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Infinite multiplicity of stable entire solutions for a semilinear elliptic equation with exponential nonlinearity

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We consider the infinite multiplicity of entire solutions for the elliptic equation $\Delta u + K(x)e^u + \mu f(x) = 0$ in $\mathbb{R}^n$, $n \geqslant 3$. Under suitable conditions on K and f, the equation with small $\mu \geqslant 0$ possesses a continuum of entire solutions with a specific asymptotic behaviour. Typically, K behaves like $|x|^\ell$ at ∞ for some $\ell > -2$ and the entire solutions behave asymptotically like $-(2+\ell)\log|x|$ near ∞. Main tools of the analysis are comparison principle for separation structure, asymptotic expansion of solutions near ∞, barrier method and strong maximum principle. The linearized operator for the equation has two characteristic behaviours related with the stability and the weak asymptotic stability of the solutions as steady states for the corresponding parabolic equation.

Keywords: semilinear elliptic equation; exponential nonlinearity; stable entire solution; separation; infinite multiplicity.

2010 *Mathematics subject classification:* 35J61; 35B08; 35B35; 35B40

1. Introduction

In this paper, we study the elliptic equation

$$\Delta u + K(x)e^u + \mu f(x) = 0, \tag{1.1}$$

where $n > 2$, $\Delta = \sum_{i=1}^n ((\partial^2)/(\partial x_i^2))$ is the Laplace operator, $\mu \geqslant 0$ is a parameter, and f as well as K is a locally Hölder continuous function in $\mathbb{R}^n \setminus \{0\}$. By an entire solution of (1.1), we mean a weak solution in $\mathbb{R}^n$ satisfying (1.1) pointwise in $\mathbb{R}^n \setminus \{0\}$. Recent studies in [**1**, **2**, **5**, **6**] considered the existence of a continuum of positive entire solutions to the equation

$$\Delta u + K(x)u^p + \mu f(x) = 0 \tag{1.2}$$

when p is sufficiently large. The positive entire solutions are stable under a topology determined by the asymptotic behaviour near infinity. Hence, an interesting question is to ask the existence of stable entire solutions to (1.1). The purpose of the paper is to establish the existence and look for a suitable topology for the stability. In order to construct such entire solutions, we make use of entire solutions of the homogeneous equation

$$\Delta u + K(x)e^u = 0. \tag{1.3}$$

The method needs detailed information on the asymptotic behaviour of entire solutions of (1.3). We first study the radial version of (1.3),

$$u_{rr} + \frac{n-1}{r}u_r + K(r)e^u = 0 \tag{1.4}$$

with $r = |x|$, under the following condition:

$$\text{(Kr)} \begin{cases} K(r) \text{ is continuous on } (0,\infty), \\ K(r) \geqslant 0 \text{ and } K(r) \not\equiv 0 \text{ on } (0,\infty), \\ \int_0 rK(r)\,\mathrm{d}r < \infty. \end{cases}$$

From now on, we assume that (Kr) contains the radial symmetry of K. It is well-known that (1.4) with $u(0) = \alpha \in \mathbb{R}$ has a unique solution $u \in C^2(0,\varepsilon) \cap C[0,\varepsilon)$ for small $\varepsilon > 0$. By $u_\alpha(r)$ we denote the unique local solution with $u_\alpha(0) = \alpha$. The typical equation of (1.4) is

$$\Delta u + c|x|^\ell e^u = 0 \tag{1.5}$$

where $c > 0$ and $\ell > -2$, and its radial version is

$$u_{rr} + \frac{n-1}{r}u_r + cr^\ell e^u = 0. \tag{1.6}$$

We denote by $\overline{u}_\alpha(r)$ the solution of (1.6) with $\overline{u}_\alpha(0) = \alpha$. It is easy to see that (1.6) has the scale invariance by

$$\overline{u}_\alpha(r) = \alpha + \overline{u}_0(e^{\alpha/(2+\ell)}r), \tag{1.7}$$

and the invariant singular solution is

$$\overline{U}_c(r) := -(2+\ell)\log r + \log(2+\ell)(n-2) - \log c.$$

We call this behaviour the self-similarity. For every α, $\overline{u}_\alpha$ has the asymptotic self-similarity, that is, $\overline{u}_\alpha(r) = \overline{U}_c(r) + o(1)$ at ∞. See [**9**] for the asymptotic behaviour when $c = 1$ and $\ell = 0$. The result holds even for (1.4) if K satisfies $r^{-\ell}K(r) \to c$ at ∞. The following assertion in [**4**, theorems 1.1, 1.2] explains the existence and the asymptotic behaviour under the condition:

(M) $r^{-\ell}K(r)$ is non-increasing in $(0,\infty)$ for given $\ell > -2$.

THEOREM A. *Let $n > 2$ with $\ell > -2$. Assume that K satisfies (Kr) and (M). For every α, (1.4) has an entire solution u_α such that $u_\alpha(r) + (2+\ell)\log r$ is bounded below. If $r^{-\ell}K(r) \to c$ at ∞ for some $c > 0$, then u_α has the asymptotic behaviour*

$$\lim_{r\to\infty}\left[u_\alpha(r) - \log\frac{(2+\ell)(n-2)}{cr^{2+\ell}}\right] = 0. \tag{1.8}$$

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The next question is whether the entire solutions are stable in a proper sense or not. For (1.5) with $\ell = 0$, the stability was studied in [**10**] which is motivated by the work in [**8**] for Lane-Emden equation

$$\Delta u + u^p = 0. \tag{1.9}$$

In [**8**], the separation of positive solutions for (1.9) is a basic tool for the stability. If $n \geqslant 10 + 4\ell$, (1.6) has the separation structure. Namely, any two distinct solutions have no intersection point. In [**4**, theorem 1.5, proposition 4.1], (M) for (1.4) turns out to be a sufficient condition to maintain the separation property.

Theorem B. *Let $\ell > -2$. Assume that K satisfies (Kr) and (M). Then, (1.4) has an entire solution for each $\alpha \in \mathbb{R}$. Moreover, entire solutions have the following property.*

(i) *For $2 < n < 10 + 4\ell$, if $r^{-\ell}K(r) \to c$ at ∞ for some $c > 0$, then two entire solutions u_α and u_β of (1.4) with $\alpha < \beta$ intersect infinitely many times.*

(ii) *For $n \geqslant 10 + 4\ell$, any two entire solutions of (1.4) do not intersect each other.*

For case (ii), it is known in [**4**, theorem 1.5] that the supremum of regular solutions is a singular solution.

