the mean-value theorem. A bound on its contributions to $\|Q\|$ now follows by reasoning similar to that in (5.6), upon use of $\lambda' \leqslant \lambda/2$. Adding up the results, we get (5.5).

We can solve (5.4a) iteratively. Let

$$\tilde{\psi}^{(0)} = \psi + Q = U\psi + Q, \tag{5.8a}$$

on account of (4.8). Then let

$$\tilde{\psi}^{(i+1)} = U\tilde{\psi}^{(i)} + Q, \quad i \geqslant 0. \tag{5.8b}$$

We have

$$\|\tilde{\psi}^{(0)} - \psi\| = \|Q\| \tag{5.9a}$$

and

$$\|\tilde{\psi}^{(1)} - \tilde{\psi}^{(0)}\| = \|U\tilde{\psi}^{(0)} - U\psi\| \leqslant K\|Q\|, \tag{5.9b}$$

where $K$ is given in (4.22). Continuing, for $i \geqslant 1$ we get

$$\|\tilde{\psi}^{(i+1)} - \tilde{\psi}^{(i)}\| = \|U\tilde{\psi}^{(i)} - U\tilde{\psi}^{(i-1)}\| \leqslant K\|\tilde{\psi}^{(i)} - \tilde{\psi}^{(i-1)}\| \leqslant K^{i+1}\|Q\|. \tag{5.10}$$

These results are summarized in the following lemma.

**Lemma 5.2** *Let* $\tilde{G}$ *satisfy the inequality* (4.28a). *Then*

$$\|\tilde{\psi} - \psi\| \leqslant 2\|Q\|. \tag{5.11}$$

Before continuing, let us recapitulate the results of the last section. For the function $\Psi$ given in (4.23):

$$\Psi(t) = \psi(t) + c_2 + \tilde{\theta}(t_0), \tag{5.12}$$

we could write

$$\dot{\Psi}(t_0) = \mathscr{F}(\Psi(t_0), t_0), \tag{5.13a}$$

where

$$|\mathscr{F}(x_1, y) - \mathscr{F}(x_2, y)| \leqslant C|x_1 - x_2| \tag{5.13b}$$

with $C$ given by (4.38),

$$\mathscr{F}(x + 2\pi, y) = \mathscr{F}(x, y), \tag{5.13c}$$

and

$$\mathscr{F}\left(x, y + \frac{2\pi}{\omega}\right) = \mathscr{F}(x, y). \tag{5.13d}$$

Integration of (5.13a) over one period of the driving frequency led to

$$\Psi\left(t+\frac{2\pi}{\omega}\right) = F(\Psi(t), t), \tag{5.14a}$$

where

$$e^{-2\pi C/\omega}(x_1-x_2) \leqslant F(x_1, y) - F(x_2, y) \leqslant e^{2\pi C/\omega}(x_1-x_2) \tag{5.14b}$$

when $x_1 \geqslant x_2$,

$$F(x+2\pi, y) = F(x, y)+2\pi, \tag{5.14c}$$

and

$$F\left(x, y+\frac{2\pi}{\omega}\right) = F(x, y). \tag{5.14d}$$

Finally, there was a non-empty set of numbers $S_0(t)$ such that for, and only for, $\sigma \in S_0(t)$,

$$F(\sigma, t) = \sigma. \tag{5.15}$$

From (5.14) and (5.15) we concluded that

$$\operatorname{dist}(F^N(x, t), S_0(t)) \underset{N\to\infty}{\longrightarrow} 0 \tag{5.16}$$

for any $x$. (In other words, by waiting a sufficient number $N$ of periods, we can get the motion as close as we want to a motion with period $2\pi/\omega$.)

In addition to $S_0(t)$, defined in (5.15), we introduce the sets

$$S_+(t_0) = \{x \,|\, F(x, t_0) > x\} \quad \text{and} \quad S_-(t_0) = \{x \,|\, F(x, t_0) < x\}. \tag{5.17}$$

Note that

$$S_a\left(t_0+\frac{2\pi}{\omega}\right) = S_a(t_0), \quad \text{where} \quad a = 0, +, \quad \text{or} \quad -. \tag{5.18}$$

It follows from (5.14) that components of $S_+(t_0)$ consist of open intervals between elements of $S_0(t_0)$. The same holds for components of $S_-(t_0)$.

