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Analyticity for a class of non-linear evolutionary pseudo-differential equations

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We study the analyticity properties of solutions for a class of non-linear evolutionary pseudo-differential equations possessing global attractors. In order to do this we utilise an analyticity criterion for spatially periodic functions, which involves the rate of growth of a suitable norm of the n^{th} derivative of the solution, with respect to the spatial variable, as n tends to infinity. This criterion can be used to a wide class of dissipative-dispersive partial differential equations, provided they possess global attractors. Using this criterion and the *spectral method* developed in Akrivis *et al.* [1] we have improved previous results.

Key words: Kuramoto-Sivashinksy equation; Dissipative-dispersive equations; Analyticity of solutions of partial differential equations; Global attractors

1 Introduction

In this work, we present analyticity properties of zero mean, spatially 2π-periodic solutions of partial differential equations of the form

$$u_t + uu_x + \mathscr{P}u = 0, \tag{1.1}$$

possessing a global attractor. Here, $\mathscr{P}$ is a linear pseudo-differential operator defined by its symbol in Fourier space, that is,

$$\left(\widehat{\mathscr{P}w}\right)_k = \lambda_k \, \hat{w}_k, \quad k \in \mathbb{Z}, \tag{1.2}$$

whenever $w(x) = \sum_{k\in\mathbb{Z}} \hat{w}_k \, \mathrm{e}^{ikx}$, and with the eigenvalues λ_k satisfying the condition

$$\operatorname{Re}\lambda_k \geqslant c_1|k|^\gamma \quad \text{for all} \quad |k| \geqslant k_0, \tag{1.3}$$

for some positive constants c_1, γ and k_0 a sufficiently large positive integer. Global existence of solutions of (1.1) has been established for $\gamma > 3/2$ (see [23]); when $\gamma \geqslant 2$, it can be deduced from [6] that equation (1.1) possesses a global attractor compact in every Sobolev norm. Analyticity of solutions of (1.1) is established when $\gamma > 5/2$, in [1]. In this work, we shall prove that the solutions of (1.1) are analytic even when $\gamma > 2$.

A special case of equation (1.1) is the dispersively modified Kuramoto-Sivashinsky (KS) equation

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} + \mathscr{D}u = 0, \tag{1.4}$$

with $\nu > 0$ and $\mathscr{D}$ a linear antisymmetric pseudo-differential operator; in Fourier space

$$(\widehat{\mathscr{D}w})_k = \mathrm{i}d_k\hat{w}_k, \quad d_{-k} = -d_k \in \mathbb{R},$$

that is, $\mathscr{D}$ is dispersive. If $\mathscr{D} \equiv 0$ in (1.4), we obtain the usual KS equation on 2π-periodic domains, which arises in a variety of applications and describes the asymptotic behaviour of many physical systems. It occurs in free surface film flows [2, 9, 19, 22], in two-phase flows in cylindrical and plane geometries [5, 18, 24], flame-front instabilities and reaction diffusion combustion dynamics [20,21], chemical physics for propagation of concentration waves [14–16] and plasma physics [3]. In (1.4) when $d_k = -k^3$, we obtain the Kawahara equation [12, 13]; another application that emerges from the dynamics of two-phase core-annular flows yields d_k in terms of modified Bessel functions of the first kind [18]. Another special case of equation (1.1) is the Hilbert transform equation

$$u_t + uu_x \pm u_{xx} + \nu u_{xxxx} + \mu\mathscr{H}[u]_{xxx} = 0, \tag{1.5}$$

with $\nu > 0$, $\mu \geqslant 0$ and $\mathscr{H}$ the Hilbert transform operator defined by

$$\mathscr{H}[f](x) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(\xi)}{x - \xi} d\xi,$$

where the integral is understood in the sense of a Cauchy principal value; in Fourier space

$$(\widehat{\mathscr{H}[w]})_k = -\mathrm{i}\,\mathrm{sgn}(\mathrm{Re}\,k)\hat{w}_k.$$

This equation was first derived by Gonzales and Castellanos [7] and recently by Tseluiko and Papageorgiou [25] using formal asymptotics. A plus sign in front of the u_{xx} term corresponds to the linearly unstable hydrodynamic regime (the modified KS equation) and a minus sign to the stable one (the modified damped KS equation). It can be deduced from [6] that the 2π-periodic solutions of (1.1) possess a global attractor, bounded in every Sobolev norm; in fact, such proofs are possible for $\gamma \geqslant 2$ in (1.3). This Sobolev norm boundedness is used in our analyticity estimates to obtain a lower bound on the band of analyticity.