Theorem C. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Assume that K satisfies (Kr) and (M). Then, (1.4) possesses a singular solution U satisfying*

$$e^{u_\alpha(r)} < e^{U(r)} \leqslant \frac{(2+\ell)(n-2)}{r^2K(r)}. \tag{1.10}$$

Moreover, U is the monotone upper limit of entire solutions as $\alpha \uparrow \infty$.

Theorems A, B and C motivate the present work to analyse further the separation property of entire solutions. In particular, we focus on entire solutions satisfying (1.8). The separation structure in (ii) is clarified by analysing the asymptotic behaviour. In order to describe the asymptotic behaviour of solutions at ∞, we introduce the following two numbers

$$\lambda_1 = \lambda_1(n,\ell) = \frac{(n-2) - \sqrt{(n-2)(n-10-4\ell)}}{2}$$

and

$$\lambda_2 = \lambda_2(n,\ell) = \frac{(n-2) + \sqrt{(n-2)(n-10-4\ell)}}{2}.$$

Note that λ_1 and λ_2 are the two real roots of the quadratic polynomial $P(z) = z^2 - (n-2)z + (\ell+2)(n-2)$ if and only if $n \geqslant 10 + 4\ell$. Then, $\lambda_1 \leqslant \lambda_2$.

Setting

$$\mathcal{D}(\alpha, r) := \begin{cases} r^{\lambda_1}\left(u_\alpha(r) - \log \frac{(2+\ell)(n-2)}{cr^{2+\ell}}\right) & \text{if} \quad n > 10+4\ell, \\ r^{\lambda_1}(\log r)^{-1}\left(u_\alpha(r) - \log \frac{(2+\ell)(n-2)}{cr^{2+\ell}}\right) & \text{if} \quad n = 10+4\ell, \end{cases} \tag{1.11}$$

we observe that $\mathcal{D}(\alpha, r)$ converges to a continuous function $D(\alpha)$ as $r \to \infty$ under the integral condition

$$\int_1^\infty |r^{-\ell}K(r) - c|r^{-1+\lambda_1}\,\mathrm{d}r < \infty \tag{1.12}$$

for some $c > 0$. For example,

$$K(r) = cr^\ell + O(r^{\ell-\lambda_1}(\log r)^{-\theta}) \quad \text{near} \quad \infty$$

for some $\theta > 1$. When u_α satisfies (1.8), (1.12) is sufficient for the existence of $D(\alpha)$. Moreover, (1.12) is also a necessary condition for any entire solution u_α to have the limit $D(\alpha)$ provided that $r^{-\ell}K(r) \geqslant c$ on $(0, \infty)$.

Theorem 1.1. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Assume that K satisfies (Kr) and $r^{-\ell}K(r) \geqslant c$ on $(0, \infty)$ for some $c > 0$. Then, (1.12) is a necessary and sufficient condition for any entire solution u_α of (1.4) to be such that $\mathcal{D}(\alpha, r)$ has finite limit $D(\alpha)$.*

The function $D(\alpha)$ for (1.6) exhibits the relation

$$D(\alpha) = e^{-((\alpha)/(2+\ell))\lambda_1}D(0) < 0 \tag{1.13}$$

due to (1.7). In the context of theorem B for $n \geqslant 10 + 4\ell$ and (1.8), the behaviour of the limit D can be described in detail. In particular, $D(\alpha)$ is *strictly* increasing as α increases.

Theorem 1.2. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Assume that K satisfies (Kr) and (1.12) for some $c > 0$. Then, there exists $\alpha^* \in (-\infty, +\infty]$ such that for each $\alpha < \alpha^*$, (1.4) has an entire solution u_α satisfying (1.8) and $u_\beta < u_\alpha$ for $\beta < \alpha < \alpha^*$. Moreover, $\mathcal{D}(\alpha, r)$ converges to a continuous and strictly increasing negative function $D(\alpha)$ in $\alpha \in (-\infty, \alpha^*)$ as $r \to \infty$. In addition, if K satisfies (M), then $\alpha^* = +\infty$.*

The continuity of D overcomes the lack of compactness due to the space $\mathbb{R}^n$. Each solution obtained in theorem 1.2 is identified by the value of D when it is strictly increasing. These two properties enable us to establish the existence of a continuum of entire solutions even for (1.3) without any sign condition of K in compact regions, and even when K is not radially symmetric.

Theorem 1.3. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Assume that K satisfies*

(K1) *$K(x) = O(|x|^\sigma)$ at $x = 0$ for some $c > 0$ and $\sigma > -2$,*

(K2) *$|x|^{-\ell}K(x) = c + O(|x|^{-\lambda_1}(\log|x|)^{-\theta})$ near ∞ for some $c > 0$ and $\theta > 1$.*

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Then, (1.3) possesses a continuum $\mathcal{C}$ of entire solutions with (1.8). Moreover, there exists an infinite subset $\mathcal{S} \subset \mathcal{C}$ such that any two in $\mathcal{S}$ do not intersect.

More generally, we consider the existence for the inhomogeneous equation. The corresponding radial equation is of the form

$$u_{rr} + \frac{n-1}{r} u_r + K(r)e^u + \mu f(r) = 0. \tag{1.14}$$

We denote by $u_{\mu,\alpha}(r)$ the solution of (1.14) with $u_{\mu,\alpha}(0) = \alpha$. In order to regard (1.14) as a perturbation of (1.4), we impose the following hypotheses on K and f:

(KR) $K(r)$ is continuous on $(0, \infty)$ and $\int_0 r|K(r)| < \infty$;

(fR1) $f(r)$ is continuous on $(0, \infty)$ and $\int_0 r|f(r)| < \infty$;

(fR2) $f(r) = O(r^{-(\lambda_1+2)}(\log r)^{-\delta})$ near ∞ for a constant $\delta > 1$.

We make use of two classes $MI(c)$ and $M(c)$ to verify the separation structure for (1.14). The first class $MI(c)$ is the set of K with (Kr) which satisfies (1.12) for some $c > 0$. The second class $M(c)$ is the subset of $MI(c)$ whose element satisfies (M) also. Set $D_\mu(\alpha)$ as the limit $\mathcal{D}_\mu(\alpha, r)$ defined in the same way as in (1.11) where u_α is replaced by $u_{\mu,\alpha}$.

