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GROUP RINGS WITH HYPERCENTRAL UNIT GROUPS

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ABSTRACT. Let KG be the group ring of a group G over a field K and let $U(KG)$ be its group of units. If K has characteristic $p > 0$ and G contains p-elements, then it is proved that $U(KG)$ is hypercentral if and only if G is nilpotent and G' is a finite p-group.

A group G is said to be *hypercentral* if the ascending central series reaches G after some, possibly infinite, ordinal. Černikov [6, 2.19] showed this to be equivalent to the property that for every sequence $x_1, x_2, \ldots$ of elements in G, there exists an index m such that the group commutator $(x_1, x_2, \ldots, x_m) = 1$. So, in particular, if m can be chosen independently of the sequence, then G is nilpotent.

The problem of classifying group rings whose unit groups have such properties as solubility or nilpotency has been studied by many authors (see [7] for an overview). Examined here are some group rings whose unit groups are hypercentral.

Let K be a field of characteristic $p > 0$, and suppose that G contains p-elements. Denote by $U(KG)$ the group of units of the group algebra KG. Inspired by the paper of Bovdi and Khripta [1] in which group algebras that are Lie T-nilpotent are classified, the goal of this paper is to describe precisely the structure of KG when $U(KG)$ is hypercentral. The main result of this article is as follows.

THEOREM. *Let p be any prime and let G be any group containing p-elements. If K is a field of characteristic p, then the following are equivalent:*

(i) $U(KG)$ is hypercentral;

(ii) $U(KG)$ is nilpotent; and

(iii) G is nilpotent and G' is a finite p-group.

Since (ii) $\Rightarrow$ (i) is trivial and (ii) $\Leftrightarrow$ (iii) is a result of Khripta [3] and [7, pp. 179–181], it is enough to show only that (i) $\Rightarrow$ (iii). The remainder of this article is devoted to proving precisely this implication.

In addition to conventional notation, the following will be used: $(x,y) = xyx^{-1}y^{-1}$ is the group commutator of group elements x and y; $\phi(G)$ denotes the FC-centre of the group G; if h is a group element of finite order, let $\hat{h} = \sum_{i=1}^{o(h)} h^i$; Dr denotes a restricted direct product; and if $N \lhd G$, then $\Delta(G,N)$ is the kernel of the natural epimorphism $KG \longrightarrow K(G/N)$ induced by $G \longrightarrow G/N$. Unless specified to the contrary, *throughout this paper, K is assumed to be a field of characteristic $p > 0$ and G is a group containing a p-element such that $U(KG)$ is hypercentral.*

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LEMMA 1. *If G is any hypercentral group containing a p-element, then $\zeta(G)$, the centre of G, contains an element of order p.*

PROOF. By [5, 12.2.4], G hypercentral implies that G is locally nilpotent. Let P be the set of p-elements in G. Then $1 \neq P \triangleleft G$ by [5, 12.1.1], so that since G is hypercentral, $P \cap \zeta(G) \neq 1$ by [6, 2.16].

The following is a generalization of an unpublished result of Jairo Goncalves.

PROPOSITION 1. *G is an FC-group.*

PROOF. Since $U(KG)$ being hypercentral implies that G is hypercentral, by Lemma 1 we may pick $h \in \zeta(G)$ such that $o(h) = p$. Set $\eta = \hat{h}$. Then $\eta^2 = 0$ so that $1 + \eta\alpha \in U(KG)$ with inverse $1 - \eta\alpha$, for any $\alpha \in KG$. Assume to the contrary that G is not an FC-group. Since G is hypercentral, the ascending central series of G reaches G [6, p. 2.19]. Thus let λ be the first ordinal such that the λth centre of G, $\zeta_\lambda(G) \not\subseteq \phi(G)$. Notice that λ cannot be a limit ordinal since if $\zeta_\alpha(G) \subseteq \phi(G)$ for all $\alpha < \lambda$, then $\zeta_\lambda(G) = \bigcup_{\alpha<\lambda} \zeta_\alpha(G) \subseteq \phi(G)$ as well. Hence, in particular, $(G, \zeta_\lambda(G)) \leq \zeta_{\lambda-1}(G)$. Thus by Poincare's theorem, if $c_1, c_2, \ldots, c_r \in \zeta_{\lambda-1}(G) \subseteq \phi(G)$, then $[G : C_G\{c_1, \ldots, c_r\}] < \infty$. Now pick any $g \in \zeta_\lambda(G) \setminus \phi(G)$.

We construct a sequence (g_n) with the following properties:

(1) $g_{n+1} \in C_G\{b_1, \ldots, b_n\}$ where $b_j = (g_j, g) \in \zeta_{\lambda-1}(G)$; and

(2) $\eta(1 - b_1)(1 - b_2)\cdots(1 - b_{n+1}) \neq 0$ for all n.

Since $[G : C_G(g)] = \infty$, we can find $g_1 \in G$ such that $\eta(1 - b_1) \neq 0$ and $b_1 = (g_1, g) \in \zeta_{\lambda-1}(G)$. For recall that the right annihilator of a nonzero element of KG contains only finitely many elements of the form $g - 1$, $g \in G$. Suppose now we have selected $g_1, g_2, \ldots, g_n$ in G satisfying (1) and (2). Let $H = C_G\{b_1, \ldots, b_n\}$. We claim $[H : C_H(g)] = \infty$. Indeed, if $[H : C_H(g)] < \infty$, it follows that $[G : C_G(g)] \leq [G : C_H(g)] = [G : H][H : C_H(g)] < \infty$, contradicting our choice of g. Hence $[H : C_H(g)] = \infty$, and we can choose an element $g_{n+1} \in H$ such that $b_{n+1} = (g_{n+1}, g) \in \zeta_{\lambda-1}(G)$ and $\eta(1 - b_1)(1 - b_2)\cdots(1 - b_{n+1}) \neq 0$. Thus the sequence may be constructed.