The analyticity of solutions of the L-periodic KS equation,

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0,$$

which is a special case of (1.4) with $\mathscr{D} \equiv 0$, is established in [4]. In particular, it is shown that at large times the solution is analytic in a strip of size

$$\gamma_L \geqslant c\,L^{-16/25}$$

around the real axis, where c is a positive constant independent of L. This provides the following estimate for the spectral density at high wavenumbers,

$$\limsup_{t\to\infty} |\hat{u}(j,t)| = \mathcal{O}(\mathrm{e}^{-c\,L^{-16/25} q|j|}),$$

where $\hat{u}(j,t)$ is the jth Fourier coefficient of $u(\cdot,t)$ and $q = 2\pi/L$.

For comparison purposes with our 2π-periodic solutions, we have repeated the analysis of Collet *et al.* [4] to cast the results in terms of ν and ν, μ, respectively, for (1.4) and (1.5) with the plus sign in front of the u_{xx} term. We find, respectively, for (1.4) and (1.5) with the plus sign in front of the u_{xx} term, that the width of the strip of analyticity, β_ν say, satisfies

$$\beta_\nu \geqslant b\,\nu^{41/50},$$

where b is a positive constant, and $\delta_{\nu,\mu}$ say, satisfies

$$\delta_{\nu,\mu} \geqslant d\left(\frac{\nu}{\mu}\right)^{41/25},$$

where d is a positive constant (see [10]).

Another case of equation (1.1) is the Burgers-Sivashinsky (BS) equation [8]

$$u_t + uu_x - u - u_{xx} = 0,$$

which superficially seems to have much in common with the KS equation. It too has low wave number instability, high wave number damping, and nonlinear stabilization via energy transfer. Despite the similarity between KS and BS, when L is large their solutions have different qualitative behaviour. KS solutions are observed to have high dimensional chaos (see [17]) while BS solutions just approach time independent steady states as $t \to \infty$.

2 An analyticity criterion

A real analytic and periodic function $f : \mathbb{R} \to \mathbb{C}$ extends holomorphically in a neighbourhood

$$\Omega_\beta = \{x + \mathrm{i}y : x, y \in \mathbb{R} \text{ and } |y| < \beta\}$$

for some $\beta > 0$. The maximum such $\beta \in (0,\infty]$ is called the *band of analyticity* of f. For completeness, we say that the band of analyticity of f is zero if and only if f is not real analytic. Next, we state an analyticity criterion for periodic functions which involves the rate of growth of suitable norms of f.

Lemma 2.1 (***Analyticity criterion***) *Let $f : \mathbb{R} \to \mathbb{C}$ be an L-periodic C^∞ function, $p \in [1,\infty]$ and*

$$\mu = \limsup_{s\to\infty} \frac{\|f\|_{p,s}^{1/s}}{s},$$

where

$$\|f\|_{p,s} = \left(\sum_{k\in\mathbb{Z}} |k|^{ps} |\hat{f}_k|^p\right)^{1/p},$$

with $\hat{f}_k = \frac{1}{L}\int_0^L f(x)\,\mathrm{e}^{-\mathrm{i}kqx}dx$ *and* $q = 2\pi/L$. *Then the band of analyticity* β *of* f *is given by*

$$\beta = \begin{cases} \infty & \text{if} \quad \mu = 0, \\ \dfrac{1}{\mathrm{e}\mu} & \text{if} \quad \mu \in (0,\infty), \\ 0 & \text{if} \quad \mu = \infty. \end{cases}$$

Proof. See Appendix.