THEOREM 1.4. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Assume (KR), (1.12) for some $c > 0$ and (fR1,2). Then, there exists $\mu^* > 0$ with the property that for fixed $0 < \mu < \mu^*$, there exists an interval $I_\mu = (\alpha_\mu, \beta_\mu)$, $-\infty \leqslant \alpha_\mu < \beta_\mu \leqslant +\infty$, such that for each $\xi \in I_\mu$, (1.14) has an entire solution $u_{\mu,\xi}$ satisfying (1.8), and any two solutions among them are separated. Moreover, the limit $D_\mu(\alpha)$ is a continuous and increasing function in $\alpha \in I_\mu$. If K is bounded above by a function $\tilde{K}$ in $M(c)$, then D_μ is strictly increasing. If $K \geqslant 0$ and $f \geqslant 0$, then $\alpha_\mu = -\infty$. In addition, if K satisfies (M), then $\beta_\mu = +\infty$, and (1.14) has a singular solution U_μ described by the monotone upper limit of $u_{\mu,\alpha}$ as $\alpha \uparrow +\infty$ and (1.10) holds for $u_{\mu,\alpha}$ and U_μ.*

Infinite multiplicity for the non-radial inhomogeneous equation improves theorem 1.3.

THEOREM 1.5. *Let $n \geqslant 10 + 4\ell$. Assume that K satisfies (K1), (K2), and f satisfies*

(f1) $f(x) = O(|x|^\tau)$ *at $x = 0$ for some $\tau > -2$.*

(f2) $f(x) = O(|x|^{-(\lambda_1+2)}(\log|x|)^{-\vartheta})$ *near ∞*

for a constant $\vartheta > 1$. Then, there exists $\mu_* > 0$ such that for every $\mu \in [0, \mu_*)$, (1.1) possesses a continuum $\mathcal{C}$ of entire solutions satisfying (1.8).

This paper is the counterpart of several works for the equation with the non-linearity of Ku^p type. When known arguments for Ku^p type can be applied properly to (1.3) and (1.1), we state the arguments and the corresponding results. This paper is organized as follows. We first consider the nonexistence of positive solutions on bounded domains, and the existence of local solutions for given intervals in § 2. In § 3, we present a comparison principle which plays a fundamental role in verifying separation of solutions for (1.14) when K has not one sign. We explain the separation structure under (M). In § 4, we describe the asymptotic behaviour of solutions of (1.4) under (1.12). In § 5, we study the infinite multiplicity for (1.3) and (1.1). In order to establish theorem 1.5, we analyse further the asymptotic behaviour. The solutions obtained in theorem 1.5 are characterized by D. This makes it possible to confirm the existence of a continuum of solutions. theorem 1.1 and a part of theorem 1.3 are verified in § 5.1 while theorems 1.2, 1.4 and 1.5 are proved in § 5.2. Finally, we study the stability property in § 6. The asymptotic behaviour suggests a weighted L^∞ norm for the weak asymptotic stability.

2. Preliminaries

We first consider a necessary condition for the existence of positive solutions to the equation

$$\Delta u + e^u + f(x) = 0 \tag{2.1}$$

on bounded domains. The following result shows that (2.1) cannot have positive solutions on a given bounded domain if f is sufficiently large.

Lemma 2.1. *Let $\Omega \neq \emptyset$ be a bounded domain with C^2 boundary such that*

$$\inf_\Omega f(x) \geqslant \begin{cases} -1 & \text{if } \quad 0 < \theta_1 \leqslant 1, \\ \theta_1(\log \theta_1 - 1) & \text{if } \quad \theta_1 > 1, \end{cases} \tag{2.2}$$

where θ_1 is the first eigenvalue of $-\Delta$ on Ω with zero Dirichlet boundary conditions. Then, (2.1) does not possess any positive solution in Ω.

Proof. Let φ_1 be the corresponding first eigenfunction which is positive and normalized by $\int_\Omega \varphi_1 = 1$. Multiplying (2.1) by φ_1 and integrating, we have

$$\theta_1 \int_\Omega u\varphi_1 + \int_{\partial\Omega} u \frac{\partial \varphi_1}{\partial \nu} = \int_\Omega f\varphi_1 + \int_\Omega e^u \varphi_1$$

where ν is the unit outward normal to $\partial\Omega$. Then, using $((\partial\varphi_1)/(\partial\nu)) < 0$ on $\partial\Omega$ and Jensen's inequality, we obtain

$$\begin{aligned} \theta_1 \int_\Omega u\varphi_1 &\geqslant \inf_\Omega f(x) - \int_{\partial\Omega} u \frac{\partial\varphi_1}{\partial\nu} + \int_\Omega e^u\varphi_1 \\ &> \inf_\Omega f(x) + \exp\left(\int_\Omega u\varphi_1\right). \end{aligned}$$

If $0 < \theta_1 \leqslant 1$, then $\theta_1 s - e^s < -1$ for $s > 0$. Hence, $\inf_\Omega f(x) < -1$. If $\theta_1 > 1$, then

$$\inf_\Omega f(x) < \max_{s>0} (\theta_1 s - e^s) = \theta_1(\log\theta_1 - 1),$$

which contradicts (2.2). □

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Now, we study the local existence of (1.14) with (KR) and (fR1). Let $u_{\mu,\alpha}(r)$ denote the unique local solution with $u_{\mu,\alpha}(0) = \alpha$ where it exists and belongs to $C^2(0,\varepsilon) \cap C[0,\varepsilon)$ for small $\varepsilon > 0$. We present the proof on the existence of local solutions in a given interval.

THEOREM 2.2. *Let $R > 0$ and $0 < \xi < 1$. Assume that continuous functions K and f on $(0, R)$ satisfy (KR) and (fR1), respectively. For each $\mu \geqslant 0$, there exists $\tilde{\alpha} < 0$ such that for each $\alpha < \tilde{\alpha}$, (1.14) has a radial solution $u_{\mu,\alpha}$ on $(0, R)$ and $(2 - \xi)\alpha \leqslant u_{\mu,\alpha}(r) \leqslant \xi\alpha$ on $[0, R]$.*

Proof. For given $\alpha < 0$ and $0 < \xi < 1$, setting a space

$$S_R := \{u \in C[0, R] \mid (2 - \xi)\alpha \leqslant u \leqslant \xi\alpha\},$$

we consider a nonlinear operator T from S_R to $C[0, R]$ by

$$T(u)(r) := \alpha - T_1(u)(r), \tag{2.3}$$

where for $r \in [0, R]$,

$$T_1(u)(r) := \int_0^r \frac{1}{s^{n-1}} \int_0^s t^{n-1} \left(K(t)e^u + \mu f(t)\right) \mathrm{d}t \, \mathrm{d}s.$$

Changing the order of integration gives that

$$T_1(u)(r) = \frac{1}{n-2} \int_0^r t \left\{1 - \left(\frac{t}{r}\right)^{n-2}\right\} \left(K(t)e^u + \mu f(t)\right) \mathrm{d}t \tag{2.4}$$

and thus,

$$||T_1(u)|| \leqslant \frac{1}{n-2} \int_0^R t \left(e^{\xi\alpha}|K(t)| + \mu|f(t)|\right) \mathrm{d}t. \tag{2.5}$$