Finally, we claim $(1 + \eta g, g_1, g_2, \ldots, g_n) = 1 + \eta(1 - b_1)(1 - b_2)\cdots(1 - b_n)g$, for all $n \geq 1$, contradicting that $U(KG)$ is hypercentral. Indeed,

$$\begin{aligned}(1 + \eta g, g_1) &= (1 + \eta g)g_1(1 - \eta g)g_1^{-1} \\ &= 1 + \eta g - \eta g_1 g g_1^{-1} = 1 + \eta(1 - b_1)g,\end{aligned}$$

so by induction we may assume the result for $n \geq 1$. Then

$$\begin{aligned}(1 + \eta g, g_1, \ldots, g_{n+1}) &= (1 + \eta(1 - b_1)\cdots(1 - b_n)g, g_{n+1}) \\ &= 1 + \eta(1 - b_1)\cdots(1 - b_n)g - \eta(1 - b_1)\cdots(1 - b_n)g_{n+1}gg_{n+1}^{-1} \\ &= 1 + \eta(1 - b_1)\cdots(1 - b_{n+1})g.\end{aligned}$$

This proves the proposition.

LEMMA 2. *For every $x, y \in G$, there is an $m \in \mathbb{N}$ (depending on x and y) such that $(x, y^{p^m}) = 1^{(*)}$.*

PROOF. [7, VI. 3.2]

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LEMMA 3. *G' is a p-group.*

PROOF. For $(x, y) \in G$, we may assume $G = \langle x, y \rangle$. Then G is a finitely generated FC-group by Proposition 1, so that G is residually finite [7, I.4.7]. Since G' is a torsion group, let $g \in G'$ have prime order q. Let $N \lhd G$ be such that G/N is finite and $g \notin N$. Since G/N is finite and satisfies $(*)$, there is an integer m such that $(G/N)^{p^m} \subseteq \zeta(G/N)$. So by Schur's lemma [7, I.4.2], then $((G/N)')^{p^t} = 1$ for some integer t, so that $(G/N)' = G'N/N$ is a p-group. Hence $p = q$ since $1 \neq gN \in G'N/N$, proving the lemma.

PROPOSITION 2. *$G' \cap \zeta(G)$ cannot contain a direct product of infinitely many nontrivial subgroups.*

PROOF. Suppose $\mathrm{Dr}_{i\in\mathbb{N}} H_i \subseteq G' \cap \zeta(G)$ with $1 \neq H_i$ for all i. Then, in particular, the p-group $G' \cap \zeta(G)$ contains a subgroup $H = \mathrm{Dr}_{i\in\mathbb{N}} C_i$, where each C_i is a cyclic group of order p.

We construct sequences (g_n) and (h_{2n-1}) in G with the following properties:

(1) $g_1, g_2 \in G$ are such that $(g_2, g_1) = b_1 \neq 1$;

(2) $h_1 \in H \setminus \langle b_1 \rangle$;

(3) $g_{2n+1}, g_{2n+2} \in C_G\{g_1, g_2, \ldots, g_{2n}\}$ are such that $(g_{2n+2}, g_{2n+1}) = b_{2n+1} \notin \langle b_1, b_3, \ldots, b_{2n-1}, h_1, h_3, \ldots, h_{2n-1} \rangle$; and

(4) $h_{2n+1} \in H \setminus \langle b_1, b_3, \ldots, b_{2n+1}, h_1, h_3, \ldots, h_{2n-1} \rangle$ for all n.

Certainly such g_1, g_2 and h_1 can be found. Suppose as well that for $n \geq 1$, $g_1, g_2, \ldots, g_{2n}$ and $h_1, h_3, \ldots, h_{2n-1}$ satisfy (1)–(4). Since G is an FC-group by Proposition 1, $[G : C] < \infty$ where $C = C_G\{g_1, g_2, \ldots, g_{2n}\}$. Hence by [8, 7.5], C' is infinite since G' is. Since G' is locally finite, so is C'. Thus C' contains infinitely many commutators $b_{2n+1} = (g_{2n+2}, g_{2n+1})$ with $g_{2n+1}, g_{2n+2} \in C$. Since $F_1 := \langle b_1, b_3, \ldots, b_{2n-1}, h_1, h_3, \ldots, h_{2n-1} \rangle \subseteq G'$ is finite, we may pick $b_{2n+1} \notin F_1$. Similarly since $F_2 := \langle b_1, b_3, \ldots, b_{2n+1}, h_1, h_3, \ldots, h_{2n-1} \rangle$ is finite and H is not, we can choose $h_{2n+1} \in H \setminus F_2$. Thus the sequences can be constructed.

Observe that since $o(h_{2n+1}) = p$, clearly (4) implies

$$\langle h_{2n+1} \rangle \cap \langle b_1, b_3, \ldots, b_{2n+1}, h_1, h_3, \ldots, h_{2n-1} \rangle = 1$$

for all $n \geq 0$. This together with the fact that $b_{2n-1} \notin \langle b_1, b_3, \ldots, b_{2n-3}, h_1, h_3, \ldots, h_{2n-3} \rangle$ for all $n \geq 1$ implies easily that

$$\hat{h}_1 \hat{h}_3 \cdots \hat{h}_{2n+1}(1 - b_1)(1 - b_3) \cdots (1 - b_{2n-1}) \neq 0$$

for all $n \geq 1$ since the coefficient of 1 is exactly 1.