3 Analyticity of certain dissipative evolutionary systems

We shall apply our analyticity criterion to 2π-periodic solutions (with zero spatial mean) of (1.1), where $\mathscr{P}$ is a linear pseudo-differential operator with a symbol in Fourier space given by (1.2). Well-posedness and global existence (in time) of solutions of (1.1) is established in [23]. Existence of a global attractor $\mathscr{X}$ can be derived from the results in [6]. In fact, when $t > 0$, every solution of (1.1) becomes C^∞ with respect to x. In particular, for every $n \in \mathbb{N}$, there exists an R_n, depending on $\mathscr{P}$, but independent of the initial data u_0, such that

$$\limsup_{t\to\infty} \|\partial_x^n u(\cdot,t)\| \leqslant R_n.$$

We follow now the approach of Akrivis *et al.* from [1]. Expressing $u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}_k(t)\,\mathrm{e}^{\mathrm{i}kx}$, equation (1.1) is transformed into the following infinite dimensional dynamical system

$$\frac{d}{dt}\hat{u}_k = -\lambda_k\hat{u}_k - \mathrm{i}k\hat{\varphi}_k, \quad k \in \mathbb{Z}, \tag{3.1}$$

with

$$\hat{\varphi}_k(t) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}u^2(x,t)\,\mathrm{e}^{-\mathrm{i}kx}\,dx = \frac{1}{2}\sum_{j=1}^{k-1}\hat{u}_j(t)\hat{u}_{k-j}(t) + \sum_{j=1}^{\infty}\hat{u}_{-j}(t)\hat{u}_{k+j}(t). \tag{3.2}$$

Clearly, (3.1) implies that

$$\hat{u}_k(t) = \mathrm{e}^{-\lambda_k t}\hat{u}_k(0) - \mathrm{i}k\int_0^t \mathrm{e}^{-\lambda_k(t-s)}\hat{\varphi}_k(s)\,ds,$$

and consequently

$$\limsup_{t\to\infty}|\hat{u}_k(t)| \leqslant \frac{|k|}{\operatorname{Re}\lambda_k}\limsup_{t\to\infty}|\hat{\varphi}_k(t)|, \tag{3.3}$$

whenever $\operatorname{Re}\lambda_k > 0$. We next define for $p > 2$

$$h_p(s) = \limsup_{t\to\infty}\left(\sum_{k=1}^{\infty} k^{ps}|\hat{u}_k(t)|^p\right)^{1/p}, \quad s \in \mathbb{R}.$$

Note that, if $n \in \mathbb{N}$ and $n \leqslant s$, then

$$2^{1/p} h_p(s) = \limsup_{t\to\infty} \left(\sum_{k\in\mathbb{Z}} |k|^{ps} |\hat{u}_k(t)|^p \right)^{1/p} \geqslant \limsup_{t\to\infty} \left(\sum_{k\in\mathbb{Z}} |k|^{pn} |\hat{u}_k(t)|^p \right)^{1/p}$$
$$= \limsup_{t\to\infty} \| u(\cdot,t)\|_{p,n}.$$

Also,

$$\limsup_{t\to\infty} |\hat{u}_m(t)| \leqslant \frac{h_p(s)}{|m|^s} \quad \text{for all } m \in \mathbb{Z} \setminus \{0\}. \tag{3.4}$$

Our target is to show the following claim.

CLAIM I. *There exist positive constants* M *and* a, *such that, for every* $s \geqslant 0$,

$$h_p(s) \leqslant M(as)^s. \tag{3.5}$$

This result in turn implies that

$$\limsup_{s\to\infty} \left(\frac{1}{s} \limsup_{t\to\infty} \|u(\cdot,t)\|_{p,s}^{1/s} \right) = \limsup_{s\to\infty} \frac{2^{1/(ps)} h_p^{1/s}(s)}{s} \leqslant \limsup_{s\to\infty} \frac{2^{1/(ps)} M^{1/s} as}{s} \leqslant a.$$

By using our analyticity criterion, we shall consequently obtain a lower bound for the band of analyticity β of solutions u in the attractor, namely

$$\beta \geqslant \frac{1}{\mathrm{e}a}.$$

The claim will be proved by the following inductive method. First, we pick $M, a > 0$, so that

$$h_p(s) \leqslant M(as)^s, \quad \text{for every } s \in [0,2].$$

Suitable values are, for example,

$$M \geqslant 2^{1/2} R_2 \geqslant 2^{1/2} \limsup_{t\to\infty} \|u_{xx}(\cdot,t)\| \quad \text{and} \quad a \geqslant 1.$$

