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Abstract

Given a compact p-adic Lie group G over a finite unramified extension $L/\mathbb{Q}_p$ let $G_{L/\mathbb{Q}_p}$ be the product over all Galois conjugates of G. We construct an exact and faithful functor from admissible G-Banach space representations to admissible locally L-analytic $G_{L/\mathbb{Q}_p}$-representations that coincides with passage to analytic vectors in the case $L=\mathbb{Q}_p$. On the other hand, we study the functor 'passage to analytic vectors' and its derived functors over general basefields. As an application we compute the higher analytic vectors in certain locally analytic induced representations.

1. Introduction

Recently, Schneider and Teitelbaum initiated a systematic study of continuous representations of p-adic Lie groups into p-adic topological vector spaces (cf. [ST01a, ST01b, ST02a, ST02b, ST03, ST05]). A central result in this theory is that in the case of a compact group G over $\mathbb{Q}_p$ the algebra of locally analytic distributions on G is a faithfully flat extension of the algebra of continuous distributions. As a consequence, passage to analytic vectors constitutes an exact and faithful functor $F_{\mathbb{Q}_p}$ from admissible Banach space G-representations to admissible locally analytic G-representations. Owing to its properties $F_{\mathbb{Q}_p}$ is a basic tool in a possible classification of admissible topologically irreducible (unitary) Banach space representations which is of particular interest in the realm of the p-adic Langlands programme. It is therefore a natural question (raised by Teitelbaum [Tei06]) to ask how can we correctly generalize the above results to groups over arbitrary base fields $\mathbb{Q}_p\subseteq L$.

Given a compact locally L-analytic group G, simple examples show that the naive analogues of the above results do not hold (for example, F_L is not exact and often zero). The reason, as we believe, is that the notion of a K-valued locally L-analytic function depends on embedding the base field L into the coefficient field K. Consequently, we introduce, at least in the case where $L/\mathbb{Q}_p$ is Galois, the various restrictions of scalars G_σ of G via $\sigma\in\mathrm{Gal}(L/\mathbb{Q}_p)$ together with their function spaces. Let $G_{L/\mathbb{Q}_p}:=\prod_\sigma G_\sigma$. Denoting as usual by $D^c(\cdot,K)$ and $D(\cdot,K)$ continuous and locally analytic K-valued distributions respectively we construct a ring extension

$$D^c(G,K)\to D(G_{L/\mathbb{Q}_p},K),$$

which reduces to the former in the case $L=\mathbb{Q}_p$ and is faithfully flat in the case where $L/\mathbb{Q}_p$ is unramified (Theorem 4.6). To obtain a well-behaved generalization of $F_{\mathbb{Q}_p}$ from this we introduce for any Banach space G-representation V the subspace $V_{\sigma\text{-an}}$ of σ-analytic vectors whose formation is functorial in V. Denoting by $\mathrm{Ban}_G^{\mathrm{adm}}(K)$ and $\mathrm{Rep}_K^a(G)$ the abelian categories

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of admissible Banach space and locally analytic representations of G over K, respectively, we construct a functor

$$F : \mathrm{Ban}_G^{\mathrm{adm}}(K) \to \mathrm{Rep}_K^a(G_{L/\mathbb{Q}_p})$$

that enjoys, in the case where $L/\mathbb{Q}_p$ is unramified, the following properties (Theorem 4.7): it is exact and faithful and coincides with $F_{\mathbb{Q}_p}$ in the case $L = \mathbb{Q}_p$. Given $V \in \mathrm{Ban}_G^{\mathrm{adm}}(K)$ the representation $F(V)$ is strongly admissible. Viewed as a $G_{\sigma^{-1}}$-representation $F(V)$ contains $V_{\sigma\text{-an}}$ functorially as a closed subrepresentation. The results obtained so far generalize two main theorems of Schneider and Teitelbaum (cf. [Sch06, Theorems 4.2 and 4.3]) to unramified extensions $L/\mathbb{Q}_p$.

Nevertheless as the functor F_L (over a general finite extension $L/\mathbb{Q}_p$) is an important construction, we continue our work by studying its derived functors. More generally, for an arbitrary locally L-analytic group G we study the functor 'passage to L-analytic vectors' $F_{\mathbb{Q}_p}^L$ from admissible locally $\mathbb{Q}_p$-analytic representations to admissible locally L-analytic representations. Then $F_L = F_{\mathbb{Q}_p}^L \circ F_{\mathbb{Q}_p}$ if G is compact and we may deduce left-exactness of F_L. It is unclear at present whether the categories of admissible locally analytic representations have enough injective objects. Nevertheless, we prove that $F_{\mathbb{Q}_p}^L$ extends to a cohomological δ-functor $R^iF_{\mathbb{Q}_p}^L$ between admissible representations vanishing in degrees $i > ([L:\mathbb{Q}_p]-1)\dim_L G$. The functors $R^iF_{\mathbb{Q}_p}^L$ turn out to be certain Ext-groups and satisfy $R^iF_L = R^iF_{\mathbb{Q}_p}^L \circ F_{\mathbb{Q}_p}$ with R^iF_L the right-derived functors of F_L.

As an application we study the interaction of the δ-functor $R^iF_{\mathbb{Q}_p}^L$ with locally analytic induction. Let $P \subseteq G$ be a closed subgroup (satisfying a mild extra condition). For all $i \geq 0$ we obtain (Theorem 7.5)

$$R^iF_{\mathbb{Q}_p}^L \circ \mathrm{Ind}_{P_0}^{G_0} = \mathrm{Ind}_P^G \circ R^iF_{\mathbb{Q}_p}^L$$

as functors on finite-dimensional locally $\mathbb{Q}_p$-analytic P-representations. Here, $(\cdot)_0$ refers to the underlying $\mathbb{Q}_p$-analytic group. In the case that G equals the L-points of a quasi-split connected reductive group over L and P a parabolic subgroup we deduce from this an explicit formula for the higher analytic vectors in principal series representations of G.

Notation. Let $|\cdot|$ be the p-adic absolute value of $\mathbb{C}_p$ normalized by $|p| = p^{-1}$. Let $\mathbb{Q}_p \subseteq L \subseteq K \subseteq \mathbb{C}_p$ be complete intermediate fields with respect to $|\cdot|$, where $L/\mathbb{Q}_p$ is a finite extension of degree n and K is discretely valued. Let $\mathfrak{o} \subseteq L$ be the valuation ring. Here G always denotes a locally L-analytic group with Lie algebra $\mathfrak{g}$. Their restriction of scalars to $\mathbb{Q}_p$ are denoted by G_0 and $\mathfrak{g}_0$. For any field F denote by Vec_F the category of F-vector spaces. For any ring R denote by $\mathcal{M}(R)$ the category of right modules. Let $\kappa = 1$ or 2 if p is odd or even, respectively. We refer to [Sch02] for all notions from non-archimedean functional analysis.

2. Fréchet–Stein algebras

In this section we recall and discuss two classes of Fréchet–Stein algebras: distribution algebras and hyperenveloping algebras. For a detailed account on abstract Fréchet–Stein algebras as well as distribution algebras as their first examples we refer to [ST03]. For all basic theory on uniform pro-p groups we refer to [DdMS99]. Throughout this work all indices r are assumed to satisfy the technical conditions $r \in p^{\mathbb{Q}}$, $p^{-1} < r < 1$ and $r \notin \{p^{-1/(p^h - p^{h-1})},\ h \in \mathbb{N}\}$.

A K-Fréchet algebra A is called (two-sided) *Fréchet–Stein* if there is a sequence $q_1 \leq q_2 \leq \cdots$ of algebra norms on A defining its Fréchet topology and such that for all $m \in \mathbb{N}$ the completion A_m

of A with respect to q_m is a left and right noetherian K-Banach algebra and a flat left and right A_{m+1}-module via the natural map $A_{m+1} \to A_m$.

Theorem 2.1 (Schneider–Teitelbaum). *Given a compact locally L-analytic group G the algebra $D(G,K)$ of K-valued locally analytic distributions on G is Fréchet–Stein.*

This is [ST03, Theorem 5.1]. We recall the construction thereby fixing some notation: choose a normal open subgroup $H_0 \subseteq G_0$ which is a uniform pro-p group. Choose a minimal set of ordered generators $h_1, \ldots, h_d$ for H_0. The bijective global chart $\mathbb{Z}_p^d \to H_0$ for the manifold H_0 given by

$$(x_1, \ldots, x_d) \mapsto h_1^{x_1} \cdots h_d^{x_d} \tag{1}$$

induces a topological isomorphism $C^{\mathrm{an}}(H_0, K) \simeq C^{\mathrm{an}}(\mathbb{Z}_p^d, K)$ for the locally convex spaces of K-valued locally analytic functions. In this isomorphism the right-hand side is a space of classical Mahler series and the dual isomorphism $D(H_0, K) \simeq D(\mathbb{Z}_p^d, K)$ therefore realizes $D(H_0, K)$ as a space of noncommutative power series. More precisely, putting $b_i := h_i - 1 \in \mathbb{Z}[G]$, $\mathbf{b}^\alpha := b_1^{\alpha_1} \ldots b_d^{\alpha_d}$ for $\alpha \in \mathbb{N}_0^d$ the Fréchet space $D(H_0, K)$ equals all convergent series

$$\lambda = \sum_{\alpha \in \mathbb{N}_0^d} d_\alpha \mathbf{b}^\alpha \tag{2}$$

with $d_\alpha \in K$ such that the set $\{|d_\alpha| r^{\kappa|\alpha|}\}_\alpha$ is bounded for all r. The family of norms $\|\cdot\|_r$ defined via $\|\lambda\|_r := \sup_\alpha |d_\alpha| r^{\kappa|\alpha|}$ defines the Fréchet topology. They are multiplicative and the corresponding completions $D_r(H_0, K)$ are K-Banach algebras exhibiting a Fréchet–Stein structure on $D(H_0, K)$. Choose representatives $g_1, \ldots, g_r$ for the cosets in $H \setminus G$ and define on $D(G_0, K) = \bigoplus_i D(H_0, K) g_i$ the norms $\|\sum_i \lambda_i g_i\|_r := \max_i \|\lambda_i\|_r$. The completions $D_r(G_0, K)$ are the desired Banach algebras for $D(G_0, K)$. Finally, $D(G, K)$ is equipped with the corresponding quotient norms $\|\cdot\|_{\bar{r}}$ coming from the quotient map $\iota' : D(G_0, K) \to D(G, K)$. The latter arises as the dual map to the embedding

$$\iota : C^{\mathrm{an}}(G, K) \subseteq C^{\mathrm{an}}(G_0, K). \tag{3}$$

Passing to the norm completions $D_r(G, K)$ yields the appropriate Banach algebras.

We mention another important feature of $D(G, K)$ in the case where G_0 is a uniform pro-p group. Each algebra $D_r(G_0, K)$ carries the filtration defined by the additive subgroups

$$F_r^s D_r(G_0, K) := \{\lambda \in D_r(G_0, K), \|\lambda\|_r \le p^{-s}\}$$

and $F_r^{s+} D_r(G_0, K)$ (defined as $F_r^s D_r(G_0, K)$ via replacing $\le$ by $<$) for $s \in \mathbb{R}$. Put

$$gr_r^\bullet D_r(G_0, K) := \bigoplus_{s \in \mathbb{R}} F_r^s D_r(G_0, K) / F_r^{s+} D_r(G_0, K)$$

for the associated graded ring. Given $\lambda \in F_r^s D_r(G_0, K) \setminus F_r^{s+} D_r(G_0, K)$ we denote by $\sigma(\lambda) = \lambda + F_r^{s+} D_r(G_0, K) \in gr_r^\bullet D_r(G_0, K)$ the *principal symbol* of λ. Note that $D_r(G, K) \simeq D_r(G_0, K)/I_r$ is endowed with the quotient filtration where $I := \ker D(G_0, K) \to D(G, K)$ and I_r denotes the closure of $I \subseteq D_r(G_0, K)$. These filtrations are exhaustive, separated, complete and quasi-integral (in the sense of [ST03, § 1]). For $\lambda \neq 0$ in $D_r(G, K)$ the principal symbol $\sigma(\lambda) \in gr_r^\bullet D_r(G, K)$ is defined analogously.

Theorem 2.2 (Schneider–Teitelbaum). *If G_0 is a d-dimensional locally $\mathbb{Q}_p$-analytic group and uniform pro-p, then there is an isomorphism of $gr^\bullet K$-algebras*

$$gr_r^\bullet D_r(G_0, K) \xrightarrow{\sim} (gr^\bullet K)[X_1, \ldots, X_d], \ \sigma(b_i) \mapsto X_i.$$

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This is [ST03, Theorem 4.5]. Since $gr_r^{\bullet} D_r(G,K)$ equals a quotient of $gr_r^{\bullet} D_r(G_0,K)$ each $D_r(G,K)$ is a complete filtered ring with noetherian graded ring and, hence, is a Zariski ring (cf. [Lv96, § II.2.2.1]).

Working over the base field L we need to impose a mild additional condition on uniform subgroups. Let G be a d-dimensional locally L-analytic group whose underlying $\mathbb{Q}_p$-analytic group is uniform. Any minimal ordered set of generators for G_0 defines a global chart $\mathbb{Z}_p^{nd} \to G_0$ and, hence, determines a $\mathbb{Q}_p$-basis of $\mathfrak{g}_0$. Note that $\mathfrak{g}_0 \simeq \mathfrak{g}_L$ canonically over $\mathbb{Q}_p$ (see [Bou67, § 5.14.5]). We call G *uniform** if the generators can be chosen in such a way that this basis has the form $v_i \mathfrak{x}_j$ for a $\mathbb{Z}_p$-basis $v_1 = 1, v_2, \ldots, v_n$ of $\mathfrak{o}_L$ and an L-basis $\mathfrak{x}_1, \ldots, \mathfrak{x}_d$ of $\mathfrak{g}_L$. By [Sch08, Corollary 4.4] each locally L-analytic group has a fundamental system of neighbourhoods of the identity consisting of normal uniform* subgroups (note that being uniform* implies the condition (L) used in [Sch08, Corollary 4.4]).