Let us now use the result (5.11) with $\lambda' = \lambda/2$. It follows from (4.5), (5.5), (5.12) and (5.13a), that

$$|\dot{\tilde{\theta}}(t_0) - \mathscr{F}(\tilde{\theta}(t_0)+c_2, t_0)| \leqslant \frac{2\omega_0^2}{\lambda}(1+|c_1|)\, e^{-\lambda t_0/2}. \tag{5.19}$$

Hence

$$\left|\tilde{\theta}\left(t_0+\frac{2\pi}{\omega}\right)+c_2-F(\tilde{\theta}(t_0)+c_2, t_0)\right| \leqslant \frac{4\pi\omega_0^2}{\omega\lambda}(1+|c_1|)\, e^{2\pi C/\omega}\, e^{-\lambda t_0/2}. \tag{5.20}$$

**Theorem 5.3** *Let $\tilde{\theta}(t)$ be a solution of* (2.5). *For $\tilde{G}$ large enough, $\tilde{\theta}$ becomes periodic in $t$ with period $2\pi/\omega$ as $t \to \infty$.*

**Proof** First, let $\tilde{G}$ be large enough that $C$, given in (4.38), satisfies $C \leqslant \lambda/4$. That is,

$$\sqrt{\tilde{G}} \geqslant \frac{16\sqrt{10\pi}\,\omega_0^2}{\omega\lambda}\frac{1}{1-e^{-\pi\lambda/\omega}}. \tag{5.21}$$

Either for all $t_0 > 0$ the distance $\epsilon$ of $\tilde{\theta}(t_0)+c_2$ from $S_0(t_0)$ satisfies

$$\epsilon < \frac{4\pi\omega_0^2}{\omega\lambda}\frac{(1+|c_1|)\,\mathrm{e}^{2\pi C/\omega}}{\mathrm{e}^{-2\pi C/\omega}-\mathrm{e}^{-\lambda\pi/\omega}}\mathrm{e}^{-\lambda t_0/2}, \tag{5.22}$$

in which case we readily conclude from (5.14) and (5.20) that $|\tilde{\theta}(t_0+2\pi/\omega)-\tilde{\theta}(t_0)|\to 0$ as $t_0\to\infty$, or there is a $t_0 > 0$ such that

$$\frac{4\pi\omega_0^2}{\omega\lambda}(1+|c_1|)\,\mathrm{e}^{2\pi C/\omega}\,\mathrm{e}^{-\lambda t_0/2} \leqslant \epsilon\,(\mathrm{e}^{-2\pi C/\omega}-\mathrm{e}^{-\lambda\pi/\omega}). \tag{5.23}$$

Let $\tilde{\theta}(t_0)+c_2$ be in the component of $S_+(t_0)$ or $S_-(t_0)$ consisting of the open interval $(\sigma_-(t_0)$, $\sigma_+(t_0))$. We have

$$\tilde{\theta}(t_0)+c_2-\sigma_-(t_0) \geqslant \epsilon, \quad \sigma_+(t_0)-\tilde{\theta}(t_0)-c_2 \geqslant \epsilon. \tag{5.24}$$

It follows from (5.14b) that

$$\mathrm{dist}\,(F(\tilde{\theta}(t_0)+c_2, t_0), \quad S_0(t_0)) \geqslant \epsilon\,\mathrm{e}^{-2\pi C/\omega}, \tag{5.25}$$

and from (5.20) that

$$\begin{aligned}\mathrm{dist}\left(\tilde{\theta}\left(t_0+\frac{2\pi}{\omega}\right)+c_2, S_0(t_0)\right) &\geqslant \epsilon\,\mathrm{e}^{-2\pi C/\omega}-\frac{4\pi\omega_0^2}{\omega\lambda}(1+|c_1|)\,\mathrm{e}^{2\pi C/\omega}\,\mathrm{e}^{-\lambda t_0/2}\\ &\geqslant \epsilon\,\mathrm{e}^{-2\pi C/\omega}-\epsilon(\mathrm{e}^{-2\pi C/\omega}-\mathrm{e}^{-\lambda\pi/\omega}) = \epsilon\,\mathrm{e}^{-\lambda\pi/\omega},\end{aligned} \tag{5.26}$$

on account of (5.23). Thus, the distance $\epsilon_1$ of $\tilde{\theta}(t_0+2\pi/\omega)+c_2$ from $S_0(t_0)$ satisfies

$$\frac{4\pi\omega_0^2}{\omega\lambda}(1+|c_1|)\,\mathrm{e}^{2\pi C/\omega}\,\mathrm{e}^{-(\lambda/2)\,(t_0+2\pi/\omega)} \leqslant \epsilon_1\,(\mathrm{e}^{-2\pi C/\omega}-\mathrm{e}^{-\lambda\pi/\omega}), \tag{5.27}$$

and similar arguments show that the distance $\epsilon_n$ of $\tilde{\theta}(t_0+2\pi n/\omega)+c_2$ from $S_0(t_0)$ satisfies

$$\frac{4\pi\,\omega_0^2}{\omega\lambda}(1+|c_1|)\,\mathrm{e}^{2\pi C/\omega}\,\mathrm{e}^{-(\lambda/2)\,(t_0+2n\pi/\omega)} \leqslant \epsilon_n\,(\mathrm{e}^{-2\pi C/\omega}-\mathrm{e}^{-\lambda\pi/\omega}), \tag{5.28}$$

for $n = 2, 3, \ldots$. Accordingly,

$$\sigma_-(t_0) < \tilde{\theta}\left(t_0+\frac{2\pi n}{\omega}\right)+c_2 < \sigma_+(t_0), \quad n = 0, 1, 2, \ldots. \tag{5.29}$$

