RECURSIVE FORMULAS FOR COMPOUND PHASE DISTRIBUTIONS – UNIVARIATE AND BIVARIATE CASES

BY

JIANDONG REN

ABSTRACT

We first present a simple matrix-based recursive formula for calculating the distribution function of compound phase-type random variables. Then we extend the results to the case when the number of claims follows a bivariate matrix negative binomial (BMNB) distribution detailed herein. Further, extending the results in Hipp (2006), we provide speedy recursive formulas for both the univariate and the bivariate models when the claim sizes follow discrete phase-type distributions. Numerical examples are provided.

1. INTRODUCTION

Let N denote the number of claims occurring in an insurance portfolio within a given period, and X_i the amount of the ith claim. Assume that X_i, $i = 1, 2, \cdots$ are i.i.d random variables independent of N, the aggregate loss random variable is given by

$$S = \sum_{i=1}^{N} X_i, \tag{1}$$

with $S = 0$ when $N = 0$. Panjer (1981) introduced a recursive method for computing the distribution of S when the distribution of N is in the so called $(a, b, 0)$ class. From then on, the recursive formula (Panjer's recursion) has been studied extensively in the risk theory literature. For example, Panjer and Wang (1993) analyzed the stability of the recursive algorithm. Willmot (1992) derived recursive formulas for compound mixed Poisson probabilities. When the claim sizes follow a phase-type distribution, Hipp (2006) developed a speedy algorithm, which reduces the computation time of the original Panjer's algorithm from $O(n^2)$ to $O(n)$. Eisele (2006) presented recursive formulas when the claim number follows a discrete phase-type distribution. Sundt (1999) extended Panjer's recursive formula to the case where one claim event can cause several losses. Specifically, he derived a recursive formula for the distribution of the aggregate loss random vector

Astin Bulletin 40(2), 615-629. doi: 10.2143/AST.40.2.2061130

$$\vec{S} = \{S_1, S_2, \cdots, S_l\} = \sum_{i=1}^{N} \vec{X}_i, \tag{2}$$

where $\vec{X}_i = \{X_{i1}, X_{i2}, \cdots, X_{il}\}$ is an l-dimensional random loss vector generated by the ith claim event. The loss random vectors $\{\vec{X}_i\}_{i \geq 1}$ are assumed to be i.i.d. and independent of the claim number random variable N.

Hesselager (1996) and Vernic (1999) focused on claim number distributions and derived recursive formulas for calculating the joint distribution of

$$(S, W) = \left(\sum_{i=1}^{K} X_i, \sum_{j=1}^{M} Y_j\right), \tag{3}$$

where $S = 0$ when $K = 0$, and $W = 0$ when $M = 0$. The claim numbers K and M may be dependent but the claim sizes $\{X_i\}_{i \geq 1}$ and $\{Y_j\}_{j \geq 1}$ are sequences of discrete, i.i.d. random variables, mutually independent, and independent of (K, M).

For a comprehensive review of this topic, one is referred to the recent book by Sundt and Vernic (2009).

As in most attempts to extending Panjer's recursion, the results in both Hipp (2006) and Eisele (2006) are based on the rationality of the characteristic functions of the claim number and/or the claim size distributions. However, in a recent development, Wu and Li (2010), introduced a matrix generalization of the $(a, b, 0)$ class of discrete (claim number) distributions, which notably includes discrete phase-type distributions, and derived matrix form recursive formulas to evaluate the corresponding compound distributions.

In this paper, we assume that the claim number follows a discrete phase-type or related distributions and the claim sizes are i.i.d. discrete random variables with supports on subsets of $\mathbb{N} \cup \{0\}$. As in Wu and Li (2009), we make use of the matrix representation of phase-type distributions and present simple matrix-based recursive formulas for aggregate loss models (1) and (2). In addition, we show that a phase-type mixture of Poisson distribution gives a discrete phase-type distribution, so its compound distributions can be readily computed with our recursive formulas. For model (1), as an extension to Hipp (2006), we present speedy algorithms when the claim sizes follow a phase-type distribution. We also provide one example to compare the computational speed of the classical Panjer recursion, Hipp's speedy recursion, as well as the fast Fourier transform (FFT) method.

For model (3), we assume that the claim numbers, (K, M), jointly follow a bivariate matrix negative binomial (BMNB) distribution, which is a nature extension of the bivariate negative binomial distributions (see for example, Chapter 5 of Kocherlakota and Kocherlakota, 1992). This enables us to unify two types of bivariate counting distributions (model A and C) in Hesselager (1996). With this assumption for the joint distribution of the claim numbers, we obtain matrix-based recursive formulas for calculating the joint compound probability functions. In addition, as another extension to Hipp (2006), we present speedy algorithms when the claim sizes follow discrete phase-type distributions. The validity of the algorithms is illustrated through an example. We argue that the bivariate recursive formulas are particularly useful, because, to our knowledge, unlike the univariate case, the accuracy of the numerical bivariate inverse fast Fourier transform is not extensively studied in the literature.

The remaining of the paper is organized as follows. Section 2 briefly introduces discrete phase-type distributions; Section 3 presents recursive formulas for the distribution of the aggregate loss random variables in models (1) and (2). Section 4 presents recursive formulas for the joint and marginal distributions of the aggregate loss random variables in model (3).

