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MULTINOMIAL VANDERMONDE CONVOLUTION VIA PERMANENT

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Abstract

We provide a generalised Laplace expansion for the permanent function and, as a consequence, we re-prove a multinomial Vandermonde convolution. Some combinatorial identities are derived by applying special matrices to the expansion.

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1. Introduction

The Vandermonde convolution is a combinatorial identity of the form

$$\binom{m+n}{r} = \sum_{k=0}^{\min\{r,m\}} \binom{m}{k}\binom{n}{r-k},$$

for any nonnegative integers r, m, n. There are many ways to prove this identity, including a proof by a combinatorial double counting principle, a geometrical proof and an algebraic proof. The identity can be extended in numerous ways. The q-binomial Vandermonde convolution form was introduced by Bender in [1], with both a partition proof and a geometric proof. Sulanke [3] extended the result of Bender to a q-multinomial Vandermonde convolution and offered a graph-theoretical proof. Zeng [4] studied multinomial convolution for a family of polynomials (including the form that appears in Corollary 3.4) and showed that multi-convolution polynomials arise as coefficients of power series in several variables.

In this article, we re-prove a multinomial Vandermonde convolution using the language of multilinear algebra. In fact, we provide a generalised Laplace expansion for the permanent function. The convolution is an immediate consequence of that expansion. An identity for the elementary symmetric functions and a relation for

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derangement numbers are given as examples of applying special matrices to the expansion.

2. Preliminaries

Let V, W be vector spaces over $\mathbb{C}$, and let $m \in \mathbb{N}$. Recall that an m-multilinear map $\varphi : \times^m V := V \times \cdots \times V \longrightarrow W$ is a map satisfying

$$\varphi(v_1, \ldots, av_i + bv'_i, \ldots, v_m) = a\varphi(v_1, \ldots, v_i, \ldots, v_m) + b\varphi(v_1, \ldots, v'_i, \ldots, v_m),$$

for $i = 1, \ldots, m$, $v_i, v'_i \in V$ and $a, b \in \mathbb{C}$. By the unique factorisation property of the m-fold tensor space $\otimes^m V$, any m-multilinear map φ will factor through $\otimes^m V$; that is, there exists a unique linear map $T : \otimes^m V \longrightarrow W$ such that

$$\varphi(v_1, \ldots, v_m) = T(v_1 \otimes \cdots \otimes v_m) \quad \text{for any } v_1, \ldots, v_m \in V.$$

An m-multilinear map $\psi : \times^m V \longrightarrow W$ is said to be *completely symmetric* if

$$\psi(v_{\sigma(1)}, \ldots, v_{\sigma(m)}) = \psi(v_1, \ldots, v_m),$$

for all $\sigma \in S_m$ (the permutation group of degree m) and $v_1, \ldots, v_m \in V$. For example, the multilinear map $\chi_\otimes : \times^m V \longrightarrow W$ defined, for each $v_1, \ldots, v_m \in V$, by

$$\chi_\otimes(v_1, \ldots, v_m) = \frac{1}{m!} \sum_{\sigma \in S_m} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)},$$

is completely symmetric. In particular, when $W = \otimes^m V$, by the unique factorisation property of $\otimes^m V$, there is a unique linear map $T(S_m, 1) : \otimes^m V \longrightarrow \otimes^m V$ such that $\chi_\otimes(v_1, \ldots, v_m) = T(S_m, 1)(v_1 \otimes \cdots \otimes v_m)$ for any $v_1, \ldots, v_m \in V$. This is a symmetriser and hence an orthogonal projection (with respect to the induced inner product defined below) on $\otimes^m V$ (see [2, Theorem 6.3]). The image of $\otimes^m V$ under the map $T(S_m, 1)$ is called the *m-fold completely symmetric space*, denoted by

$$T(S_m, 1)(\otimes^m V) := \vee^m V,$$

and its elements are linear combinations of vectors of the form

$$v_1 \vee \cdots \vee v_m := T(S_m, 1)(v_1 \otimes \cdots \otimes v_m) = \frac{1}{m!} \sum_{\sigma \in S_m} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}.$$

Of course,

$$v_{\sigma(1)} \vee \cdots \vee v_{\sigma(m)} = v_1 \vee \cdots \vee v_m \tag{2.1}$$

for any $v_1, \ldots, v_m \in V$ and any $\sigma \in S_m$. Again, the space $\vee^m V$ is equipped with the following unique factorisation property.