Finally, by an easy induction (see [7, p. 181] and [1, Lemma 5]), we have for all $n \geq 1$

$$\begin{aligned}
&(1 + \hat{h}_1 g_1,\ 1 + \hat{h}_3 g_2 g_3,\ 1 + \hat{h}_5 g_4 g_5, \ldots, 1 + \hat{h}_{2n+1} g_{2n} g_{2n+1}) \\
&\quad = 1 + \hat{h}_1 \hat{h}_3 \cdots \hat{h}_{2n+1}(g_1, g_2 g_3, \ldots, g_{2n} g_{2n+1}) \\
&\quad = 1 + \hat{h}_1 \hat{h}_3 \cdots \hat{h}_{2n+1}(1 - b_1)(1 - b_3) \cdots (1 - b_{2n-1}) g_1 g_2 \cdots g_{2n+1} \\
&\quad \neq 1.
\end{aligned}$$

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This contradicts the fact that $U(KG)$ is hypercentral, and the proposition is proved.

DEFINITION. If G is a possibly nonabelian group, let us say that a p-element g_0 in G has *infinite p-height* if we can find a g_1 in G such that $g_1^p = g_0$ and recursively a g_{i+1} in G satisfying $g_{i+1}^p = g_i$ for each $i \geq 1$.

In particular notice that if $j \geq i$ then $g_j^{p^{j-i}} = g_i$, so that g_i and g_j commute. Also observe $o(g_i) = p^i o(g_0)$. Thus since $o(g_0)$ is assumed to be a power of p, for $i \geq 1$ the g_i also have infinite p-height. It now becomes apparent that the union of the subgroups

$$\langle g_0 \rangle \leq \langle g_1 \rangle \leq \langle g_2 \rangle \leq \cdots$$

is isomorphic to the Prüfer group $Z(p^\infty)$ (see [5, p. 23]). The converse is easily seen to be true as well: if G contains a subgroup isomorphic to $Z(p^\infty)$, then G contains a nontrivial element of infinite p-height.

LEMMA 4. *Let G be any FC-group, and define $Q(G) = \{g \in G : g$ has infinite p-height$\}$. Then*

(i) $Q(G)$ is a central subgroup of G;
(ii) $Q(G/Q(G)) = 1$;
(iii) if F is a central subgroup of G of finite exponent a power of p, then $Q(G) = 1$ implies $Q(G/F) = 1$; and
(iv) if we further assume G is nilpotent and F is a finite normal p-subgroup of G, then $Q(G) = 1$ implies $Q(G/F) = 1$.

PROOF. (i) Suppose G contains an element g_0 of infinite p-height. Then for each $i \geq 1$, we can choose g_i inductively such that $g_i^p = g_{i-1}$. If $g_0 \notin \zeta(G)$, then we can find an $x \notin C_G(g_0)$. Then since $g_i^{p^i} = g_0$, certainly $x \notin C_G(g_i)$ for all $i \geq 0$. By assumption g_0 is a p-element, so let $o(g_0) = p^r$. Since G is FC, the set $\{x^{g_i} : i \geq 0\}$ is finite. Thus we can produce integers $i, j \geq 0$ such that $j > i + r$ and satisfying $x^{g_i} = x^{g_j}$, that is, $x \in C_G(g_i^{-1}g_j)$. Notice that since $o(g_0) = p^r$, $o(g_i) = p^r p^i = p^{i+r}$. Hence we get $(g_i^{-1}g_j)^{p^{i+r}} = g_j^{p^{i+r}} = g_{j-(i+r)}$, so that $x \in C_G(g_{j-(i+r)})$, a contradiction. This shows g_0 was in fact central.

It remains to show $Q(G)$ is a subgroup of G. Indeed, if g_0 and h_0 both have infinite p-height, there exists $\{g_i\}$ and $\{h_i\}$ of G such that $g_i^p = g_{i-1}$ and $h_i^p = h_{i-1}$ for each $i \geq 1$. Since each g_i and h_i is of infinite p-height, they lie in $Q(G)$ and so are central as shown above. Hence $(g_ih_i)^p = g_i^p h_i^p = g_{i-1}h_{i-1}$ for each $i \geq 1$. Also since g_0 and h_0 are commuting p-elements, g_0h_0 too is a p-element. We now see that g_0h_0 has infinite p-height.

(ii) Suppose $g_0Q(G) \in G/Q(G)$ has infinite p-height. Then there exists h_1 in G such that $\big(h_iQ(G)\big)^p = g_0Q(G)$. Thus $g_0 = h_1^p a_0$ for some a_0 in $Q(G)$. Since $a_0 \in Q(G)$, we can find an a_1 in $Q(G)$ such that $a_1^p = a_0$. Then since $Q(G)$ is central, $g_0 = h_1^p a_0 = h_1^p a_1^p = (h_1a_1)^p$. Let $g_1 = h_1a_1$. Then $g_0 = g_1^p$. Now since $g_1Q(G) = h_1a_1Q(G) = h_1Q(G)$ has infinite p-height in $G/Q(G)$, we can similarly find an h_2 in G and b_1 in $Q(G)$ such that $g_1 = h_2^p b_1$. Since $b_1 \in Q(G)$, we can find $b_2 \in Q(G)$ such that $b_2^p = b_1$. Hence

$g_1 = (h_2b_2)^p$. Let $g_2 = h_2b_2$. Then $g_1 = g_2^p$. Proceeding in this manner shows that we can find recursively a g_i in G such that $g_i^p = g_{i-1}$ for each $i \geq 1$. Finally notice that since $g_0Q(G)$ is a p-element and elements of $Q(G)$ are p-elements, necessarily g_0 is a p-element. Thus $g_0 \in Q(G)$, and so $g_0Q(G) = 1$.