2. Discrete phase-type distributions

Standard definition of discrete phase-type distributions can be found in for example, Neuts (1981), Section A5 of Asmussen (2000), and Section 2.5 of Latouche and Ramaswami (1999). Consider a terminating discrete-time Markov chain J with state space $\{E_0, E_1, \cdots, E_m\}$, where state E_0 is absorbing and $E_1, \cdots, E_m$ are transient. Let the transition probability matrix be denoted by

$$\mathbf{Q} = \begin{pmatrix} 1 & \vec{0}^\top \\ \vec{b} & \mathbf{B} \end{pmatrix}, \tag{4}$$

and the initial probability vector given by $(\alpha_0, \vec{\alpha}^\top)$, where α_0 denotes the probability of starting from the absorbing state E_0 and the row vector $\vec{\alpha}^\top$ denotes the initial state distribution in $\{E_1, \cdots, E_m\}$. The number of transitions, N, needed for the process J to get absorbed into state E_0 is said to follow a phase-type distribution with representation $(\vec{\alpha}^\top, \mathbf{B})$. So the probability function is given by

$$p(0) = \mathbb{P}_{\vec{\alpha}}(N = 0) = \alpha_0, \tag{5}$$

$$p(n) = \mathbb{P}_{\vec{\alpha}}(N = n) = \vec{\alpha}^\top \mathbf{B}^{n-1} \vec{b}, \ \text{for } n \geq 1, \tag{6}$$

where the subscripts $\vec{\alpha}$ indicates the condition under which the probability operation is taken, $\vec{b} = \vec{e} - \mathbf{B}\vec{e}$ and $\vec{e}$ is a column vector of ones. The probability generating function (p.g.f.) is given by

$$P_N(z) = \mathbb{E}_{\vec{\alpha}}\left[z^N\right] = \sum_{i=0}^{\infty} z^i p(i) = \alpha_0 + z\vec{\alpha}^\top (\mathbf{I} - z\mathbf{B})^{-1} \vec{b}. \tag{7}$$

Two special cases of the initial distributions are worth noticing:

- If $\alpha_0 = 0$ and $\vec{\alpha}^\top \vec{e} = 1$, then the phase-type distribution is defined only on positive integers.

- If the initial distribution is given by $\vec{\alpha} = \vec{\tilde{\alpha}}^\top \mathbf{B}$ with $\vec{\tilde{\alpha}}^\top \vec{e} = 1$ and so $\alpha_0 = 1 - \vec{\tilde{\alpha}}^\top \mathbf{B} \vec{e} = \vec{\tilde{\alpha}}^\top \vec{b}$, then the distribution can be viewed as that of the number of transitions within the set $\{E_1, \cdots, E_m\}$ needed to reach state E_0 given initial distribution $\vec{\alpha}^\top$. In this case,

$$p(n) = \mathbb{P}_{\vec{\alpha}}(N = n) = \vec{\tilde{\alpha}}^\top \mathbf{B}^n \vec{b}, \text{ for } n \geq 0 \tag{8}$$

and

$$P_N(z) = \mathbb{E}_{\vec{\alpha}}[z^N] = \vec{\tilde{\alpha}}^\top (\mathrm{I} - z\mathbf{B})^{-1} \vec{b}, \tag{9}$$

which are somewhat simpler than definitions (5) and (6) and the p.g.f. expression in (7).

The two special cases correspond to the two commonly used definitions of Geometric random variables — the number of epoches needed to reach the first success (zero probability at zero) versus the number of failures before the first success (positive probability at zero).

Notice that the definition in (5) and (6) allows arbitrary probability mass at zero, i.e, any zero-modified discrete phase-type distribution is again phase-type.

3. Compound phase-type distribution

This section derives formulas for calculating the probability functions of the loss random variables in models (1) and (2).

3.1. Univariate case

Consider model (1), let N follow a phase-type distribution with representation $(\vec{\alpha}^\top, \mathbf{B})$. Let the probability function of the i.i.d. discrete claim size random variables X_i, $i = 1, 2, \cdots$, be $f(n) = \mathbb{P}(X_1 = n)$, where $n \geq 0$.

Theorem 3.1. *For $n \geq 0$, let $\vec{g}(n)$ be a vector with the ith element $g_i(n) = \mathbb{P}(S = n \mid J(0) = E_i)$, where $i = 1, 2, \cdots, m$. Then*

$$\vec{g}(0) = (\mathbf{I} - f(0)\mathbf{B})^{-1} f(0) \vec{b}, \tag{10}$$

and

$$\vec{g}(n) = (\mathbf{I} - f(0)\mathbf{B})^{-1} \left[\mathbf{B} \left(\sum_{i=0}^{n-1} f(n-i) \vec{g}(i) \right) + f(n) \vec{b} \right], \quad n \geq 1. \tag{11}$$

Moreover, let $g(n) = \mathbb{P}_{\vec{\alpha}}(S = n)$ denote the probability distribution of S given initial distribution $\vec{\alpha}$ of the Markov chain J. Then, $g(0) = \alpha_0 + \vec{\alpha}^\top \vec{g}(0)$ and $g(n) = \vec{\alpha}^\top \vec{g}(n)$, for $n \geq 1$.

https://doi.org/10.2143/AST.40.2.2061130 Published online by Cambridge University Press

This result follows directly from Theorem 1 of Wu and Li (2010), to which readers are referred for a proof. Note however that the number of computation required by (11) to calculate $g(n)$ is $O(n^2)$. Inspired by Hipp (2006), we next present an algorithm that reduces the required number of computations to $O(n)$ when the claim sizes also follow a discrete phase-type distribution.

