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The symbolic dynamics of multidimensional tiling systems

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Abstract. We prove a multidimensional version of the theorem that every shift of finite type has a power that can be realized as the same power of a tiling system. We also show that the set of entropies of tiling systems equals the set of entropies of shifts of finite type.

1. *Introduction*

In [**CGST**], Coven *et al* consider a class of subshifts of $\mathcal{A}^{\mathbb{Z}}$ defined by finite subsets of the integers. These subshifts are called tiling systems. The main result in [**CGST**] shows that given a one-dimensional shift of finite type, there is a tiling system and a positive integer m, such that the mth powers of the shift of finite type and of the tiling system are topologically conjugate. It is also shown that every tiling system is sofic. It then follows that the set of entropies of tiling systems is equal to the set of entropies of shifts of finite type. For a thorough introduction to one-dimensional symbolic dynamical systems, see either [**K**] or [**LM**]. In this paper we extend the main result to higher dimensions. In higher dimensions it is not known whether the set of entropies of shifts of finite type and of sofic shifts are equal, but we show that the set of entropies of tiling systems equals the set of entropies of shifts of finite type (Theorem 4.1).

We begin with some definitions. Let $\mathcal{A}$ be a finite alphabet, and consider the compact metric space $\mathcal{A}^{\mathbb{Z}^d}$. For $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $w \in \mathbb{Z}^d$, we will write x_w for $x(w)$.

Let $\sigma : \mathbb{Z}^d \to \text{Homeo}(\mathcal{A}^{\mathbb{Z}^d})$ be the continuous $\mathbb{Z}^d$-action (the homomorphism from $\mathbb{Z}^d$ to the group of homeomorphisms of $\mathcal{A}^{\mathbb{Z}^d}$) defined by

$$(\sigma_v(x))_w = x_{w+v}$$

for all $v, w \in \mathbb{Z}^d$ and all $x \in \mathcal{A}^{\mathbb{Z}^d}$. Thus σ is defined by d commuting homeomorphisms σ_{e_i}, $i = 1, 2, \ldots, d$, where e_i is the d-tuple consisting of all 0's except for a 1 in the ith position. We call σ the *d-dimensional shift*. For an integer m we define the *mth power*, σ^m, of σ to be the $\mathbb{Z}^d$-action given by the composition homomorphism from $\mathbb{Z}^d$ to $\text{Homeo}(\mathcal{A}^{\mathbb{Z}^d})$, which is defined by $e_i \mapsto me_i \mapsto \sigma_{me_i}$; that is,

$$(\sigma_v^m(x))_w = x_{w+mv}.$$

A subset $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ is said to be *σ-invariant* if for all $v \in \mathbb{Z}^d$, $\sigma_v(X) \subseteq X$, or equivalently, if $\sigma_{e_i}(X) \subseteq X$ for all $i = 1, \ldots, d$. If X is a closed, σ-invariant subset of $\mathcal{A}^{\mathbb{Z}^d}$, we say that (X, σ) is a subshift of $(\mathcal{A}^{\mathbb{Z}^d}, \sigma)$.

We define a d-dimensional tiling system in the following way. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_K\}$ be a finite collection of finite subsets of $\mathbb{Z}^d$, called *prototiles*. Normalize the prototiles so that the lexicographically smallest point is the origin. A *tile* is a translate of a prototile, and a *tiling of $\mathbb{Z}^d$ by $\mathcal{P}$* is an expression of $\mathbb{Z}^d$ as a disjoint union of tiles, $\mathbb{Z}^d = \bigcup (t_j + P_{k_j})$. Corresponding to a tiling is a point $x \in \mathcal{A}^{\mathbb{Z}^d}$, $\mathcal{A} = \{1, 2, \ldots, K\}$, where $x_v = r$ if v lies in a tile that is a translate of P_r. Formally, for every $x \in \mathcal{A}^{\mathbb{Z}^d}$ and for every $v \in \mathbb{Z}^d$, $x_v = r$ if and only if $v \in P_{k_j} + t_j$ and $k_j = r$. Let $T(\mathcal{P})$ denote the set of all points in $\mathcal{A}^{\mathbb{Z}^d}$ corresponding to tilings of $\mathbb{Z}^d$ by $\mathcal{P}$. The set $T(\mathcal{P})$ is a σ-invariant, closed subset of the full shift $\mathcal{A}^{\mathbb{Z}^d}$, i.e. a subshift of $\mathcal{A}^{\mathbb{Z}^d}$. We call $(T(\mathcal{P}), \sigma)$ a *tiling system*.

A *d-dimensional shift of finite type* is a subshift (X, σ) of $(\mathcal{A}^{\mathbb{Z}^d}, \sigma)$ defined by a finite list of allowable d-dimensional cubes (see [**LM**, p. 467]). By moving to a higher block presentation if necessary, a d-dimensional shift of finite type can be described as a closed, shift-invariant subspace Σ of $\mathcal{A}^{\mathbb{Z}^d}$, given by d directed graphs, $G_1, \ldots, G_d$, each of whose edges is labeled with distinct elements of the alphabet $\mathcal{A}$. For each $1 \leq i \leq d$, the mapping from the edges in G_i to the symbols in $\mathcal{A}$ is one-to-one. A $\mathbb{Z}^d$-array $x = (x_v)_{v \in \mathbb{Z}^d} \in \mathcal{A}^{\mathbb{Z}^d}$ is in Σ if and only if for each $v \in \mathbb{Z}^d$ and for each $i = 1, 2, \ldots, d$, the initial vertex of x_{v+e_i} in the graph G_i is the terminal vertex of x_v in G_i.

A *d-dimensional sofic shift* is a factor of a d-dimensional shift of finite type. The factor map is a block map that, without loss of generality, is defined on d-dimensional cubes. Again, by going to a higher block representation if necessary, every d-dimensional sofic shift is an image of a shift of finite type through a $(1 \times \overset{\leftarrow d \rightarrow}{1 \times \cdots \times} 1)$-block map. Unfortunately, no good graph representations of multidimensional sofic shifts are known. The straightforward extension of the one-dimensional case, changing the labels in the edge representation of the shift of finite type, does not necessarily represent the sofic shift (see Example 2.1).

For ease of notation, we state and prove the main theorem only in two dimensions. An outline of the proof for higher dimensions is included at the end of the paper. In two dimensions, the graphs representing a given shift of finite type will be denoted by G_{H} and G_{V} (for horizontal and vertical moves), instead of G_1 and G_2. Similarly, we denote σ_{e_1} and σ_{e_2} by σ_{H} and σ_{V}.