(iii) Suppose $1 \neq g_0F$ has infinite p-height. Then we can find inductively $g_i \in G$ and $f_i \in F$ such that $g_i^p = g_{i-1}f_i$ for $i \geq 1$. Pick j large enough that g_jF has order exceeding p^n, the exponent of F. Set $h_i = g_{j+i}^{p^n}$ for $i \geq 0$. Then $h_i^p = g_{j+i}^{p^{n+1}} = \left(g_{j+i-1}f_{j+i}\right)^{p^n} = g_{j+i-1}^{p^n} = h_{i-1}$ for $i \geq 1$ since F is central. Since g_0F is a p-element and F has exponent p^n, then g_0 and hence $h_0 = g_j^{p^n} \neq 1$ is a p-element. Thus we have shown $1 \neq h_0 \in Q(G)$.

(iv) We use induction on $|F|$. If $|F| = 1$, the result is trivial. So suppose now that the result holds for all nilpotent groups G_1 and finite normal p-subgroups F_1 of G_1 such that $|F_1| < |F|$. Since G is nilpotent and $F \neq 1, F \cap \zeta G \neq 1$. Hence if $F_1 = \frac{F}{F \cap \zeta(G)}$, then $|F_1| < |F|$. Now if $G_1 = \frac{G}{F \cap \zeta(G)}$, then $Q(G_1) = 1$ by (iii), so that by the induction hypothesis, $Q(G_1/F_1) = 1$. But clearly $G_1/F_1 \cong G/F$.

LEMMA 5. *(i) If A is any central p-subgroup of a group G, then $\Delta(G,A)$ is a nil ideal of KG. (ii) If I is any nil ideal of KG, then the natural projection $KG \longrightarrow KG/I$ induces an epimorphism $U(KG) \longrightarrow U(KG/I)$.*

PROOF. (i) By [7, p. 2], we know $\Delta(G,A) = \Delta(A,A)KG$, and any $\delta \in \Delta(G,A)$ can be written as $\delta = \sum_{i=1}^n (a_i - 1)\alpha_i$ where $a_i \in A$ and $\alpha_i \in KG$. Let $p^m = \max_i\{o(a_i)\}$. Then since A is central, we have by Frobenius' theorem $\delta^{p^m} = \sum_{i=1}^n (a_i^{p^m} - 1)\alpha_i^{p^m} = 0$.

(ii) The proof of [7, VI.3.3] uses only the fact that I is nil.

PROPOSITION 3. *Under our assumptions, if G' contains no subgroups isomorphic to the quasicyclic p-group, $Z(p^\infty)$, that is $Q(G') = 1$, then G is nilpotent and G' is a finite p-group.*

PROOF. Let us show first that G is nilpotent. Thus we can assume $G' \neq 1$. By [8, 1.16] since G is FC and hypercentral, $\gamma_\omega(G) \subseteq \zeta(G)$. If $\gamma_\omega(G)$ is infinite, then $G' \cap \zeta(G)$ contains a countably infinite abelian p-group C without nontrivial elements of infinite p-height. By [5, 4.3.15] then C is the direct product of (infinitely many) cyclic groups. However, this cannot be by Proposition 2, so that necessarily $\gamma_\omega(G)$ is finite.

Now since $\gamma_\omega(G)$ is a finite central p-subgroup of G, $Q\big((G/\gamma_\omega(G))'\big) = Q\big(G'/\gamma_\omega(G)\big) = 1$ by Lemma 4(iii). Also by Lemma 5, $\Delta(G,\gamma_\omega(G))$ is nilpotent and hence $U\Big(KG/\Delta\big(G,\gamma_\omega(G)\big)\Big) \cong U\Big(K\big(G/\gamma_\omega(G)\big)\Big)$ is hypercentral. Clearly $G/\gamma_\omega(G)$ has p-elements as G' is a p-group by Lemma 3. Finally since $\gamma_\omega(G) \subseteq \zeta(G)$, and so $G/\gamma_\omega(G)$ is nilpotent if and only if G is nilpotent, we need only examine the case when $\gamma_\omega(G) = 1$.

If G is not nilpotent, then $\gamma_n(G) \neq 1$ for every $n \geq 1$. Thus we can find $1 \neq c_n \in \zeta(G) \cap \gamma_n(G)$ (since G is hypercentral and $1 \neq \gamma_n(G) \triangleleft G$) for every n. Then $\{c_n : n \geq 1\}$ is infinite for otherwise there exists an $i \geq 1$ such that $c_i \in \gamma_n(G)$ for infinitely many n, and hence $1 \neq c_i \in \gamma_\omega(G)$. But then $C = \langle c_n : n \geq 1 \rangle$ is a countably infinite abelian subgroup of $G' \cap \zeta(G)$ without elements of infinite p-height. This contradicts Proposition 2 as above; hence, G is indeed nilpotent.