Proposition 2.1 [2, Theorem 6.14]. *Let V and W be vector spaces over $\mathbb{C}$, and let $\phi : \times^m V \longrightarrow W$ be a completely symmetric multilinear map. Then, there is a unique*

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linear transformation $T_1 : \vee^m V \longrightarrow W$ *such that* $\phi(v_1, \ldots, v_m) = T_1(v_1 \vee \cdots \vee v_m)$, *for any* $v_1, \ldots, v_m \in V$.

Let V be an inner product space with inner product $(\,,)$. Let $E = \{e_1, \ldots, e_n\}$ be an orthonormal ordered basis for V. It is well known that

$$E^{\otimes} := \{e^{\otimes}_{\alpha} := e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(m)} \mid \alpha \in \Gamma_{m,n}\}$$

is an orthonormal ordered (lexicographic order) basis for $\otimes^m V$ under the induced inner product on $\otimes^m V$, defined by

$$\langle v_1 \otimes \cdots \otimes v_m, u_1 \otimes \cdots \otimes u_m \rangle = \prod_{i=1}^{m} (v_i, u_i) \quad \text{for all } v_i, u_i \in V,$$

where

$$\Gamma_{m,n} = \{\alpha := (\alpha(1), \ldots, \alpha(m)) \in \mathbb{N}^m \mid 1 \leq \alpha(i) \leq n, \ \text{for } i = 1, \ldots, m\}.$$

Then, $\vee^m V$ is spanned by

$$E^{\vee} = \{T(S_m, 1)(e^{\otimes}_{\alpha}) := e^{\vee}_{\alpha} := e_{\alpha(1)} \vee \cdots \vee e_{\alpha(m)} \mid \alpha \in \Gamma_{m,n}\}.$$

Since $T(S_m, 1)^2 = T(S_m, 1) = T(S_m, 1)^*$, for $\alpha, \beta \in \Gamma_{m,n}$,

$$\begin{aligned}\langle e^{\vee}_{\alpha}, e^{\vee}_{\beta} \rangle &= \langle T(S_m, 1)(e^{\otimes}_{\alpha}), T(S_m, 1)(e^{\otimes}_{\beta}) \rangle = \langle T(S_m, 1)(e^{\otimes}_{\alpha}), e^{\otimes}_{\beta} \rangle \\ &= \Big\langle \frac{1}{m!} \sum_{\sigma \in S_m} e_{\alpha(\sigma(1))} \otimes \cdots \otimes e_{\alpha(\sigma(m))}, e_{\beta(1)} \otimes \cdots \otimes e_{\beta(m)} \Big\rangle \\ &= \frac{1}{m!} \sum_{\sigma \in S_m} \prod_{i=1}^{m} (e_{\alpha(\sigma(i))}, e_{\beta(i)}).\end{aligned}$$

That is,

$$\langle e^{\vee}_{\alpha}, e^{\vee}_{\beta} \rangle = \frac{1}{m!} \sum_{\sigma \in S_m} \delta_{\alpha\sigma, \beta} \quad \text{for any } \alpha, \beta \in \Gamma_{m,n}. \tag{2.2}$$

Define the right action of S_m on $\Gamma_{m,n}$ by $\alpha \cdot \sigma := (\alpha(\sigma^{-1}(1)), \ldots, \alpha(\sigma^{-1}(m)))$ for each $\sigma \in S_m$ and $\alpha \in \Gamma_{m,n}$. Then, the orbit of α is

$$\Gamma_{\alpha} := \{\alpha \cdot \sigma \mid \sigma \in S_m\} = \{\alpha \cdot \sigma^{-1} \mid \sigma \in S_m\} = \{\alpha\sigma \mid \sigma \in S_m\},$$

and the stabiliser of α is

$$G_{\alpha} := \{\sigma \in S_m \mid \alpha \cdot \sigma = \alpha\} = \{\sigma \in S_m \mid \alpha \cdot \sigma^{-1} = \alpha\} = \{\sigma \in S_m \mid \alpha\sigma = \alpha\}.$$