We turn to another closely related class of Fréchet–Stein algebras. Let G be a locally L-analytic group of dimension d. Let $U(\mathfrak{g})$ be the enveloping algebra and let $C_1^{\mathrm{an}}(G,K)$ be the stalk at $1 \in G$ of the sheaf of K-valued locally L-analytic functions on G. It is a topological algebra with augmentation whose underlying locally convex K-vector space is of compact type (for the basic properties of such spaces we refer to [ST02a, § 1]). Denote by $U(\mathfrak{g},K) := C_1^{\mathrm{an}}(G,K)'_b$ its strong dual, the *hyperenveloping algebra* (cf. [Pir06, § 8]). The notation reflects that, up to isomorphism, $U(\mathfrak{g},K)$ depends only on $\mathfrak{g}$. It is a topological algebra with augmentation on a nuclear Fréchet space. There is a canonical algebra embedding $U(\mathfrak{g}) \subseteq U(\mathfrak{g},K)$ with dense image compatible with the augmentations. The formation of $C_1^{\mathrm{an}}(G,K)$ and $U(\mathfrak{g},K)$ (as locally convex topological algebras) is functorial in G and converts direct products into (projectively) completed tensor products taken over K. Dualizing the strict surjection $C^{\mathrm{an}}(G,K) \to C_1^{\mathrm{an}}(G,K)$ yields an injective continuous algebra map $U(\mathfrak{g},K) \to D(G,K)$ which is a topological embedding with closed image.

Theorem 2.3 (Kohlhaase). *Suppose that G_0 is a uniform pro-p group. Denote by $h_{11}, \ldots, h_{nd}$ a minimal set of ordered generators and put $b_{ij} = h_{ij} - 1$. Denote by $U_r(\mathfrak{g},K)$ the closure of $U(\mathfrak{g},K) \subseteq D_r(G,K)$. There is a number $\epsilon(r,p) \in \mathbb{N}$ depending only on r and p such that the (left or right) $U_r(\mathfrak{g},K)$-module $D_r(G,K)$ is finite free on the basis $\mathcal{R} := \{\mathbf{b}^{\alpha}, \alpha_{ij} < \epsilon(r,p)$ for all $(i,j) \in \{1,\ldots,n\} \times \{1,\ldots,d\}\}$. Letting $\mathfrak{x}_1, \ldots, \mathfrak{x}_d$ be an L-basis of $\mathfrak{g}$ and $\mathfrak{X}^{\beta} := \mathfrak{x}_1^{\beta_1} \cdots \mathfrak{x}_d^{\beta_d}$ one has as K-vector spaces*

$$U_r(\mathfrak{g},K) = \Big\{ \sum_{\beta \in \mathbb{N}_0^d} d_{\beta} \mathfrak{X}^{\beta},\ d_{\beta} \in K,\ \|d_{\beta}\mathfrak{X}^{\beta}\|_{\bar{r}} \to 0 \text{ for } |\beta| \to \infty \Big\},$$

where the power series expansions are determined uniquely. The Banach algebras $U_r(\mathfrak{g},K)$ exhibit $U(\mathfrak{g},K)$ as a Fréchet–Stein algebra.

Proof. This is extracted from (the proof of) [Koh07a, Theorem 1.4.2]. The number $\epsilon(r,p)$ equals the (unique) value of t where the supremum $\sup_{t \in \mathbb{N}} |1/t| r^{\kappa t}$ is attained. □

In [Koh07a, Theorem 1.4.2] the noetherian and the flatness property of the family $U_r(\mathfrak{g},K)$ is immediately deduced from the commutative diagram

$$\begin{array}{ccc} U_r(\mathfrak{g},K) & \longrightarrow & U_{r'}(\mathfrak{g},K) \\ \downarrow & & \downarrow \\ D_r(G,K) & \longrightarrow & D_{r'}(G,K) \end{array}$$

for $r' \leq r$ in which the lower horizontal arrow is a flat map between noetherian rings and the vertical arrows are, by the first statement in the theorem, finite free ring extensions. We also remark that the first statement in the theorem in case $L = \mathbb{Q}_p$ is due to Frommer [Fro03, 1.4, Lemma 3, Corollaries 1–3].

PROPOSITION 2.4. *Suppose that G_0 is uniform* and let $\mathfrak{x}_1, \ldots, \mathfrak{x}_d$ be the corresponding basis of $\mathfrak{g}_L$. Endowing the extension $U_r(\mathfrak{g}, K) \subseteq D_r(G, K)$ with the $\|\cdot\|_{\bar{r}}$-norm filtration the map $gr_r^\bullet U_r(\mathfrak{g}, K) \to gr_r^\bullet D_r(G, K)$ is finite free on the basis $\sigma(\mathcal{R})$. Moreover, $gr_r^\bullet U_r(\mathfrak{g}, K)$ equals a polynomial ring in $\sigma(\mathfrak{x}_1), \ldots, \sigma(\mathfrak{x}_d)$ and $\|\cdot\|_{\bar{r}}$ is multiplicative on $U_r(\mathfrak{g}, K)$. For any $\lambda = \sum_{\beta \in \mathbb{N}_0^d} d_\beta \mathfrak{X}^\beta \in U_r(\mathfrak{g}, K)$ one has*

$$\|\lambda\|_{\bar{r}} = \sup_\beta |d_\beta| c_r^{|\beta|}$$

where $c_r \in \mathbb{R}_{>0}$ depends only on r and p.

Proof. Let $v_1, \ldots, v_n$ be a corresponding $\mathbb{Z}_p$-basis of $\mathfrak{o}_L$ for the uniform* group G. Then $h_{ij} := \exp(v_i \mathfrak{x}_j)$ are a minimal set of topological generators for G. Put as usual $b_{ij} := h_{ij} - 1$. Now $v_i \mathfrak{x}_j = \log(1 + b_{ij})$ is a $\mathbb{Q}_p$-basis for $\mathfrak{g}_0$ and a short calculation yields $\sigma(v_i \mathfrak{x}_j) = \sigma(p)^{-h} \sigma(b_{ij})^{p^h}$ with h depending only on r and p. Hence, Theorem 2.2 translates the map $gr_r^\bullet U_r(\mathfrak{g}_0, K) \to gr_r^\bullet D_r(G_0, K)$ into the inclusion $(gr^\bullet K)[X_{11}^{p^h}, \ldots, X_{nd}^{p^h}] \subseteq (gr^\bullet K)[X_{11}, \ldots, X_{nd}]$. For $L = \mathbb{Q}_p$ we obtain from this all statements together with the fact that the $U_r(\mathfrak{g}_0, K)$-module basis $\mathcal{R}$ for $D_r(G_0, K)$ is, in fact, orthogonal with respect to $\|\cdot\|_r$. By [Sch08, Lemma 5.3 and Proposition 5.5], the graded ideal $gr_r^\bullet I_r$ where $I_r = \ker\,(D_r(G_0, K) \to D_r(G, K))$ is generated by the elements $X_{ij}^{p^h} - \bar{v}_i X_{1j}^{p^h}$ where $\bar{v}_i \in gr^\bullet K$ equals the residue class of v_i. By similar arguments the same holds true for $gr_r^\bullet J_r$ where $J_r = \ker\,(U_r(\mathfrak{g}_0, K) \to U_r(\mathfrak{g}, K))$. It follows that $I_r = \bigoplus_{g \in \mathcal{R}} J_r g$. By orthogonality the quotient norm on $U_r(\mathfrak{g}, K)$ with respect to $\|\cdot\|_r$ and $U_r(\mathfrak{g}_0, K) \to U_r(\mathfrak{g}, K)$ equals precisely $\|\cdot\|_{\bar{r}}$. In other words, $gr_r^\bullet U_r(\mathfrak{g}_0, K)/gr J_r \simeq gr_r^\bullet U_r(\mathfrak{g}, K)$ and since $gr_r^\bullet I_r \cap gr_r^\bullet U_r(\mathfrak{g}_0, K) = gr_r^\bullet J_r$ the first statement follows. Now $gr_r^\bullet U_r(\mathfrak{g}_0, K)/gr J_r$ is readily seen to be a polynomial ring in the residue classes of the $X_{1j}^{p^h}$, $j = 1, \ldots, d$, which correspond to $\sigma(\mathfrak{x}_j) \in gr_r^\bullet U_r(\mathfrak{g}, K)$. This implies that $\|\cdot\|_{\bar{r}}$ is multiplicative on $U_r(\mathfrak{g}, K)$, that the topological K-basis $\mathfrak{X}^\beta$, $\beta \in \mathbb{N}_0^d$ for $U_r(\mathfrak{g}, K)$ is, in fact, an orthogonal basis with respect to $\|\cdot\|_{\bar{r}}$ and that $c_r := \|\mathfrak{x}_j\|_{\bar{r}} = \|\mathfrak{x}_j\|_r = \|\log(1 + b_{1j})\|_r = \sup_{t \in \mathbb{N}} |1/t| r^{\kappa t}$ depends only on r and p. $\square$

Next we prove a proposition on the compatibility of two Fréchet–Stein structures. This will be used in the proof of Theorem 7.5.

LEMMA 2.5. *Let G be a compact locally L-analytic group of dimension d and $P \subseteq G$ a closed subgroup of dimension $l \leq d$. There is an open normal subgroup $G' \subseteq G$ with the following properties: it is uniform* with respect to bases $\mathfrak{x}_1, \ldots, \mathfrak{x}_d$ and $v_1, \ldots, v_n$. Furthermore, $P' := P \cap G'$ is uniform* with respect to the bases $\mathfrak{x}_1, \ldots, \mathfrak{x}_l$ and $v_1, \ldots, v_n$.*

Proof. Denote the Lie algebras of G and P by $\mathfrak{g}_L$ and $\mathfrak{p}_L$, respectively. Denote by G_0, P_0 the underlying locally $\mathbb{Q}_p$-analytic groups. Applying [DdMS99, Proposition 3.9 and Theorems 4.2 and 4.5] we see that P contains a uniform subgroup P_1 such that every open normal subgroup of P lying in P_1 is uniform itself. After this preliminary remark we choose, according to [Sch08, Proposition 4.3 and its proof] and [Sch08, Corollary 4.4], a locally L-analytic group G' open normal in G with the following properties: it is uniform* with an L-basis $\mathfrak{x}_1, \ldots, \mathfrak{x}_d$ of $\mathfrak{g}_L$ such that $\mathfrak{x}_1, \ldots, \mathfrak{x}_l$ is an L-basis of $\mathfrak{p}_L$. Furthermore, we may arrange that $P' := G' \cap P \subseteq P_1$. Since $\mathfrak{x}_1, \ldots, \mathfrak{x}_l$ is an L-basis of $\mathfrak{p}_L$ and exp may be viewed an exponential map for P the nl elements $\exp(v_i \mathfrak{x}_j)$, $i = 1, \ldots, n$, $j = 1, \ldots, l$, are part of a minimal generating system for G' and lie in $G' \cap P = P'$. Since they are pairwise different modulo G'^p, hence modulo P'^p, it follows from $\dim_L P' = l$ that they form a minimal generating system for the uniform group P'. Thus, P' is uniform* with the required bases. □

Proposition 2.6. *Let G be a compact locally L-analytic group and $P \subseteq G$ a closed subgroup. There is a family of norms $(\| \cdot \|_r)$ on $D(G_0, K)$ with the following properties: it defines the Fréchet–Stein structure on $D(G_0, K)$ as well as on the subalgebra $D(P_0, K)$. For each r the completion $D_r(G_0, K)$ is flat as a $D_r(P_0, K)$-module. The family of quotient norms $(\| \cdot \|_{\bar{r}})$ defines the Fréchet–Stein structure on $D(G, K)$ as well as on the subalgebra $D(P, K)$. For each r the completion $D_r(G, K)$ is flat as a $D_r(P, K)$-module.*

Proof. Apply the preceding lemma to $P \subseteq G$ to find an open normal subgroup $G' \subseteq G$ which is uniform* with respect to bases $\mathfrak{x}_j$ and v_i. The nd elements $\exp(v_i \mathfrak{x}_j)$ are then topological generators for G' where the first nl elements ($l := \dim_L P$) generate the uniform group $P' := P \cap G'$. Endow $D(G_0, K)$ and $D(G, K)$ with the Fréchet–Stein structures constructed in the beginning of this section and restrict the norms to $D(P_0, K)$ and $D(P, K)$. Then [ST05, Proposition 6.2] yields all statements over $\mathbb{Q}_p$. By definition, the quotient map $D(G_0, K) \to D(G, K)$ restricts to the quotient map $D(P_0, K) \to D(P, K)$ whence it is easy to see that the restricted norms $\| \cdot \|_{\bar{r}}$ on $D(P, K)$ equal the quotient norms. Hence, they realize a Fréchet–Stein structure on $D(P, K)$ and only the last claim remains to be justified. By the argument given at the end of [ST05, Proposition 6.2] it suffices to prove it for the pair $P' \subseteq G'$. Applying Proposition 2.4 to $D(P', K)$ and $D(G', K)$ we obtain a commutative diagram of commutative algebras

$$\begin{array}{ccc} gr_r^{\bullet} U_r(\mathfrak{p}, K) & \longrightarrow & gr_r^{\bullet} U_r(\mathfrak{g}, K) \\ \downarrow & & \downarrow \\ gr_r^{\bullet} D_r(P', K) & \longrightarrow & gr_r^{\bullet} D_r(G', K) \end{array}$$

in which the vertical arrows are finite free ring extensions on bases $\sigma(\mathcal{R}(\mathfrak{p}))$ and $\sigma(\mathcal{R}(\mathfrak{g}))$, respectively. By our assumptions and by the explicit shape of these bases there is a set $S \subseteq \sigma(\mathcal{R}(\mathfrak{g}))$ such that $\sigma(\mathcal{R}(\mathfrak{g})) = \{st, s \in S, \ \ t \in \sigma(\mathcal{R}(\mathfrak{p}))\}$. It follows that the map of $gr_r^{\bullet} D_r(P', K)$-modules

$$\bigoplus_{g \in S} gr_r^{\bullet} D_r(P', K) \otimes_{gr_r^{\bullet} U_r(\mathfrak{p}, K)} gr_r^{\bullet} U_r(\mathfrak{g}, K) \to gr_r^{\bullet} D_r(G', K)$$

induced by $(\lambda \otimes \mu)_g \mapsto \sum_g \lambda \mu g$ is bijective. Using [ST03, Proposition 1.2] we are hence reduced to prove the flatness of the upper horizontal arrow. However, this equals the inclusion of a polynomial ring over $gr^{\bullet} K$ in l variables into one of d variables (again by Proposition 2.4) which is clearly flat. □

In case $L = \mathbb{Q}_p$ this result is precisely [ST05, Proposition 6.2]. The proof of our proposition was simplified by a remark from J. Kohlhaase.