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THEOREM 1.1. *Let (Σ, σ) be a two-dimensional shift of finite type. Then there is a positive integer m and a two-dimensional tiling system $(T(\mathcal{P}), \sigma)$ such that:*

(1) *$T(\mathcal{P}) = \bigcup_{0 \le i,j \le m-1} T_{i,j}$, where each $T_{i,j}$ is a closed subset of $T(\mathcal{P})$ and the collection $\{T_{i,j}\}$ is permuted by σ:*

$$\sigma_{(k,l)}(T_{i,j}) = T_{i+k,j+l},$$

where subscript addition is modulo m;

(2) *(Σ, σ^m) is topologically conjugate to $(T_{i,j}, \sigma^m)$ for every (i, j).*

Using this theorem, we show that the set of topological entropies of multidimensional tiling systems equals the set of topological entropies of multidimensional shifts of finite type (Theorem 4.1). In one dimension, both sets are equal to the set of logarithms of roots of Perron numbers [**LM**] and to the set of entropies of sofic shifts. On the other hand, little is known about the set of topological entropies of multidimensional shifts of finite type and multidimensional sofic shifts.

The outline of the paper is as follows. In §2, we start with an example which illustrates some of the difficulties in working in more than one dimension and conclude with a proof that multidimensional tiling systems are sofic. In §3, we show that every shift of finite type can be realized as a power of a tiling system (Theorem 3.1). Although this result is not difficult, it helps to introduce the more difficult constructions needed for the proof of the main theorem. Section 4 contains the proof of Theorem 1.1, including a description of the prototiles of the two-dimensional tiling system. The modifications needed for the proof of Theorem 1.1 in higher dimensions are found in §5.

2. *Multidimensional tiling and sofic systems*

In this section, we show that every tiling system is sofic. The one-dimensional case is proved in [**CGST**] using 'subscripted tiling systems' and the 'drop the subscripts' map. The proof for the multidimensional case can be done similarly. The idea of 'dropping the subscripts' and the difficulty in presenting two-dimensional sofic shifts with graphs are illustrated in the example below.

Example 2.1. Let $\mathcal{A} = \{a, b\}$. Consider $\Sigma \subseteq \mathcal{A}^{\mathbb{Z}^2}$, a two-dimensional shift of finite type such that in every point $x \in \Sigma$, every appearance of b is surrounded by a's. Then Σ is given by the following set of allowable (2×2) blocks:

$$p := \begin{matrix} a & a \\ a & a \end{matrix} \quad q_1 := \begin{matrix} b & a \\ a & a \end{matrix} \quad q_2 := \begin{matrix} a & b \\ a & a \end{matrix} \quad q_3 := \begin{matrix} a & a \\ b & a \end{matrix} \quad q_4 := \begin{matrix} a & a \\ a & b \end{matrix}.$$

By going to the (2×2) block presentation, Σ is topologically conjugate to $\hat{\Sigma} \subseteq \{p, q_1, q_2, q_3, q_4\}^{\mathbb{Z}^2}$, where $\{p, q_1, q_2, q_3, q_4\}$ is the set of five allowable (2×2) blocks in Σ. Edge graphs representing $\hat{\Sigma}$ may be obtained in the standard way: for the graph G_{V} associated to vertical movement, let the set of vertices be all allowed rows in the list of allowed (2×2) blocks. There is an edge from x_1x_2 to y_1y_2, if $\begin{smallmatrix} y_1 & y_2 \\ x_1 & x_2 \end{smallmatrix}$ is one of the allowed blocks. This edge is given the label of that block. The graph G_{H} associated to horizontal movement is given similarly, using allowed columns as the vertices. The graphs for $\hat{\Sigma}$ are given in Figure 1.

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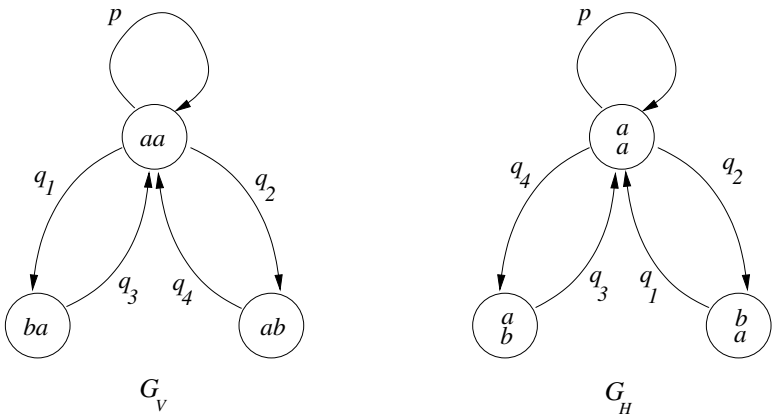

FIGURE 1. Graphs G_{V} and G_{H} representing the SFT $\hat{\Sigma}$.

Now consider the sofic system S obtained as a factor of $\hat{\Sigma}$ using the (1×1) block map $p \mapsto p$ and $q_i \mapsto q$ for $i = 1, \ldots, 4$. In [**CGST**], this map is called the drop the subscripts map.

What do the points in S look like? We can answer this by first looking at the points in $\hat{\Sigma}$. Note that if the symbol q_1 appears at a point in $\hat{\Sigma}$, then it must be preceded horizontally by q_2. The symbol q_2 must be followed vertically by q_4, and q_4 must be followed horizontally by q_3. Hence, every appearance of the (1×1) block q_1 occurs in an appearance of the (2×2) block $\begin{smallmatrix} q_4 & q_3 \\ q_2 & q_1 \end{smallmatrix}$. We come to the same conclusion if we consider the appearance of any other q_i $(i = 2, 3, 4)$. Thus the set of points in $\hat{\Sigma}$ is equivalent to the set of points obtained by 'concatenating' p's and the block $\begin{smallmatrix} q_4 & q_3 \\ q_2 & q_1 \end{smallmatrix}$ in a $\mathbb{Z}^2$-array. It then follows that the set of points in S is the same as the set obtained by concatenating p's and the block $\begin{smallmatrix} q & q \\ q & q \end{smallmatrix}$ in a $\mathbb{Z}^2$-array. So S is the tiling system defined by two prototiles, a p-tile, $P = \{(0, 0)\}$, and a q-tile, $Q = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

Note that S is *not* represented by the graphs obtained by dropping the subscripts of the edge labels of the graphs G_{H} and G_{V} in Figure 1. That presentation allows the following point, which is not in S (p's are bold to make them easier to distinguish from q's).