3.2. A speedy algorithm

Let the claim sizes follow a discrete phase-type distribution with representation $(\vec{\beta}^\top, \mathbf{T})$, where $\vec{\beta}^\top$ is an s dimensional row vector and $\mathbf{T}$ an $s \times s$ dimensional matrix. Then $f(0) = 1 - \vec{\beta}^\top \vec{e}$ and $f(n) = \vec{\beta}^\top \mathbf{T}^{n-1} \vec{t}$, for $n \geq 1$, where $\vec{t} = \vec{e} - \mathbf{T}\vec{e}$. In this case, (11) becomes

$$\vec{g}(n) = (\mathbf{I} - f(0)\mathbf{B})^{-1}[\mathbf{B}\mathbf{U}(n)\vec{t} + (\vec{\beta}^\top \mathbf{T}^{n-1}\vec{t})\vec{b}], \tag{12}$$

where

$$\mathbf{U}(n) = \sum_{i=0}^{n-1} \vec{g}(i)\vec{\beta}^\top \mathbf{T}^{n-i-1}. \tag{13}$$

Since

$$\mathbf{U}(n+1) = \sum_{i=0}^{n} \vec{g}(i)\vec{\beta}^\top \mathbf{T}^{n-i} = \vec{g}(n)\vec{\beta}^\top + \mathbf{U}(n)\mathbf{T}, \tag{14}$$

the recursion (11) simplifies to the following two steps:

- Set $\vec{g}(0)$ using (10) and set $\mathbf{U}(1) = \vec{g}(0)\vec{\beta}^\top$.
- For $n \geq 1$, using (12) and (14) recursively to evaluate $\vec{g}(n)$ and $\mathbf{U}(n+1)$.

Remark 1: This speedy algorithm requires $O(n)$ matrix multiplications of sizes $(m \times s)$ and $(s \times s)$ to obtain $\vec{g}(n)$, whereas the recursion in (11) requires $O(n^2)$ multiplications of a vector with size m and a scalar.

Remark 2: As shown in Theorem 2.2.6 in Neuts (1981), when the claim number and the claim sizes are both phase-type, the resulting compound loss random variable also follows a phase-type distribution. However, as was pointed there, with the dimensions of claim numbers and claim sizes being $m \times m$ and $s \times s$ respectively, the dimension of the phase-type distribution resulting from the compounding is $ms \times ms$, which grows very fast and could cause computational inefficiency. In the speedy algorithm provided here, the computations only involves matrices of sizes $m \times s$ and $s \times s$.

3.3. A numerical example

In this example, we assume that the claim number N is the summation of two Geometrically distributed random variables, N_1 and N_2, with probability functions

$$\mathbb{P}(N_1 = i) = p_1(1-p_1)^i, \ \ i = 0, 1, \cdots,$$

and

$$\mathbb{P}(N_2 = i) = p_2(1-p_2)^i, \ \ i = 0, 1, \cdots$$

respectively. So the distribution of N has phase-type representation $(\vec{\alpha}^\top, \mathbf{B})$, where

$$\vec{\alpha}^\top = [1-p_1, p_1(1-p_2)] \ \text{ and } \ \mathbf{B} = \begin{pmatrix} 1-p_1 & p_1(1-p_2) \\ 0 & 1-p_2 \end{pmatrix}.$$

The claim size distribution is assumed to be a mixture of exponentials and has a continuous phase-type representation $(\vec{\beta}_c^\top, \mathbf{T}_c)$, where

$$\vec{\beta}_c^\top = [0.6635948, 0.3114878, 0.02405664, 0.0008425574, 0.00001823254]$$

and

$$\mathbf{T}_c = -diag[3.675472, 0.7116063, 0.09447445, 0.009322980, 0.0004965620].$$

As such, the probability density function of the claim sizes is given by

$$f(x) = \vec{\beta}_c^\top e^{\mathbf{T}_c x} \vec{t}_c,$$

where $\vec{t}_c = -\mathbf{T}_c \vec{e}$.

This claim size distribution was used in Thorin and Wikstad (1973) to approximate a Pareto distribution.

To discretize the claim size distribution, we use the standard rounding method (p. 232 Klugman et al., 2008):

$$f_0 = F\left(\frac{h}{2}\right)$$

and

$$f_j = F\left(jh + \frac{h}{2}\right) - F\left(jh - \frac{h}{2}\right), \ \ j = 1, 2, \cdots.$$

With this, the discretized distribution of the claim size has a discrete phase-type representation $(\vec{\beta}^\top, \mathbf{T})$, where $\vec{\beta}^\top = \vec{\beta}_c^\top e^{\frac{h}{2}\mathbf{T}_c}$ and $\mathbf{T} = e^{h\mathbf{T}_c}$.

Using values $p_1 = p_2 = 0.5$ and $h = 0.1$, we calculated probabilities $g(0), g(0.1), \cdots, g(199.9)$, and $g(200)$ using recursion (11), speedy recursion (12) and the FFT method. The actual values of probabilities calculated with the recursive and the speedy recursive method agree up to the 17th decimal points and can be considered as the same. They agree with the values generated by the FFT method up to the 7th decimal point. The average computation time needed was 0.09, 0.06 and 0.02 seconds respectively. The speedy algorithm is indeed faster than the original Panjer's algorithm, however FFT was by far the fastest. The problem with the FFT method (without tilting) is that it suffers the so called "alias error" (see for example, Embrechts and Frei, 2009). For a detailed comparison of Panjer's recursion and the FFT method in evaluating compound distributions, one is referred to Embrechts and Frei (2009).

Remark: Discretizing a continuous phase-type distribution with the simple scheme shown in the example results in a discrete phase-type distribution. Therefore, the speedy algorithm is applicable to the example. In addition, for many commonly used claim size distributions such as mixture of exponentials and generalized Erlang, the matrix $\mathbf{T}$ associated with the resultant discrete phase-type distribution is highly structured (in the first case, it is diagonal; in the second case, it is triangular). The structures should be considered in programming to reduce computation time.