In order to have $T(S_R) \subset S_R$, we need the inequality

$$\frac{1}{n-2} \int_0^R t \left(e^{\xi\alpha}|K(t)| + \mu|f(t)|\right) \mathrm{d}t \leqslant (\xi - 1)\alpha. \tag{2.6}$$

We may regard (2.6) as the inequality of the form

$$Ae^{\xi\alpha} + B\mu \leqslant -(1 - \xi)\alpha, \tag{2.7}$$

which holds for α near $-\infty$. Combining (2.3) and (2.5), we have $(2 - \xi)\alpha \leqslant T(u)(r) \leqslant \xi\alpha$ and $T(S_R) \subset S_R$. If α is near $-\infty$, we may choose $0 < \delta < 1$ such that

$$||T(u_2) - T(u_1)|| \leqslant \frac{1}{n-2}\int_0^R te^{\xi\alpha}|K(t)|\,\mathrm{d}t\,||u_2-u_1|| \leqslant \delta||u_2-u_1||.$$

Hence, T is a contraction mapping in S_R and thus T has a unique fixed point $\tilde{u}_\alpha$. In other words, $\tilde{u}_\alpha$ satisfies

$$\tilde{u}_\alpha(r) = \alpha - \int_0^r\int_0^s \left(\frac{t}{s}\right)^{n-1}\left(K(t)e^{u_\alpha(t)} + \mu f(t)\right)\,\mathrm{d}t\,\mathrm{d}s.$$

Then, it is easy to see that $\tilde{u}_\alpha$ is also a solution of (1.14) on $(0,R)$ with $\tilde{u}_\alpha(0)=\alpha$. Hence, we have $\tilde{u}_\alpha = u_{\mu,\alpha}$, which completes the proof. □

Remark 2.3. Let α be given. By (KR) and (fR1), the operator norm of T_1 in (2.4) can be arbitrary small on $[0,R]$ if $R>0$ is sufficiently small. Then, we apply the contraction mapping principle to T. Hence (1.14) has a local solution u_α on $(0,R)$. For $r>0$ sufficiently small, we have

$$r^{n-1}u'_{\mu,\alpha}(r) = -\int_0^r s^{n-1}\left(K(s)e^{u_{\mu,\alpha}} + \mu f(s)\right)\,\mathrm{d}s,$$

$$\left|r^{n-1}u'_{\mu,\alpha}(r)\right| \leqslant r^{n-2}\int_0^r s\left(e^{\alpha+1}|K(s)| + \mu|f(s)|\right)\,\mathrm{d}s,$$

where $u'_{\mu,\alpha}(r) = ((\mathrm{d})/(\mathrm{d}r))u_{\mu,\alpha}(r)$. Then, (KR) and (fR1) imply that $\lim_{r\to 0} ru'_{\mu,\alpha}(r)=0$. In addition, if

$$\lim_{r\to 0} r^{1-n}\int_0^r s^{n-1}K(s)\,\mathrm{d}s = 0 = \lim_{r\to 0} r^{1-n}\int_0^r s^{n-1}f(s)\,\mathrm{d}s,$$

then $u'_{\mu,\alpha}(0)=0$. We use the notation u' instead of u_r whenever some subscripts are employed to specify u.

Let $R(\alpha)$ be the supremum of $R>0$, where u_α satisfies the result of theorem 2.2 in $B(R)$. It follows from (2.7) that $R(\alpha)\to\infty$ as $\alpha\to-\infty$. Indeed, (2.7) holds for α near $-\infty$, even if A, B in (2.7) are very large. We state the fact separately in the following lemma.

Lemma 2.4. *$R(\alpha)\to\infty$ as $\alpha\to-\infty$.*

3. Separation

In this section, we consider the separation of solutions of (1.14). By $\tilde{u}_\alpha$ with $\tilde{u}_\alpha(0)=\alpha$, we denote the solution of the equation

$$u_{rr} + \frac{n-1}{r}u_r + \tilde{K}(r)e^u = 0, \tag{3.1}$$

where $\tilde{K}$ satisfies (Kr). Separation of solutions for (1.14) may follow from the existence of two separated solutions of the homogeneous equation. We refer the

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reader to the arguments in [**2**, lemma 4.1]. For the sake of completeness, we provide the proof.

Lemma 3.1. *Assume that K and f satisfy (KR) and (fR1), respectively, and $K \leqslant \tilde{K}$, and moreover, for some $\xi > \beta$ there exist two entire solutions $\tilde{u}_\xi$, $\tilde{u}_\beta$ of (3.1) satisfying $\tilde{u}_\beta(0) = \beta$, $\tilde{u}_\xi(0) = \xi$ and $\tilde{u}_\beta < \tilde{u}_\xi$. If for $\alpha < \eta < \beta$, u_α and u_η are the solutions of (1.14) satisfying $u_\eta \leqslant \tilde{u}_\beta$ in $(0, R_\eta)$ for some $R_\eta > 0$, then $u_\alpha < u_\eta$ in $(0, R_\eta)$.*

Proof. Suppose that u_η meets u_α at some $0 < R < R_\eta$ and $w_1 := u_\eta - u_\alpha$ is positive in $[0, R)$. Then, w_1 satisfies

$$\begin{cases} \Delta w_1 + k_1 w_1 = 0 \quad \text{in } B(R), \\ w_1 > 0 \text{ in } B(R) \text{ and } w_1|_{\partial B(R)} = 0, \end{cases}$$

where

$$k_1 := K \frac{e^{u_\eta} - e^{u_\alpha}}{u_\eta - u_\alpha} \leqslant \tilde{K} e^{u_\eta}$$

in $B(R)$. We note $w_1'(R) \leqslant 0$. On the other hand, we have $w_2 := \tilde{u}_\xi - \tilde{u}_\beta > 0$ in $[0, \infty)$ and w_2 satisfies

$$\Delta w_2 + k_2 w_2 = 0$$

in $\mathbb{R}^n$, where

$$k_2 := \tilde{K} \frac{e^{\tilde{u}_\xi} - e^{\tilde{u}_\beta}}{\tilde{u}_\xi - \tilde{u}_\beta} > \tilde{K} e^{\tilde{u}_\beta}.$$

It follows from Green's identity that

$$\begin{aligned} \omega_n R^{n-1} w_1'(R) w_2(R) &= \int_{B(R)} (w_2 \Delta w_1 - w_1 \Delta w_2) \\ &\geqslant \int_{B(R)} (k_2 - k_1) w_1 w_2 \ > 0, \end{aligned}$$

where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$. We reach a contradiction, $w_1'(R) > 0$. Hence, u_α cannot touch u_η in $(0, R_\eta)$. □

In general, lemma 3.1 leads to *partial separation*, that is, any two solutions in a special set of initial data do not intersect. The whole separation needs stronger conditions. For instance, when (1.4) has the whole separation on $(-\infty, +\infty)$, it follows from theorem 2.1 in [**5**] that if (1.14) with $f \geqslant 0$ has an entire solution $u_{\mu,\alpha}$ with $u_{\mu,\alpha}(0) = \alpha$, then $u_{\mu,\alpha} \leqslant u_\alpha$ and $u_{\mu,\alpha} < u_{\mu,\beta} < u_{\mu,\gamma} \leqslant u_\gamma$ for any $\alpha < \beta < \gamma$. The monotonicity of entire solutions in initial data implies the existence of a singular solution.