$$
\begin{array}{cccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \boldsymbol{p} & q & q & \boldsymbol{p} & q & q & \boldsymbol{p} & q & \cdot \\
\cdot & q & q & \boldsymbol{p} & q & q & \boldsymbol{p} & q & q & \cdot \\
\cdot & q & \boldsymbol{p} & q & q & \boldsymbol{p} & q & q & \boldsymbol{p} & \cdot \\
\cdot & \boldsymbol{p} & q & q & \boldsymbol{p} & q & q & \boldsymbol{p} & q & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

This example illustrates another difference between the drop the subscripts map in one-dimensional tiling systems and in multidimensional tiling systems. In one dimension, this map is finite-to-one and thus entropy-preserving. Our example shows that in more than one dimension, it can be infinite-to-one. The fixed point of all q's is in S and has an infinite number of preimages in $\hat{\Sigma}$. In fact, S cannot be an image of a shift of finite type by a

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finite-to-one map [**Q**]. However, Theorem 4.1 shows that the drop the subscripts map is entropy-preserving.

THEOREM 2.2. *Every multidimensional tiling system is sofic. There is a two-dimensional shift of finite type that is not topologically conjugate to a tiling system.*

Proof. As mentioned before, the proof is essentially the same as the proof of the one-dimensional case [**CGST**]. We present an outline for two dimensions.

Let $T(\mathcal{P})$ be a two-dimensional tiling system with $\mathcal{P} = \{P_1, P_2, \ldots, P_K\}$. Consider the closed, shift-invariant subset $\tilde{X}$ of $\{k_{i,j} : 1 \le k \le K \text{ and } (i, j) \in P_k\}^{\mathbb{Z}^2}$, defined by $\tilde{x} \in \tilde{X}$ if and only if there is a tiling of $\mathbb{Z}^2$ by $\mathcal{P}$ such that, for every $(r, s) \in \mathbb{Z}^2$, $\tilde{x}_{(r,s)} = k_{i,j}$, where (r, s) is in the (i, j)th location in P_k. (This is the two-dimensional version of the one-dimensional subscripted tiling system.) As in the one-dimensional case, $\tilde{X}$ is a shift of finite type, and the (1×1)-block map $k_{i,j} \mapsto k$ maps $\tilde{X}$ onto $T(\mathcal{P})$. (This is the two-dimensional version of the drop the subscripts map.) Therefore $T(\mathcal{P})$ is a sofic system.

Consider the shift of finite type $\Sigma \subseteq \{0, 1\}^{\mathbb{Z}^2}$, having exactly two points, x and y, such that $\sigma_{\mathrm{H}}(x) = y$, $\sigma_{\mathrm{H}}(y) = x$, $\sigma_{\mathrm{V}}(x) = x$, and $\sigma_{\mathrm{V}}(y) = y$. Then for every k, j, $x_{2k,j} = 0 = y_{2k+1,j}$ and $x_{2k+1,j} = 1 = y_{2k,j}$, or the same is true with x and y interchanged. Notice that, except for interchanging the symbols, the only system topologically conjugate to Σ is Σ itself. Suppose that T is a tiling system that contains two points x_T and y_T such that $\sigma_{\mathrm{H}}(x_T) = y_T$, $\sigma_{\mathrm{H}}(y_T) = x_T$, $\sigma_{\mathrm{V}}(x_T) = x_T$, and $\sigma_{\mathrm{V}}(y_T) = y_T$. These two points are obtained from a set of two prototiles, a '0-tile' and a '1-tile'. The 0-tile can cover all points $(i, j) \in \mathbb{Z}^2$ with i even and therefore it tiles $\mathbb{Z}^2$. Thus T has to contain the fixed point of all 0's, and it cannot be conjugate to Σ. □

3. *Construction of the two-dimensional tiling systems*

From Theorem 2.2, there is a two-dimensional shift of finite type that is not topologically conjugate to a tiling system. In this section, we show that every two-dimensional shift of finite type is topologically conjugate to a power of a subsystem of a tiling system. The proof of the following theorem also introduces the notion of marking the edges of a tile to reflect the allowed transitions of the shift of finite type. This idea is modified in the proof of Theorem 1.1.

THEOREM 3.1. *Let (Σ, σ) be a two-dimensional shift of finite type. Then there is a positive integer m and a two-dimensional tiling system $(T(\mathcal{P}), \sigma)$ such that:*

(1) *$T(\mathcal{P}) = \bigcup_{0 \le i,j \le m-1} T_{i,j}$, where each $T_{i,j}$ is a closed subset of $T(\mathcal{P})$ and the collection $\{T_{i,j}\}$ is permuted by σ:*

$$\sigma_{(k,l)}(T_{i,j}) = T_{i+k,j+l},$$

where subscript addition is modulo m;

(2) *(Σ, σ) is topologically conjugate to $(T_{i,j}, \sigma^m)$ for every (i, j).*

Proof. The prototiles for the tiling system consist of squares, one associated to each symbol of $\mathcal{A}$, with 'markings' on the sides to indicate which squares can appear next to each other.

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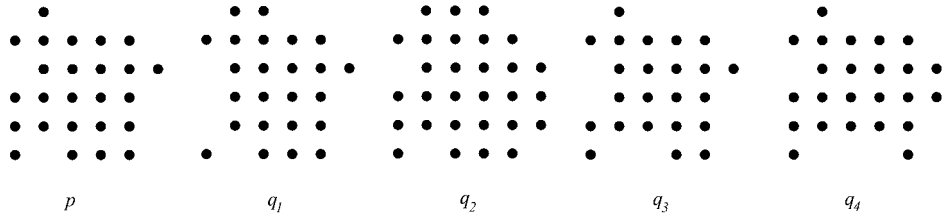

FIGURE 2. Prototiles associated to the SFT presented in Figure 1. The vertices $\begin{smallmatrix} a \\ a \end{smallmatrix}$, $\begin{smallmatrix} a \\ b \end{smallmatrix}$, $\begin{smallmatrix} b \\ a \end{smallmatrix}$ in G_{H} and vertices aa, ba, ab in G_{V} are enumerated 1, 2, 3 respectively.

For convenience in the construction, number the vertices $1, 2, \ldots, k_{\mathrm{H}}$ in the horizontal graph G_{H} and $1, 2, \ldots, k_{\mathrm{V}}$ in the vertical graph G_{V}. Let $m \geq 2 + \max(k_{\mathrm{H}}, k_{\mathrm{V}})$.