3.4. Phase-type mixture of Poisson claim numbers

Let Θ be a random variable with continuous phase-type distribution associated with a continuous time Markov chain J_c defined on the state space $\{E_0, E_1, \cdots, E_m\}$. It is assumed to have representation $(\vec{\beta}^\top, \mathbf{C})$ and thus its probability density function is given by $f_\Theta(\theta) = \vec{\beta}^\top e^{\mathbf{C}\theta}\vec{c}$, where $\vec{c} = -\mathbf{C}\vec{e}$. Suppose that conditional on $\Theta = \theta$, the number of claims, N, follows a Poisson distribution with mean $\lambda\theta$. Then the unconditional probability generating function of N is given by

$$\begin{aligned} P_N(z) &= \int_0^\infty e^{\lambda\theta(z-1)}\vec{\beta}^\top e^{\mathbf{C}\theta}\vec{c}\,d\theta \\ &= \vec{\beta}^\top\left[\lambda(1-z)\mathbf{I}-\mathbf{C}\right]^{-1}\vec{c} \qquad (15) \\ &= \vec{\beta}^\top(\lambda\mathbf{I}-\mathbf{C})^{-1}\left[\mathbf{I}-z\lambda(\lambda\mathbf{I}-\mathbf{C})^{-1}\right]^{-1}\vec{c}. \end{aligned}$$

Let $\mathbf{D} = \lambda(\lambda\mathbf{I}-\mathbf{C})^{-1}$. Then as shown in Example 2.5.4 of Latouche and Ramaswami (1999), it is the transition probability matrix among states $E_1, \cdots, E_m$ of the process J_c during a random time interval with an exponential distribution of rate λ. Consequently, it is a sub-probability matrix. Let $\vec{d} = \vec{e} - \mathbf{D}\vec{e}$, then it is easy to verify that $\vec{d} = (\lambda\mathbf{I}-\mathbf{C})^{-1}\vec{c}$. Therefore, (15) can be written as

$$P_N(z) = \vec{\beta}^\top[\mathbf{I}-z\mathbf{D}]^{-1}\vec{d}. \qquad (16)$$

https://doi.org/10.2143/AST.40.2.2061130 Published online by Cambridge University Press

Comparing with (9), this shows that N has phase-type distribution with representation $(\vec{\beta}^\top \mathbf{D}, \mathbf{D})$. We remark that this result coincides with that in Example 2.5.4 of Latouche and Ramaswami (1999), which was derived in the context of finding the distribution of the number of Poisson events before a phase-type horizon.

Since the phase-type mixture of a Poisson distribution results in a discrete phase-type distribution, the corresponding compound distributions can be computed using results in the previous sections.

3.5. Multivariate claim sizes

This section presents a recursive formula for the distribution of the multivariate aggregated loss random vector $\vec{S}$ described in model (2). We again assume that the claim number N follows a phase-type distribution with representation $(\vec{\alpha}^\top, \mathbf{B})$. Let the joint probability function of the i.i.d. discrete claim size random vectors $\{\vec{X}_i,\ i = 1, 2, \cdots, l\}$ be $f(\vec{n}) = \mathbb{P}(\vec{X}_1 = \vec{n})$, where $\vec{n}$ denotes a vector of non-negative integers.

Theorem 3.2. *For $\vec{n} \geq \vec{0}$, let $\vec{g}(\vec{n})$ be a vector with the ith element $g_i(\vec{n}) = \mathbb{P}(\vec{S} = \vec{n} \mid J(0) = E_i)$ for $i = 1, \cdots, m$. Then*

$$\vec{g}(\vec{0}) = (\mathbf{I} - f(\vec{0})\mathbf{B})^{-1} f(\vec{0})\vec{b}, \tag{17}$$

and for $\vec{n} > \vec{0}$,

$$\vec{g}(\vec{n}) = (\mathbf{I} - f(\vec{0})\mathbf{B})^{-1} \left[\mathbf{B} \left(\sum_{0 \leq \vec{i} < \vec{n}} f(\vec{n} - \vec{i})\, \vec{g}(\vec{i}) \right) + f(\vec{n})\vec{b} \right], \tag{18}$$

where by $\vec{i} \geq \vec{0}$ we mean that the jth elements $i_j \geq 0$ for $j = 1, \cdots, l$ and by $\vec{i} < \vec{n}$ we mean that $i_j \leq n_j$ for $j = 1, \cdots, l$ with strict inequality for at least one j.

Moreover, let $g(\vec{n}) = \mathbb{P}_{\vec{\alpha}}(\vec{S} = \vec{n})$ denote the probability distribution of $\vec{S}$ given initial distribution $\vec{\alpha}$ of the Markov chain J. Then $g(\vec{0}) = \alpha_0 + \vec{\alpha}^\top \vec{g}(\vec{0})$ and $g(\vec{n}) = \vec{\alpha}^\top \vec{g}(\vec{n})$ for $\vec{n} > \vec{0}$.

Similar procedures to the proof of Theorem 3.1 lead to Theorem 3.2. Details are omitted here.

4. Bivariate claim numbers

This section considers the bivariate claim numbers and their compound distributions, i.e, model (3). Hesselager (1996) discussed three types (A, B and C) of bivariate claim number distributions, from which we choose to study model A

https://doi.org/10.2143/AST.40.2.2061130 Published online by Cambridge University Press

and C because they can be nicely unified as shown in the later parts of this section.