Indeed, noting that

$$(as)^s \geqslant \mathrm{e}^{-1/(\mathrm{e}a)} > \frac{1}{2}, \quad \text{for all } a \geqslant 1 \text{ and } s \geqslant 0,$$

we obtain

$$M(as)^s > \frac{M}{2} \geqslant \frac{1}{\sqrt{2}} \limsup_{t\to\infty} \|u_{xx}(\cdot,t)\| = \limsup_{t\to\infty} \left(\sum_{k=1}^{\infty} k^4 |\hat{u}_k(t)|^2 \right)^{1/2}$$
$$= \limsup_{t\to\infty} \left(\sum_{k=1}^{\infty} \left(k^2 |\hat{u}_k(t)|\right)^2 \right)^{1/2} \geqslant \limsup_{t\to\infty} \left(\sum_{k=1}^{\infty} \left(k^2 |\hat{u}_k(t)|\right)^p \right)^{1/p} = h_p(2) \geqslant h_p(s),$$

for all $s \in [0,2]$, since $p > 2$. Next we shall prove (by selecting a possibly larger a) that (3.5) holds for every $s \in [\sigma, \sigma+1]$, provided that the same inequality holds for every

$s \in [0,\sigma]$ and $\sigma \geqslant 2$. This in turn establishes that (3.5) holds for every $s \geqslant 0$. It suffices to show the following claim.

Claim II. *If* (3.5) *holds for every* $s \in [0,\sigma]$ *and* $\sigma \geqslant 1$, *then it also holds for* $s = \sigma + \sigma_1$, *where* $\sigma_1 \in (0, \gamma - \frac{2p+1}{p})$.

Proof of Claim II. For every $j = 1, \ldots, k-1$, we have, by virtue of (3.4),

$$\limsup_{t\to\infty} |\hat{u}_j(t)| \leqslant \frac{h_p\left(\frac{\sigma j}{k}\right)}{j^{\frac{\sigma j}{k}}} \leqslant \frac{M\left(a\frac{\sigma j}{k}\right)^{\frac{\sigma j}{k}}}{j^{\frac{\sigma j}{k}}},$$

and thus, the first sum on the right-hand side of (3.2) is estimated as follows:

$$\begin{aligned}
\limsup_{t\to\infty} \sum_{j=1}^{k-1} |\hat{u}_j(t)|\,|\hat{u}_{k-j}(t)| &\leqslant \sum_{j=1}^{k-1} \frac{h_p\left(\frac{\sigma j}{k}\right)}{j^{\frac{\sigma j}{k}}} \cdot \frac{h_p\left(\frac{\sigma(k-j)}{k}\right)}{(k-j)^{\frac{\sigma(k-j)}{k}}} \\
&\leqslant \sum_{j=1}^{k-1} \frac{M\left(a\frac{\sigma j}{k}\right)^{\frac{\sigma j}{k}}}{j^{\frac{\sigma j}{k}}} \cdot \frac{M\left(a\frac{\sigma(k-j)}{k}\right)^{\frac{\sigma(k-j)}{k}}}{(k-j)^{\frac{\sigma(k-j)}{k}}} \\
&= \frac{(k-1)M^2(a\sigma)^\sigma}{k^\sigma} \leqslant \frac{M^2(a\sigma)^\sigma}{k^{\sigma-1}}.
\end{aligned} \tag{3.6}$$

For the second sum in the right-hand side of (3.2), using inequality (3.4) and the fact that $|\hat{u}_{-j}(t)| = |\hat{u}_j(t)|$, we obtain that

$$\begin{aligned}
\limsup_{t\to\infty} \sum_{j=1}^{\infty} |\hat{u}_j(t)|\,|\hat{u}_{k+j}(t)| &\leqslant \limsup_{t\to\infty} \left(\sum_{j=1}^{\infty} |\hat{u}_j(t)|^p\right)^{1/p} \limsup_{t\to\infty} \left(\sum_{j=1}^{\infty} |\hat{u}_{k+j}(t)|^q\right)^{1/q} \\
&\leqslant h_p(0) \left(\sum_{j=1}^{\infty} \frac{h_p^q(\sigma)}{(k+j)^{q\sigma}}\right)^{1/q} \leqslant M\, h_p(\sigma) \left(\int_0^\infty \frac{dx}{(x+k)^{q\sigma}}\right)^{1/q} \\
&\leqslant M^2(a\sigma)^\sigma \left(\frac{1}{q\sigma - 1} \cdot \frac{1}{k^{q\sigma-1}}\right)^{1/q} \\
&= \frac{M^2(a\sigma)^\sigma}{(q\sigma-1)^{1/q} k^{\sigma-(1/q)}} \leqslant \frac{M^2(a\sigma)^\sigma}{(q-1)^{1/q} k^{\sigma-1}},
\end{aligned} \tag{3.7}$$