Theorem 3.2. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Assume that K satisfies (Kr) and (M) while f is continuous on $(0, \infty)$, $f \geqslant 0, \not\equiv 0$ and $rf(r)$ is integrable near 0. Then, for every $\mu > 0$ and $\alpha \in \mathbb{R}$, (1.14) has an entire solution $u_{\mu,\alpha}$ with $u_{\mu,\alpha}(0) = \alpha$.*

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Moreover, any two solutions of (1.14) do not intersect each other, and for each $\mu > 0$ there exists a singular solution U_μ which is the monotone upper limit of entire solutions as $\alpha \uparrow +\infty$ and satisfies

$$e^{u_{\mu,\alpha}(r)} < e^{U_\mu(r)} \leqslant \frac{(2+\ell)(n-2)}{r^2 K(r)}. \tag{3.2}$$

Furthermore, U_μ is monotonically decreasing as μ increases.

Proof. The separation of solutions follows from theorem B(ii) and [**5**, theorem 2.1]. Moreover, we have $u_{\mu_1,\alpha} \geqslant u_{\mu_2,\alpha}$ for $\mu_2 \geqslant \mu_1 > 0$. Then, the bound for $u_{\mu,\alpha}$ in (3.2) follows from (1.10) in theorem C. Combining (3.2) and the fact that $r^{-\ell}K(r)$ is non-increasing, we have

$$\begin{aligned} -u'_{\mu,\alpha}(r) &= \frac{1}{r^{n-1}} \int_0^r \left(K(s)e^{u_{\mu,\alpha}} + \mu f(s)\right) s^{n-1}\,\mathrm{d}s \\ &\leqslant \frac{(2+\ell)(n-2)}{r^{n-1}} \int_0^r s^{n-3}\,\mathrm{d}s + \frac{1}{r^{n-1}} \int_0^r \mu f(s) s^{n-1}\,\mathrm{d}s \\ &= \frac{2+\ell}{r} + \frac{\mu}{r} \int_0^r s f(s)\,\mathrm{d}s. \end{aligned} \tag{3.3}$$

Hence, $u'_{\mu,\alpha}$ is bounded on any compact subset of $(0,\infty)$, uniformly with respect to $\alpha \in \mathbb{R}$ and consequently, $\{u_{\mu,\alpha}\}$ is equicontinuous on any compact subset. Since $u_{\mu,\alpha}$ is monotonically increasing in α, it follows from the Arzelà-Ascoli theorem that $U_\mu(r) := \lim_{\alpha\to+\infty} u_{\mu,\alpha}(r)$ is well-defined and continuous on $(0,\infty)$. Consider the equation

$$u''_{\mu,\alpha} = -\frac{n-1}{r} u'_{\mu,\alpha} - K e^{u_{\mu,\alpha}} - \mu f. \tag{3.4}$$

It follows from (3.2) and (3.3) that $u''_{\mu,\alpha}$ is bounded uniformly in α on any compact subset of $(0,\infty)$. The Arzelà-Ascoli theorem implies that there is a subsequence of $\{\alpha_j\}$ such that u'_{μ,α_j} converges uniformly on any compact subset of $(0,\infty)$. Then, U_μ is differentiable on $(0,\infty)$ and $u'_{\mu,\alpha_j} \to U'_\mu$ uniformly on any compact subset of $(0,\infty)$. Then, by (3.4), u''_{μ,α_j} converges also uniformly on compact subsets. Hence, U'_μ is differentiable on $(0,\infty)$ and $u''_{\mu,\alpha_j} \to U''_\mu$ uniformly on any compact subset of $(0,\infty)$. By (3.4) again as $j\to\infty$, U_μ is a singular solution of (1.14). For $\mu_2 > \mu_1 > 0$, we have $u_{\mu_1,\alpha} \geqslant u_{\mu_2,\alpha}$ for every α and thus $U_{\mu_1} \geqslant U_{\mu_2}$. Since $u_{\mu,\alpha} \leqslant u_\alpha$ and $U_\mu \leqslant U$ where u_α and U are the solutions of (1.4), (3.2) for U_μ follows from (1.10), and the proof is complete. □

Theorem 3.2 is the second part in theorem 1.4. The first part of theorem 1.4 requires several observations to obtain the asymptotic behaviour under extra conditions of K and f near ∞. In particular, $D_\mu(\alpha)$ defined by the limit of $\mathcal{D}_\mu(\alpha, r)$ as $r\to\infty$ is useful in identifying $u_{\mu,\alpha}$.

4. Asymptotic behaviour

We begin by reviewing the asymptotic behaviour of entire solutions to (1.6) with $c = 1$. Let $V(t) = \overline{u}_\alpha(r) - \log b/r^{2+\ell}$, $t = \log r$, with $b = (2+\ell)(n-2)$. Then, V

satisfies that

$$V_{tt} + aV_t - b(1 - e^V) = 0, \tag{4.1}$$

where $a = n - 2$. For $\ell = 0$, Tello in [**10**, lemma 2.1] studied (4.1) to obtain the asymptotic behaviour when $n \geqslant 10$. The following result for $\ell > -2$ and $n \geqslant 10 + 4\ell$ is the corresponding asymptotic behaviour.

LEMMA 4.1. *Let $n \geqslant 10 + 4\ell$. Let $\overline{u}_\alpha$ be an entire solution to (1.6). Then, the following limit $D(\alpha)$,*

$$D(\alpha) = \lim_{r\to\infty} \begin{cases} r^{\lambda_1}\left(\overline{u}_\alpha(r) - \log \frac{(2+\ell)(n-2)}{cr^{2+\ell}}\right) & \text{if} \quad n > 10 + 4\ell, \\ r^{\lambda_1}(\log r)^{-1}\left(\overline{u}_\alpha(r) - \log \frac{(2+\ell)(n-2)}{cr^{2+\ell}}\right) & \text{if} \quad n = 10 + 4\ell, \end{cases} \tag{4.2}$$

exists and (1.13) holds.