4.1. Model A

This is also model A in Hesselager (1996). Here we assume that the claim number N follows a phase-type distribution with representation $(\vec{\alpha}^\top, \mathbf{B})$. However, the claims can be separated into types I and II, the number of which are denoted by K and M respectively. Conditional on $N = n$, K follows a Binomial distribution with parameters (n, ρ) and M follows a Binomial distribution with parameters $(n, 1 - \rho)$.

Following Hesselager (1996), the joint probability generating function of (K, M) is given by

$$\begin{aligned} P_{K,M}(z_1, z_2) &= \mathbb{E}\left[z_1^K z_2^M\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[z_1^K z_2^M \mid N\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[z_2^N \left(\frac{z_1}{z_2}\right)^K \mid N\right]\right] \\ &= \mathbb{E}\left[z_2^N \left(\rho \frac{z_1}{z_2} + (1-\rho)\right)^N\right] \\ &= \mathbb{E}\left[\left(\rho z_1 + (1-\rho) z_2\right)^N\right] \\ &= P_N\left(\rho z_1 + (1-\rho) z_2\right) \\ &= \alpha_0 + \left(\rho z_1 + (1-\rho) z_2\right) \vec{\alpha}\left[\mathbf{I} - \left(\rho z_1 + (1-\rho) z_2\right)\mathbf{B}\right]^{-1} \vec{b}. \end{aligned} \tag{19}$$

In the literature (for example, Chapter 5 of Kocherlakota and Kocherlakota, 1992), a bivariate discrete distribution with p.g.f.

$$P_{BG}(z_1, z_2) = [1 - (\rho z_1 + (1 - \rho) z_2) p]^{-1},$$

where p is a constant between 0 and 1, is referred to as a bivariate Geometric distribution. The p.g.f. (19) is a matrix generalization of $P_{BG}(z_1, z_2)$. So we name a distribution with p.g.f. (19) a bivariate matrix negative binomial (BMNB) distribution with representation $(\rho, \vec{\alpha}^\top, \mathbf{B})$.

4.2. Model B

This is model C in Hesselager (1996). As in Section 3.4, let Θ be a continuous phase-type random variable having representation $(\vec{\beta}^\top, \mathbf{C})$. Assume that conditional on $\Theta = \theta$, the number of type I claims K follows a Poisson distribution with mean $\lambda_1\theta$; the number of type II claims M follows a Poisson distribution with mean $\lambda_2\theta$. Then the unconditional p.g.f. of (K, M) is given by

$$\begin{aligned} P_{K,M}(z_1,z_2) &= \int_0^\infty e^{\lambda_1\theta(z_1-1)} e^{\lambda_2\theta(z_2-1)}\vec{\beta}^\top e^{\mathbf{C}\theta}\vec{c}d\theta \\ &= \vec{\beta}^\top\left[\lambda_1(1-z_1)\mathbf{I}+\lambda_2(1-z_2)\mathbf{I}-\mathbf{C}\right]^{-1}\vec{c} \\ &= \vec{\beta}^\top\left((\lambda_1+\lambda_2)\mathbf{I}-\mathbf{C}\right)^{-1} \\ &\quad \left[\mathbf{I}-\lambda_1 z_1\left((\lambda_1+\lambda_2)\mathbf{I}-\mathbf{C}\right)^{-1}-\lambda_2 z_2\left((\lambda_1+\lambda_2)\mathbf{I}-\mathbf{C}\right)^{-1}\right]^{-1}\vec{c}, \end{aligned}$$

A similar argument to (16) leads to

$$P_{K,M}(z_1,z_2) = \vec{\beta}^\top\left[\mathbf{I}-\frac{\lambda_1}{\lambda_1+\lambda_2}z_1\mathbf{D}-\frac{\lambda_1}{\lambda_1+\lambda_2}z_2\mathbf{D}\right]^{-1}\vec{d}, \tag{20}$$

where $\mathbf{D} = (\lambda_1+\lambda_2)((\lambda_1+\lambda_2)\mathbf{I}-\mathbf{C})^{-1}$ is a sub-probability matrix and $\vec{d} = \vec{e} - \mathbf{D}\vec{e}$.

Comparing with (19), we see that (20) is the p.g.f. of a BMNB distribution with representation $(\frac{\lambda_1}{\lambda_1+\lambda_2}, \vec{\beta}^\top\mathbf{D}, \mathbf{D})$. Thus, Model A and C in Hessellager (1996) are unified in this setting. As such, we will treat them as one and discuss the marginal distributions in the next subsection.

Remark: To our knowledge, the BMNB distribution proposed here is new. It is much simpler in form than the famous multivariate phase-type distribution (MPH) introduced in Assaf et al. (1984), however, the latter is more general. For recursive formulas of compound MPH, one is referred to Eisele (2008).

4.3. Marginal distributions

When the pair (K, M) has p.g.f. (19), the marginal p.g.f. of K is given by

$$\begin{aligned} P_K(z) &= P_{K,M}(z,1) \\ &= \alpha_0+(\rho z+(1-\rho))\vec{\alpha}[\mathbf{I}-(\rho z+(1-\rho))\mathbf{B}]^{-1}\vec{b} \\ &= \alpha_0+\vec{\alpha}\mathbf{B}^{-1}\left[(\mathbf{I}-(\rho z+(1-\rho))\mathbf{B})^{-1}-\mathbf{I}\right]\vec{b} \\ &= \alpha_0-\vec{\alpha}\mathbf{B}^{-1}\vec{b}+\vec{\alpha}\mathbf{B}^{-1}(\mathbf{I}-(\rho z+(1-\rho))\mathbf{B})^{-1}\vec{b} \\ &= \alpha_0-\vec{\alpha}\mathbf{B}^{-1}\vec{b}+\vec{\alpha}\mathbf{B}^{-1}(\mathbf{I}-(1-\rho)\mathbf{B})^{-1}[\mathbf{I}-z\rho(\mathbf{I}-(1-\rho)\mathbf{B})^{-1}\mathbf{B}]^{-1}\vec{b}. \end{aligned}$$

https://doi.org/10.2143/AST.40.2.2061130 Published online by Cambridge University Press