assuming that $p, q \in (1,\infty)$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. In arriving at the result above, we have used the fact

$$\limsup_{t\to\infty} \sum_{j=1}^{\infty} |\hat{u}_{k+j}(t)|^q \leqslant \sum_{j=1}^{\infty} \limsup_{t\to\infty} |\hat{u}_{k+j}(t)|^q \leqslant \sum_{j=1}^{\infty} \frac{h_p^q(\sigma)}{(k+j)^{q\sigma}},$$

along with (3.4). Also, we note that

$$\int_0^\infty \frac{dx}{(x+k)^{q\sigma}} = \lim_{t\to\infty} \int_0^t (x+k)^{-q\sigma} dx = \frac{1}{(q\sigma-1)k^{q\sigma-1}},$$

since $q\sigma \geqslant q > 1$. Finally, notice that

$$q\sigma - 1 \geqslant q - 1 \iff (q\sigma - 1)^{1/q} \geqslant (q-1)^{1/q} \quad \text{and} \quad \sigma - \frac{1}{q} > \sigma - 1 \iff k^{\sigma-(1/q)} > k^{\sigma-1}.$$

Now, from (1.3), we have

$$\operatorname{Re}\lambda_k \geqslant c_1 k^{\gamma} \quad \text{for } k \geqslant k_0. \tag{3.8}$$

Combination of (3.3), (3.6), (3.7) and (3.8) provides that

$$\limsup_{t\to\infty} |\hat{u}_k(t)| \leqslant \frac{\big(2 + (q-1)^{1/q}\big) M^2 (a\sigma)^{\sigma}}{2c_1 (q-1)^{1/q}\, k^{\sigma+\gamma-2}} \quad \text{for } k \geqslant k_0.$$

Thus,

$$\begin{aligned}
\limsup_{t\to\infty} \sum_{k=1}^{\infty} k^{p\sigma+p\sigma_1} |\hat{u}_k(t)|^p &\leqslant \sum_{k=k_0}^{\infty} \frac{k^{p\sigma+p\sigma_1}\big(2 + (q-1)^{1/q}\big)^p M^{2p}(a\sigma)^{p\sigma}}{(2c_1)^p (q-1)^{p/q}\, k^{p\sigma+p\gamma-2p}} \\
&\quad + \limsup_{t\to\infty} \sum_{k=1}^{k_0-1} k^{p\sigma+p\sigma_1} |\hat{u}_k(t)|^p \\
&\leqslant \frac{\big(2 + (q-1)^{1/q}\big)^p M^{2p}(a\sigma)^{p\sigma}}{(2c_1)^p (q-1)^{p/q}} \sum_{k=k_0}^{\infty} \frac{1}{k^{p(\gamma-2-\sigma_1)}} + (k_0-1)^{p\sigma+p\sigma_1-2p} \limsup_{t\to\infty} \sum_{k=1}^{k_0-1} k^{2p} |\hat{u}_k(t)|^p \\
&= \frac{\big(2 + (q-1)^{1/q}\big)^p M^{2p}(a\sigma)^{p\sigma}}{(2c_1)^p (q-1)^{p/q}} \sum_{k=k_0}^{\infty} \frac{1}{k^{p(\gamma-2-\sigma_1)}} + (k_0-1)^{p\sigma+p\sigma_1-2p} \\
&\quad \times \limsup_{t\to\infty} \sum_{k=1}^{k_0-1} \big(k^4 |\hat{u}_k(t)|^2\big)^{p/2} \\
&\leqslant \frac{\big(2 + (q-1)^{1/q}\big)^p M^{2p}(a\sigma)^{p\sigma}}{(2c_1)^p (q-1)^{p/q}} \sum_{k=k_0}^{\infty} \frac{1}{k^{p(\gamma-2-\sigma_1)}} + (k_0-1)^{p\sigma+p\sigma_1-2p} \\
&\quad \times \limsup_{t\to\infty} \left(\sum_{k=1}^{k_0-1} k^4 |\hat{u}_k(t)|^2 \right)^{p/2} \\
&\leqslant \frac{\big(2 + (q-1)^{1/q}\big)^p M^{2p}(a\sigma)^{p\sigma}}{(2c_1)^p (q-1)^{p/q}} \int_{k_0-1}^{\infty} \frac{dx}{x^{p(\gamma-2-\sigma_1)}} + (k_0-1)^{p\sigma+p\sigma_1-2p} R_2^p \\
&= \frac{\big(2 + (q-1)^{1/q}\big)^p M^{2p}(a\sigma)^{p\sigma}}{(2c_1)^p (q-1)^{p/q}} \frac{1}{\big(p(\gamma-2-\sigma_1)-1\big)(k_0-1)^{p(\gamma-2-\sigma_1)-1}} \\
&\quad + (k_0-1)^{p\sigma+p\sigma_1-2p} R_2^p,
\end{aligned}$$