Let Δ be the set comprising the first element (ordered by lexicographic order) in each orbit Γ_{α} of $\Gamma_{m,n}$. For each $\alpha \in \Delta$, S_m can be partitioned as $S_m = \sqcup_{i=1}^{s} G_{\alpha}\tau_i$, where $S = \{\tau_1, \ldots, \tau_s\}$ is the set of representatives of the right cosets of G_{α} in S_m. So,

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$\Gamma_\alpha = \{\alpha\tau_1, \ldots, \alpha\tau_s\}$ and $|\{\sigma \in S_m \mid \alpha\sigma = \alpha\tau_j\}| = |G_\alpha|$ for each $j = 1, \ldots, s$. Hence,

$$\sum_{\gamma\in\Gamma_{m,n}} f(\gamma) = \sum_{\alpha\in\Delta}\sum_{\gamma\in\Gamma_\alpha} f(\gamma) = \sum_{\alpha\in\Delta}\frac{1}{|G_\alpha|}\sum_{\sigma\in S_m} f(\alpha\sigma), \tag{2.3}$$

for any function $f : \Gamma_{m,n} \longrightarrow W$.

By (2.2), if $\beta \notin \Gamma_\alpha$, then $\langle e^\vee_\alpha, e^\vee_\beta\rangle = 0$. Also,

$$\| e^\vee_\alpha \|^2 = \frac{1}{m!}\sum_{\sigma\in S_m}\delta_{\alpha\sigma,\alpha} = \frac{|G_\alpha|}{m!}.$$

Then, $e^\vee_\alpha = 0$ if and only if $|G_\alpha| = 0$. Since G_α contains at least one element (the identity element), $e^\vee_\alpha \neq 0$ for all $\alpha \in \Gamma_{m,n}$. Let $\vee^m_\alpha(V) := \mathrm{Span}\{e^\vee_{\alpha\sigma} \mid \sigma \in G\}$ be the orbital subspace of $\vee^m V$ associated with $\alpha \in \Delta$. Then $\vee^m V = \oplus^\perp_{\alpha\in\Delta} V^m_\alpha(V)$ (orthogonal direct sum). By Freese's theorem [2, Theorem 6.34], $\dim(\vee^m_\alpha(V)) = 1$ for each $\alpha \in \Delta$. Note that

$$\Delta = G_{m,n} := \{\alpha := (\alpha(1), \ldots, \alpha(m)) \in \Gamma_{m,n} \mid \alpha(1) \leq \cdots \leq \alpha(m)\}. \tag{2.4}$$

Thus, $\bar{E}^\vee = \{e^\vee_\alpha \mid \alpha \in G_{m,n}\}$ is an orthogonal ordered basis for $\vee^m V$.

Furthermore, for each $\alpha \in G_{m,n}$, if $\alpha = (l_1, \ldots, l_1, l_2, \ldots, l_2, \ldots, l_k \ldots, l_k)$, where the multiplicity of l_i is m_i for each $i = 1, \ldots, k$, we write α as $\alpha := (l_1^{m_1}, l_2^{m_2}, \ldots, l_k^{m_k})$, where $1 \leq l_1 < \cdots < l_k \leq n$ and $m_1 + \cdots + m_k = m$. Note also that each element of G_α can only permute an entry of α among l_i entries. Thus, $G_\alpha \cong S_{m_1} \times \cdots \times S_{m_k}$ and hence

$$|G_\alpha| = m_1! \cdots m_k! := \nu(\alpha). \tag{2.5}$$

For any matrix $A = (a_{ij}) \in M_n(\mathbb{C})$ and $\alpha, \beta \in \Gamma_{m,n}$, denote by $A[\alpha|\beta]$ the matrix of size $m \times m$ constructed from A using rows and columns of A indexed by α and β, respectively. This matrix need not be a submatrix of A unless $\alpha, \beta \in Q_{m,n}$, where

$$Q_{m,n} := \{\alpha \in \Gamma_{m,n} \mid \alpha(1) < \cdots < \alpha(m)\} \subset G_{m,n} \subset \Gamma_{m,n}.$$