It remains only to show that G' is finite. Suppose to the contrary that G' is infinite. Since G is nilpotent there then exists an integer r such that $\gamma_r(G)$ is infinite but $\gamma_{r+1}(G)$ is finite. Therefore since $\Delta(G, \gamma_{r+1}(G))$ is nilpotent by [7, I.2.21], by Lemma 5 (ii) $U\big(K(G/\gamma_{r+1}(G))\big)$ is hypercentral and $G/\gamma_{r+1}(G)$ has p-elements. Also $Q(G') = 1$ implies $Q\big(G'/\gamma_{r+1}(G)\big) = 1$ by Lemma 4(iv). Thus since $\big(G/\gamma_{r+1}(G)\big)' = G'/\gamma_{r+1}(G)$ is finite if and only if G' is finite, we may assume $\gamma_{r+1}(G) = 1$. Hence $\gamma_r(G)$ is an infinite central subgroup of G'. As above, this contradicts Proposition 2, so that indeed G' is finite.

COROLLARY. *(i) $(G/Q(G))' \cong G'/G' \cap Q(G)$ is a finite p-group and $G/Q(G)$ is nilpotent.*

(ii) G'' is a finite p-group and G is nilpotent.

PROOF. (i) Since $(G/Q(G))' = G'Q(G)/Q(G) \cong G'/G' \cap Q(G)$ is a p-group by Lemma 3, either $G/Q(G)$ has p-elements or it is abelian. So suppose without loss that $G/Q(G)$ has p-elements. By Lemma 4(ii), $Q(G/Q(G)) = 1$, so that in particular $Q\big((G/Q(G))'\big) = 1$. By Lemmas 4(i) and 5 we also have that $U\big(KG/\Delta(G, Q(G))\big) \cong U\big(K(G/Q(G))\big)$ is hypercentral. Hence by Proposition 3, we are done.

(ii) Since $G' \cap Q(G) \subseteq G' \cap \zeta(G) \subseteq \zeta(G')$, by (i), $G'/\zeta(G')$ is finite. Hence by Schur's lemma, G'' is finite, too. Finally, since $Q(G) \subseteq \zeta(G)$ by Lemma 4(i), $G/Q(G)$ being nilpotent implies that G itself is nilpotent.

LEMMA 6. *If x is an element of order $p^n > 1$ in a group G, then $(1-x)^{p^n-1} = \hat{x}$ in KG.*

PROOF. $(1-x)^{p^n-1} = \sum_{k=0}^{p^n-1} \binom{p^n-1}{k}(-1)^k x^k$ by the binomial theorem. But

$$\begin{aligned}\binom{p^n-1}{k} &= \frac{(p^n-1)!}{k!\,(p^n-1-k)!}\\ &= \frac{(p^n-1)(p^n-2)\cdots(p^n-k)}{k!} \equiv \frac{(-1)^k k!}{k!}\\ &= (-1)^k \qquad \text{modulo } p.\end{aligned}$$

Hence $(1-x)^{p^n-1} = \sum_{k=0}^{p^n-1} x^k = \hat{x}$ as claimed.

PROPOSITION 4. *G' is finite.*

PROOF. Suppose to the contrary that G' is infinite. We first show that it is sufficient to consider only the case when $G' \cong Z(p^\infty)$.

Since G'' is finite by the preceding corollary, we may assume G' is abelian by factoring by G''. Indeed $(G/G'')' = G'/G''$ is finite if and only if G' is finite. Also as G'' is a finite p-group, $\Delta(G, G'')$ is nilpotent by [7, I.2.21]; hence by Lemma 5(ii), $UK(G/G'')$ is hypercentral. Clearly G/G'' has p-elements as G' is a p-group.

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Now by the structure theorem for divisible abelian groups, $G' \cap Q(G)$ is a direct product of copies of $Z(p^\infty)$ [5, 4.1.5]. There can be only finitely many such factors by Lemma 4(i) and Proposition 2. Thus since $G'/G' \cap Q(G)$ is finite by the preceding corollary, G' is an abelian Černikov group, and hence by [5, 5.4.23 and 4.2.11] we have $G' = \left(\mathrm{Dr}_{i=1}^t Q_i\right) \times H$ where each $Q_i \cong Z(p^\infty)$ and H is finite. Since G' was assumed to be infinite, necessarily $t \geq 1$.

We now make the following reductions. Let $A = \mathrm{Dr}_{i=2}^t Q_i$. Then by Lemma 4(i), A is a central p-subgroup of G, so that by Lemma 5, $UK(G/A)$ is hypercentral. Further, $(G/A)' = G'/A \cong Q_1 \times H$. So to derive a contradiction, it suffices to consider $G' = Q_1 \times H$. Next let m be the exponent of H, that is, $H^{p^m} = 1$. Set $B = \{x \in G' : x^{p^m} = 1\}$. Then $B \supseteq H$ and B is a finite characteristic subgroup of G'. Hence $B \triangleleft G$. Since B is a finite normal p-subgroup of G, $UK(G/B)$ is hypercentral as above with G'' replaced by B. But clearly $(G/B)' = G'/B \cong Z(p^\infty)$. Thus indeed it is enough to consider $G' \cong Z(p^\infty)$.

First we consider the case when p is odd. Since $G' \cong Z(p^\infty)$, for each $j \in \mathbb{N}$ choose $q_{2j+1} \in G'$ inductively such that $o(q_1) = p$ and $q_{2j+1}^p = q_{2j-1}$. We now construct a sequence $(g_n) \subseteq G$ with the following properties:

(1) $g_1, g_2 \in G$ with $b_1 = (g_2, g_1) \notin \langle q_1 \rangle$; and

(2) $g_{2n+1}, g_{2n+2} \in C_G\{g_1, g_2, \ldots, g_{2n}\}$ with $b_{2n+1} = (g_{2n+2}, g_{2n+1}) \notin \langle b_{2n-1}, q_{2n+1} \rangle$, for all $n \geq 1$.