The prototile associated to the symbol a is an $m \times m$ square, modified as follows: start with the $m \times m$ square $\{(i, j) : 0 \leq i, j \leq m - 1\}$ in $\mathbb{Z}^2$. Above the north edge of the square, add the points $(1, m), (2, m), \ldots, (t_{\mathrm{V}}(a), m)$, where $t_{\mathrm{V}}(a)$ is the number associated to the terminal vertex of edge a in the vertical edge shift. To the right of the east edge, add the points $(m, m - 2), (m, m - 3), \ldots, (m, t_{\mathrm{H}}(a))$, where $t_{\mathrm{H}}(a)$ is the number associated to the terminal vertex of edge a in the horizontal edge shift. From the south face, remove the points $(1, 0), (2, 0), \ldots, (i_{\mathrm{V}}(a), 0)$, where $i_{\mathrm{V}}(a)$ is the number associated to the initial vertex of edge a in the vertical edge shift. From the west face, remove the points $(0, m - 2), (0, m - 3), \ldots, (0, i_{\mathrm{H}}(a))$, where $i_{\mathrm{H}}(a)$ is the number associated to the initial vertex of edge a in the horizontal edge shift. We end up with a modified square that (when connected) looks like a jigsaw puzzle piece. This construction gives a one-to-one correspondence between the symbols in $\mathcal{A}$ and the set of prototiles. For the tiling system considered in Example 2.1, the corresponding prototiles are depicted in Figure 2.

Let $(T(\mathcal{P}), \sigma)$ be the resulting tiling system, and for $0 \leq i, j \leq m - 1$, let $T_{i,j}$ be the set of points in $T(\mathcal{P})$ such that the southwest corners of the tiles appear at locations $(mk+i, ml+j)$. Let $\varphi : T_{i,j} \to \Sigma$ be defined by $(\varphi(t))_{k,l} = a$, if the tile whose southwest corner is at location $(mk + i, ml + j)$ is the tile associated to symbol a. We leave it to the reader to verify that φ is a topological conjugacy of $(T_{i,j}, \sigma^m)$ onto (Σ, σ). □

4. *The Proof of Theorem 1.1*

The prototiles necessary for the tiling system described in Theorem 1.1 must be more complicated than the ones described in Theorem 3.1. The authors in [**CGST**] use two kinds of prototiles, called *barbells* and *racks*. Here, we also use two kinds of prototiles, barbells and two-dimensional versions of racks.

4.1. *Barbells and one-dimensional racks.*

In [**CGST**], racks have three sections: a head, a center, and a tail. Barbells are used to fill the empty places in the center sections of the racks, and the number of ways this can be done is equal to the number of blocks of a fixed length in the given one-dimensional shift of finite type. The heads and tails of the racks are used to control which racks may sit side-by-side, i.e. the transition rules.

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For the proof of Theorem 1.1 we also use barbells, and the center sections of the racks are used in the centers of the two-dimensional racks.

For fixed n, the center section of a rack has the form

$$\bullet^{3n+i} \underset{\_\cdots\_}{\overset{\leftarrow 2k \rightarrow}{}} \bullet \underset{\_\cdots\_}{\overset{\leftarrow 2k \rightarrow}{}} \bullet^{8n-4k-1-i}$$

where $k = 1, \ldots, n$ and $i = 0, \ldots, n$. In this notation, $\bullet$ is a point in the lattice of the two-dimensional integers, $\bullet^s$ stands for $\bullet \overset{\leftarrow s \rightarrow}{\bullet \cdots \bullet} \bullet$ and '_' denotes an empty space. We will denote this center section of a rack by $r(k, i)$.

Barbells are of the following form:

$$\bullet\bullet \underset{\_\cdots\_}{\overset{\leftarrow 1+2p \rightarrow}{}} \bullet\bullet$$

where $0 \leq p \leq 2n - 2$. The gaps in $r(k, i)$ may be filled in exactly $k!$ ways by barbells. To see this, first imagine partitioning the set of empty places on the left-hand side of the center into pairs of consecutive places. Then think of the placement of barbells in the rack as a one-to-one map from this partition onto the corresponding partition of the set of empty places on the right-hand side of the center. There are exactly $k!$ such maps. By varying i between 0 and n, we obtain $n + 1$ different racks, and the centers of each of these racks can be filled in $k!$ ways.

4.2. *Two-dimensional racks.* Besides barbells, for the two-dimensional case we use prototiles that are modified $m \times m$ squares, which we call two-dimensional racks. The first modification uses the idea of side 'markings' as described in the proof of Theorem 3.1. These markings control how the square tiles fit together on each side. An $m \times m$ square tile, intended to correspond to an $(m \times m)$ block in the original shift of finite type Σ, has its sides marked in such a way that it can be adjacent to another $m \times m$ square tile with marked edges if and only if the symbols on the sides of the corresponding $(m \times m)$ blocks in Σ can appear side-by-side. We call this first modification 'side setting'.

The second modification involves leaving spaces in the center of the squares to be filled in a variety of ways by barbells. The spaces are chosen so that there is a one-to-one correspondence between the ways of filling these spaces with barbells and the $(m \times m)$ blocks in Σ. Such a correspondence naturally leads to a block map conjugacy between the tiling system we construct and the original shift of finite type Σ. This second modification is called 'center setting'.

Step 1: side setting. Recall that the shift of finite type Σ is described by two edge graphs, G_{H} and G_{V}. Thus, symbol a can be followed by symbol b in the horizontal or vertical direction if the terminal vertex of a equals the initial vertex of b in the appropriate graph. Now consider the $(m \times m)$ blocks over alphabet $\mathcal{A}$ which are allowed in Σ. The block BB', where both B and B' are allowed $(m \times m)$ blocks, is an allowed block in Σ if the rightmost column of B is 'compatible' with the leftmost column of B'. That means: if the rightmost column of B is $(a_1, \ldots, a_m)^{\mathrm{t}}$ and the leftmost column of B' is $(a'_1, \ldots, a'_m)^{\mathrm{t}}$, then a_i can be followed by a'_i in the horizontal direction for all $i = 1, \ldots, m$. Let $\mathcal{V}_{\mathrm{H}}$ and $\mathcal{V}_{\mathrm{V}}$ be the sets of vertices of G_{H} and G_{V}, and $\mathcal{V}_{\mathrm{H}}^m$ and $\mathcal{V}_{\mathrm{V}}^m$ the set of m-tuples of vertices from these sets. Set $\mathcal{V}^m = \mathcal{V}_{\mathrm{H}}^m \cup \mathcal{V}_{\mathrm{V}}^m$.