The permanent of A is defined as $\mathrm{per}(A) := \sum_{\sigma\in S_n}\prod_{i=1}^n A_{i\sigma(i)}$ and thus

$$\mathrm{per}(A[\alpha|\beta]) = \sum_{\sigma\in S_m}\prod_{i=1}^m A_{\alpha(i)\beta(\sigma(i))},$$

for any $\alpha, \beta \in \Gamma_{m,n}$.

3. Main results

Let V be an inner product space over $\mathbb{C}$ equipped with an orthonormal basis $E := \{e_1, \ldots, e_n\}$. Let $1 \leq m \leq n$, and let $u_1, \ldots, u_{n-m} \in V$ be arbitrary fixed elements. Consider a multilinear map $\Psi : \times^m V \longrightarrow \vee^n V$ defined by

$$\Psi(v_1, \ldots, v_m) := v_1 \vee \cdots \vee v_m \vee u_1 \vee \cdots \vee u_{n-m} \quad \text{for } v_1, \ldots, v_m \in V.$$

By (2.1), it turns out that $\Psi(v_{\sigma(1)}, \ldots, v_{\sigma(m)}) = \Psi(v_1, \ldots, v_m)$ for any $v_1, \ldots, v_m \in V$. Thus, Ψ is a completely symmetric multilinear map. By Proposition 2.1, there is a

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unique linear transformation $T_1 : \vee^m V \longrightarrow \vee^n V$ satisfying

$$T_1(v_1 \vee \cdots \vee v_m) = v_1 \vee \cdots \vee v_m \vee u_1 \vee \cdots \vee u_{n-m},$$

for each $v_1 \vee \cdots \vee v_m \in \vee^m V$.

Let $A = (a_{ij}) \in M_n(\mathbb{C})$ and $\alpha \in \Gamma_{m,n}$. Define

$$\alpha_m := (\alpha(1), \ldots, \alpha(m)) \in \Gamma_{m,n} \quad \text{and} \quad \alpha_m^c := (\alpha(m+1), \ldots, \alpha(n)) \in \Gamma_{n-m,n}, \tag{3.1}$$

and further define

$$v_i := \sum_{j=1}^{n} a_{\alpha_m(i)j} e_j \ (1 \le i \le m) \quad \text{and} \quad u_k := \sum_{j=1}^{n} a_{\alpha_m^c(k)j} e_j \ (1 \le k \le n-m).$$

Denote $v_1 \vee \cdots \vee v_m \vee u_1 \vee \cdots \vee u_{n-m} := v^\vee \vee u^\vee$. We now calculate:

$$\begin{aligned}
v^\vee \vee u^\vee &= \Big(\sum_{j=1}^{n} a_{\alpha_m(1)j} e_j\Big) \vee \cdots \vee \Big(\sum_{j=1}^{n} a_{\alpha_m(m)j} e_j\Big) \vee \Big(\sum_{j=1}^{n} a_{\alpha_m^c(1)j} e_j\Big) \vee \cdots \\
&\qquad \vee \Big(\sum_{j=1}^{n} a_{\alpha_m^c(n-m)j} e_j\Big) \\
&= \sum_{\gamma \in \Gamma_{n,n}} \prod_{i=1}^{n} a_{\alpha(i)\gamma(i)} e_\gamma^\vee \quad \text{(by the multilinear property of } \vee) \\
&= \sum_{\beta \in G_{n,n}} \frac{1}{\nu(\beta)} \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{\alpha(i)\beta\sigma(i)} e_{\beta\sigma}^\vee \quad \text{(by (2.3), (2.4) and (2.5))} \\
&= \sum_{\beta \in G_{n,n}} \frac{1}{\nu(\beta)} \Big(\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{\alpha(i)\beta\sigma(i)}\Big) e_\beta^\vee \quad \text{(by (2.1))}.
\end{aligned}$$