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We finish this section with some results on the Lie algebra cohomology of $U(\mathfrak{g}, K)$. Recall the homological standard complex of free $U(\mathfrak{g})$-modules $U(\mathfrak{g}) \otimes_L \dot{\bigwedge}\mathfrak{g}$ whose differential is given via

$$\partial(\lambda \otimes \mathfrak{x}_1 \wedge \cdots \wedge \mathfrak{x}_q) = \sum_{s<t} (-1)^{s+t} \lambda \otimes [\mathfrak{x}_s, \mathfrak{x}_t] \wedge \mathfrak{x}_1 \wedge \cdots \wedge \widehat{\mathfrak{x}_s} \wedge \cdots \wedge \widehat{\mathfrak{x}_t} \wedge \cdots \wedge \mathfrak{x}_q \\ + \sum_s (-1)^{s+1} \lambda \mathfrak{x}_s \otimes \mathfrak{x}_1 \wedge \cdots \wedge \widehat{\mathfrak{x}_s} \wedge \cdots \wedge \mathfrak{x}_q.$$

Composing with the augmentation $U(\mathfrak{g}) \to L$ yields a finite free resolution of the $U(\mathfrak{g})$-module L and $H^*(\mathfrak{g}, V) := h^*(\mathrm{Hom}_L(\dot{\bigwedge}\mathfrak{g}, V))$ and $H_*(\mathfrak{g}, V) := h_*(V \otimes_L \dot{\bigwedge}\mathfrak{g})$, respectively, as objects in Vec_L for any $\mathfrak{g}$-module V. Assume that $V \in \mathrm{Vec}_K$ is nuclear Fréchet or of compact type such that $\mathfrak{g}$ acts by continuous K-linear operators. Endow each $V \otimes_L \bigwedge^q \mathfrak{g}$ and $\mathrm{Hom}_L(\bigwedge^q \mathfrak{g}, V)$ with the projective tensor product topology and the strong topology, respectively. The obvious map $V_b' \otimes_L \bigwedge^q \mathfrak{g} \to (\mathrm{Hom}_L(\bigwedge^q \mathfrak{g}, V))_b'$ is a topological isomorphism and identifies $V_b' \otimes_L \dot{\bigwedge}\mathfrak{g}$ with the strong dual of $\mathrm{Hom}_L(\dot{\bigwedge}\mathfrak{g}, V)$ (see, e.g., [Pir06, § 1.4]). Always endow $H^*(\mathfrak{g}, V)$ and $H_*(\mathfrak{g}, V)$, respectively, with the induced topologies.

LEMMA 2.7. *Let V be a nuclear Fréchet space with continuous $\mathfrak{g}$-action. Suppose that the differential in $V \otimes_L \dot{\bigwedge}\mathfrak{g}$ is strict. There are isomorphisms of locally convex K-vector spaces*

$$H^*(\mathfrak{g}, V_b') \simeq H_*(\mathfrak{g}, V)_b' \tag{4}$$

natural in V.

Proof. Since $V \otimes_L \dot{\bigwedge}\mathfrak{g}$ consists of Fréchet spaces the differential has closed image. By [ST02a, Theorem 1.1 and Proposition 1.2] the complex $(V \otimes_L \dot{\bigwedge}\mathfrak{g})_b'$ consists of spaces of compact type and has a strict differential with closed image. Thus, we may substitute in the proof of [Koh07b, Lemma 3.6] all weak topologies by the strong topologies and obtain K-linear bijections $H_*(\mathfrak{g}, V)_b' = (h_*(V \otimes_L \dot{\bigwedge}\mathfrak{g}))_b' \simeq h^*((V \otimes_L \dot{\bigwedge}\mathfrak{g})_b')$ which are readily seen to be topological. Since $\mathrm{Hom}_L(\dot{\bigwedge}\mathfrak{g}, V_b')$ consists of spaces of compact type the remark preceding the lemma implies $(V \otimes_L \dot{\bigwedge}\mathfrak{g})_b' \simeq \mathrm{Hom}_L(\dot{\bigwedge}\mathfrak{g}, V_b')$ topologically whence the claim follows. □

PROPOSITION 2.8. *One has $\bigoplus_* H_*(\mathfrak{g}, U(\mathfrak{g}, K)) = H_0(\mathfrak{g}, U(\mathfrak{g}, K)) = K$.*

Proof. Over the complex numbers this follows from [Pir06, Theorem 8.6]. In our setting our results allow us to give a proof along the lines of [ST05, Proposition 3.1]. The case $* = 0$ is clear. Now being a Fréchet–Stein algebra (Theorem 2.3) the topology on $U(\mathfrak{g}, K)$ is nuclear Fréchet and the differential in $U(\mathfrak{g}, K) \otimes_L \dot{\bigwedge}\mathfrak{g}$ is strict [ST03, § 3]. By Lemma 2.7 it suffices to prove $H^*(\mathfrak{g}, C_1^{\mathrm{an}}(G, K)) = 0$ for $* > 0$. By [BW80, VII.1.1] the complex $\mathrm{Hom}_L(\dot{\bigwedge}\mathfrak{g}, C_1^{\mathrm{an}}(G, K))$ equals (up to sign) the stalk at $1 \in G$ of the de Rham complex of K-valued global locally analytic differential forms on the manifold G. By the usual Poincaré lemma the latter is acyclic. □

3. Continuous representations and analytic vectors

We recall some definitions and results from continuous representation theory relying on [Sch06]. We introduce the notion of analytic vector and prove some basic properties.

A *locally analytic G-representation* is a barrelled locally convex Hausdorff K-vectorspace V equipped with a G-action via continuous operators such that for all $v \in V$ the orbit map

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$o_v : G \to V$, $g \mapsto g^{-1}v$ lies in $C^{\mathrm{an}}(G,V)$, the space of V-valued locally analytic functions on G. With continuous K-linear G-maps these representations form a category $\mathrm{Rep}_K(G)$. Endowing $C^{\mathrm{an}}(G,V)$ with the left regular action ($(g\cdot f)(h) = f(g^{-1}h)$) yields $C^{\mathrm{an}}(G,V) \in \mathrm{Rep}_K(G)$ and a G-equivariant embedding $o : V \to C^{\mathrm{an}}(G,V)$, $v \mapsto o_v$. Let $\mathrm{Rep}_K^a(G) \subseteq \mathrm{Rep}_K(G)$ be the full subcategory of *admissible* representations. Denote the abelian category of coadmissible modules by $\mathcal{C}_G$. There is an anti-equivalence $\mathrm{Rep}_K^a(G) \simeq \mathcal{C}_G$ via $V \mapsto V_b'$. In particular, any $M \in \mathcal{C}_G$ has a nuclear Fréchet topology (the *canonical* topology). If G is compact $\mathcal{C}_G$ contains all finitely presented modules. In this case, a $V \in \mathrm{Rep}_K^a(G)$ such that V' is finitely generated is called *strongly admissible*.

Now let G be compact and let $K/\mathbb{Q}_p$ be finite. A *Banach space representation* of G is a K-Banach space V with a linear action of G such that $G \times V \to V$ is continuous. Let $D^c(G,K)$ be the algebra of continuous K-valued distributions on G. Denote by $\mathrm{Ban}_G^{\mathrm{adm}}(K)$ the abelian category of admissible representations and by $\mathcal{M}^{fg}(D^c(G,K))$ the finitely generated modules. There is an anti-equivalence $\mathrm{Ban}_G^{\mathrm{adm}}(K) \simeq \mathcal{M}^{fg}(D^c(G,K))$ via $V \mapsto V'$. In particular, $\mathrm{Ban}_G^{\mathrm{adm}}(K)$ has enough injective objects. If $V \in \mathrm{Ban}_G^{\mathrm{adm}}(K)$, then $v \in V$ is called a *locally L-analytic* vector if the orbit map $g \mapsto gv$ lies in $C^{\mathrm{an}}(G,V)$. The subspace $V_{\mathrm{an}} \subseteq V$ consisting of all of these vectors has an induced continuous G-action and is endowed with the subspace topology arising from the embedding $o : V_{\mathrm{an}} \to C^{\mathrm{an}}(G,V)$. By [Eme, Proposition 2.1.26] the inclusion $C(G,K) \subseteq C^{\mathrm{an}}(G,K)$ is continuous and dualizes therefore to an algebra map $D^c(G,K) \to D(G,K)$.

Theorem 3.1 (Schneider–Teitelbaum). *Let G be compact and $K/\mathbb{Q}_p$ be finite. Suppose that $L = \mathbb{Q}_p$. The map*

$$D^c(G,K) \to D(G,K) \tag{5}$$

is faithfully flat. Given $V \in \mathrm{Ban}_G^{\mathrm{adm}}(K)$ the representation V_{an} is a strongly admissible locally analytic representation and $V_{\mathrm{an}} \subseteq V$ is norm-dense. The functor $F_{\mathbb{Q}_p} : V \mapsto V_{\mathrm{an}}$ between $\mathrm{Ban}_G^{\mathrm{adm}}(K)$ and $\mathrm{Rep}_K^a(G)$ is exact. The dual functor equals base extension.

This is [Sch06, Theorems 4.2 and 4.3]. Given the exactness statement $V_{\mathrm{an}} \subseteq V$ being dense is equivalent to the functor being faithful. However, in general, the functor is not full (cf. the end of § 3 of [Eme04]). Furthermore, if $L \neq \mathbb{Q}_p$, it is generally not exact and can be zero on objects. For example (cf. [Eme, § 3]), let $G = (\mathfrak{o}_L, +)$ and suppose that $\psi : G \to K$ is $\mathbb{Q}_p$-linear but not L-linear. The two-dimensional representation of G given by the matrix $\left(\begin{smallmatrix} 1 & \psi \\ & 1 \end{smallmatrix}\right)$ is an extension of the trivial representation by itself but not locally L-analytic.

To study F_L we have to introduce another functor. Let G be an *arbitrary* locally L-analytic group and $\mathbb{Q}_p \subseteq K$ be discretely valued. Given $V \in \mathrm{Rep}_K(G_0)$ we call $v \in V$ a *locally L-analytic vector* if $o_v \in C^{\mathrm{an}}(G_0, V)$ lies in the subspace $C^{\mathrm{an}}(G,V)$. Denote the space of these vectors, endowed with the subspace topology from V, by V_{an}. Since translation on G is locally L-analytic V_{an} has an induced continuous G-action. In the following we show that the correspondence $V \mapsto V_{\mathrm{an}}$ induces a functor

$$F_{\mathbb{Q}_p}^L : \mathrm{Rep}_K^a(G_0) \to \mathrm{Rep}_K^a(G).$$

Given $V \in \mathrm{Rep}_K(G)$ the Lie algebra $\mathfrak{g}$ acts on V via continuous endomorphisms

$$\mathfrak{x}v := \frac{d}{dt}\, \exp(t\mathfrak{x})v|_{t=0}$$

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for $\mathfrak{x} \in \mathfrak{g}$, $v \in V$. Denote by $C_1^{\mathrm{an}}(G, K)$ the local ring at $1 \in G$ as introduced before. Given $f \in C^{\mathrm{an}}(G, K)$ denote its image in $C_1^{\mathrm{an}}(G, K)$ by $[f]$. Viewing $\mathfrak{g}_0$ as point derivations on $C_1^{\mathrm{an}}(G_0, K)$ restricting derivations to $C_1^{\mathrm{an}}(G, K)$ induces a map $L \otimes_{\mathbb{Q}_p} \mathfrak{g}_0 \to \mathfrak{g}$. Denote its kernel by $\mathfrak{g}^0$. Let $(\mathfrak{g}^0)$ denote the two-sided ideal generated by $\mathfrak{g}^0$ inside $U(\mathfrak{g}_0, K)$ as well as in $D(G_0, K)$. It equals the kernel of the quotient maps $U(\mathfrak{g}_0, K) \to U(\mathfrak{g}, K)$ as well as $D(G_0, K) \to D(G, K)$ (by straightforward generalizations of [Sch08, Lemma 5.1]).

LEMMA 3.2. *An element $f \in C^{\mathrm{an}}(G_0, K)$ is locally L-analytic at $1 \in G$ if and only if the space of derivations $\mathfrak{g}^0$ annihilates $[f]$.*

Proof. The function f is locally L-analytic at 1 if and only if this is true for $[f]$. By Theorem 2.3 $U(\mathfrak{g}_0, K)$ is Fréchet–Stein. Hence, $(\mathfrak{g}^0) \subseteq U(\mathfrak{g}_0, K)$ being finitely generated is closed. By [Bou03, IV.2.2 Corollary] the natural map $C_1^{\mathrm{an}}(G_0, K)/C_1^{\mathrm{an}}(G, K) \to (\mathfrak{g}^0)'$ is an isomorphism whence the claim follows. □

LEMMA 3.3. *Given $V \in \mathrm{Rep}_K(G_0)$ of compact type one has $V_{\mathrm{an}} = V^{\mathfrak{g}^0}$ as subspaces of V. In particular, $V_{\mathrm{an}} \subseteq V$ is closed.*

Proof. This follows also from [Eme, Proposition 3.6.19] but we give a proof in the present language. We may assume that G is compact. Suppose first that $V = C^{\mathrm{an}}(G_0, K)$. The inclusion $V_{\mathrm{an}} \subseteq V^{\mathfrak{g}^0}$ is clear from Lemma 3.2. Let $f \in V^{\mathfrak{g}^0}$ and $g \in G$. Denote by Ad the adjoint action of G. Since $\mathfrak{g}^0$ is $L \otimes_{\mathbb{Q}_p} \mathrm{Ad}(g)$-stable the identity $g\mathfrak{x}g^{-1} \cdot v = \mathrm{Ad}(g)\mathfrak{x} \cdot v$ for $v \in V$ implies that $g \cdot f$ (left regular action) lies in $V^{\mathfrak{g}^0}$, thus is locally analytic at $1 \in G$ by Lemma 3.2. This settles the case $V = C^{\mathrm{an}}(G_0, K)$. For general $V \in \mathrm{Rep}_K(G_0)$ of compact type equipping $C^{\mathrm{an}}(G_0, K)\hat{\otimes}_K V$ with the diagonal action (trivial on the second factor), the topological vector space isomorphism $C^{\mathrm{an}}(G_0, K) \hat{\otimes}_K V \xrightarrow{\sim} C^{\mathrm{an}}(G_0, V)$ (see [Eme, Proposition 2.1.28]) becomes G-equivariant. By continuity one obtains $C^{\mathrm{an}}(G_0, V)^{\mathfrak{g}^0}$ equals the closure of $C^{\mathrm{an}}(G_0, K)^{\mathfrak{g}^0} \otimes_K V$ which, by the first step, equals $C^{\mathrm{an}}(G, V)$. Hence, for $v \in V$ we have $\mathfrak{g}^0 v = 0 \Leftrightarrow \mathfrak{g}^0 o_v = 0 \Leftrightarrow o_v \in C^{\mathrm{an}}(G, V)$, thus $V^{\mathfrak{g}^0} = V_{\mathrm{an}}$. □

We remark that, by definition, admissible locally analytic representations are, in particular, vector spaces of compact type.