Let $\tilde{\mathbf{B}} = \rho(\mathbf{I} - (1-\rho)\mathbf{B})^{-1}\mathbf{B}$ and $\vec{\tilde{b}} = \vec{e} - \tilde{\mathbf{B}}\vec{e}$, then after some algebraic simplification, we have

$$\begin{aligned} P_K(z) &= \alpha_0 - \vec{\alpha}\mathbf{B}^{-1}\vec{b} + \vec{\alpha}\mathbf{B}^{-1}(\mathbf{I} - z\tilde{\mathbf{B}})^{-1}\vec{\tilde{b}} \\ &= \alpha_0 - \vec{\alpha}\mathbf{B}^{-1}\vec{b} + \vec{\alpha}\mathbf{B}^{-1}\vec{\tilde{b}} + z\vec{\alpha}\mathbf{B}^{-1}\tilde{\mathbf{B}}(\mathbf{I} - z\tilde{\mathbf{B}})^{-1}\vec{\tilde{b}} \\ &= \tilde{\alpha}_0 + z\vec{\alpha}\rho(\mathbf{I} - (1-\rho)\mathbf{B})^{-1}(\mathbf{I} - z\tilde{\mathbf{B}})^{-1}\vec{\tilde{b}}, \end{aligned} \tag{21}$$

where

$$\tilde{\alpha}_0 = \alpha_0 - \vec{\alpha}\mathbf{B}^{-1}\vec{b} + \vec{\alpha}\mathbf{B}^{-1}\vec{\tilde{b}} = 1 - \vec{\alpha}\rho(\mathbf{I} - (1-\rho)\mathbf{B})^{-1}\vec{e}.$$

This indicates that the marginal distribution of K is discrete phase-type with representation $(\vec{\alpha}\rho(\mathbf{I} - (1-\rho)\mathbf{B})^{-1}, \tilde{\mathbf{B}})$. The marginal distribution of M may be obtained by symmetry. Since both marginal distributions are phase-type, their compound distributions can be computed using results in Section 3.1.

4.4. Compound distribution with bivariate claim numbers

Let the number of type I and II claims (K, M) follow a BMNB distribution with representation $(\rho, \vec{\alpha}^\top, \mathbf{B})$, where $0 < \vec{\alpha}^\top\vec{e} \le 1$. Let $\alpha_0 = 1 - \vec{\alpha}^\top\vec{e}$. Let the type I and type II claim size random variables be independent i.i.d. sequences $\{X_i\}_{i\ge1}$ and $\{Y_i\}_{i\ge1}$ with discrete probability functions f_x and f_y respectively. The sequences $\{X_i\}_{i\ge1}$ and $\{Y_i\}_{i\ge1}$ are assumed to be independent of the claim number random variables (K, M).

Theorem 4.1. *For n_1, $n_2 \ge 0$, let $\vec{g}_{sw}(n_1, n_2)$ be a vector with the ith element $g_{sw,i}(n_1, n_2) = \mathbb{P}(S = n_1,\ W = n_2 \,|\, J(0) = E_i)$, where $i = 1, \cdots, m$. Then*

$$\vec{g}_{sw}(0,0) = (\mathbf{I} - (\rho f_x(0) + (1-\rho)f_y(0))\mathbf{B})^{-1}(\rho f_x(0) + (1-\rho)f_y(0))\vec{b}, \tag{22}$$

and for $(n_1, n_2) > (0,0)$,

$$\begin{aligned} \vec{g}_{sw}(n_1,n_2) = {} & \left[\mathbf{I} - (\rho f_x(0) + (1-\rho)f_y(0))\mathbf{B}\right]^{-1} \\ & \left[\rho\mathbf{B}\sum_{i=0}^{n_1-1} f_x(n_1 - i)\,\vec{g}_{sw}(i, n_2) + (1-\rho)\,\mathbf{B}\sum_{i=0}^{n_2-1} f_y(n_2 - i)\,\vec{g}_{sw}(n_1, i)\right. \\ & \left. + \left(\rho f_x(n_1)\,\mathbb{I}(n_2 = 0) + (1-\rho)f_y(n_2)\,\mathbb{I}(n_1 = 0)\right)\vec{b}\right], \end{aligned} \tag{23}$$

where $\mathbb{I}(\cdot)$ is an indicator function.

Moreover, let $g_{sw}(n_1,n_2) = \mathbb{P}_{\vec{\alpha}}(S = n_1,\ W = n_2)$*, where* $\vec{\alpha}$ *is as in the representation of the BMNB distribution and gives the initial distribution of the Markov chain J. Then,* $g_{sw}(0,0) = \alpha_0 + \vec{\alpha}^\top \vec{g}_{sw}(0,0)$ *and* $g_{sw}(n_1,n_2) = \vec{\alpha}^\top \vec{g}_{sw}(n_1,n_2)$ *for* $(n_1,n_2) > (0,0)$.