Indeed, since $G' \cong Z(p^\infty)$, G' is locally finite. Thus since G' is infinite, there exist infinitely many commutators (x, y), x and y in G. Since $\langle q_1 \rangle$ is finite, such g_1 and g_2 exist. Next suppose $g_1, g_2, \ldots, g_{2n}$ have already been chosen satisfying properties (1) and (2). Let $C = C_G\{g_1, g_2, \ldots, g_{2n}\}$. Then since G is FC, $[\,G : C\,] < \infty$ by Poincare's Theorem. Hence $|G'| = \infty$ implies that $|C'| = \infty$ by [8, 7.5]. Thus C has infinitely many commutators (x, y) with $x, y \in C$ since $C' \subseteq G'$ is locally finite. So now since $\langle b_{2n-1}, q_{2n+1} \rangle$ is finite, $g_{2n+1}, g_{2n+2} \in C$ exist, and such a sequence can be constructed.

We use this sequence in G to construct a new one in $U(KG)$. For each $i \in \mathbb{N}$, set $e_{2i+1} = p - 1$ if there exists no integer $j(i)$ such that $o(b_{2j(i)+1}) = p^{i+1} = o(q_{2i+1})$. If such a $j(i)$ does exist, set $e_{2i+1} = p - 2$. Notice that in the latter case, this $j(i)$ is necessarily unique and less than i by the construction of (g_n). Also as $b_{2j(i)+1} \in G' \cong Z(p^\infty)$, then $b_{2j(i)+1} = q_{2i+1}^{r_{2i+1}}$ for some integer r_{2i+1}, $p \nmid r_{2i+1}$.

We make the following claim:

(i) $\alpha_{2n+1} := \prod_{i=0}^n (1 - q_{2i+1})^{e_{2i+1}} \prod_{i=0}^{n-1} (1 - b_{2i+1}) \neq 0$; and

(ii) $\left(1 - (1 - q_1)^{e_1} g_1,\ 1 - (1 - q_3)^{e_3} g_2 g_3,\ \ldots,\ 1 - (1 - q_{2n+1})^{e_{2n+1}} g_{2n} g_{2n+1}\right) = 1 - (-1)^n \alpha_{2n+1} g_1 g_2 \cdots g_{2n+1}$, for all $n \geq 1$.

Assuming the claim can be established, we then see immediately that $U(KG)$ is not hypercentral, a contradiction that implies G' is in fact finite as required.

PROOF OF CLAIM. First recall by Lemma 4, $G' \cong Z(p^\infty)$ implies that $G' \subseteq \zeta(G)$. Since for $n \geq 1$, $\left((1 - q_{2n+1})^{e_{2n+1}}\right)^{p^{n+1}} = \left(1 - q_{2n+1}^{p^{n+1}}\right)^{e_{2n+1}} = 0$, it is clear that each $1 - (1 - q_{2n+1})^{e_{2n+1}} g_{2n} g_{2n+1} \in U(KG)$. Also since necessarily $e_1 = p - 1$, and thus $2e_1 \geq p$, we have $(1 - (1 - q_1)^{e_1} g_1)^{-1} = 1 + (1 - q_1)^{e_1} g_1$.

Now we show $\alpha_{2n+1} \neq 0$ for all $n \geq 1$. To this end, let $p^m = o(b_{2n-1})$. Then $m-1 \geq n$ since $b_{2n-1} \notin \langle q_{2n-1} \rangle$ and $o(q_{2n-1}) = p^n$. Let $I = \{\, i \mid e_{2i+1} = p-2 \text{ and } 0 \leq i \leq n-1 \,\}$. Then by previous remarks, i lies in I precisely when there exists a unique $j(i)$ such that $0 \leq j(i) < i \leq n-1$ and $b_{2j(i)+1} = q_{2i+1}^{r_{2i+1}}$ with $p \not| r_{2i+1}$. Moreover,

$$\begin{aligned} 1 - b_{2j(i)+1} &= 1 - q_{2i+1}^{r_{2i+1}} \\ &= (1 - q_{2i+1})\big(1 + q_{2i+1} + q_{2i+1}^2 + \cdots + q_{2i+1}^{r_{2i+1}-1}\big). \end{aligned}$$

Also notice for $0 \leq i \leq m-1$, $o(q_{2i+1}) \mid p^m$, so that if $e_{2i+1} = p-2$, then $j(i) \leq n-1$ since $o(b_{2n-1}) = p^m$. Hence

$$\begin{aligned} \beta &:= \prod_{i=0}^{m-1} (1 - q_{2i+1})^{e_{2i+1}} \prod_{i=0}^{n-1} (1 - b_{2i+1}) \\ &= \prod_{i=0}^{m-1} (1 - q_{2i+1})^{p-1} \prod_{i \in I} \big(1 + q_{2i+1} + q_{2i+1}^2 + \cdots + q_{2i+1}^{r_{2i+1}-1}\big) \\ &= \Big(\prod_{i \in I} r_{2i+1}\Big) \hat{q}_{2m-1} \neq 0 \end{aligned}$$

since

$$\begin{aligned} \prod_{i=0}^{m-1} (1 - q_{2i+1})^{p-1} &= \prod_{i=0}^{m-1} (1 - q_{2m-1})^{p^{m-i-1}(p-1)} \\ &= (1 - q_{2m-1})^{(p-1)\sum_{i=0}^{m-1} p^i} = (1 - q_{2m-1})^{p^m - 1} = \hat{q}_{2m-1} \end{aligned}$$

by Lemma 6. But since $m-1 \geq n$, α_{2n+1} divides β, so that $\alpha_{2n+1} \neq 0$ either.