because of the fact that

$$p(\gamma - 2 - \sigma_1) > 1 \iff \sigma_1 < \gamma - \frac{2p+1}{p}.$$

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Since

$$h_p^p(\sigma+\sigma_1) = \limsup_{t\to\infty} \sum_{k=1}^{\infty} k^{p\sigma+p\sigma_1} |\hat{u}_k(t)|^p,$$

we have

$$h_p(\sigma+\sigma_1) \leqslant CM^2(a\sigma)^\sigma + (k_0-1)^{\sigma+\sigma_1-2}M,$$

where

$$C = \frac{2+(q-1)^{1/q}}{2c_1(q-1)^{1/q}\Big(\big(p(\gamma-2-\sigma_1)-1\big)(k_0-1)^{p(\gamma-2-\sigma_1)-1}\Big)^{1/p}}.$$

In arriving at the result above, we have used the fact

$$(\vartheta+\varphi)^{1/p} \leqslant \vartheta^{1/p}+\varphi^{1/p} \quad \text{for all } \vartheta,\varphi>0 \text{ and } p\geqslant 1.$$

This inductive step is complete if we can find positive constants M and a satisfying

$$CM^2(a\sigma)^\sigma + (k_0-1)^{\sigma+\sigma_1-2}M \leqslant M\big(a(\sigma+1)\big)^{\sigma+1} \quad \text{for every} \quad \sigma \geqslant 1. \tag{3.9}$$

Clearly, for every $M>0$, there exists an $a_0>0$, such that (3.9) holds for every $a\geqslant a_0$. We have proved the following:

Theorem 3.1 *Let $\mathscr{X}$ be the global attractor of the equation*

$$u_t + uu_x + \mathscr{P}u = 0,$$

with 2π-periodic initial data in L^2, where $\mathscr{P}$ is a linear pseudo-differential operator defined by its symbol in Fourier space, that is,

$$\big(\widehat{\mathscr{P}w}\big)_k = \lambda_k \hat{w}_k, \quad k\in\mathbb{Z},$$

whenever $w(x) = \sum_{k\in\mathbb{Z}} \hat{w}_k\, \mathrm{e}^{\mathrm{i}kx}$, and with the eigenvalues λ_k satisfying the condition

$$\operatorname{Re}\lambda_k \geqslant c_1|k|^\gamma \quad \textit{for all} \quad |k|\geqslant k_0,$$

for some positive constants c_1, $\gamma>2$ and k_0 a sufficiently large positive integer. Then, every $w\in\mathscr{X}$ extends to a holomorphic function in Ω_β, for a suitable $\beta>0$. □

4 Conclusions

In this work we have established analyticity for $2\pi-$periodic solutions of equation (1.1), provided that these solutions are attracted by a compact set and that the eigenvalues of $\mathscr{P}$ satisfy condition (1.3), for $\gamma>2$. It is noteworthy that our numerical experiments for a special model of equation (1.1), namely the equation

$$u_t + uu_x - |\partial_x|^\alpha u + |\partial_x|^\beta u = 0, \tag{4.1}$$

for spatially 2π-periodic data and $\beta > \alpha \geqslant 0$, suggest that for $\beta > 3/2$ the solutions are attracted by a compact set and they are analytic with the band of analyticity tending to zero as $\beta \searrow 3/2$. We are currently investigating extensions of the results presented here to classes of equations such as (4.1) that are characterised by $\gamma \leqslant 2$. Also, we have been investigating possible extensions of out theory to multi-dimensional dissipative PDEs, such as the two–dimensional Kuramoto-Sivashinsky equation

$$u_t + \Delta^2 u + u_{xx} + uu_x = 0,$$

on doubly periodic domains [11].