By the definition of the permanent function on the matrix $A[\alpha|\beta]$, we conclude that

$$v^\vee \vee u^\vee = \sum_{\beta \in G_{n,n}} \frac{1}{\nu(\beta)} \operatorname{per}(A[\alpha|\beta]) e_\beta^\vee. \tag{3.2}$$

On the other hand, by an analogous calculation to the one given above,

$$v^\vee = v_1 \vee \cdots \vee v_m = \sum_{\theta \in G_{m,n}} \frac{1}{\nu(\theta)} \operatorname{per}(A[\alpha_m|\theta]) e_\theta^\vee.$$

Applying T_1 to both sides,

$$\begin{aligned}
v^\vee \vee u^\vee &= \sum_{\theta \in G_{m,n}} \frac{1}{\nu(\theta)} \operatorname{per}(A[\alpha_m|\theta]) T_1(e_\theta^\vee) \\
&= \sum_{\theta \in G_{m,n}} \frac{1}{\nu(\theta)} \operatorname{per}(A[\alpha_m|\theta]) (e_\theta^\vee \vee u_1 \vee \cdots \vee u_{n-m})
\end{aligned}$$

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$$= \sum_{\theta \in G_{m,n}} \frac{1}{\nu(\theta)} \operatorname{per}(A[\alpha_m|\theta])\Big(e_\theta^\vee \vee \sum_{\gamma \in G_{n-m,n}} \frac{1}{\nu(\gamma)} \operatorname{per}(A[\alpha_m^c|\gamma])e_\gamma^\vee\Big).$$

Let $\beta = (\beta(1), \ldots, \beta(n)) \in G_{n,n}$, $\theta \in G_{m,n}$ and $\gamma \in G_{n-m,n}$. By (2.1), any permutation on the subscripts of $e_\theta^\vee \vee e_\gamma^\vee = e_{\theta(1)} \vee \cdots \vee e_{\theta(m)} \vee e_{\gamma(1)} \vee \cdots \vee e_{\gamma(n-m)}$ does not give a new element in $\vee^n V$. Then, $e_\beta^\vee = e_\theta^\vee \vee e_\gamma^\vee$ if and only if $I_\beta = I_\theta \cup I_\gamma$ as multisets, where $I_\beta := \{\beta(1), \ldots, \beta(n)\}$, $I_\theta := \{\theta(1), \ldots, \theta(m)\}$ and $I_\gamma := \{\gamma(1), \ldots, \gamma(n-m)\}$. Define

$$G(\beta) := \{(\theta, \gamma) \in G_{m,n} \times G_{(n-m,n)} \mid I_\theta \cup I_\gamma = I_\beta \text{ as multisets}\}. \tag{3.3}$$

Then,

$$v^\vee \vee u^\vee = \sum_{\beta \in G_{n,n}} \Big(\sum_{(\theta,\gamma) \in G(\beta)} \frac{1}{\nu(\theta)\nu(\gamma)} \operatorname{per}(A[\alpha_m|\theta]) \operatorname{per}(A[\alpha_m^c|\gamma]) \Big) e_\beta^\vee. \tag{3.4}$$

Since $\bar{E}^\vee = \{e_\beta^\vee \mid \beta \in G_{n,n}\}$ is a basis for $\vee^n V$, comparing (3.2) and (3.4), yields a generalised Laplace expansion for the permanent function.

THEOREM 3.1. *Let $A \in M_n(\mathbb{C})$ and $m \in \mathbb{N}$ such that $m \leq n$. Then, for each $\alpha, \beta \in G_{n,n}$,*

$$\operatorname{per}(A[\alpha|\beta]) = \sum_{(\theta,\gamma) \in G(\beta)} \frac{\nu(\beta)}{\nu(\theta)\nu(\gamma)} \operatorname{per}(A[\alpha_m|\theta]) \operatorname{per}(A[\alpha_m^c|\gamma]),$$

where $G(\beta)$ is defined as in (3.3) and α_m, α_m^c are defined as in (3.1).