PROPOSITION 3.4. *The correspondence $V \mapsto V_{\mathrm{an}}$ induces a left exact functor*

$$F_{\mathbb{Q}_p}^L : \mathrm{Rep}_K^a(G_0) \to \mathrm{Rep}_K^a(G).$$

The dual functor equals base extension.

Proof. By Lemma 3.3, $V_{\mathrm{an}} \subseteq V$ is closed and hence of compact type. If $H \subseteq G$ is a compact open subgroup, then since $D(H_0, K)/(\mathfrak{g}^0) = D(H, K)$ the strong dual $(V_{\mathrm{an}})'_b$ lies in $\mathcal{C}_{H_0} \cap \mathcal{M}(D(H, K)) = \mathcal{C}_H$. Thus, $V_{\mathrm{an}} \in \mathrm{Rep}_K^a(G)$. It is immediate that $V \mapsto V_{\mathrm{an}}$ is a functor which is left exact (Lemma 3.3). Putting $\mathfrak{h} := \mathfrak{g}^0$ in Proposition 5.5 below the last claim follows. □

COROLLARY 3.5. *Let G be compact and $K/\mathbb{Q}_p$ be finite. One has $F_L = F_{\mathbb{Q}_p}^L \circ F_{\mathbb{Q}_p}$, thus F_L is left exact and the dual functor equals base extension.*

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Proof. Given $V \in \mathrm{Ban}_G^{\mathrm{adm}}(K)$ the identity $F_L(V) = F_{\mathbb{Q}_p}^L \circ F_{\mathbb{Q}_p}(V)$ as abstract $K[G]$-modules is clear from the definitions. Since $C^{\mathrm{an}}(G, V) \subseteq C^{\mathrm{an}}(G_0, V)$ is a closed topological embedding (by a straightforward generalization of [ST01b, Lemma 1.2]) the topology on the space $F_L(V) = F_{\mathbb{Q}_p}(V) \cap C^{\mathrm{an}}(G, V)$ coincides with that induced by $F_{\mathbb{Q}_p}(V)$. By definition, this equals the topology of $F_{\mathbb{Q}_p}^L \circ F_{\mathbb{Q}_p}(V)$. The last claim follows from Theorem 3.1 and the last proposition by associativity of the tensor product. □

Remarks. (1) Since F_L is not exact we obtain from the corollary that as a rule, the map $D^c(G, K) \to D(G, K)$ is not flat for $L \neq \mathbb{Q}_p$ (but see Theorem 4.6). In view of the characterization as certain Lie invariants (Lemma 3.3) one may ask whether flatness holds when G is semisimple. This is answered negatively by Corollary 7.8 below.

(2) On certain interesting subcategories of $\mathrm{Rep}_K^a(G_0)$ the functor $F_{\mathbb{Q}_p}^L$ may very well be exact. To give an example, recall that $V \in \mathrm{Rep}_K^a(G)$ is called *locally* $U(\mathfrak{g})$*-finite* if, for all $x \in V_b'$, the orbit $U(\mathfrak{g})x$ is contained in a finite-dimensional K-subspace of V_b'. These representations are studied in [ST01a]. Let $\mathrm{Rep}_K^{a,f}(G)$ denote the full abelian subcategory of $\mathrm{Rep}_K^a(G)$ consisting of these representations. We claim that if G is semisimple then passage to analytic vectors is an *exact* functor $\mathrm{Rep}_K^{a,f}(G_0) \to \mathrm{Rep}_K^{a,f}(G)$. Indeed, let $V \in \mathrm{Rep}_K^{a,f}(G_0)$. It suffices to see that $H^1(\mathfrak{g}^0, V) = 0$. The $\mathfrak{g}^0$-module V_b' is a direct limit of finite-dimensional modules W. Since $\mathfrak{g}_0$ and thus $\mathfrak{g}^0$ are semisimple Lie algebras the first Whitehead lemma together with Lemma 2.7 for $* = 1$ yields $H_1(\mathfrak{g}^0, W) = 0$. Since $H_1(\mathfrak{g}^0, \cdot)$ commutes with direct limits, using Lemma 2.7 again we obtain $H^1(\mathfrak{g}^0, V) = 0$.

4. σ-analytic vectors

In this section G denotes a compact d-dimensional locally L-analytic group. Under the assumption that $L/\mathbb{Q}_p$ is unramified we prove a generalization to Theorem 3.1.

So let us first *assume* that $L/\mathbb{Q}_p$ is Galois. We start with a result on the Fréchet–Stein structure on $U(\mathfrak{g}, K)$. Given $\sigma \in \mathrm{Gal}(L/\mathbb{Q}_p)$ let G_σ be the scalar restriction of G via $\sigma : L \to L$ (see [Bou67, § 5.14.1]). It is a compact locally L-analytic group. Denote by $\mathfrak{g}_\sigma$ its Lie algebra. Of course, $(G_\sigma)_0 = G_0$, $(\mathfrak{g}_\sigma)_0 = \mathfrak{g}_0$ since σ is $\mathbb{Q}_p$-linear. Put $G_{L/\mathbb{Q}_p} := \prod_\sigma G_\sigma$ and $\mathfrak{g}_{L/\mathbb{Q}_p}$ for its Lie algebra. There is a commutative diagram of locally convex K-vector spaces

$$\begin{array}{ccc} C^{\mathrm{an}}(G_{L/\mathbb{Q}_p}, K) & \xrightarrow{\Delta^* \circ \iota} & C^{\mathrm{an}}(G_0, K) \\ \downarrow{[.]} & & \downarrow{[.]} \\ C_1^{\mathrm{an}}(G_{L/\mathbb{Q}_p}, K) & \xrightarrow{\Delta^* \circ \iota} & C_1^{\mathrm{an}}(G_0, K) \end{array}$$

where the horizontal arrows are induced functorially from the diagonal map $\Delta : G_0 \mapsto (\prod_\sigma G_\sigma)_0$ together with the canonical embedding ι (cf. (3)).

LEMMA 4.1. *The lower horizontal map is bijective.*

Proof. We may assume that G admits a global chart ϕ, that $L = K$ and that $\dim_L G = 1$. Using ϕ and the induced global chart for G_σ and G_0, respectively, we arrive at a map $C_1^{\mathrm{an}}(\prod_\sigma L_\sigma, L) \to C_1^{\mathrm{an}}(\mathbb{Q}_p^n, L) \simeq C_1^{\mathrm{an}}(L^n, L)$, where the first map depends on a choice of $\mathbb{Q}_p$-basis v_i of L and the second identification as rings is obvious. Tracing through the definitions shows that this map is induced by the L-linear isomorphism $\prod_\sigma L_\sigma \simeq L \otimes_{\mathbb{Q}_p} L \simeq \bigoplus_i L v_i \simeq L^n$ and, hence, is bijective. □

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Passing to strong duals yields the following commutative diagram of topological algebras.

$$\begin{array}{ccc} U(\mathfrak{g}_0, K) & \xrightarrow{\sim} & U(\mathfrak{g}_{L/\mathbb{Q}_p}, K) \\ \downarrow{\scriptstyle [\cdot]'} & & \downarrow{\scriptstyle [\cdot]'} \\ D(G_0, K) & \xrightarrow{\iota' \circ \Delta_*} & D(G_{L/\mathbb{Q}_p}, K) \end{array}$$

In the following we use the abbreviation $\varphi := \iota' \circ \Delta_*$.

Lemma 4.2. *Suppose that G is uniform*. For each r sufficiently close to one the above diagram extends to a commutative diagram*

$$\begin{array}{ccc} U_r(\mathfrak{g}_0, K) & \longrightarrow & U_r(\mathfrak{g}_{L/\mathbb{Q}_p}, K) \\ \downarrow & & \downarrow \\ D_r(G_0, K) & \xrightarrow{\varphi_r} & D_r(G_{L/\mathbb{Q}_p}, K) \end{array}$$

of Banach algebras where the upper horizontal map is injective, norm-decreasing and has dense image.

Proof. Let $\|\cdot\|_r$ be a fixed norm on $D(G_0, K) = D((G_\sigma)_0, K)$ and let $\|\cdot\|_{\bar{r}}^{\sigma}$ be the quotient norm on $D(G_\sigma, K)$ under $\iota'_\sigma : D((G_\sigma)_0, K) \to D(G_\sigma, K)$. Let $\mathfrak{x}_j$ and v_i be bases for the uniform* group G. In particular, $\mathfrak{x}_1, \ldots, \mathfrak{x}_d$ is an L-basis for $\mathfrak{g}$ and $v_1 = 1, \ldots, v_n$ is a $\mathbb{Z}_p$-basis for $\mathfrak{o}_L$. The elements $h_{ij} := \exp(v_i \mathfrak{x}_j)$ are a minimal set of topological generators for G_0. Putting $b_{ij} := h_{ij} - 1$ we obtain from (2) that $D(G_0, K)$ consists of certain series $\lambda = \sum_{\alpha \in \mathbb{N}_0^{nd}} d_\alpha \mathbf{b}^\alpha$, where $\|\lambda\|_r = \sup_\alpha |d_\alpha| r^{\kappa\alpha}$. The product version of these considerations yields norms $\|\cdot\|_r^{(\sigma)}$ and quotient norms $\|\cdot\|_{\bar{r}}^{(\sigma)}$ on $D(\prod_\sigma G_0, K)$ and $D(G_{L/\mathbb{Q}_p}, K)$, respectively. Now φ is induced from Δ and $D((\prod_\sigma G_\sigma)_0, K) \to D(\prod_\sigma G_\sigma, K)$ where the second map is certainly norm-decreasing with respect to $\|\cdot\|_r^{(\sigma)}$ and $\|\cdot\|_{\bar{r}}^{(\sigma)}$. Furthermore, $\Delta_*(b_{ij}) = \Delta(h_{ij}) - 1$. Since all nd elements $\Delta(h_{ij})$ are pairwise different modulo the first step $(\prod_\sigma G_0)^p$ in the lower p-series of the uniform group $\prod_\sigma G_0$, the discussion in [DdMS99, 4.2] shows that they may be completed to a minimal ordered system of generators for $\prod_\sigma G_0$. Since the norm $\|\cdot\|_r^{(\sigma)}$ does not depend on a particular choice of such system (cf. the discussion after Theorem 4.10 in [ST03]) one obtains $\|\Delta(h_{ij}) - 1\|_r^{(\sigma)} = \|(h_{ij}, 1, \ldots) - 1\|_r^{(\sigma)} = \|h_{ij} - 1\|_r$ whence it easily follows that Δ_* is an isometry. Then φ is norm-decreasing with respect to $\|\cdot\|_r$ and $\|\cdot\|_{\bar{r}}^{(\sigma)}$ which yields the completed diagram and its commutativity. It is clear that the upper horizontal map is norm-decreasing with dense image whence it remains to establish injectivity. We first show that the inverse map

$$\varphi^{-1} : U(\mathfrak{g}_{L/\mathbb{Q}_p}, K) \to U(\mathfrak{g}_0, K)$$

is norm-decreasing when both sides are given suitable norm topologies. By Theorem 2.3 the rings $U_r(\mathfrak{g}_0, K)$ and $U_r(\mathfrak{g}_\sigma, K)$ are certain noncommutative power series rings in the 'variables' $\partial_{ij} := v_i \mathfrak{x}_j \in \mathfrak{g}_0$ and $\partial_{\sigma,j} := \mathfrak{x}_j \in \mathfrak{g}_\sigma$, respectively. More precisely, $U_r(\mathfrak{g}_\sigma, K)$ consists of all formal series $\sum_{\beta \in \mathbb{N}_0^d} d_\beta \partial_\sigma^\beta$ where $d_\beta \in K$, $\partial_\sigma^\beta := \partial_{\sigma,1}^{\beta_1} \cdots \partial_{\sigma,d}^{\beta_d}$ and $\|d_\beta \partial_\sigma^\beta\|_{\bar{r}}^{\sigma} \to 0$ for $|\beta| \to \infty$. By Proposition 2.4, $\|\cdot\|_{\bar{r}}^{\sigma}$ is multiplicative, the topological K-basis ∂_σ^β for $U_r(\mathfrak{g}_\sigma, K)$ is even orthogonal with respect to $\|\cdot\|_{\bar{r}}^{\sigma}$ and $\|\partial_{\sigma,j}\|_{\bar{r}}^{\sigma} = \|\partial_{1j}\|_r = c_r$. Given a generic element of $U(\mathfrak{g}_{L/\mathbb{Q}_p}, K)$, say $\lambda := \sum_{\beta \in \mathbb{N}_0^{nd}} d_\beta \prod_\sigma \partial_\sigma^{\beta_{\sigma\bullet}}$ with $d_\beta \in K$, $\partial_\sigma^{\beta_{\sigma\bullet}} = \partial_{\sigma,1}^{\beta_{\sigma,1}} \cdots \partial_{\sigma,d}^{\beta_{\sigma,d}}$ we have $\|\lambda\|_{\bar{r}}^{(\sigma)} = \sup_\beta |d_\beta| (c_r)^{|\beta|}$ since the elements $\prod_\sigma \partial_\sigma^{\beta_{\sigma\bullet}}$ are orthogonal with respect to $\|\cdot\|_{\bar{r}}^{(\sigma)}$ and this