Appendix

Proof of Lemma 2.1

The proof that follows is along the lines of the proof of Theorem 1 in [1]. Clearly, if $1 \leqslant p \leqslant \infty$, then there exist positive constants C_1 and C_2, such that

$$C_1 \|f\|_{p,n} \leqslant \|f^{(n)}\|_\infty \leqslant C_2 \|f\|_{p,n+1}, \tag{A.1}$$

for every $n \geqslant 1$ and $f \in C^\infty(\mathbb{R})$, which is L-periodic. It is readily seen that (A.1) implies

$$\limsup_{n\to\infty} \frac{\|f\|_{p,n}^{1/n}}{n} = \limsup_{n\to\infty} \frac{\|f^{(n)}\|_\infty^{1/n}}{n}. \tag{A.2}$$

Formula (A.2) implies that it suffices to show the lemma for the $\|f\|_{p,\infty}$-norm, instead of the $\|f\|_{p,s}$-norm. Due to Stirling's formula we have that

$$\lim_{n\to\infty} \frac{n}{(n!)^{1/n}} = \mathrm{e},$$

which in combination with (A.2), yields that

$$\tilde{\mu} = \limsup_{n\to\infty} \left(\frac{\|f\|_{p,\infty}}{n!}\right)^{1/n} = \limsup_{n\to\infty} \frac{n}{(n!)^{1/n}} \cdot \frac{\|f\|_{p,\infty}^{1/n}}{n} = \mathrm{e}\mu.$$

Therefore, in order to prove our analyticity criterion it suffices to establish the following two claims.

Claim I. *If* $\tilde{\mu} < \infty$ *and* $\gamma := \begin{cases} \infty & \text{if } \tilde{\mu} = 0, \\ \frac{1}{\tilde{\mu}} & \text{if } \tilde{\mu} > 0, \end{cases}$ *then* f *extends holomorphically in* Ω_γ.

Claim II. *If* $\gamma \in (0, \infty)$ *and* f *extends holomorphically in* Ω_γ, *then* $\tilde{\mu} \leqslant 1/\gamma$.

Proof of Claim I. It can be readily seen that the function

$$F(x + \mathrm{i}y) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (\mathrm{i}y)^n$$

is well defined (cf. n^{th}-root test for series) and differentiable (in fact C^∞), with respect to both x and y, for every $(x, y) \in \mathbb{R} \times (-\gamma, \gamma)$, and satisfies the Cauchy-Riemann equations, i.e., $F_y = \mathrm{i}F_x$. Therefore, F is holomorphic in Ω_γ, and since $F(x) = f(x)$, for $x \in \mathbb{R}$, then f extends holomorphically in Ω_γ.

Proof of Claim II. Assume that F is holomorphic in Ω_γ and also that it agrees with f in $\mathbb{R}$, and let $\varepsilon \in (0, \gamma)$. Set

$$M_\varepsilon = \max\{|F(x + \mathrm{i}y)| : x \in [0, L] \text{ and } |y| \leqslant \gamma - \varepsilon\}.$$

We have

$$M_\varepsilon = \sup_{z \in \overline{\Omega}_{\gamma-\varepsilon}} |F(z)|,$$

since F is L-periodic as well. Also, for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$f^{(n)}(x) = F^{(n)}(x) = \frac{n!}{2\pi \mathrm{i}} \int_{|z-x|=\gamma-\varepsilon} \frac{F(z)}{(z-x)^{n+1}}\, dz.$$

Therefore

$$|f^{(n)}(x)| \leqslant \frac{n! M_\varepsilon}{(\gamma - \varepsilon)^n},$$

and thus

$$\tilde{\mu} = \limsup_{n\to\infty} \left(\frac{\|f^{(n)}\|_\infty}{n!} \right)^{1/n} \leqslant \frac{1}{\gamma - \varepsilon}$$

for every $\varepsilon \in (0, \gamma)$. Consequently, $\tilde{\mu} \leqslant 1/\gamma$. □

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