Proof: Let $\vec{p}_{km}(n_1,n_2)$ be a vector with the ith element $p_{km,i}(n_1,n_2) = \mathbb{P}(K = n_1,\ M = n_2 \,|\, J(0) = E_i)$. Let $\vec{P}_{SW}(z_1,z_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} z_1^i z_2^j\, \vec{g}_{sw}(i,j)$, $\vec{P}_{KM}(z_1,z_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} z_1^i z_2^j\, \vec{p}_{km}(i,j)$, $P_X(z) = \sum_{i=0}^{\infty} z^i f_x(i)$, and $P_Y(z) = \sum_{i=0}^{\infty} z^i f_y(i)$ be the probability generating functions of probability functions $\vec{g}_{sw}$, $\vec{p}_{km}$, f_x and f_y respectively. Then following (19), we have

$$\begin{aligned}\vec{P}_{SW}(z_1,z_2) &= \vec{P}_{K,M}(P_X(z_1), P_Y(z_2)) \\ &= (\rho P_X(z_1) + (1-\rho)\, P_Y(z_2))\,[\mathbf{I} - (\rho P_X(z_1) + (1-\rho)\, P_Y(z_2))\mathbf{B}]^{-1}\vec{b}\end{aligned}$$

Pre-multiplying both sides by $[\mathbf{I} - (\rho P_X(z_1) + (1-\rho)\, P_Y(z_2))\mathbf{B}]$ and rearranging, we have

$$\begin{aligned}&\vec{P}_{SW}(z_1,z_2) = \\ &\quad (\rho P_X(z_1) + (1-\rho)\, P_Y(z_2))\mathbf{B}\vec{P}_{SW}(z_1,z_2) + (\rho P_X(z_1) + (1-\rho)\, P_Y(z_2))\,\vec{b}.\end{aligned}$$

In terms of the polynomials of z, this is

$$\begin{aligned}\sum_{i,j=0}^{\infty} z_1^i z_2^j\, \vec{g}_{sw}(i,j) &= \left(\rho \sum_{i=0}^{\infty} z_1^i f_x(i) + (1-\rho)\sum_{i=0}^{\infty} z_2^i f_y(i)\right)\mathbf{B}\sum_{i,j=0}^{\infty} z_1^i z_2^j\, \vec{g}_{sw}(i,j) \\ &\quad + \left(\rho \sum_{i=0}^{\infty} z_1^i f_x(i) + (1-\rho)\sum_{i=0}^{\infty} z_2^i f_y(i)\right)\vec{b}.\end{aligned}$$

Comparing the coefficients of $z_1^i z_2^j$ for $i, j \geq 0$, we have

$$\vec{g}_{sw}(0,0) = \big(\rho f_x(0) + (1-\rho) f_y(0)\big)\,\mathbf{B}\,\vec{g}_{sw}(0,0) + \big(\rho f_x(0) + (1-\rho) f_y(0)\big)\,\vec{b},$$

implying (22), and

$$\begin{aligned}\vec{g}_{sw}(n_1,n_2) &= \rho\mathbf{B}\sum_{i=0}^{n_1} \vec{g}_{sw}(i,n_2) f_x(n_1 - i) \\ &\quad + (1-\rho)\,\mathbf{B}\sum_{i=0}^{n_2} \vec{g}_{sw}(n_1, i) f_y(n_2 - i) \\ &\quad + (\rho f_x(n_1)\,\mathbb{I}(n_2 = 0) + (1-\rho) f_y(n_2)\,\mathbb{I}(n_1 = 0))\,\vec{b},\end{aligned}$$

implying (23).

The second statement of the Theorem can be obtained by the law of total probability. ■

As in Section 3.2, when the claim size distributions f_x and f_y are discrete phase-type with representations, say, $(\vec{\beta}_1^\top, \mathbf{T}_1)$ and $(\vec{\beta}_2^\top, \mathbf{T}_2)$ respectively, then the recursion in (23) can be carried out with the following speedy scheme:

1. Set $\vec{g}_{sw}(0,0)$ using (22).

2. For $n_1 > 0$ and $n_2 > 0$,

$$\vec{g}_{sw}(n_1,0) = \left[\mathbf{I} - (\rho f_x(0) + (1-\rho) f_y(0))\mathbf{B}\right]^{-1} \left[\rho \mathbf{B}\mathbf{U}_1(n_1,0)\vec{t}_1 + \rho f_x(n_1)\vec{b}\right], \tag{24}$$

$$\vec{g}_{sw}(0,n_2) = \left[\mathbf{I} - (\rho f_x(0) + (1-\rho) f_y(0))\mathbf{B}\right]^{-1} \left[(1-\rho)\mathbf{B}\mathbf{U}_2(0,n_2)\vec{t}_2 + (1-\rho) f_y(n_2)\vec{b}\right], \tag{25}$$

and

$$\begin{aligned}\vec{g}_{sw}(n_1,n_2) = {} & \left[\mathbf{I} - (\rho f_x(0) + (1-\rho) f_y(0))\mathbf{B}\right]^{-1} \\ & \left[\rho \mathbf{B}\mathbf{U}_1(n_1,n_2)\vec{t}_1 + (1-\rho)\mathbf{B}\mathbf{U}_2(n_1,n_2)\vec{t}_2\right],\end{aligned} \tag{26}$$

where

$$\mathbf{U}_1(n_1,n_2) = \sum_{i=0}^{n_1-1} \vec{g}_{sw}(i,n_2)\,\vec{\beta}_1^\top \mathbf{T}_1^{n_1-i-1} \tag{27}$$

and

$$\mathbf{U}_2(n_1,n_2) = \sum_{i=0}^{n_2-1} \vec{g}_{sw}(n_1,i)\,\vec{\beta}_2^\top \mathbf{T}_2^{n_2-i-1}. \tag{28}$$