It remains to show (ii). To see this, let us first prove $(1 - q_{2n+1})^{e_{2n+1}} \alpha_{2n+1} = 0$. Let I be as above. Then

$$\begin{aligned} &(1 - q_{2n+1})^{e_{2n+1}} \alpha_{2n+1} \\ &= (1 - q_{2n+1})^{2e_{2n+1}} \prod_{i=0}^{n-1} (1 - q_{2i+1})^{p-1} \prod_{i \in I} \big(1 + q_{2i+1} + q_{2i+1}^2 + \cdots + q_{2i+1}^{r_{2i+1}-1}\big) \prod_{\substack{i=0 \\ i \notin I}}^{n-1} (1 - b_{2i+1}). \end{aligned}$$

But as above $\prod_{i=0}^{n-1} (1 - q_{2i+1})^{p-1} = \hat{q}_{2n-1}$. We have two cases to consider. If $e_{2n+1} = p-1$, then $2e_{2n+1} \geq p$, and hence $(1 - q_{2n+1})^{2e_{2n+1}} = (1 - q_{2n-1})\gamma$, say, annihilates $\hat{q}_{2n-1}$. If $e_{2n+1} = p-2$, then there exists $j(n) < n$ not in I (by its uniqueness) such that $b_{2j(n)+1} = q_{2n+1}^{r_{2n+1}}$ for some $p \not| r_{2n+1}$. Then $(1 - q_{2n+1})^{e_{2n+1}} \alpha_{2n+1}$ contains the factor

$$\begin{aligned} &\hat{q}_{2n-1}(1 - q_{2n+1})^{2e_{2n+1}}(1 - b_{2j(n)+1}) \\ &\qquad = \hat{q}_{2n-1}(1 - q_{2n+1})^{2(p-2)}\big(1 - q_{2n+1})(1 + q_{2n+1} + q_{2n+1}^2 + \cdots + q_{2n+1}^{r_{2n+1}-1}\big) = 0 \end{aligned}$$

since $2p - 3 \geqq p$ for $p \geq 3$. This concludes the proof that $(1 - q_{2n+1})^{e_{2n+1}} \alpha_{2n+1} = 0$.

We now prove (ii) by induction on $n \geq 1$. For $n = 1$, we have

$$\begin{aligned}
&(1-(1-q_1)^{e_1}g_1,\ 1-(1-q_3)^{e_3}g_2g_3)\\
&\quad = (1-(1-q_1)^{e_1}g_1-(1-q_3)^{e_3}g_2g_3+(1-q_1)^{e_1}(1-q_3)^{e_3}g_1g_2g_3)\cdot\\
&\qquad\qquad (1+(1-q_1)^{e_1}g_1)(1+(1-q_3)^{e_3}g_2g_3+(1-q_3)^{2e_3}g_2^2g_3^2+\cdots)\\
&\quad = (1-(1-q_3)^{e_3}g_2g_3+(1-q_1)^{e_1}(1-q_3)^{e_3}(1-b_1)g_1g_2g_3)\cdot\\
&\qquad\qquad (1+(1-q_3)^{e_3}g_2g_3+(1-q_3)^{2e_3}g_2^2g_3^2+\cdots)\\
&\quad = 1+(1-q_1)^{e_1}(1-q_3)^{e_3}(1-b_1)g_1g_2g_3,
\end{aligned}$$

as required, since $(1-q_3)^{e_3}\alpha_3 = 0$. Now consider $n > 1$. By induction we have

$$\begin{aligned}
&(1-(1-q_1)^{e_1}g_1,\ 1-(1-q_3)^{e_3}g_2g_3,\ldots,1-(1-q_{2n+3})^{e_{2n+3}}g_{2n+2}g_{2n+3})\\
&\quad = (1-(-1)^n\alpha_{2n+1}g_1g_2\cdots g_{2n+1},\ 1-(1-q_{2n+3})^{e_{2n+2}}g_{2n+2}g_{2n+3})\\
&\quad = (1-(-1)^n\alpha_{2n+1}g_1g_2\cdots g_{2n+1}-(1-q_{2n+3})^{e_{2n+3}}g_{2n+2}g_{2n+3}\\
&\qquad\qquad +(-1)^n(1-q_{2n+3})^{e_{2n+3}}\alpha_{2n+1}g_1g_2\cdots g_{2n+3})\cdot\\
&\quad (1+(-1)^n\alpha_{2n+1}g_1g_2\cdots g_{2n+1})\cdot(1-(1-q_{2n+3})^{e_{2n+3}}g_{2n+2}g_{2n+3})^{-1}\\
&\quad = (1-(1-q_{2n+3})^{e_{2n+3}}g_{2n+2}g_{2n+3}\\
&\qquad\qquad +(-1)^n(1-q_{2n+3})^{e_{2n+3}}\alpha_{2n+1}g_1g_2\cdots g_{2n+3}\\
&\qquad\qquad -(-1)^n(1-q_{2n+3})^{e_{2n+3}}\alpha_{2n+1}g_{2n+2}g_{2n+3}g_1g_2\cdots g_{2n+1})\cdot\\
&\quad (1-(1-q_{2n+3})^{e_{2n+3}}g_{2n+2}g_{2n+3})^{-1}\\
&\quad = (1-(1-q_{2n+3})^{e_{2n+3}}g_{2n+2}g_{2n+3}\\
&\qquad\qquad +(-1)^n(1-q_{2n+3})^{e_{2n+3}}\alpha_{2n+1}(1-b_{2n+1})g_1g_2\cdots g_{2n+3})\cdot\\
&\qquad\qquad (1+(1-q_{2n+3})^{e_{2n+3}}g_{2n+2}g_{2n+3}+(1-q_{2n+3})^{2e_{2n+3}}g_{2n+2}^2g_{2n+3}^2+\cdots)\\
&\quad = 1-(-1)^{n+1}\alpha_{2n+3}g_1g_2\cdots g_{2n+3}
\end{aligned}$$