In fact, this theorem also holds when $G_{n,n} \subsetneq \Gamma_{n,n}$ is replaced by $\Gamma_{n,n}$, because per is invariant under permuting rows or columns.

In particular, if $\beta = (1, \ldots, n)$, then θ and γ must have union $\{1, \ldots, n\}$. Only one of θ, γ can be chosen freely, say $\theta \in G_{m,n}$. But then $\theta \in Q_{m,n}$ (because the entries of θ are parts of β with no multiplicities) and thus $\gamma = \theta^c$. That is,

$$G(\beta) = \{(\theta, \theta^c) \mid \theta \in Q_{m,n}\}.$$

Moreover, $\nu(\beta) = \nu(\theta) = \nu(\theta^c) = 1$ and if $\alpha_m \in Q_{m,n}$, then $A[\alpha_m^c|\theta^c] = A(\alpha_m|\theta)$ is a submatrix of A obtained by deleting rows and columns indexed by α_m and θ, respectively. Thus, by Theorem 3.1, we obtain the standard form of the Laplace expansion for the permanent function.

COROLLARY 3.2. *Let $A \in M_n(\mathbb{C})$ and $\rho \in Q_{m,n}$, where $1 \leq m \leq n$. Then,*

$$\operatorname{per} A = \sum_{\theta \in Q_{m,n}} \operatorname{per}(A[\rho|\theta]) \operatorname{per}(A(\rho|\theta)).$$

As another point of view, we consider each $\beta \in G_{n,n}$ in the form

$$\beta = (l_1^{n_1}, \ldots, l_k^{n_k}) \in G_{n,n},$$

where $l_1 < \cdots < l_k$, $n_i \in \mathbb{N}$, for $1 \leq i \leq k$ and $n_1 + \cdots + n_k = n$. Then, each $\theta \in G(\beta)$ must be in the form $\beta_{\bar{m}} = (l_1^{m_1}, \ldots, l_k^{m_k})$, where $0 \leq m_i \leq \min\{n_i, m\} \in \mathbb{Z}$ for $1 \leq i \leq k$ and $m_1 + \cdots + m_k = m$. The corresponding $\gamma \in G_{n-m,m}$ (since $(\theta, \gamma) \in G(\beta)$) must have

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the form $\gamma = (l_1^{n_1-m_1}, \dots, l_k^{n_k-m_k}) := \beta_{\bar{m}^c}$. Thus,

$$\bar{G}(\beta) := \Big\{\bar{m} := (m_1, \dots, m_k) \in \mathbb{Z}^k \mid 0 \le m_i \le \min\{n_i, m\} \text{ for } i = 1, \dots, k \text{ and } \sum_{i=1}^{k} m_i = m\Big\} \tag{3.5}$$

is in one-to-one correspondence with $G(\beta)$. Now, $\nu(\beta) = n_1! \cdots n_k!$, $\nu(\theta) = m_1! \cdots m_k!$ and $\nu(\gamma) = (n_1 - m_1)! \cdots (n_k - m_k)!$. This information leads to the following alternative version of Theorem 3.1.

THEOREM 3.3. *Let $A \in M_n(\mathbb{C})$ and $\alpha \in G_{n,n}$. Then, for each $\beta = (l_1^{n_1}, \dots, l_k^{n_k}) \in G_{n,n}$ and $1 \le m \le n$,*

$$\frac{\mathrm{per}(A[\alpha|\beta])}{\prod_{j=1}^{k} n_j!} = \sum_{\bar{m}\in\bar{G}(\beta)} \bigg(\frac{\mathrm{per}(A[\alpha_m|\beta_{\bar{m}}])}{\prod_{j=1}^{k} m_j!}\bigg)\bigg(\frac{\mathrm{per}(A[\alpha_m^c|\beta_{\bar{m}^c}])}{\prod_{j=1}^{k}(n_j - m_j)!}\bigg),$$

where $\beta_{\bar{m}} := (l_1^{m_1}, \dots, l_k^{m_k})$, $\beta_{\bar{m}^c} := (l_1^{n_1-m_1}, \dots, l_k^{n_k-m_k})$ and $\bar{G}(\beta)$ is as in (3.5).