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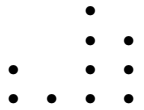

FIGURE 3. $C(\pi_\alpha)$ for $n = 4$ and $\pi_\alpha = \left(\begin{smallmatrix}1&2&3&4\\2&1&4&3\end{smallmatrix}\right)$.

FIGURE 4. $C^{\leftarrow}(\pi_\alpha)$ and $\hat{C}^{\rightarrow}(\pi_\alpha)$ for $\pi_\alpha = \left(\begin{smallmatrix}1&2&3&4\\2&1&4&3\end{smallmatrix}\right)$.

Since the number of vertices is at most one more than the number of edges, we have $|\mathcal{V}^m| \leq 2|\mathcal{A}+1|^m < (2|\mathcal{A}+1|)^m$. Set $s = 2|\mathcal{A}+1|$. There are at most $|\mathcal{A}|^{m^2} < s^{m^2}$ allowable blocks of size $m \times m$ in Σ. Because factorial growth is faster than exponential growth, we may choose n so that $s^{13n} < (n!)$. In fact, choose n so that $|\mathcal{A}|^{(13n)^2} < s^{(13n)^2} < (n!)^n$. Set $m = 13n$.

Let S_n be the group of permutations of n elements, and note that $|S_n| = n!$. Since $|\mathcal{V}^m| < s^m < n!$, there is an injection π from $\mathcal{V}^m$ into S_n. Denote the permutation associated with the m-tuple $\alpha \in \mathcal{V}^m$ by π_α. We use π_α to create a 'comb' in the following way: let $C(\pi_\alpha)$ be the subset of $\mathbb{Z}^2$ consisting of the set of points $\{(i, j) : 1 \leq i \leq n, 1 \leq j \leq \pi_\alpha(i)\}$. In other words, $C(\pi_\alpha)$ consists of n columns such that the ith column has height $\pi_\alpha(i)$. We call the set $\{(i, 1) : 1 \leq i \leq n\}$ the *base* of $C(\pi_\alpha)$.

Figure 3 shows a comb facing north, which we will denote by $C^{\uparrow}(\pi_\alpha)$. We will also need combs facing south, east and west, which we will denote by $C^{\downarrow}(\pi_\alpha)$, $C^{\rightarrow}(\pi_\alpha)$, and $C^{\leftarrow}(\pi_\alpha)$, respectively.

Associated to each comb $C(\pi_\alpha)$ is its complement $\hat{C}(\pi_\alpha) = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq n - \pi_\alpha(i)\}$ (see Figure 4). Note that $C^{\uparrow}(\pi_\alpha) \cup \hat{C}^{\downarrow}(\pi_\beta)$ creates an $n \times n$ square if and only if $\pi_\alpha = \pi_\beta$, that is if and only if the m-tuples of vertices α and β are equal.

Given the boundary $\mathcal{B}$ of an $(m \times m)$ block occurring in Σ, we use combs to 'mark' the edges of an $m \times m$ square tile as follows. Let $\alpha_{\mathrm{N}}, \alpha_{\mathrm{S}}, \alpha_{\mathrm{E}}, \alpha_{\mathrm{W}}$ be the m-tuples of vertices associated to the one-dimensional m-blocks that make up the north, south, east, and west sides of $\mathcal{B}$. So $\alpha_{\mathrm{N}}, \alpha_{\mathrm{S}} \in \mathcal{V}_{\mathrm{V}}^m$ and $\alpha_{\mathrm{E}}, \alpha_{\mathrm{W}} \in \mathcal{V}_{\mathrm{H}}^m$.

As described above, $\pi_{\alpha_{\mathrm{N}}}$, $\pi_{\alpha_{\mathrm{S}}}$, $\pi_{\alpha_{\mathrm{E}}}$, $\pi_{\alpha_{\mathrm{W}}}$ are permutations and each has associated to it a comb shape. Suppose $P \subseteq \mathbb{Z}^2$ denotes an $m \times m$ square with corner vertices $(0, 0)$, $(m-1, 0)$, $(0, m-1)$, and $(m-1, m-1)$. The sides of P are altered by combs $C^{\uparrow}(\pi_{\alpha_{\mathrm{N}}})$, $C^{\rightarrow}(\pi_{\alpha_{\mathrm{E}}})$, $\hat{C}^{\leftarrow}(\pi_{\alpha_{\mathrm{W}}})$, and $\hat{C}^{\downarrow}(\pi_{\alpha_{\mathrm{S}}})$. We place $C^{\uparrow}(\pi_{\alpha_{\mathrm{N}}})$ so that its base is at points $(6n, m), \ldots, (7n-1, m)$. (Recall that $m = 13n$.) Place $C^{\rightarrow}(\pi_{\alpha_{\mathrm{E}}})$ so that its base is at points $(m, 6n), \ldots, (m, 7n-1)$. Place $\hat{C}^{\leftarrow}(\pi_{\alpha_{\mathrm{W}}})$ and $\hat{C}^{\downarrow}(\pi_{\alpha_{\mathrm{S}}})$ so that the base of $\hat{C}^{\leftarrow}(\pi_{\alpha_{\mathrm{W}}})$ is at points $(n-1, 6n), \ldots, (n-1, 7n-1)$, and the base of $\hat{C}^{\downarrow}(\pi_{\alpha_{\mathrm{S}}})$ is at

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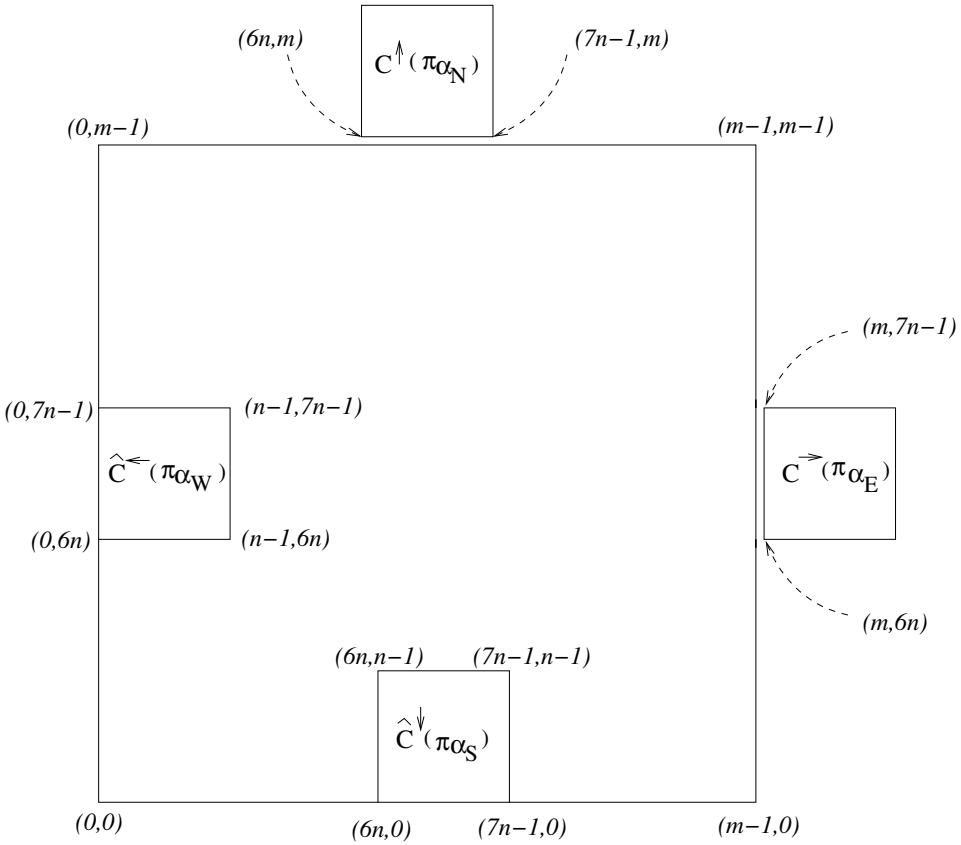