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latter norm is multiplicative. After these remarks consider the map $L\otimes_{\mathbb{Q}_p} L \to \prod_\sigma L_\sigma$. Let $\sum_{a_\sigma,b_\sigma} a_\sigma\otimes b_\sigma$ be the inverse image of $1\in L_\sigma$ where we may assume that $b_\sigma \in \mathfrak{o}_L$. Choosing $b_\sigma^{(i)}\in\mathbb{Z}_p$ such that $\sum_i b_\sigma^{(i)} v_i = b_\sigma$ we put

$$s := \sup_{\sigma,a_\sigma,b_\sigma,i} |a_\sigma b_\sigma^{(i)}|.$$

We have, by definition of the map φ, that

$$\varphi^{-1}(\partial_{\sigma,j}) = \sum_{a_\sigma,b_\sigma^{(i)},i} a_\sigma b_\sigma^{(i)} \partial_{ij}.$$

Hence,

$$\|\varphi^{-1}(\partial_{\sigma,j})\|_r = \sup_i \Bigg| \sum_{a_\sigma,b_\sigma^{(i)},i} a_\sigma b_\sigma^{(i)} \Bigg| c_r \le s c_r$$

using that the ∂_{ij} are orthogonal. Given $\lambda\in U(\mathfrak{g}_{L/\mathbb{Q}_p},K)$ as above, define another norm on $U(\mathfrak{g}_{L/\mathbb{Q}_p},K)$ via

$$\|\lambda\|_{(\bar r)}^{(\sigma)} := \sup_\beta |d_\beta|(sc_r)^{|\beta|}$$

and let $U_{(r)}(\mathfrak{g}_{L/\mathbb{Q}_p},K)$ be the completion. We obtain

$$\|\varphi^{-1}(\lambda)\|_r \le \sup_\beta |d_\beta| \prod_{\sigma,j} \|\varphi^{-1}(\partial_{\sigma,j})\|_r^{\beta_{\sigma,j}} = \sup_\beta |d_\beta|(sc_r)^{|\beta|} = \|\lambda\|_{(\bar r)}^{(\sigma)}$$

using that $\|\cdot\|_r$ is multiplicative. Thus, φ^{-1} is norm-decreasing with respect to the indicated norms. Now suppose that r is sufficiently close to one. Then there exists $r'\le r$ such that $sc_{r'}\le c_r$ (note that $c_r\uparrow\infty$ for $r\uparrow 1$) whence $U_r(\mathfrak{g}_{L/\mathbb{Q}_p},K)\subseteq U_{(r')}(\mathfrak{g}_{L/\mathbb{Q}_p},K)$. A simple approximation argument shows that the map

$$U_r(\mathfrak{g}_0,K)\xrightarrow{\varphi_r} U_r(\mathfrak{g}_{L/\mathbb{Q}_p},K)\xrightarrow{\subseteq} U_{(r')}(\mathfrak{g}_{L/\mathbb{Q}_p},K)\xrightarrow{\varphi_{r'}^{-1}} U_{r'}(\mathfrak{g}_0,K)$$

equals the inclusion $U_r(\mathfrak{g}_0,K)\subseteq U_{r'}(\mathfrak{g}_0,K)$. This finishes the proof. □

For the rest of this section we assume that $L/\mathbb{Q}_p$ is *unramified*.

Lemma 4.3. *Let $x\in\mathfrak{o}_L^\times$ be a lift of a primitive element $\bar x$ for the residue field extension of $L/\mathbb{Q}_p$. Suppose that G is uniform* with a $\mathbb{Z}_p$-basis $v_1=1,\dots,v_n$ of $\mathfrak{o}_L$ of the form $v_i=x^{i-1}$. In the situation of the last lemma the map*

$$\varphi_r : U_r(\mathfrak{g}_0,K)\to U_r(\mathfrak{g}_{L/\mathbb{Q}_p},K)$$

is an isometry.

Proof. Source and target of our map are filtered through the respective norm and it suffices to see that the associated graded map $gr_r^\bullet\varphi_r : gr_r^\bullet U_r(\mathfrak{g}_0,K)\to gr_r^\bullet U_r(\mathfrak{g}_{L/\mathbb{Q}_p},K)$ is injective. We use the notation of the preceding proof. Let X_{ij} respectively $X_{\sigma,j}$ be the principal symbol of ∂_{ij} respectively $\partial_{\sigma,j}$. Then $gr_r^\bullet U_r(\mathfrak{g}_0,K) = (gr^\bullet K)[X_{11},\dots,X_{nd}]$ and $U_r(\mathfrak{g}_{L/\mathbb{Q}_p},K) = (gr^\bullet K)[X_{\sigma_1,1},\dots,X_{\sigma_n,d}]$. Denote for each $\sigma\in\mathrm{Gal}(L/\mathbb{Q}_p)$ by $\bar\sigma$ the induced Galois automorphism on residue fields. Let F_j be the ring homomorphism

$$F_j : (gr^\bullet K)[X_{1j},\dots,X_{nj}]\to (gr^\bullet K)[X_{\sigma_1,j},\dots,X_{\sigma_n,j}],\quad X_{ij}\mapsto \sum_\sigma \bar\sigma^{-1}\bar v_i\cdot X_{\sigma,j}.$$

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Then $gr^{\bullet}_r\varphi_r$ equals, in the obvious sense, $F_1 \otimes_{gr^{\bullet}K} \cdots \otimes_{gr^{\bullet}K} F_d$ whence, by induction, it suffices to prove bijectivity of F_1. Now F_1 respects the grading by total degree whence it is enough to prove bijectivity on homogeneous components. Since each latter is free of finite rank over the principal ideal domain $gr^{\bullet}K$ it suffices to prove surjectivity in each component or, since F_1 is a ring homomorphism, in the degree-one component. However, the representing matrix $(a_{ij})_{i,j=1,\dots,n}$ of the degree-one part of F_1 with respect to the $gr^{\bullet}K$-bases X_i, $i=1,\dots,n$, and X_σ, $\sigma \in \mathrm{Gal}(L/\mathbb{Q}_p)$ on the source and target, respectively, has the shape $a_{ij} = (\bar{\sigma}_i^{-1}\bar{x})^{j-1}$ and, hence, determinant $\prod_{j>i}(\bar{\sigma}_j^{-1}\bar{x} - \bar{\sigma}_i^{-1}\bar{x})$. Since $L/\mathbb{Q}_p$ is unramified and $\bar{x}$ generates the residue extension this determinant is nonzero. □

Lemma 4.4. *Let r be sufficiently close to one. The ring extension*

$$\varphi_r : D_r(G_0, K) \longrightarrow D_r(G_{L/\mathbb{Q}_p}, K)$$

is faithfully flat.

Proof. We prove only the left version (the right version follows similarly). Suppose that $H \subseteq G$ is a normal open subgroup which is uniform*. Endow $D(G,K)$ with the Fréchet–Stein structure induced by $D(H,K)$ as explained in § 2. One obtains the following commutative diagram.

$$\begin{array}{ccc} D_r(H_0,K) & \longrightarrow & D_r(H_{L/\mathbb{Q}_p},K) \\ \downarrow & & \downarrow \\ D_r(G_0,K) & \xrightarrow{\varphi_r} & D_r(G_{L/\mathbb{Q}_p},K) \end{array}$$

Let $\mathcal{R}$ be a system of coset representatives for G/H. Choose a system of coset representatives $\mathcal{R}'$ in $G_{L/\mathbb{Q}_P}$ for coset representatives of the cokernel of the inclusion $G/H \xrightarrow{\Delta} \prod_\sigma G_\sigma/H_\sigma = G_{L/\mathbb{Q}_p}/H_{L/\mathbb{Q}_p}$. Then $\varphi_r(\mathcal{R})\mathcal{R}'$ equals a system of coset representatives for $G_{L/\mathbb{Q}_p}/H_{L/\mathbb{Q}_p}$ and so the vertical arrows in the above diagram are finite free ring extensions on $\mathcal{R}$ and $\varphi_r(\mathcal{R})\mathcal{R}'$, respectively. Consider the map of left $D_r(G_0,K)$-modules

$$\bigoplus_{g' \in \mathcal{R}'} D_r(G_0,K) \otimes_{D_r(H_0,K)} D_r(H_{L/\mathbb{Q}_p},K) \longrightarrow D_r(G_{L/\mathbb{Q}_p},K) \qquad (6)$$

induced by $(\lambda \otimes \mu)_{g' \in \mathcal{R}'} \mapsto \sum_{g' \in \mathcal{R}'} \varphi_r(\lambda)\mu g'$. On the level of vector spaces this map factors through

$$\bigoplus_{g \in \mathcal{R},\ g' \in \mathcal{R}'} g D_r(H_{L/\mathbb{Q}_p},K) \xrightarrow{\sim} \sum_{g,g'} \varphi_r(g) D_r(H_{L/\mathbb{Q}_p},K) g' = D_r(G_{L/\mathbb{Q}_p},K)$$

and, hence, is bijective (using that $g' D_r(H_{L/\mathbb{Q}_p},K) = D_r(H_{L/\mathbb{Q}_p},K)g'$ by the normality of $H_{L/\mathbb{Q}_p}$ in $G_{L/\mathbb{Q}_p}$). It therefore suffices to establish the claim for H. In other words, we may assume in the following that G is uniform*. By the construction of uniform* subgroups (cf. [Sch08, Corollary 4.4]) we may assume that the associated $\mathbb{Z}_p$-basis $v_1 = 1, \dots, v_n$ is as described in Lemma 4.3. This lemma together with Lemma 4.2 then yield the commutative diagram

$$\begin{array}{ccc} U_r(\mathfrak{g}_0,K) & \longrightarrow & U_r(\mathfrak{g}_{L/\mathbb{Q}_p},K) \\ \downarrow & & \downarrow \\ D_r(G_0,K) & \xrightarrow{\varphi_r} & D_r(G_{L/\mathbb{Q}_p},K) \end{array} \qquad (+)$$

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where the upper vertical arrow is an isometry with dense image. Let $\mathfrak{X} := \{\mathfrak{x}_1, \dots, \mathfrak{x}_d\}$ be the associated L-basis of $\mathfrak{g}_L$ for G. Then the nd elements $\exp(v_i\mathfrak{x}_j)$ are a minimal set S of topological generators for G. Hence, according to Theorem 2.3, the left vertical arrow is finite free on a basis $\mathcal{R}$ in $\mathbb{Z}[G]$. Putting $\mathbf{b}^\alpha = b_{11}^{\alpha_{11}} \cdots b_{nd}^{\alpha_{nd}}$, $b_{ij} = \exp(v_i\mathfrak{x}_j) - 1 \in \mathbb{Z}[G]$ one has $\mathcal{R} = \{\mathbf{b}^\alpha, \alpha_{ij} < \epsilon(r,p) \text{ for all } ij\}$ with $\epsilon(r,p)$ depending only on r and p. Apply this to the group $G_{L/\mathbb{Q}_p}$ as well. More precisely, $G_{L/\mathbb{Q}_p}$ is uniform* with $v_1, \dots, v_n$ and $n = |\mathrm{Gal}(L/\mathbb{Q}_p)|$ copies of $\mathfrak{X}$ as bases. By a previous argument the set $\varphi_r(S) \subseteq G_{L/\mathbb{Q}_p}$ may be completed to a minimal set S' of generators for the uniform group $\prod_\sigma G_0$. Choose an ordering $h'_1, \dots, h'_{ndn}$ of S' such that $\varphi_r(S) = \{h'_1, \dots, h'_{nd}\}$. Put $b'_k = h'_k - 1$ and form the set $\mathcal{R}' := \{\mathbf{b}'^\alpha, \alpha_k < \epsilon(r,p) \text{for all } k\}$. Again by Theorem 2.3 the right vertical arrow is finite free on the basis $\mathcal{R}'$. Let $\mathcal{R}'_< = \{\mathbf{b}'^\alpha \in \mathcal{R}', \alpha_k = 0 \text{ for all } k > nd\}$ and $\mathcal{R}'_> = \{\mathbf{b}'^\alpha \in \mathcal{R}', \alpha_k = 0 \text{ for all } k \le nd\}$. The chosen ordering of S' implies that $\varphi_r(\mathcal{R}) = \mathcal{R}'_<$. Now recall that $D_r(G_0, K)$ and $D_r(G_{L/\mathbb{Q}_p}, K)$ are Zariski rings (cf. § 2) with respect to the norm filtrations and φ_r is norm-preserving (Lemma 4.2), thus is a filtered morphism. Thus, it suffices to prove faithful flatness of the graded map $gr_r^\bullet\varphi_r : gr_r^\bullet D_r(G_0, K) \to gr_r^\bullet D_r(G_{L/\mathbb{Q}_p}, K)$ (see [Lv96, Proposition II.1.2.2]). To do this consider the graded version of (+) (where the upper objects are formed with respect to the induced filtrations via the vertical inclusions of (+))

$$\begin{array}{ccc} gr_r^\bullet U_r(\mathfrak{g}_0, K) & \xrightarrow{\sim} & gr_r^\bullet U_r(\mathfrak{g}_{L/\mathbb{Q}_p}, K) \\ \downarrow & & \downarrow \\ gr_r^\bullet D_r(G_0, K) & \xrightarrow{gr_r^\bullet\varphi_r} & gr_r^\bullet D_r(G_{L/\mathbb{Q}_p}, K) \end{array}$$

Recall that all rings occurring are commutative. For the rest of this proof $\sigma(\cdot)$ always denotes the principal symbol map. By Proposition 2.4 the vertical arrows are finite free on the basis $\sigma(\mathcal{R})$ and $\sigma(\mathcal{R}')$, respectively, and $\sigma(\mathcal{R}') = \{st,\ s \in \sigma(\mathcal{R}'_<),\ t \in \sigma(\mathcal{R}'_>)\}$. Since $gr_r^\bullet\varphi_r$ is injective on $gr_r^\bullet U_r(\mathfrak{g}_0, K)$ one has $gr_r^\bullet\varphi_r(\sigma(b_{ij})) \neq 0$ for all ij, thus $gr_r^\bullet\varphi_r(\sigma(\mathcal{R})) = \sigma(\varphi_r(\mathcal{R})) = \sigma(\mathcal{R}'_<)$. Then the map of $gr_r^\bullet D_r(G_0, K)$-modules

$$\bigoplus_{x' \in \sigma(\mathcal{R}'_>)} gr_r^\bullet D_r(G_0, K) \otimes_{gr_r^\bullet U_r(\mathfrak{g}_0, K)} gr_r^\bullet U_r(\mathfrak{g}_{L/\mathbb{Q}_p}, K) \to gr_r^\bullet D_r(G_{L/\mathbb{Q}_p}, K)$$

induced by $(\lambda \otimes \mu)_{x' \in \sigma(\mathcal{R}'_>)} \mapsto \sum_{x' \in \sigma(\mathcal{R}'_>)} gr_r^\bullet\varphi_r(\lambda)\mu x'$ is bijective. It follows that $gr_r^\bullet D_r(G_{L/\mathbb{Q}_p}, K)$ is a finite free $gr_r^\bullet D_r(G_0, K)$-module (of rank $\#\mathcal{R}'_>$). □

Corollary 4.5. *The map φ is injective.*

Proof. Since all φ_r are injective (for r sufficiently close to one) and compatible with transition with respect to $r' \le r$ the map φ is injective by left-exactness of the projective limit. □

For the rest of this section we assume that $K/\mathbb{Q}_p$ is finite. Consider the faithfully flat algebra map $D^c(G, K) \to D(G_0, K)$ stated in (5). We compose it with φ and show our first main result.