The matrices $\mathbf{U}_1$ and $\mathbf{U}_2$ can be recursively evaluated by

$$\mathbf{U}_1(n_1+1, n_2) = \vec{g}_{sw}(n_1,n_2)\,\vec{\beta}_1^\top + \mathbf{U}_1(n_1,n_2)\,\mathbf{T}_1, \tag{29}$$

and

$$\mathbf{U}_2(n_1, n_2+1) = \vec{g}_{sw}(n_1,n_2)\,\vec{\beta}_2^\top + \mathbf{U}_2(n_1,n_2)\,\mathbf{T}_2. \tag{30}$$

This iteration avoids the global iteration required by formula (23), reducing computation time when n_1 and n_2 are large.

https://doi.org/10.2143/AST.40.2.2061130 Published online by Cambridge University Press

4.5. Example Continued

In this example, we assume that the distribution of the total claim number N is given in Section 3.3. Conditional on $N = n$, the number of type I claims, K, follows a Binomial distribution with parameters $(n, 0.5)$ and the number of type II claims, M, follows a Binomial distribution with parameters $(n, 0.5)$. The distribution of the type I claim sizes is assumed to be the same as the claim size distribution in Section 3.3. The type II claim sizes are assumed to be five times larger and so the distribution has the phase-type representation $(\vec{\beta}_{2c}^{\top}, \mathbf{T}_{2c})$, where

$$\vec{\beta}_{2c}^{\top} = [0.6635948,\ 0.3114878,\ 0.02405664,\ 0.0008425574,\ 0.00001823254]$$

and

$$\mathbf{T}_{2c} = -0.2 \times diag[3.675472,\ 0.7116063,\ 0.09447445,\ 0.009322980,\ 0.0004965620].$$

Thus the probability density function is given by

$$f_y(x) = \vec{\beta}_{2c}^{\top} e^{\mathbf{T}_{2c} x} \vec{t}_{2c},$$

where $\vec{t}_{2c} = -\mathbf{T}_{2c} \vec{e}$.

After discretizing claim size distributions in a similar fashion as in Section 3.3, a probability matrix for the bivariate aggregate losses

$$(S, W) = (0, 0.1, \cdots, 200;\ 0, 0.1, \cdots, 200)$$

are computed using recursion (23) and the speedy recursive (26), consuming on average 574 and 386 seconds respectively. Multidimensional FFT calculation is not implemented for this example because the method is much less studied than the one-dimensional FFT in the literature. In fact, we propose to compare the recursive and the FFT methods for computing multivariate probability functions in a future project.

Acknowledgments

The author acknowledges the support of the Natural Science and Engineering Research Council of Canada. The author is also grateful to two anonymous referees for helpful suggestions.

References

Asmussen, S. (2000) *Ruin Probabilities*. World Scientific, Singapore.

Eisele, K.T. (2006) Recursions for compound phase distributions. *Insurance: Mathematics and Economics*, **38**, 149-156.

EISELE, K.T. (2008) Recursions for multivariate compound phase distributions. *Insurance: Mathematics and Economics*, **42**, 65-72.

EMBRECHTS, P. and FREI, M. (2009) Panjer recursion versus FFT for compound distributions. *Mathematical Methods of Operations Research*, **69**, 497-508.

HESSELAGER, O. (1994) A recursive procedure for calculation of some compound distributions. *Astin Bulletin*, **24**, 19-32.

HESSELAGER, O. (1996) Recursions for certain bivariate counting distributions and their compound distributions. *Astin Bulletin*, **26**, 35-52.

HIPP, C. (2006) Speedy convolution algorithms and Panjer recursions for phase-type distributions. *Insurance: Mathematics and Economics*, **38**, 176-188.

KLUGMAN, S.A., PANJER, H.H., and WILLMOT, G.E. (2008) *Loss Models: From Data to Decisions*. Wiley, New Jersey, 3rd edn.

KOCHERLAKOTA, S. and KOCHERLAKOTA, K. (1992) *Bivariate Discrete Distributions*. Dekker.

LATOUCHE, G. and RAMASWAMI, V. (1999) *Introduction to Matrix Analytic Methods in Stochastic Modeling*. SIAM, Philadelphia.

NEUTS, M.F. (1981) *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*. Dover, New York.

PANJER, H. (1981) Recursive evaluation of a family of compound distributions. *Astin Bulletin*, **12**, 22-26.

SUNDT, B. (1992) On some extensions of Panjer's class of counting distributions. *Astin Bulletin*, **22**, 61-80.

SUNDT, B. (1999) On multivariate Panjer recursions. *Astin Bulletin*, **29**, 29-45.

SUNDT, B. and VERNIC, R. (2009) *Recursions for Convolutions and Compound Distributions with Insurance Applications*. Springer Verlag: Berlin-Heidelberg.

THORIN, O. and WIKSTAD, N. (1973) Numerical evaluation of ruin probabilities. *Astin Bulletin*, **7**, 137-153.

VERNIC, R. (1999) Recursive evaluation of some bivariate compound distributions. *Astin Bulletin*, **29**, 315-325.

WILLMOT, G. (1993) On recursive evaluation of mixed Poisson probabilities and related quantities. *Scandinavian Actuarial Journal*, pp. 114-133.

WU, X. and LI, S. (2010) Recursive evaluation of compound phase-type distributions. *Astin Bulletin*, **40**, 351-368.

JIANDONG REN
Department of Statistical and Actuarial Sciences,
University of Western Ontario
1150 Richmond Street North
Western Science Centre, Room 262
London, Ontario, N6A 5B7, Canada.
Tel: 001-(519) 661-2111 Ext. 88209
Fax: (519) 661-3813
E-Mail: jren@stats.uwo.ca

https://doi.org/10.2143/AST.40.2.2061130 Published online by Cambridge University Press