See the arguments in [**8**, theorem 2.5] for the proof. Moreover, for each α, (1.6) has a super-solution bigger than but close sufficiently to $\overline{u}_\alpha$ in the following sense.

PROPOSITION 4.2. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Then, for each α, there exists a radial super-solution $\overline{u}^+_\alpha(r)$ of (1.6) such that $\overline{u}^+_\alpha(r) > \overline{u}_\alpha(r)$ for $r \in [0, \infty)$ and $\overline{u}^+_\alpha(r) - \overline{u}_\alpha(r) = O(r^{-\lambda_2})$ as $r \to \infty$.*

See [**8**, theorem 4.1] and [**10**, proposition 4.1] for the construction. Now, we study the existence of $D(\alpha)$ to improve lemma 4.1.

LEMMA 4.3. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Assume that K satisfies (KR) and (1.12) for some $c > 0$. Let u_α be a solution of (1.4) satisfying (1.8). Then the limit $D(\alpha)$ exists. This is valid for any solution of (1.4) near ∞ satisfying (1.8).*

Sketch of the proof. Setting $W(\alpha, t) = u_\alpha(r) - \log b/r^{2+\ell} + \log c$, $t = \log r$, we see that

$$W_{tt} + aW_t + bW + bg(W) + h(e^t)e^{-\ell t}e^W = 0, \tag{4.3}$$

where $h(r) := b/c(K(r) - cr^\ell)$ and

$$g(s) := e^s - 1 - s = \frac{1}{2}s^2 + O(s^3)$$

for s near 0. In order to conclude lemma 4.3, we argue in the same way as in the proof of [**1**, lemma 3.4]. Here, (4.3) is compared with [**1**, (3.1)]. More precisely, when u_α satisfies the equation $\Delta u + Ku^p = 0$, [**1**, (3.1)] is obtained by setting

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$W(\alpha, t) = r^m u_\alpha(r) - L$, $t = \log r$, so that

$$W_{tt} + (n-2-2m)W_t + c(p-1)L^{p-1}W + cg(W) + h(e^t)e^{-\ell t}(W+L)^p = 0, \tag{4.4}$$

where $h(r) := K(r) - cr^\ell$ and

$$g(s) := (s+L)^p - L^p - pL^{p-1}s = \frac{p(p-1)}{2}L^{p-2}s^2 + O(s^3).$$

The constant coefficients and the last term in the left-hand side of (4.4) can be changed as in (4.3) under (1.12). Then, the proof of [**1**, lemma 3.4] works for (4.3) to verify the existence of $D(\alpha)$. In fact, lemma 4.3 is true for any solution of (1.4) near ∞ satisfying (1.8). □

Importantly, $D(\alpha)$ has the following integral representation.

PROPOSITION 4.4. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Assume that K satisfies (KR) and (1.12) for some $c > 0$. Every entire solution u_α of (1.4) satisfying (1.8) has the integral representation*

$$D(\alpha) = \frac{-1}{\lambda_2 - \lambda_1}\int_0^\infty \left[r^2Ke^{u_\alpha} - b - b\left(u_\alpha - \log\frac{b}{cr^{2+\ell}}\right)\right] r^{-1+\lambda_1}\,\mathrm{d}r$$

for $n > 10 + 4\ell$,

$$D(\alpha) = -\int_0^\infty \left[r^2Ke^{u_\alpha} - b - b\left(u_\alpha - \log\frac{b}{cr^{2+\ell}}\right)\right] r^{-1+\lambda_1}\,\mathrm{d}r$$

for $n = 10 + 4\ell$.

Sketch of the proof. Proposition 4.4 is verified in the similar way as in [**1**, (3.11) and (3.13)], where the analogous result for Ku^p is obtained.

Case 1. Let $n > 10 + 4\ell$. Then, $D(\alpha, t) = e^{\lambda_1 t}W(\alpha, t)$ holds

$$D_{tt} + (\lambda_2 - \lambda_1)D_t + e^{\lambda_1 t}\left[bg(W) + h(e^t)e^{-\ell t}e^W\right] = 0. \tag{4.5}$$

Integrating (4.5) over $[t, +\infty)$ and letting $t \to -\infty$, and, we obtain that

$$\begin{aligned} D(\alpha) &= \frac{-1}{\lambda_2 - \lambda_1}\int_{-\infty}^{+\infty} e^{\lambda_1 s}\left[bg(W) + h(e^s)e^{-\ell s}e^W\right]\,\mathrm{d}s \\ &= \frac{-1}{\lambda_2 - \lambda_1}\int_0^\infty \left[r^2Ke^{u_\alpha} - b - b\left(u_\alpha - \log\frac{b}{cr^{2+\ell}}\right)\right] r^{-1+\lambda_1}\,\mathrm{d}r. \end{aligned}$$

Case 2. Let $n = 10 + 4\ell$. Then, $D(\alpha, t) = t^{-1}e^{\lambda_1 t}W(\alpha, t)$ satisfies

$$D_{tt} + \frac{2}{t}D_t + \frac{e^{\lambda_1 t}}{t}\left[bg(W) + h(e^t)e^{-lt}e^W\right] = 0. \tag{4.6}$$

Integrating (4.6) and reasoning similarly as in [**1**, (3.13)], we obtain that

$$D(\alpha) = -\int_{-\infty}^{+\infty} e^{\lambda_1 s}\left[bg(W) + h(e^s)e^{-\ell s}e^W\right]\mathrm{d}s$$
$$= -\int_0^\infty \left[r^2 K e^{u_\alpha} - b - b\left(u_\alpha - \log\frac{b}{cr^{2+\ell}}\right)\right] r^{-1+\lambda_1}\,\mathrm{d}r.$$

This completes the proof. □

We perturb K only in a compact region and find two solutions of (1.4) near ∞ with the same limit D. One is a solution of (1.4) in $\mathbb{R}^n$ and the other is a solution of (1.4) only near ∞. Then, the asymptotic behaviour of the difference of two solutions plays a crucial role in analysing the stability. For our convenience, we use β, α to denote distinct solutions of (1.4) near ∞ in the following assertion.

Lemma 4.5. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Assume that K satisfies (Kr) and (1.12) for some $c > 0$. If u_β and u_α are two solutions of (1.4) near ∞ satisfying (1.8) such that $u_\beta > u_\alpha$ and $D(\beta) = D(\alpha)$, then*

$$\lim_{r\to\infty} r^{\lambda_2}(u_\beta(r) - u_\alpha(r)) = d$$

for some $d > 0$.

The arguments reveal that r^{λ_2} is the *maximal* weight to define the distance between two separated solutions near ∞ by a weighted uniform norm. Namely, it is impossible for two separated solutions u_β and u_α satisfying (1.8) to have the asymptotic behaviour, $r^{\lambda_2}(u_\beta(r) - u_\alpha(r)) = o(1)$ at ∞. See the proof of [**1**, lemma 3.6] for the details. Here, we provide the proof briefly.