Theorem 4.6. *The map $D^c(G, K) \to D(G_{L/\mathbb{Q}_p}, K)$ is faithfully flat.*

Proof. We show only left faithful flatness. For flatness we are reduced, by the usual argument, to show that the map $D(G_{L/\mathbb{Q}_p}, K) \otimes_{D^c(G,K)} J \to D(G_{L/\mathbb{Q}_p}, K)$ is injective for any left ideal $J \subseteq D^c(G, K)$. The ring $D^c(G, K)$ being noetherian the left-hand side is a coadmissible $D(G_{L/\mathbb{Q}_p}, K)$-module. By left exactness of the projective limit we are thus reduced to show

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that $D_r(G_{L/\mathbb{Q}_p}, K) \otimes_{D^c(G,K)} J \to D_r(G_{L/\mathbb{Q}_p}, K)$ is injective for all r (sufficiently close to one). This is clear since

$$D^c(G, K) \to D(G_0, K) \to D_r(G_0, K) \xrightarrow{\varphi_r} D_r(G_{L/\mathbb{Q}_p}, K)$$

is flat (the second map by [ST03, Remark 3.2]).

For faithful flatness we have to show $D(G_{L/\mathbb{Q}_p}, K) \otimes_{D^c(G,K)} M \neq 0$ for any nonzero left $D^c(G, K)$-module M. By the first step we may assume that M is finitely generated. Then $D(G_{L/\mathbb{Q}_p}, K) \otimes_{D^c(G,K)} M$ is coadmissible, thus we are reduced, by the equivalence of categories between coadmissible modules and coherent sheafs [ST03, Corollary 3.3], to find an index r such that

$$D_r(G_{L/\mathbb{Q}_p}, K) \otimes_{D^c(G,K)} M \neq 0.$$

Put $N := D(G_0, K) \otimes_{D^c(G,K)} M \in \mathcal{C}_{G_0}$. Then $N \neq 0$ whence $N_r \neq 0$ for some r (sufficiently close to one). It follows that $D_r(G_{L/\mathbb{Q}_p}, K) \otimes_{D^c(G,K)} M$ equals

$$D_r(G_{L/\mathbb{Q}_p}, K) \otimes_{D(G_0,K)} N = D_r(G_{L/\mathbb{Q}_p}, K) \otimes_{D_r(G_0,K)} N_r$$

and the right-hand side is nonzero by faithful flatness of φ_r. □

Each choice $\sigma \in \mathrm{Gal}(L/\mathbb{Q}_p)$ gives rise to the locally L-analytic manifold K_σ arising from restriction of scalars via σ. The space $C^{\mathrm{an}}(G, K_\sigma) = C^{\mathrm{an}}(G_{\sigma^{-1}}, K)$ is called the space of *locally σ-analytic* functions [Bou67, § 5.14.3]. This motivates the following definition: given $V \in \mathrm{Rep}_K(G)$ an element $v \in V$ is called a *locally σ-analytic vector* if $o_v \in C^{\mathrm{an}}(G_{\sigma^{-1}}, V)$. Let $V_{\sigma\text{-an}}$ denote the subspace of all of these vectors in V.

Consider the Lie algebra map $L \otimes_{\mathbb{Q}_p} \mathfrak{g}_0 \simeq \prod_\sigma \mathfrak{g}_\sigma \to \mathfrak{g}_{\sigma^{-1}}$. The kernel $\mathfrak{g}_\sigma^0$ acts on V, thus one deduces (as in the case $\sigma = id$) that $V_{\sigma\text{-an}} = V^{\mathfrak{g}_\sigma^0}$, functorial in V and that passage to σ-analytic vectors is a left exact functor $\mathrm{Rep}_K^a(G_0) \to \mathrm{Rep}_K^a(G_{\sigma^{-1}})$. Note also that given $\sigma \neq \tau$ one has $V_{\sigma\text{-an}} \cap V_{\tau\text{-an}} = V^{\mathfrak{g}_0} = V^\infty$, the smooth vectors in V.

We consider the base extension functor $M \mapsto M \otimes_{D^c(G,K)} D(G_{L/\mathbb{Q}_p}, K)$ on finitely generated $D^c(G, K)$-modules and pull back to representations. This yields a functor

$$F : \mathrm{Ban}_G^{\mathrm{adm}}(K) \longrightarrow \mathrm{Rep}_K^a(G_{L/\mathbb{Q}_p}),$$

which is exact and faithful according to Theorem 4.6. Given $V \in \mathrm{Ban}_G^{\mathrm{adm}}(K)$ note that, by exactness and since $D^c(G, K)$ is noetherian, the coadmissible module associated with $F(V)$ is even finitely presented. We deduce the second main result.

Theorem 4.7. *The functor F is exact and faithful. Given $V \in \mathrm{Ban}_G^{\mathrm{adm}}(K)$ the representation $F(V)$ is strongly admissible. Viewed as a $G_{\sigma^{-1}}$-representation, $\sigma \in \mathrm{Gal}(L/\mathbb{Q}_p)$, it contains $V_{\sigma\text{-an}}$ as a closed subrepresentation and functorial in V. In the case $L = \mathbb{Q}_p$ the functor coincides with $F_{\mathbb{Q}_p}$.*

Proof. It remains to see the latter statements. The projection $pr_\sigma : \prod_\sigma G_\sigma \to G_\sigma$ induces a continuous inclusion $pr_\sigma^* : C^{\mathrm{an}}(G_\sigma, K) \to C^{\mathrm{an}}(G_{L/\mathbb{Q}_p}, K)$. By definition of the locally convex topologies on both sides it is a compact locally convex inductive limit of isometries and so, according to [Eme, Proposition 1.1.41], a topological embedding with closed image. Dualizing we obtain a continuous algebra surjection $(pr_\sigma)_* : D(G_{L/\mathbb{Q}_p}, K) \to D(G_\sigma, K)$ exhibiting $D(G_\sigma, K)$

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as coadmissible $D(G_{L/\mathbb{Q}_p}, K)$-module. It follows from [ST03, Lemma 3.8] and its proof that $V_b' \otimes_{D^c(G,K)} D(G_\sigma, K) \in \mathcal{C}_{G_\sigma}$ lies in $\mathcal{C}_{G_{L/\mathbb{Q}_p}}$ and that the two canonical topologies coincide. The $D(G_{L/\mathbb{Q}_p}, K)$-linear surjection

$$\mathrm{id} \otimes (pr_\sigma)_* : V_b' \otimes_{D^c(G,K)} D(G_{L/\mathbb{Q}_p}, K) \to V_b' \otimes_{D^c(G,K)} D(G_\sigma, K)$$

then lies in $\mathcal{C}_{G_{L/\mathbb{Q}_p}}$ and is therefore continuous and strict. It is also $D(G_\sigma, K)$-linear and the right-hand side equals $(V_{\sigma^{-1}\text{-an}})_b'$ according to the σ^{-1}-analytic version of Proposition 3.4 (cf. the remarks above). Passing to strong duals yields a closed G_σ-equivariant topological embedding $V_{\sigma^{-1}\text{-an}} \to F(V)$. It is natural in V and clearly onto in the case $L = \mathbb{Q}_p$. □

Remarks. Let $H \subseteq G$ be a compact subgroup and denote the versions of the functor F relative to H and G by F_H and F_G, respectively. The natural map

$$D^c(G, K) \otimes_{D^c(H,K)} D(H_{L/\mathbb{Q}_p}, K) \to D(G_{L/\mathbb{Q}_p}, K)$$

being an isomorphism of bimodules is equivalent to $L = \mathbb{Q}_p$ (cf. (6)). It follows that in the case $L \neq \mathbb{Q}_p$ the functors F_H and F_G do not commute with the restriction functors induced by $H \subseteq G$ and $H_{L/\mathbb{Q}_p} \subseteq G_{L/\mathbb{Q}_p}$, respectively. Therefore, there is no naive generalization of the functor F to noncompact groups in the case $L \neq \mathbb{Q}_p$ and we leave this matter as an open problem.

5. Standard resolutions

Turning back to a general extension $L/\mathbb{Q}_p$ (not necessarily Galois) the functors $F^L_{\mathbb{Q}_p}$ and F_L defined previously remain interesting in themselves. We begin their study with some general analysis of certain base extension functors between coadmissible modules.

Let G be a locally L-analytic group. For the rest of this section we *fix* an ideal $\mathfrak{h}$ of the Lie algebra $L \otimes_{\mathbb{Q}_p} \mathfrak{g}_0$ stable under $L \otimes \mathrm{Ad}(g)$ for all $g \in G$ where Ad refers to the adjoint action of G. Denote by $(\mathfrak{h})$ the two-sided ideal generated by $\mathfrak{h}$ in $L \otimes_{\mathbb{Q}_p} U(\mathfrak{g}_0)$ as well as in $D(G_0, K)$. Put $D := D(G_0, K)/(\mathfrak{h})$, $\mathcal{C}_D := \mathcal{C}_{G_0} \cap \mathcal{M}(D)$. Then $\mathcal{C}_D$ (with D-linear maps) is an abelian category. If G is compact, then since $(\mathfrak{h})$ is closed D is a Fréchet–Stein algebra and if $(\mathfrak{h})_r$ denotes the closure of $(\mathfrak{h}) \subseteq D_r(G_0, K)$, the coherent sheaf associated to D equals $D_r := D_r(G_0, K)/(\mathfrak{h})_r$ (see [ST03, Proposition 3.7]). If we base extend the standard complex $U(\mathfrak{h}) \otimes_L \bigwedge^{\cdot} \mathfrak{h}$ via $U(\mathfrak{h}) \subseteq D(G_0, K)$, then the complex

$$D(G_0, K) \otimes_L \bigwedge^{l} \mathfrak{h} \to \cdots \to D(G_0, K) \otimes_L \bigwedge^{0} \mathfrak{h} \to D, \tag{7}$$

$l := \dim_L \mathfrak{h}$, consists of finite free left $D(G_0, K)$-modules.

Lemma 5.1. *The complex (7) is a free resolution of the left $D(G_0, K)$-module D by $D(G_0, K)$-bimodules.*

Proof. Let us prove that $D(G_0, K) \otimes_L \bigwedge^{\cdot} \mathfrak{h}$ is acyclic. Choosing $H \subseteq G$ compact open and using $\mathfrak{h}$-invariant decompositions

$$D(G_0, K) = \bigoplus_{g \in G/H} g D(H_0, K), \quad D = \bigoplus_{g \in G/H} g(D(H_0, K)/(\mathfrak{h}))$$

we are reduced to compact G. The complex (7) consists then of coadmissible left $D(G_0, K)$-modules whence acyclicity may be tested on coherent sheafs. It thus suffices to see that $D_r(G_0, K) \otimes_L \bigwedge^{\cdot} \mathfrak{h}$ is exact for a fixed radius r. The maps $U(\mathfrak{g}_0, K) \to U_r(\mathfrak{g}_0, K) \to D_r(G_0, K)$

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are flat, the first by [ST03, Remark 3.2] and the second by Theorem 2.3. We are thus reduced to showing that base extending the standard complex via $U(\mathfrak{h}) \subseteq U(\mathfrak{h}, K) \subseteq U(L \otimes_{\mathbb{Q}_p} \mathfrak{g}_0, K) = U(\mathfrak{g}_0, K)$ is exact. By Lemma 2.8 this holds for the extension $U(\mathfrak{h}) \subseteq U(\mathfrak{h}, K)$ and $U(\mathfrak{h}, K) \subseteq U(L \otimes_{\mathbb{Q}_p} \mathfrak{g}_0, K) = U(\mathfrak{g}_0, K)$ is clearly a free ring extension. This proves acyclicity. Using the fact that $\mathfrak{h}$ is Ad-stable we may endow the complex (7) with a right $D(G_0, K)$-module structure as follows. The adjoint action of G_0 on $\mathfrak{h}$ is locally analytic and extends functorially to a continuous right action on $\bigwedge^q \mathfrak{h}$ given explicitly via $(\mathfrak{x}_1 \wedge \cdots \wedge \mathfrak{x}_q)g = \mathrm{Ad}(g^{-1})\mathfrak{x}_1 \wedge \cdots \wedge \mathrm{Ad}(g^{-1})\mathfrak{x}_q$. Letting G_0 act on $D(G_0, K)$ by right multiplication we give $D(G_0, K) \otimes \bigwedge^q \mathfrak{h}$ the right diagonal G_0-action which extends to a separately continuous right $D(G_0, K)$-module structure (cf. [ST05, Appendix]). The identity $g\mathfrak{x}g^{-1} = \mathrm{Ad}(g)\mathfrak{x}$ in $D(G_0, K)$ implies that the differential ∂ of (7) respects the diagonal right G_0-action. Since $K[G_0] \subseteq D(G_0, K)$ is dense [ST02a, Lemma 3.1], ∂ respects the right $D(G_0, K)$-module structure by continuity. □

Proposition 5.2. *Given $X \in \mathcal{C}_{G_0}$ one has $\mathrm{Tor}_*^{D(G_0,K)}(X, D) \in \mathcal{C}_D$. Furthermore, there is an isomorphism $\mathrm{Tor}_*^{D(G_0,K)}(X, D) \simeq H_*(\mathfrak{h}, X)$ in Vec_K natural in X which is topological with respect to the canonical topology on the left-hand side. In particular, $\mathrm{Tor}_*^{D(G_0,K)}(\cdot, D) = 0$ in degrees $* > \dim_L \mathfrak{h}$.*

Proof. Let $X \in \mathcal{C}_{G_0}$ be given. By the above lemma $\mathrm{Tor}_*^{D(G_0,K)}(X, D) \simeq h_*(X \otimes_L \dot{\bigwedge}\mathfrak{h})$ in Vec_K. By the usual argument with double complexes, the right module structure on $X \otimes_L \dot{\bigwedge}\mathfrak{h}$ makes the isomorphism right $D(G_0, K)$-equivariant. Now X is coadmissible and $\bigwedge^q \mathfrak{h}$ is finite dimensional. Hence, dualizing [Eme, Proposition 6.1.5] yields that each right module $X \otimes_L \bigwedge^q \mathfrak{h}$ is coadmissible and its canonical topology coincides with the tensor product topology. Since $\mathcal{C}_{G_0}$ is abelian we obtain $\mathrm{Tor}_*^{D(G_0,K)}(X, D) \in \mathcal{C}_{G_0} \cap \mathcal{M}(D) = \mathcal{C}_D$. The remaining statements are now clear. □

In the compact case the associated coherent sheafs are easily computed.