since $\alpha_{2n+1}^2 = 0$ and $(1-q_{2n+3})^{e_{2n+3}}\alpha_{2n+3} = 0$. This concludes the proof of the proposition for the odd p case.

Finally consider when $p = 2$. Recall that we can assume $G' \cong Z(2^\infty)$. Let $q \in G'$ be such that $o(q) = 2$. Construct a sequence (g_n) in G as follows:

(1) $g_1, g_2 \in G$ with $b_1 = (g_2, g_1) \notin \langle q \rangle$; and

(2) $g_{2n+1}, g_{2n+2} \in C_G\{g_1, g_2, \cdots, g_{2n}\}$ with $b_{2n+1} = (g_{2n+2}, g_{2n+1}) \notin \langle b_{2n-1} \rangle^{1/2}$.

In a similar fashion to the odd case, this can be done by using the fact that G is an FC-group implies that G' is locally finite, so that G and hence $C_G\{g_1, g_2, \ldots, g_{2n}\}$ contain infinitely many commutators, while $\langle b_{2n-1} \rangle^{1/2} \subseteq Z(2^\infty)$ is finite.

We construct another sequence (q_{2n+1}) in G'. Let $q_1 = q$. For $n \geq 1$, pick $q_{2n+1} \in G'$ such that $q_{2n+1}^2 = q_{2n-1}$ unless there exists a $j(n) \in \mathbb{N}$ such that $o(b_{2j(n)-1}) = 2o(q_{2n-1})$, in which case choose q_{2n+1} such that $q_{2n+1}^4 = q_{2n-1}$. Observe that if $o(b_{2j(n)-1}) = 2o(q_{2n-1})$, then $o(b_{2j(n)+1}) \neq 4o(q_{2n-1})$ for otherwise $b_{2j(n)+1} \in \langle b_{2j(n)-1} \rangle^{1/2}$, contrary to (2). Thus for each $n \geq 1$ there is precisely one "q" or "b" with order 2^n. Also (1) and (2) imply $2o(q_{2i+1}) \leq o(b_{2j+1})$ and $4o(b_{2i+1}) \leq o(b_{2j+1})$ for all $i < j$.

We claim now that

(i) $\alpha_{2n+1} := \prod_{i=0}^{n}(1+q_{2i+1})\prod_{i=0}^{n-1}(1+b_{2i+1}) \neq 0$ and

(ii) $(1+(1+q_1)g_1,\ 1+(1+q_3)g_2g_3, \ldots, 1+(1+q_{2n+1})g_{2n}g_{2n+1}) = 1+\alpha_{2n+1}g_1g_2\cdots g_{2n+1}$, for all $n \geq 1$.

Obviously, this would contradict that $U(KG)$ is hypercentral, and so G' would be finite.

To prove the claim, first observe that if m is minimal such that $2o(b_{2n-1}) \leq o(q_{2m+1})$, then $\beta = \prod_{i=0}^{m}(1+q_{2i+1})\prod_{i=0}^{n-1}(1+b_{2i+1})$ is a nonzero scalar multiple of $\hat{q}_{2m+1}$. Indeed q_{2m+1} is the most primitive element appearing in the expression for β, and by construction precisely one element of each lesser order also appears. Then if x is one of the q's or b's appearing in the expression, $1+x = 1+q_{2m+1}^{2^j r}$ for some integers r and j, where r is odd and j is unique to x. Hence $1+x = (1+q_{2m+1})^{2^j}(1+q_{2m+1}^{2^j}+\cdots+q_{2m+1}^{2^j(r-1)})$. Now reasoning as in the odd case, Lemma 6 gives us $\beta = s\hat{q}_{2m+1} \neq 0$ for some integer s. In particular, since by the remarks above $m \geq n$, we see that α_{2n+1} divides β and thus cannot be zero.

It remains to prove (ii). To do this, we first show that $(1+q_{2n+1})\alpha_{2n+1} = 0$. Let m be maximal such that $2o(b_{2m-1}) \leq o(q_{2n+1})$, and let $\gamma = \prod_{i=0}^{n}(1+q_{2i+1})\prod_{i=0}^{m-1}(1+b_{2i+1})$, where the latter product is considered to be 1 if no such m exists. Then $\gamma = s\hat{q}_{2n+1}$ for some integer s by precisely the same arguments used on β above. Since necessarily $m \leq n$, γ divides α_{2n+1}. Thus since $(1+q_{2n+1})\gamma = 0$, we see $(1+q_{2n+1})\alpha_{2n+1} = 0$. An easy induction using this fact proves (ii).

This finishes the proof of the proposition.

Finally, notice that the proof of the theorem now follows at once by combining the results of Lemma 3 and Propositions 3 and 4.

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