Proof. Setting $W(\alpha, t) := u_\alpha(r) - \log b/r^{2+\ell} + \log c$, $t = \log r$, we see that $W_\alpha(t) := W(\alpha, t)$ satisfies (4.3). Set $\varphi(t) := D(\beta, t) - D(\alpha, t)$. Let $G(\beta, \alpha) := bg(W_\beta) - bg(W_\alpha)$ and $\Upsilon(\beta, \alpha) := e^{W_\beta} - e^{W_\alpha}$.

Case 1. Let $n > 10 + 4\ell$. Then, $D(\alpha, t) = e^{\lambda_1 t}W(\alpha, t)$ holds (4.5) and $\varphi(t) = e^{\lambda_1 t}(W_\beta(t) - W_\alpha(t)) = e^{\lambda_1 t}(u_\beta(r) - u_\alpha(r))$ satisfies that

$$(\lambda_2 - \lambda_1)\varphi(t) = -\varphi_t(t) + \int_t^{+\infty}\left[e^{\lambda_1 s}G(\beta, \alpha) + e^{(\lambda_1 - \ell)s}h(e^s)\Upsilon(\beta, \alpha)\right]\mathrm{d}s$$
$$\leqslant -\varphi_t(t) + C\int_t^{+\infty}\left[1 + e^{(\lambda_1 - \ell)s}|h(e^s)|\right]e^{-\lambda_1 s}\varphi(s)\,\mathrm{d}s.$$

Since $\varphi(t) \to 0$ as $t \to +\infty$, we see that for given $\epsilon > 0$,

$$(\lambda_2 - \lambda_1)\varphi(t) \leqslant -\varphi_t(t) + \epsilon e^{-\lambda_1 t} \tag{4.7}$$

if t is large enough. Multiplying (4.7) by $e^{(\lambda_2 - \lambda_1)t}$ and integrating over $[T, t]$ with T large, we have

$$\varphi(t) = \begin{cases} O(e^{-(\lambda_2 - \lambda_1)t}) & \text{if} \quad \lambda_2 < 2\lambda_1, \\ o(te^{-(\lambda_2 - \lambda_1)t}) & \text{if} \quad \lambda_2 = 2\lambda_1, \\ o(e^{-\lambda_1 t}) & \text{if} \quad \lambda_2 > 2\lambda_1 \end{cases}$$

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near $+\infty$. We make use of these decay estimates again to obtain that

$$\varphi(t) \leqslant \begin{cases} O(e^{-(\lambda_2-\lambda_1)t}) & \text{if} \quad \lambda_2 < 3\lambda_1, \\ o(te^{-(\lambda_2-\lambda_1)t}) & \text{if} \quad \lambda_2 = 3\lambda_1, \\ o(e^{-2\lambda_1 t}) & \text{if} \quad \lambda_2 > 3\lambda_1. \end{cases}$$

After finite iterations, we conclude that $\varphi(t) = O(e^{-(\lambda_2-\lambda_1)t})$ near $+\infty$. In other words, $u_\beta(r) - u_\alpha(r) = O(r^{-\lambda_2})$ at ∞. In order to derive a finer asymptotic behaviour, we consider the function $\Gamma(\alpha, t) := e^{\lambda_2 t}W(\alpha, t)$. Then, $\Gamma(\alpha, t)$ satisfies

$$\Gamma_{tt} - (\lambda_2 - \lambda_1)\Gamma_t + e^{\lambda_2 t}\left[bg(W) + h(e^t)e^{-\ell t}e^W\right] = 0.$$

The difference $\psi(t) := \Gamma(\beta, t) - \Gamma(\alpha, t)$ is represented by

$$\begin{aligned} \psi(t) &= C_1(T) + C_2(T)e^{(\lambda_2-\lambda_1)t} \\ &\quad + \frac{1}{\lambda_2 - \lambda_1}\int_T^t \left[e^{\lambda_2 s}G(\beta,\alpha) + e^{(\lambda_2-\ell)s}h(e^s)\Upsilon(\beta,\alpha)\right] \mathrm{d}s \\ &\quad - \frac{e^{(\lambda_2-\lambda_1)t}}{\lambda_2-\lambda_1}\int_T^t \left[e^{\lambda_1 s}G(\beta,\alpha) + e^{(\lambda_1-\ell)s}h(e^s)\Upsilon(\beta,\alpha)\right] \mathrm{d}s. \end{aligned}$$

Since $e^{(\lambda_1-\lambda_2)t}\psi(t) = \varphi(t) \to 0$ as $t \to +\infty$, we have

$$\begin{aligned} \psi(t) &= C_1(T) + \frac{1}{\lambda_2 - \lambda_1}\int_T^t \left[e^{\lambda_2 s}G(\beta,\alpha) + e^{(\lambda_2-\ell)s}h(e^s)\Upsilon(\beta,\alpha)\right] \mathrm{d}s \\ &\quad + \frac{e^{(\lambda_2-\lambda_1)t}}{\lambda_2-\lambda_1}\int_t^{+\infty} \left[e^{\lambda_1 s}G(\beta,\alpha) + e^{(\lambda_1-\ell)s}h(e^s)\Upsilon(\beta,\alpha)\right] \mathrm{d}s. \end{aligned} \tag{4.8}$$

Combining (1.12) and the fact that $\psi(t) = O(1)$ at $+\infty$, we observe by (4.8) that

$$\int_T^{+\infty} e^{\lambda_2 s}G(\beta,\alpha)\,\mathrm{d}s < \infty.$$

Hence, $\psi(t)$ converges to a constant d as $t \to +\infty$ and

$$\begin{aligned} \psi(t) &= d - \frac{1}{\lambda_2 - \lambda_1}\int_t^{+\infty} \left[e^{\lambda_2 s}G(\beta,\alpha) + e^{(\lambda_2-\ell)s}h(e^s)\Upsilon(\beta,\alpha)\right] \mathrm{d}s \\ &\quad + \frac{e^{(\lambda_2-\lambda_1)t}}{\lambda_2-\lambda_1}\int_t^{+\infty} \left[e^{\lambda_1 s}G(\beta,\alpha) + e^{(\lambda_1-\ell)s}h(e^s)\Upsilon(\beta,\alpha)\right] \mathrm{d}s. \end{aligned}$$

Suppose $\psi(t) \to 0$ as $t \to +\infty$, i.e., $d = 0$. Then, we have $\psi(t) = o(e^{-\lambda_1 t})$ at $+\infty$. Repeating the process with finer estimates, we can show that for any positive integer q, $\psi(t) = o(e^{-q\lambda_1 t})$ at $+\infty$. In other words, $u_\beta(r) - u_\alpha(r) = o(r^{-\lambda_2 - q\lambda_1})$ at ∞. In particular, for some $\varepsilon > 0$, $u_\beta(r) - u_\alpha(r) = o(r^{2-n-\varepsilon})$ at ∞. However, this is impossible since $u_\beta - u_\alpha$ is a positive superharmonic function near ∞.