Corollary 5.3. *Let G be compact. Given $X \in \mathcal{C}_{G_0}$ then $\mathrm{Tor}_*^{D(G_0,K)}(X, D)_r = \mathrm{Tor}_*^{D_r(G_0,K)}(X_r, D_r)$. Also, $\mathrm{Tor}_*^{D_r(G_0,K)}(X_r, D_r) \simeq H_*(\mathfrak{h}, X_r)$ in Vec_K.*

Proof. Let $P_\bullet \to X$ be a projective resolution in $\mathcal{M}(D(G_0, K))$. By flatness of $D(G_0, K) \to D_r(G_0, K)$ the complex $P_\bullet \otimes_{D(G_0,K)} D_r(G_0, K) \to X_r$ is a projective resolution of the $D_r(G_0, K)$-module X_r. Since $D_r \simeq D \otimes_{D(G_0,K)} D_r(G_0, K))$ as $(D(G_0, K), D_r(G_0, K))$-bimodules [ST03, Corollary 3.1] we have as right $D_r(G_0, K)$-modules

$$\begin{aligned}\mathrm{Tor}_*^{D_r(G_0,K)}(X_r, D_r) &\simeq h_*(P_\bullet \otimes_{D(G_0,K)} D_r)\\ &\simeq h_*(P_\bullet \otimes_{D(G_0,K)} D) \otimes_{D(G_0,K)} D_r(G_0, K)\\ &\simeq \mathrm{Tor}_*^{D(G_0,K)}(X, D) \otimes_{D(G_0,K)} D_r(G_0, K).\end{aligned}$$ □

Since $\mathcal{C}_{G_0} \subseteq \mathcal{M}(D(G_0, K))$ is a full embedding (and similarly for $\mathcal{C}_D$) we have the following result.

Corollary 5.4. *The functors $\mathrm{Tor}_*^{D(G_0,K)}(\cdot, D)$ form a homological δ-functor between $\mathcal{C}_{G_0}$ and $\mathcal{C}_D$.*

Proposition 5.5. *There is the following commutative (up to natural isomorphism) diagram of functors.*

$$\begin{array}{ccc} \mathrm{Rep}^a_K(G_0) & \xrightarrow{V\mapsto V^{\mathfrak{h}}} & \mathrm{Rep}^a_K(G_0) \\ {\scriptstyle V\mapsto V'_b}\downarrow & & \downarrow{\scriptstyle V\mapsto V'_b} \\ \mathcal{C}_{G_0} & \xrightarrow{M\mapsto M\otimes_{D(G_0,K)}D} & \mathcal{C}_{G_0} \end{array}$$

Proof. Suppose first that G is compact. By continuity of the Lie action and since $\mathfrak{h}$ is Ad-stable, $V\mapsto V^{\mathfrak{h}}$ is an auto-functor of $\mathrm{Rep}^a_K(G_0)$. The complex $V'_b\otimes_L \dot{\bigwedge}\mathfrak{h}$ consists of coadmissible right modules whence the differential is strict. Hence, Lemma 2.7 for $*=0$ implies that restriction of functionals yields a G_0-isomorphism $V'_b/V'_b\mathfrak{h}\simeq (V^{\mathfrak{h}})'_b$ of topological vector spaces, functorial in V. By local analyticity this extends to an isomorphism of right $D(G_0,K)$-modules $V'_b\otimes_{D(G_0,K)} D\simeq (V^{\mathfrak{h}})'_b$ natural in V. This settles the compact case. If G is arbitrary the result follows easily from the compact case by choosing a compact open subgroup. □

6. Higher analytic vectors

We are mainly interested in the choice $\mathfrak{h}:=\mathfrak{g}^0$ where, as before, $\mathfrak{g}^0=\ker(L\otimes_{\mathbb{Q}_p}\mathfrak{g}_0\to\mathfrak{g})$. Hence, $D=D(G_0,K)/(\mathfrak{g}^0)\simeq D(G,K)$ and $\mathcal{C}_D=\mathcal{C}_{G_0}\cap\mathcal{M}(D(G,K))=\mathcal{C}_G$.

Theorem 6.1. *Passage to analytic vectors $F^L_{\mathbb{Q}_p}$ extends to a cohomological δ-functor $(R^iF^L_{\mathbb{Q}_p})_{i\geq 0}$ with $R^iF^L_{\mathbb{Q}_p}=0$ for $i>([L:\mathbb{Q}_p]-1)\dim_L G$.*

Proof. We may clearly replace the right upper and lower corners in Proposition 5.5 by $\mathrm{Rep}^a_K(G)$ and $\mathcal{C}_G$, respectively, without changing the statement. Both vertical arrows are anti-equivalences between abelian categories and therefore exact functors. By direct calculation pulling back the functors $\mathrm{Tor}^{D(G_0,K)}_*(\cdot,D(G,K)):\mathcal{C}_{G_0}\to\mathcal{C}_G$ (cf. Corollary 5.4) yields a cohomological δ-functor extending $F^L_{\mathbb{Q}_p}$. Finally, $\dim_L\mathfrak{g}^0=([L:\mathbb{Q}_p]-1)\dim_L G$. □

The functors $R^iF^L_{\mathbb{Q}_p}$ can be expressed without referring to coadmissible modules. Endowing $V\in\mathrm{Rep}^a_K(G_0)$ with the uniquely determined separately continuous left $D(G_0,K)$-module structure we may consider the G-representation

$$\mathrm{Ext}^i_{D(G_0,K)}(D(G,K),V) \qquad (*)$$

where the left G-action comes from right multiplication on $D(G,K)$.

Corollary 6.2. *The G-representation $(*)$ lies in $\mathrm{Rep}^a_K(G)$ and there is a natural isomorphism*

$$R^iF^L_{\mathbb{Q}_p}(V)\simeq\mathrm{Ext}^i_{D(G_0,K)}(D(G,K),V)$$

of admissible G-representations.

Proof. Taking cohomology on the complex $\mathrm{Hom}_{D(G_0,K)}(D(G_0,K)\otimes_L\dot{\bigwedge}\mathfrak{g}^0,V)=\mathrm{Hom}_L(\dot{\bigwedge}\mathfrak{g}^0,V)$ yields an isomorphism $\mathrm{Ext}^*_{D(G_0,K)}(D(G,K),V)\simeq H^*(\mathfrak{g}^0,V)$ in Vec_K natural in V. We endow the Ext group with the locally convex topology of compact type of the right-hand side. Using Lemma 2.7 as well as Proposition 5.2 we obtain a natural isomorphism in Vec_K

$$\mathrm{Tor}^{D(G_0,K)}_*(V'_b,D(G,K))'_b\simeq\mathrm{Ext}^*_{D(G_0,K)}(D(G,K),V)$$

which is topological. To check that it is G-equivariant amounts to checking the G-equivariance of the isomorphisms of complexes $(h_*(V' \otimes_L \dot{\bigwedge}\mathfrak{g}^0))' \simeq h^*((V' \otimes_L \dot{\bigwedge}\mathfrak{g}^0)')$ appearing in the proof of Lemma 2.7 and $(V' \otimes_L \dot{\bigwedge}\mathfrak{g}^0)' \simeq \mathrm{Hom}_L(\dot{\bigwedge}\mathfrak{g}^0, V)$ (see the remark before Lemma 2.7). These are direct computations. □

Let G be compact and $K/\mathbb{Q}_p$ be finite. Using the notation of § 3 recall that $F_{\mathbb{Q}_p}$ is exact and preserves injective objects (Theorem 3.1). Hence, we have the following direct application of our results to Banach space representations.

Proposition 6.3. *Let G be compact and $K/\mathbb{Q}_p$ be finite. Then*

$$R^i F_L = R^i F^L_{\mathbb{Q}_p} \circ F_{\mathbb{Q}_p} \quad \text{and} \quad R^i F_L = 0, \quad i > ([L:\mathbb{Q}_p]-1)\dim_L G$$

for the right-derived functors of F_L.

We conclude with two further applications when varying $\mathfrak{h}$.

(1) *The σ-analytic representations.* Assume that $L/\mathbb{Q}_p$ is Galois. Letting $\mathfrak{h} := \mathfrak{g}^0_\sigma$ in Corollary 5.4 and pulling back to representations yields, for each $\sigma \in \mathrm{Gal}(L/\mathbb{Q}_p)$, that the left-exact functor $\mathrm{Rep}^a_K(G_0) \to \mathrm{Rep}^a_K(G_{\sigma^{-1}})$, $V \mapsto V_{\sigma\text{-an}}$ extends to a δ-functor. The higher functors vanish in degrees $> ([L:\mathbb{Q}_p]-1)\dim_L G$.

(2) *Smooth representations.* Let $\mathrm{Rep}^{\infty,a}_K(G) \subseteq \mathrm{Rep}^a_K(G_0)$ denote the full abelian subcategory of smooth-admissible representations [ST03, § 6]. The equivalence $\mathrm{Rep}^a_K(G_0) \simeq \mathcal{C}_{G_0}$ induces an equivalence $\mathrm{Rep}^{\infty,a}_K(G) \simeq \mathcal{C}_\infty$ where $\mathcal{C}_\infty = \mathcal{C}_{G_0} \cap \mathcal{M}(D^\infty(G,K))$ and $D^\infty(G,K) = D(G_0,K)/(\mathfrak{g}_0)$ denotes the algebra of smooth distributions [ST05, § 1]. Since $V^\infty = H^0(\mathfrak{g}_0, V)$ we may put $\mathfrak{h} := L \otimes_{\mathbb{Q}_p} \mathfrak{g}_0$, apply the results of the preceding section and obtain the fact that passage to smooth vectors $\mathrm{Rep}^a_K(G_0) \to \mathrm{Rep}^{\infty,a}_K(G)$, $V \mapsto V^\infty$ extends to a δ-functor vanishing in degrees $> [L:\mathbb{Q}_p]\dim_L G$.

7. Analytic vectors in induced representations

As an application we study the interaction of the functors $R^i F^L_{\mathbb{Q}_p}$ with locally analytic induction. This implies an explicit formula for the higher analytic vectors in principal series representations. As usual G denotes a locally L-analytic group.

We let $P \subseteq G$ be a closed subgroup with Lie algebra $\mathfrak{p}$. Let Ind^G_P denote the locally analytic induction viewed as a functor from admissible P-representations (W,ρ), finite dimensional over K, to admissible G-representations. Explicitly,

$$\mathrm{Ind}^G_P(W) := \{f \in C^{\mathrm{an}}(G,W),\ f(gb) = \rho(b)^{-1} f(g) \text{ for all } g \in G,\ b \in P\}$$

and G acts by left translations. One has the isomorphism of right $D(G,K)$-modules

$$W'_b \otimes_{D(P,K)} D(G,K) \xrightarrow{\sim} (\mathrm{Ind}^G_P W)'_b \tag{8}$$

mapping $\phi \otimes \lambda$ to the functional $f \mapsto \lambda(g \mapsto \phi(f(g)))$ (see [OS06, 2.4]).

Recall that $\mathfrak{g}^0 := \ker(L \otimes_{\mathbb{Q}_p} \mathfrak{g}_0 \to \mathfrak{g})$. We *assume* for the rest of this section that there is a compact open subgroup $G' \subseteq G$ such that an 'Iwasawa decomposition'

$$G = G'P \tag{9}$$

holds.

Lemma 7.1. *Given $X \in \mathcal{M}(D(G_0, K))$ there is a natural isomorphism*

$$\mathrm{Tor}_*^{D(G_0', K)}(X, D(G', K)) \simeq \mathrm{Tor}_*^{D(G_0, K)}(X, D(G, K))$$

as right $D(G', K)$-modules.

Proof. Let $* = 0$. For $X = D(G_0, K)$ the claim follows from the fact that $G' \subseteq G$ is open, thus $\mathfrak{g}^0 \subseteq D(G_0', K)$ with $D(G', K) = D(G_0', K)/(\mathfrak{g}^0)$. The case of arbitrary X follows from this. In general, a projective resolution $P_\bullet \to X$ of X as $D(G_0, K)$-module remains a projective resolution of X as $D(G_0', K)$-module since $D(G_0, K)$ is free over $D(G_0', K)$. The claim then follows from the case $* = 0$. □

Now put $P' := P \cap G'$. Using (9), restriction of functions induces a topological G'-isomorphism

$$\mathrm{Ind}_P^G W \xrightarrow{\sim} \mathrm{Ind}_{P'}^{G'} W \tag{10}$$

where the right-hand side has the obvious meaning [Fea99, Satz 4.1.4].

Lemma 7.2. *We have the following results.*

(i) *For all $Y \in \mathcal{M}(D(P, K))$ the natural map*

$$Y \otimes_{D(P',K)} D(G', K) \to Y \otimes_{D(P,K)} D(G, K)$$

is an isomorphism of $D(G', K)$-modules.