In particular, if $A := J_n \in M_n(\mathbb{C})$ is the $n \times n$ matrix all of whose entries are 1, then $\mathrm{per}(A[\alpha|\beta]) = n!$, $\mathrm{per}(A[\alpha_m|\beta_{\bar{m}}]) = m!$ and $\mathrm{per}(A[\alpha_m^c|\beta_{\bar{m}^c}]) = (n - m)!$, for any $\bar{m} \in \bar{G}(\beta)$. Using the notation

$$\binom{n}{n_1, \dots, n_k} := \frac{n!}{n_1! \cdots n_k!}$$

and Theorem 3.3, we reach a multinomial Vandermonde convolution.

COROLLARY 3.4. *Let $m, n, n_1, \dots, n_k \in \mathbb{N}$ be such that $m \le n$ and $n_1 + \cdots + n_k = n$. Then*

$$\binom{n}{n_1, \dots, n_k} = \sum_{\substack{0\le m_i\le\min\{n_i,m\}\in\mathbb{Z},\\ m_1+\cdots+m_k=m}} \binom{m}{m_1, \dots, m_k}\binom{n-m}{n_1 - m_1, \dots, n_k - m_k}.$$

Some combinatorial identities similar to the Vandermonde convolution can also be derived by applying a matrix $A \in M_n(\mathbb{C})$ and sequences $\alpha, \beta \in G_{n,n}$ to Theorem 3.3.

EXAMPLE 3.5. Let $A \in M_n(\mathbb{C})$ be the matrix constructed from J_n by changing the first column of J_n to the vector $(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are indeterminates. Let α, β be in $G_{n,n}$, with the form $\alpha := (1, 2, \dots, n)$ and $\beta := (1^m, 2, 3, \dots, n - m + 1)$, where $1 \le m \le n$ is an integer. Let $\bar{m}_k := (1^{m-k}, m_1, \dots, m_k) \in \bar{G}(\beta)$, where $m_1, \dots, m_k \in \{2, 3, \dots, n - m + 1\}$ for each $k = 0, \dots, \min\{m, n - m\}$. Then,

$$\mathrm{per}(A[\alpha_m|\beta_{\bar{m}_k}]) = \mathrm{per}\left(\begin{bmatrix} x_1 & \cdots & x_1 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_m & \cdots & x_m & 1 & \cdots & 1 \end{bmatrix}\right) = (m - k)!\, E_{m-k}(x_1, \dots, x_m),$$

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where $E_{m-k}(x_1,\ldots,x_m) = \sum_{1\le i_1<\cdots<i_{m-k}\le m} x_{i_1}\cdots x_{i_{m-k}}$ is the elementary symmetric polynomial of degree $m-k$ in the variables $x_1,\ldots,x_m$, and

$$\mathrm{per}(A[\alpha_m^c|\beta_{\bar{m}_k^c}]) = \mathrm{per}\left(\begin{bmatrix} x_{m+1} & \cdots & x_{m+1} & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_n & \cdots & x_n & 1 & \cdots & 1 \end{bmatrix}\right) = k!\,E_k(x_{m+1},\ldots,x_n),$$

where $E_k(x_{m+1},\ldots,x_n)$ is the elementary symmetric polynomial of degree k in the variables $x_{m+1},\ldots,x_n$. Also, $\mathrm{per}(A[\alpha|\beta]) = m!\,E_n(x_1,\ldots,x_n)$. For each k, the matrix $A[\alpha_m|\beta_{\bar{m}_k}]$ does not depend on the $\binom{n-m}{k}$-choices of the columns indexed by $m_1,\ldots,m_k$ in $\{2,3,\ldots,n-m+1\}$, because they are all the columns containing only 1. Substituting into Theorem 3.3, we obtain an elementary symmetric polynomial identity:

$$E_m(x_1,\ldots,x_n) = \sum_{k=0}^{\min\{m,n-m\}} E_{m-k}(x_1,\ldots,x_m)E_k(x_{m+1},\ldots,x_n). \tag{3.6}$$