FIGURE 5. Side setting; sides of P are marked with combs ($m = 13n$).

points $(6n, n-1), \ldots, (7n-1, n-1)$. This is illustrated in Figure 5 (but is not drawn to scale).

We construct a square prototile, with edge combs as described above, for each of the boundaries $\mathcal{B}$ that occur in $(m \times m)$ blocks in Σ. It is clear that these tiles can 'fit together' horizontally (or vertically) if and only if the corresponding boundaries of $(m \times m)$ blocks in Σ can occur horizontally (or vertically) adjacent to one another for some point in Σ.

Step 2: center setting. There may be many $(m \times m)$ blocks in Σ with the same boundary, and we make our second modification to reflect this. As in the one-dimensional construction, we introduce spaces into the central portion of the modified tile. This is done so that there are exactly as many ways of filling in the central spaces in the prototiles associated with boundary $\mathcal{B}$ as there are $(m \times m)$ blocks in Σ with boundary $\mathcal{B}$. The tile P, once it is modified with 'side setting' and 'center setting', is a two-dimensional rack.

Given a boundary $\mathcal{B}$ of an $(m \times m)$ block occurring in Σ, let $N_{\mathcal{B}}$ be the number of allowable $(m \times m)$ blocks in Σ with boundary $\mathcal{B}$. By our choice of n, it is clear that

$$N_{\mathcal{B}} \leq |\mathcal{A}|^{(13n)^2} < s^{(13n)^2} < (n!)^n,$$

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and thus $N_{\mathcal{B}}$ can be written uniquely (in base $n!$) as follows:

$$N_{\mathcal{B}} = d_0 + d_1(n!) + d_2(n!)^2 + \cdots + d_{n-1}(n!)^{n-1},$$

where $0 \le d_j < n!$ for $0 \le j \le n-1$. As in [**CGST**], each d_j can be written uniquely as

$$d_j = c_{j,1}(1!) + c_{j,2}(2!) + \cdots + c_{j,n-1}(n-1)!,$$

where $0 \le c_{j,k} \le k$ and $1 \le k \le n-1$. Thus

$$N_{\mathcal{B}} = \sum_{j=0}^{n-1} d_j (n!)^j = \sum_{j=0}^{n-1} \sum_{k=1}^{n-1} c_{j,k}(k!)(n!)^j.$$

Begin with the prototile P, altered with combs corresponding to $\mathcal{B}$ as illustrated in Figure 5. For each $0 \le j \le n-1$ and each $1 \le k \le n-1$, form a two-dimensional rack $P_i(j,k)$ obtained from P in the following way. Consider a rectangular section of size $11n \times (j+1)$ such that the bottom j rows are filled with one-dimensional racks $r(n,0)$ and the top row with the rack $r(k,i)$. The two-dimensional rack $P_i(j,k)$ is obtained from P by replacing the section with corners $(n, n+1)$, $(12n-1, n+1)$, $(n, n+j+1)$, $(12n-1, n+j+1)$ with this $11n \times (j+1)$ rectangular shape.

Observe that by construction of the one-dimensional racks, the bottom j rows can be filled by barbells in exactly $n!$ ways and the top row can be filled in exactly $k!$ ways.

Thus, the modified prototile $P_i(j,k)$ can have the empty spaces filled with barbells in exactly $(k!)(n!)^j$ ways. By varying i between 1 and $c_{j,k}$, we obtain $c_{j,k}$ prototiles whose empty spaces can be filled with barbells in $(k!)(n!)^j$ ways.

So, the empty places in the racks $P_1(j,k), \ldots, P_{c_{j,k}}(j,k)$ can together be filled with barbells in $c_{j,k}(k!)(n!)^j$ ways. Let $\mathcal{P}(j,k) = \{P_i(j,k) : 1 \le i \le c_{j,k}\}$. Then the set of two-dimensional racks that correspond to the boundary $\mathcal{B}$ is $\mathcal{P}_{\mathcal{B}} = \bigcup_{j=0}^{n-1} \bigcup_{k=1}^{n-1} \mathcal{P}(j,k)$. Hence, the empty spaces of the racks in $\mathcal{P}_{\mathcal{B}}$ can be filled by barbells in exactly $N_{\mathcal{B}}$ ways.

4.3. *Proof of Theorem 1.1.* Let (Σ, σ) be the given two-dimensional shift of finite type. Choose n large enough so that $|\mathcal{A}|^{(13n)^2} < (n!)^n$, and set $m = 13n$. Take Σ^m to be the $(m \times m)$ power of Σ. As in the one-dimensional case, Σ^m is defined by $(m \times m)$ blocks that are allowable in Σ, i.e. $\Sigma^m \subseteq (A^{m^2})^{\mathbb{Z}^2}$. Hence (Σ^m, σ) is topologically conjugate to (Σ, σ^m). We show that the tiling system constructed above with barbells and two-dimensional racks as prototiles is topologically conjugate to (Σ^m, σ).

First, observe that the barbells can only be used to fill the spaces in the centers of rack tiles. The barbells, just by themselves, cannot be used to tile any rectangular region, because they have an odd number of empty spaces in the middle and they cover an even number of spaces. Clearly, they cannot be used to fill the empty spaces in the combs marking the edges of the rack tiles. Thus, the only way to tile $\mathbb{Z}^2$ with these tiles is for racks to sit next to each other with their comb markings compatible, and for barbells to fill the spaces inside the racks.