(ii) *We have a commutative diagram*

$$\begin{array}{ccc} W' \otimes_{D(P',K)} D(G', K) & \longrightarrow & (\mathrm{Ind}_{P'}^{G'} W)' \\ \downarrow & & \downarrow \\ W' \otimes_{D(P,K)} D(G, K) & \longrightarrow & (\mathrm{Ind}_P^G W)' \end{array}$$

of right $D(G', K)$-modules in which all four maps are isomorphisms.

Proof. This is a straightforward generalization of [ST05, Lemma 6.1] using (9). □

Recall from the last section that $F_{\mathbb{Q}_p}^L$ extends to a δ-functor $(R^i F_{\mathbb{Q}_p}^L)_{i \geq 0}$ (Theorem 6.1). Assume that we are given a finite-dimensional locally $\mathbb{Q}_p$-analytic P-representation W. We abbreviate as follows

$$V := \mathrm{Ind}_{P_0}^{G_0} W \in \mathrm{Rep}_K^a(G_0)$$

and study the admissible G-representations $R^i F_{\mathbb{Q}_p}^L(V)$. Let $Q_\bullet = D(P_0, K) \otimes_L \dot{\bigwedge} \mathfrak{p}^0$ and $\tilde{Q}_\bullet = D(G_0, K) \otimes_L \dot{\bigwedge} \mathfrak{g}^0$ denote the standard resolutions for the bimodules $D(P, K)$ and $D(G, K)$, respectively, as referred to in Lemma 5.1. We have the natural morphism of complexes of $D(P_0, K)$-bimodules $Q_\bullet \to \tilde{Q}_\bullet$ induced by $D(P_0, K) \to D(G_0, K)$ and $\mathfrak{p}^0 \to \mathfrak{g}^0$. Tensoring with the right $D(P_0, K)$-equivariant map $W' \to V'$, $w \mapsto w \otimes 1$ arising from (8) gives rise to a morphism of complexes of right $D(P_0, K)$-modules

$$W' \otimes_{D(P_0,K)} Q_\bullet \to V' \otimes_{D(G_0,K)} \tilde{Q}_\bullet.$$

Taking homology and extending scalars yields a map

$$f : \mathrm{Tor}_*^{D(P_0)}(W', D(P)) \otimes_{D(P)} D(G) \to \mathrm{Tor}_*^{D(G_0)}(V', D(G))$$

of right $D(G)$-modules where we have abbreviated $D(G) := D(G, K)$, $D(P) := D(P, K)$, etc.

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PROPOSITION 7.3. *The map f is an isomorphism.*

Proof. First note that the right-hand side is *a priori* coadmissible by Proposition 5.2. We now have bijective maps of right $D(G')$-modules

$$\mathrm{Tor}_*^{D(P_0)}(W', D(P)) \otimes_{D(P)} D(G) \xrightarrow{\sim} \mathrm{Tor}_*^{D(P_0)}(W', D(P)) \otimes_{D(P')} D(G')$$

(Lemma 7.2(i) applied to $Y := \mathrm{Tor}_*^{D(P_0)}(W', D(P))$) and

$$\mathrm{Tor}_*^{D(P_0)}(W', D(P)) \otimes_{D(P')} D(G') \xrightarrow{\sim} \mathrm{Tor}_*^{D(P_0')}(W', D(P')) \otimes_{D(P')} D(G')$$

(Lemma 7.1 applied to $P' \subseteq P$ and $W' \in \mathcal{M}(D(P_0, K))$. Their composite fits into the diagram of right $D(G')$-modules

$$\begin{array}{ccc} \mathrm{Tor}_*^{D(P_0)}(W', D(P)) \otimes_{D(P)} D(G) & \xrightarrow{f} & \mathrm{Tor}_*^{D(G_0)}(V', D(G)) \\ \downarrow & & \downarrow \\ \mathrm{Tor}_*^{D(P_0')}(W', D(P')) \otimes_{D(P')} D(G') & \longrightarrow & \mathrm{Tor}_*^{D(G_0')}(V', D(G')) \end{array}$$

where the right-hand vertical arrow is due to Lemma 7.1 and bijective. The lower horizontal arrow is defined analogously to the upper horizontal arrow using Lemma 7.2(ii). Tracing through the definitions of the maps involved this diagram commutes. We may thus assume that G is compact. Then both sides of our map are coadmissible: $\mathrm{Tor}_*^{D(P_0)}(W', D(P))$ is finite dimensional over K (Proposition 5.2), hence is a finitely presented $D(P)$-module. We introduce another map of right $D(G)$-modules

$$f' : \mathrm{Tor}_*^{D(P_0)}(W', D(P)) \otimes_{D(P)} D(G) \to \mathrm{Tor}_*^{D(G_0)}(V', D(G)) \tag{11}$$

as follows. Choose a projective resolution $P_\bullet \to W'$ by right $D(P_0)$-modules according to Lemma 7.4 below. Then $\tilde{P}_\bullet := P_\bullet \otimes_{D(P_0)} D(G_0)$ is a projective resolution for $W' \otimes_{D(P_0)} D(G_0) = V'$, thus the natural map

$$P_\bullet \otimes_{D(P_0)} D(P) \to \tilde{P}_\bullet \otimes_{D(G_0)} D(G)$$

induced by $m \otimes \lambda \mapsto m \otimes 1 \otimes \lambda$ for $m \in P_n$, $m \otimes 1 \in \tilde{P}_n$, $\lambda \in D(P) \subseteq D(G)$ gives our map f'. We claim that it is bijective: by coadmissibility this may be tested on coherent sheafs. Let us realize $D(G_0)$ and $D(G)$ as Fréchet–Stein algebras via the families of norms appearing in Proposition 2.6. In particular, $D_r(P) \to D_r(G)$ is flat for all r. Denote by W_r' and $V_r' = W_r' \otimes_{D_r(P_0)} D_r(G_0)$ the coherent sheafs associated with W' and V', respectively. Then the coherent sheafs associated to both sides of (11) are given by $\mathrm{Tor}_*^{D_r(P_0)}(W_r', D_r(P)) \otimes_{D_r(P)} D_r(G)$ and $\mathrm{Tor}_*^{D_r(G_0)}(V_r', D_r(G))$, respectively, according to Corollary 5.3. Put $P_{r\bullet} := P_\bullet \otimes_{D(P_0)} D_r(P_0)$ and $\tilde{P}_{r\bullet} := \tilde{P}_\bullet \otimes_{D(G_0)} D_r(G_0)$. By [ST03, Remark 3.2], $P_{r\bullet} \to W_r'$ and $\tilde{P}_{r\bullet} \to V_r'$ are projective resolutions of W_r' and V_r', respectively, and the map

$$f' \otimes_{D(G)} D_r(G) : \mathrm{Tor}_*^{D_r(P_0)}(W_r', D_r(P)) \otimes_{D_r(P)} D_r(G) \to \mathrm{Tor}_*^{D_r(G_0)}(V_r', D_r(G))$$

coincides with that induced by

$$P_{r\bullet} \otimes_{D_r(P_0)} D_r(P) \to \tilde{P}_{r\bullet} \otimes_{D_r(G_0)} D_r(G).$$

Since $D_r(P) \to D_r(G)$ is flat, $f' \otimes_{D(G)} D_r(G)$ is bijective and since this holds for all r the map f' is bijective. Using a standard double complex argument now shows that $f = f'$ and, hence, the proposition is proved. □

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The following lemma was used in the preceding proof.

Lemma 7.4. *Assume that G is compact. There is a projective resolution $P_\bullet \to W'$ by right $D(P_0)$-modules such that $P_\bullet \otimes_{D(P_0)} D(G_0)$ is acyclic.*

Proof. By density of analytic vectors (Theorem 3.1), W' being finite dimensional over K implies that $W' \otimes_{D^c(P_0)} D(P_0) = W'$. Now choose a finite free resolution $P'_\bullet$ of the $D^c(P_0)$-module W'. By the flatness result (5) $P_\bullet := P'_\bullet \otimes_{D^c(P_0)} D(P_0)$ is a finite free resolution of the $D(P_0)$-module W' and it remains to prove the last statement. Now every kernel $K_n \subseteq P_n$ of the differential in $P_\bullet$ is a finitely presented $D(P_0)$-module, thus the morphism $K_n \otimes_{D(P_0)} D(G_0) \to P_n \otimes_{D(P_0)} D(G_0)$ lies in C_{G_0}. Its injectivity follows therefore on coherent sheafs from flatness of $D_r(P_0) \to D_r(G_0)$ and left-exactness of the projective limit using the Fréchet–Stein structure of Proposition 2.6. □

Theorem 7.5. *The functors $R^i F^L_{\mathbb{Q}_p}$ commute with induction: given a finite-dimensional locally $\mathbb{Q}_p$-analytic P-representation W one has an isomorphism*

$$R^i F^L_{\mathbb{Q}_p} \circ \mathrm{Ind}^{G_0}_{P_0}(W) \simeq \mathrm{Ind}^G_P \circ R^i F^L_{\mathbb{Q}_p}(W)$$

as admissible G-representations functorial in W.

Proof. This follows from dualizing the isomorphism in the preceding proposition which is, by construction, functorial in W. □

The functor Ind^G_P is nonzero on objects [Fea99, Satz 4.3.1], thus we have the following result.

Corollary 7.6. *We have $R^i F^L_{\mathbb{Q}_p}(\mathrm{Ind}^{G_0}_{P_0} W) \neq 0$ if and only if $R^i F^L_{\mathbb{Q}_p}(W) \neq 0$. In particular, $R^i F^L_{\mathbb{Q}_p}(\mathrm{Ind}^{G_0}_{P_0} W) = 0$ for all $i > ([L:\mathbb{Q}_p]-1)\dim_L \mathfrak{p}$.*

The above results apply, in particular, when G equals the L-points of a connected reductive group over L and $P \subseteq G$ is a parabolic subgroup. If W is a one-dimensional P-representation K_χ given by a locally $\mathbb{Q}_p$-analytic character $\chi : P \to K^\times$ we may determine the vector space $R^i F^L_{\mathbb{Q}_p}(K_\chi)$ completely. For simplicity we assume that G is quasi-split and let $P := B$ be a Borel subgroup with Lie algebra $\mathfrak{b}$. Then $\mathfrak{b} = \mathfrak{t}\mathfrak{u}$ (semidirect product) where $\mathfrak{t}$ is a maximal toral subalgebra and $\mathfrak{u} = [\mathfrak{b}, \mathfrak{b}]$. Let $\mathfrak{b}^0$ be the kernel of $K \otimes_{\mathbb{Q}_p} \mathfrak{b}_0 \to K \otimes_L \mathfrak{b}$ and define $\mathfrak{t}^0$ and $\mathfrak{u}^0$ analogously. Then K_χ is a $\mathfrak{b}^0$-module via $K \otimes_{\mathbb{Q}_p} \mathrm{d}\chi$ where $\mathrm{d}\chi : \mathfrak{b}_0 \to K$ denotes the differential of χ. We denote this module as well as the induced (note that $[\mathfrak{b}^0, \mathfrak{b}^0] = \mathfrak{u}^0$) $\mathfrak{t}^0$-module by $K_{\mathrm{d}\chi}$.

Corollary 7.7. *There is an isomorphism in Vec_K*

$$R^i F^L_{\mathbb{Q}_p}(K_\chi) \simeq \sum_{j+k=i} \bigwedge^j \mathfrak{t}^0 \otimes_K H^k(\mathfrak{u}^0, K_{\mathrm{d}\chi})^{\mathfrak{t}^0}.$$

Proof. By Proposition 5.2 we have that $R^i F^L_{\mathbb{Q}_p}(K_\chi) = H^i(\mathfrak{b}^0, K_{\mathrm{d}\chi})$ in Vec_K. The algebras $\mathfrak{b}^0$, $\mathfrak{t}^0$ and $\mathfrak{u}^0$ are direct products of scalar extensions of $\mathfrak{b}$, $\mathfrak{t}$ and $\mathfrak{u}$, respectively. In particular, $\mathfrak{t}^0 \subseteq \mathfrak{b}^0$ is a reductive subalgebra whence [HS53, Theorem 12] implies that there is an isomorphism

$$H^i(\mathfrak{b}^0, K_{\mathrm{d}\chi}) \simeq \sum_{j+k=i} H^j(\mathfrak{t}^0, K) \otimes_K H^k(\mathfrak{b}^0, \mathfrak{t}^0, K_{\mathrm{d}\chi})$$

in Vec_K where $H^*(\mathfrak{b}^0, \mathfrak{t}^0, K_{\mathrm{d}\chi})$ is the relative Lie algebra cohomology with respect to $\mathfrak{t}^0$. Now $\mathfrak{t}^0$ is even toral whence the argument preceding [HS53, Theorem 13] implies that $H^*(\mathfrak{b}^0, \mathfrak{t}^0, K_{\mathrm{d}\chi}) \simeq$

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$H^*(\mathfrak{u}^0, K_{d\chi})^{\mathfrak{t}^0}$. Finally, since $\mathfrak{t}^0$ is abelian, the differential in $\mathrm{Hom}_K(\dot{\bigwedge}\mathfrak{t}^0, K)$ vanishes identically and so one obtains $H^j(\mathfrak{t}^0, K) = \mathrm{Hom}_K(\bigwedge^j \mathfrak{t}^0, K)$. □

Remarks. One has $H^k(\mathfrak{u}^0, K_{d\chi}) = H^k(\mathfrak{u}^0, K)$ in Vec_K and since $\mathfrak{u}^0$ is nilpotent [Dix95, Theorem 2] implies that $\dim_K H^k(\mathfrak{u}^0, K) \geq 1$ for $0 \leq k \leq \dim_K \mathfrak{u}^0$.

Corollary 7.8. *If χ is smooth one has $R^1 F^L_{\mathbb{Q}_p}(\mathrm{Ind}^{G_0}_{P_0} K_\chi) \neq 0$.*

Proof. We have $d\chi = 0$, thus $H^0(\mathfrak{u}^0, K_{d\chi})^{\mathfrak{t}^0} = K$. By the corollary there is an injection $\bigwedge^1 \mathfrak{t}^0 \to R^1 F^L_{\mathbb{Q}_p}(K_\chi)$, thus the claim follows from Corollary 7.6. □

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