EXAMPLE 3.6. Let $A \in M_n(\mathbb{C})$ be the matrix constructed from J_n by changing the diagonal of J_n to the vector $(1,0,\ldots,0)$. Suppose that α,β in $G_{n,n}$ have the form $\alpha := (1,2,\ldots,n)$ and $\beta := (1^m,2,3,\ldots,n-m+1)$, where $1\le m\le n$ is an integer. Let $\bar{m}_k \in \bar{G}_\beta$ have the form $(1^{m-k},m_1,\ldots,m_k)$, where $m_1,\ldots,m_k \in \{2,3,\ldots,n-m+1\}$ for each $k = 0,\ldots,\min\{m,n-m\}$. After permuting suitable rows or columns,

$$\mathrm{per}(A[\alpha_m|\beta_{\bar{m}_k}]) = \mathrm{per}\left(\begin{bmatrix} 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{bmatrix}\right) := S_k^m(0),$$

where $S_k^m(0)$ denotes the permanent of the $m\times m$ matrix $J(m,k)$, all of whose entries are 1 except for the first k diagonal entries which are 0. Using similar notation, it turns out that $\mathrm{per}(A[\alpha_m^c|\beta_{\bar{m}_k^c}]) = S_{n-m-k}^{n-m}(0)$ and $\mathrm{per}(A[\alpha|\beta]) = S_{n-m}^n(0)$. Note that the number of zeros in the matrix $A[\alpha_m|\beta_{\bar{m}_k}]$ (and hence $A[\alpha_m^c|\beta_{\bar{m}_k^c}]$) does not depend on the $\binom{n-m}{k}$-choices of the columns indexed by $m_1,\ldots,m_k \in \{2,3,\ldots,n-m+1\}$, because each column contains exactly one zero. Substituting into Theorem 3.3 gives the combinatorial identity:

$$S_{n-m}^n(0) = \sum_{k=0}^{\min\{m,n-m\}} \binom{n-m}{k}\binom{m}{k} S_k^m(0)S_{n-m-k}^{n-m}(0). \tag{3.7}$$

In fact, the value of $S_k^m(0)$ in the above example can be calculated explicitly in terms of the rencontres numbers. The rencontres number $D_{n,r}$ is the number of permutations

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of $[n] := \{1, 2, \ldots, n\}$ with exactly r fixed points. It is well known that

$$D_{n,r} = \frac{n!}{r!} \sum_{j=0}^{n-r} \frac{(-1)^j}{j!} \quad \text{and} \quad D_{n,r} = \binom{n}{r} D_{n-r,0}.$$

Since there are $\binom{n}{r}$ sets of r points $\{i_1, \ldots, i_r\}$ from $[n]$ and each permutation is one-to-one, there are exactly $D_{n,r}/\binom{n}{r}$ permutations that have $\{i_1, \ldots, i_r\}$ as their fixed points. Note also that for the matrix $J(m, k) = (J_{ij})$, the permutations $\sigma \in S_m$ for which $\prod_{i=1}^{m} J_{i\sigma(i)} \neq 0$ are those with no fixed points or fixed points that are subsets of $\{k + 1, \ldots, m\}$. Thus,

$$S_k^m(0) = D_{m,0} + \frac{\binom{m-k}{1}}{\binom{m}{1}} D_{m,1} + \frac{\binom{m-k}{2}}{\binom{m}{2}} D_{m,2} + \cdots + \frac{\binom{m-k}{m-k}}{\binom{m}{m-k}} D_{m,m-k}$$

or, equally, for each $m, k \in \mathbb{N}$ such that $k \leq m$,

$$S_k^m(0) = \sum_{i=0}^{m-k} \binom{m-k}{i} D_{m-i,0}. \tag{3.8}$$

By (3.8), the identity (3.7) becomes a relation of derangement numbers.

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