Let T be the tiling system with the collection of prototiles $\mathcal{P}$ being the barbells and racks. Let $T_{0,0}$ denote the collection of all tilings in T that consist of racks occurring with their southwest corners at points $\{(km, jm) : k, j \in \mathbb{Z}\}$. Clearly $T = \bigcup_{0 \le i,j \le m-1} T_{i,j}$, where $T_{i,j} = \sigma_{(i,j)} T_{0,0}$. This gives statement (1) of Theorem 1.1.

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To prove statement (2) of Theorem 1.1, let G_{H}^m be the directed graph with vertices $\mathcal{V}_{\mathrm{H}}^m$, given by m-tuples of vertices in $\mathcal{V}_{\mathrm{H}}$. Let $x \in \Sigma$. For each $x_{i,j}$, let $I(x_{i,j})$ and $J(x_{i,j})$ denote the initial and terminal vertices of the edge in G_{H} with label $x_{i,j}$. For every rack P with the comb marking on its west side constructed using $(I(x_{0,0}), \ldots, I(x_{0,m-1})) \in \mathcal{V}_{\mathrm{H}}$ and the comb marking on its east side constructed using $(J(x_{m-1,0}), \ldots, J(x_{m-1,m-1}))$, put an edge from vertex $(I(x_{0,0}), \ldots, I(x_{0,m-1}))$ to vertex $(J(x_{m-1,0}), \ldots, J(x_{m-1,m-1}))$. Label this edge P.

Similarly, let G_{V}^m be the directed graph with vertices $\mathcal{V}_{\mathrm{V}}^m$, the m-tuples of vertices in $\mathcal{V}_{\mathrm{V}}$. In this graph, the edges are determined by the north and south sides of the racks P.

Then G_{H}^m and G_{V}^m correspond exactly to the directed graphs of (Σ^m, σ). Thus $(T_{i,j}, \sigma^m)$ is topologically conjugate to the $(m \times m)$ power (Σ^m, σ), which is topologically conjugate to (Σ, σ^m), as desired. □

THEOREM 4.1. [**B**]

$$\{h(\Sigma, \sigma) : \Sigma \textit{ is a shift of finite type}\} = \{h(T, \sigma) : T \textit{ is a tiling system}\}$$
$$\subseteq \{h(S, \sigma) : S \textit{ is a sofic system}\},$$

where $h(\cdot)$ denotes two-dimensional topological entropy.

Proof. From Theorem 1.1 and Proposition 13.1 in [**S**], we have that $\{h(\Sigma, \sigma) : \Sigma$ is a shift of finite type$\} \subseteq \{h(T, \sigma) : T$ is a tiling system$\}$. The inclusion $\{h(T, \sigma) : T$ is a tiling system$\} \subseteq \{h(S, \sigma) : S$ is a sofic system$\}$ follows from Theorem 2.2. The statement of the theorem will follow if we show that the drop the subscripts map is entropy-preserving. Again we prove the statement in two dimensions only. The higher-dimensional case follows similarly.

Let $T = T(\mathcal{P})$ be a two-dimensional tiling system defined by a set of prototiles $\mathcal{P} = \{P_1, \ldots, P_K\}$. Let $\hat{T}$ be the closed, shift-invariant subset of $\{k_{i,j} : 1 \le k \le K$ and $(i, j) \in P_k\}^{\mathbb{Z}^2}$ described in the proof of Theorem 2.2, and let $\delta : \hat{T} \to T$ be the (1×1)-block drop the subscripts map. The prototiles that define $\hat{T}$ are subscripted prototiles $\hat{\mathcal{P}} = \{\hat{P}_1, \ldots, \hat{P}_K\}$. Set $s = \max\{|i_1 - i_2|, |j_1 - j_2| : (i_1, j_1), (i_2, j_2) \in P_r, r = 1, \ldots, K\}$. Then $\hat{T}$ is a shift of finite type defined by $(s \times s)$ blocks.

Consider an allowable $((n+2s) \times (n+2s))$ block B in T and $((n+2s) \times (n+2s))$ blocks $\hat{B}_1, \hat{B}_2$ in $\hat{T}$ such that $\delta(\hat{B}_1) = \delta(\hat{B}_2) = B$. Let $C, \hat{C}_1$ and $\hat{C}_2$ be the central $n \times n$ portions of $B, \hat{B}_1, \hat{B}_2$ respectively. We show that if $\hat{B}_1$ and $\hat{B}_2$ have equal boundaries of thickness s, then their central portions must be equal (in other words, the drop the subscripts map 'has no diamonds' (see [**QT**], [**K**])).

Since $\delta(\hat{B}_1) = \delta(\hat{B}_2) = B$, it is clear that the entries in the central portions $\hat{C}_1$ and $\hat{C}_2$ can differ only in their subscripts. In other words, each entry must be in a translate of the same prototile. If that translate intersects the s-thick boundary of either $\hat{B}_i$, then the subscripts clearly must be equal. If neither translated prototile intersects the s-thick boundary and yet the subscripts of the two entries are different, there must be two different ways to tile a portion of the centers of the $\hat{B}_i$'s with the translated prototile.

Let $P \in \mathcal{P}$ and $t_i, r_i \in \mathbb{Z}^2$ for $i = 1, \ldots, k$. Let $T_k = \bigcup_{i=1}^k (t_i + P)$ and $R_k = \bigcup_{i=1}^k (r_i + P)$, where the unions are disjoint. Then

$$T_k = R_k \text{ implies } \{t_1, \ldots, t_k\} = \{r_1, \ldots, r_k\}. \qquad (*)$$

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We show $(*)$ by induction on k. The case $k = 1$ is trivial. Assume that $T_k = R_k$. As mentioned in §1, the prototile P is chosen so that the lexicographically smallest point in P is $(0, 0)$, and so the smallest point in $t_i + P$ is t_i. Thus, if v is the lexicographically smallest point in $T_k = R_k$, v is in both $\{t_1, \ldots, t_k\}$ and $\{r_1, \ldots, r_k\}$. Without loss of generality, $v = t_k = r_k$. However, then $T_{k-1} = R_{k-1}$, and $(*)$ follows from the induction hypothesis.