Thus, the set $\{\tilde{\theta}(t_0+2\pi n/\omega)+c_2\,|\,n = 0, 1, 2, \ldots\}$ has at least one limit point. The limit point is unique, because of (5.20) and the fact that either $F(x, t_0) > x$ or $F(x, t_0) < x$ for all $x\in(\sigma_-(t_0), \sigma_+(t_0))$. The limit point cannot be in $(\sigma_-(t_0), \sigma_+(t_0))$, where $F(x, t_0) \neq x$, without violating (5.20). So the limit point is either $\sigma_+(t_0)$ (for $(\sigma_-(t_0), \sigma_+(t_0)) \subset S_+(t_0)$) or $\sigma_-(t_0)$ (for $(\sigma_-(t_0), \sigma_+(t_0)) \subset S_-(t_0)$). In either case, the theorem is proved.

## 6 Numbers and conclusions

In their experiments with the Daedalon EM-50 Chaotic Pendulum [3] Hammel and coworkers [7] have used the values $\lambda = 1.85\ \mathrm{sec}^{-1}$, $\omega_0 = 8.975979\ \mathrm{sec}^{-1}$, and $\omega = 6.102\ \mathrm{sec}^{-1}$. Hinczewski [8] performed his computations for these same parameter values. For these

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values, we shall express numerically the various conditions we have imposed on $\tilde{G}$ in order to prove the theorems and lemmas of §§2, 4 and 5.

Theorem 2.2 and Lemma 2.1 required only the conditions (2.8) and (2.9). Of these conditions, (2.8) and (2.9 b) are satisfied by choosing $t$ large enough. Equation (2.9 a) is

$$\tilde{G} \geqslant 0.0919176. \tag{6.1}$$

Lemma 4.2 depended upon satisfaction of (4.28 a), which is

$$\tilde{G} \geqslant 7716881.4. \tag{6.2}$$

Theorem 5.3 was proven when $\tilde{G}$ satisfied (5.21), or when

$$\tilde{G} \geqslant 1085909.3. \tag{6.3}$$

With regard to how long we have to wait for the estimates in §2 to hold, we note that (2.8) requires

$$t \geqslant \frac{2}{\lambda} \ln\left(\frac{\omega\sqrt{\tilde{G}}}{\lambda}\right). \tag{6.4}$$

For the case $\dot{\theta}(0) = 0$, (2.4 b) and (2.9 b) give

$$\frac{\sqrt{\tilde{G}}\,\omega}{\sqrt{\omega^2+\lambda^2}} \leqslant e^{\lambda t/2}, \tag{6.5}$$

a condition which is always satisfied when (2.8) is satisfied. Thus, for example, if $G = 10^6\ \text{sec}^{-2}$, (6.4) becomes

$$t \geqslant 6.7790\ \text{sec}, \tag{6.6 a}$$

and if $G = 10^9\ \text{sec}^{-2}$, we get

$$t \geqslant 10.5129\ \text{sec} \tag{6.6 b}$$

as sufficient conditions for the bound (2.25) to hold.

Equation (2.25) takes the form

$$|\dot{\tilde{\theta}}(t)| \leqslant \frac{1060.5401}{\sqrt{\tilde{G}}}\,\text{sec}^{-1}. \tag{6.7}$$

For $G = 10^6\ \text{sec}^{-2}$, (6.7) gives

$$|\dot{\tilde{\theta}}(t)| \leqslant 6.6152579\ \text{sec}^{-1} \tag{6.8 a}$$

and for $G = 10^9\ \text{sec}^{-2}$, (6.7) gives

$$|\dot{\tilde{\theta}}(t) \leqslant 0.20919282\ \text{sec}^{-1}. \tag{6.8 b}$$

Some of Hinczewski's computations [8] were for $G = 10^6\ \text{sec}^{-2}$ and $G = 10^9\ \text{sec}^{-2}$, and that is why we have given numerical values for the bound (2.25) in these cases.

Our conclusions are these: (1) We do not know if the motion of the damped, sinusoidally driven, simple nonlinear pendulum ever becomes chaotic, but we do know that, at high amplitudes of the driving force (cf. (6.2)), it is not chaotic, and in fact becomes periodic with the period of the driving force; (2) for values of the driving amplitude which do not have to be excessively large (cf. (6.1)), the effect of the nonlinear term $\omega_0^2 \sin\theta$ in (1.1) on the angular frequency $\dot{\theta}$ of the pendulum motion is of the order of the inverse square root of the driving amplitude for late times.


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