Now let $r_i \in \mathbb{Z}^2$ $(i = 1, \ldots, k)$ be such that for some prototile P, $r_i + \hat{P} \subseteq \hat{B}_1$. Similarly, let $t_i \in \mathbb{Z}^2$ $(i = 1, \ldots, k')$ be such that $t_i + \hat{P} \subseteq \hat{B}_2$. Then r_i and t_i are either in the central portions $\hat{C}_1$ and $\hat{C}_2$ or in the boundaries with thickness s of $\hat{B}_1$ and $\hat{B}_2$. But $\hat{B}_1$ and $\hat{B}_2$ have the same boundary and $\delta(\hat{B}_1) = \delta(\hat{B}_2) = B$. So we must have $k = k'$ and $\bigcup_{i=1}^{k}(r_i + \hat{P}) = \bigcup_{i=1}^{k}(t_i + \hat{P})$, which by $(*)$ implies $\{r_1, \ldots, r_k\} = \{t_1, \ldots, t_k\}$. Thus the central portions $\hat{C}_1$ and $\hat{C}_2$ are equal.

Now, let $\hat{\mathcal{N}}_n$ be the number of $(n \times n)$ blocks in $\hat{T}$ $(n > 2s)$, $\hat{K}$ be the number of symbols in $\hat{T}$, and $\mathcal{N}_n$ the number of $(n \times n)$ blocks in T. The number of s-thick boundaries for an $(n \times n)$ block in $\hat{T}$ is at most $\hat{K}^{4s(n-s)}$. Then the theorem follows from the inequality

$$\mathcal{N}_n \leq \hat{\mathcal{N}}_n \leq \hat{K}^{4s(n-s)}\mathcal{N}_{n-2s}$$

and the fact that since s is fixed, the exponent of $\hat{K}$ is linear in n. □

5. *Higher dimensions*

The construction of the two-dimensional racks described in the previous section can be extended inductively to d-dimensional racks. Given a boundary $\mathcal{B}$ of an $(m \times m \times \cdots \times m)$ block occurring in a d-dimensional shift of finite type Σ, let $N_{\mathcal{B}}$ be the number of $(m \times m \times \cdots \times m)$ blocks with boundary $\mathcal{B}$. The boundary $\mathcal{B}$ consists of the $(d-1)$-dimensional faces of the block; there are at most $|\mathcal{A}+1|^{m^{d-1}}$ possibilities for each of these faces. Choose $n > 0$ large enough so that

$$N_{\mathcal{B}} \leq |\mathcal{A}|^{(13n)^d} < |d(\mathcal{A}+1)|^{(13n)^d} < (n!)^{(n^{d-1})}.$$

Thus, we may write $N_{\mathcal{B}}$ uniquely (in base $(n!)^{n^{d-2}}$) as follows:

$$N_{\mathcal{B}} = b_0 + b_1((n!)^{n^{d-2}}) + b_2((n!)^{n^{d-2}})^2 + \cdots + b_{n-1}((n!)^{n^{d-2}})^{n-1},$$

where $0 \leq b_j < (n!)^{n^{d-2}}$ for $0 \leq j \leq n-1$. For each b_j, write

$$b_j = c_{j,1} + c_{j,2}((n!)^{n^{d-3}}) + \cdots + c_{j,n}((n!)^{n^{d-3}})^{n-1},$$

where $0 \leq c_{j,k} < (n!)^{n^{d-3}}$. Thus,

$$N_{\mathcal{B}} = \sum_{j=0}^{n-1}\sum_{k=1}^{n-1} c_{j,k}((n!)^{d-3})^{k-1}((n!)^{n^{d-2}})^{j}.$$

Set $m = 13n$.

In the construction of the prototiles at the previous stage, dimension $d-1$, the spaces in the center sections of the $(d-1)$-dimensional racks can be filled in with barbells in b ways, where b is any number between 0 and $(n!)^{n^{d-2}}$. These center sections have

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size $11n \times (n \times \overleftarrow{n \times \cdots \times n}^{d-2\rightarrow} \times n)$. (Note that the n at the previous stage may have been smaller; there is no difficulty in increasing it.) Thus, in d dimensions, for each term in the sum above, we construct a collection of $(m \times \overleftarrow{m \times \cdots \times m}^{d\rightarrow})$-block prototiles. For each j, $0 \leq j \leq n-1$, we replace a block of size $11n \times (n \times \overleftarrow{n \times \cdots \times n}^{d-2\rightarrow}) \times (j+1)$ in the center portion of this prototile. The first j 'sheets' of size $11n \times (n \times \overleftarrow{n \times \cdots \times n}^{d-2\rightarrow})$ are constructed as in the $(d-1)$-dimensional case, each to be filled with barbells in exactly $(n!)^{n^{d-2}}$ ways. So, all together they can be filled in $((n!)^{n^{d-2}})^j$ ways. The last 'sheet' of size $11n \times (n \times \overleftarrow{n \times \cdots \times n}^{d-2\rightarrow})$ is constructed as in the $(d-1)$-dimensional case, to be filled with barbells in exactly $((n!)^{n^{d-3}})^{k-1}$ ways. By varying k, we obtain $c_{j,k}$ different d-dimensional racks with center sections that can be filled by barbells in $c_{j,k}((n!)^{n^{d-3}})^{k-1}((n!)^{n^{d-2}})^j$ ways.

We do this for each j, $0 \leq j \leq n-1$, and thus the centers of this collection of $(m \times m \times \cdots \times m)$ prototiles can be filled with barbells in exactly $N_{\mathcal{B}}$ ways.

It only remains to mark the bounding faces of these prototiles with combs appropriate for $\mathcal{B}$. We note that, due to our choice of n, the number of possibilities for a $(d-1)$-dimensional face of an $(m \times m \times \cdots \times m)$ block in Σ is bounded by $d|\mathcal{A}+1|^{(13n)^{d-1}} < (d(|\mathcal{A}|+1))^{(13n)^{d-1}} < (n!)^{(n^{d-2})}$. Thus there is an injection from the set of faces $\bigcup_{i=1}^{d} \mathcal{V}_i^{(13n)^{d-1}}$ (where $\mathcal{V}_i$ is the set of vertices in the ith graph representing Σ) into the set S_n^{d-2} (i.e. the set of $(d-2)$-tuples of elements of S_n). We will mark each $(d-1)$-dimensional face of the prototile with the n^{d-2} combs associated with that face, each of which represents a permutation in S_n.

It is clear that, as in two dimensions, the barbells can only be used to fill the spaces in the centers of the d-dimensional racks. Thus the only way to tile the d-dimensional integer lattice with these tiles is for the d-dimensional racks to sit next to each other with their comb markings compatible, and for the barbells to fill the spaces inside the d-dimensional racks. The proof of the theorem in d-dimensions then follows analogously.

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