Case 2. Let $n = 10 + 4\ell$. Then, $D(\alpha, t) = t^{-1}e^{\lambda_1 t}W(\alpha, t)$ holds (4.6) and $\varphi(t) = t^{-1}e^{\lambda_1 t}(u_\beta(r) - u_\alpha(r))$ satisfies that

$$\begin{aligned}\varphi(t) &= -\varphi_t(t) + \int_t^{+\infty} \left[e^{\lambda_1 s}G(\beta, \alpha) + e^{(\lambda_1-\ell)s}h(e^s)\Upsilon(\beta, \alpha)\right] \mathrm{d}s \\ &\leqslant -\varphi_t(t) + C\int_t^{+\infty} \left[1 + e^{(\lambda_1-\ell)s}|h(e^s)|\right] se^{-\lambda_1 s}\varphi(s)\,\mathrm{d}s \\ &\leqslant -\varphi_t(t) + \epsilon te^{-\lambda_1 t}. \end{aligned} \tag{4.9}$$

It follows from (4.9) that for given $\epsilon > 0$, $(t\varphi)_t \leqslant \epsilon te^{-\lambda_1 t}$ for t large. Then for large T,

$$\varphi(t) \leqslant \frac{T}{t}\varphi(T) + \epsilon t^{-1}\int_T^t se^{-\lambda_1 s}\,\mathrm{d}s.$$

Hence, $\varphi(t) = O(t^{-1})$ at ∞ and $u_\beta(r) - u_\alpha(r) = O(r^{-\lambda_1})$ at ∞. We observe that $\psi(t) := e^{\lambda_1 t}(W_\beta - W_\alpha) = e^{\lambda_1 t}(u_\beta - u_\alpha)$ is represented by

$$\begin{aligned}\psi(t) = C_1(T) &+ C_2(T)t \\ &+ \int_T^t s\left[e^{\lambda_1 s}G(\beta, \alpha) + e^{(\lambda_1-\ell)s}h(e^s)\Upsilon(\beta, \alpha)\right] \mathrm{d}s \\ &- t\int_T^t \left[e^{\lambda_1 s}G(\beta, \alpha) + e^{(\lambda_1-\ell)s}h(e^s)\Upsilon(\beta, \alpha)\right] \mathrm{d}s.\end{aligned}$$

Since $t^{-1}\psi(t) = \varphi(t) \to 0$ as $t \to +\infty$, we have

$$\begin{aligned}\psi(t) = C_1(T) &+ \int_T^t s\left[e^{\lambda_1 s}G(\beta, \alpha) + e^{(\lambda_1-\ell)s}h(e^s)\Upsilon(\beta, \alpha)\right] \mathrm{d}s \\ &+ t\int_t^{+\infty} \left[e^{\lambda_1 s}G(\beta, \alpha) + e^{(\lambda_1-\ell)s}h(e^s)\Upsilon(\beta, \alpha)\right] \mathrm{d}s.\end{aligned} \tag{4.10}$$

Combining (1.12) and the fact that $\psi(t) = O(1)$ at $+\infty$, we observe by (4.10) that

$$\int_T^{+\infty} se^{\lambda_1 s}G(\beta, \alpha)\,\mathrm{d}s < \infty.$$

Hence, $\psi(t)$ converges to a constant d as $t \to +\infty$ and

$$\begin{aligned}\psi(t) = d &- \int_t^{+\infty} s\left[e^{\lambda_1 s}G(\beta, \alpha) + e^{(\lambda_1-\ell)s}h(e^s)\Upsilon(\beta, \alpha)\right] \mathrm{d}s \\ &+ t\int_t^{+\infty} \left[e^{\lambda_1 s}G(\beta, \alpha) + e^{(\lambda_1-\ell)s}h(e^s)\Upsilon(\beta, \alpha)\right] \mathrm{d}s.\end{aligned}$$

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Suppose $\psi(t) \to 0$ as $t \to +\infty$, that is, $d = 0$. Let $\varepsilon > 0$ be given. Then,

$$\begin{aligned}\psi(t) &\leqslant C\left[\int_t^{+\infty} se^{\lambda_1 s}e^{-2\lambda_1 s}\psi(s)\,\mathrm{d}s + \int_t^{+\infty} se^{(\lambda_1-\ell)s}|h(e^s)|e^{-\lambda_1 s}\psi(s)\,\mathrm{d}s\right]\\ &\quad + Ct\left[\int_t^{+\infty} e^{\lambda_1 s}e^{-2\lambda_1 s}\psi(s)\,\mathrm{d}s + \int_t^{+\infty} e^{(\lambda_1-\ell)s}|h(e^s)|e^{-\lambda_1 s}\psi(s)\,\mathrm{d}s\right]\\ &\leqslant \varepsilon t e^{-\lambda_1 t}\end{aligned}$$

for t large. Hence, $\psi(t) = o(te^{-\lambda_1 t})$ at $+\infty$. Applying this finer estimate, we have $\psi(t) = o(t^2 e^{-2\lambda_1 t})$ at $+\infty$. Similar arguments as in Case 1 lead to a contradiction. □

Remark 4.6. Assume that $\varphi(t) \to 0$ and $\psi(t) \to d$ as $t \to +\infty$.

For $n > 10 + 4\ell$, we have by (4.8),

$$\psi(T) = C_1(T) + \frac{e^{(\lambda_2-\lambda_1)T}}{\lambda_2-\lambda_1}\int_T^{+\infty}\left[e^{\lambda_1 s}G(\beta,\alpha) + e^{(\lambda_1-\ell)s}h(e^s)\Upsilon(\beta,\alpha)\right]\,\mathrm{d}s,$$

which implies that

$$\begin{aligned}d &= \psi(T) + \frac{1}{\lambda_2-\lambda_1}\int_T^{+\infty}\left[e^{\lambda_2 s}G(\beta,\alpha) + e^{(\lambda_2-\ell)s}h(e^s)\Upsilon(\beta,\alpha)\right]\,\mathrm{d}s\\ &\quad - \frac{e^{(\lambda_2-\lambda_1)T}}{\lambda_2-\lambda_1}\int_T^{+\infty}\left[e^{\lambda_1 s}G(\beta,\alpha) + e^{(\lambda_1-\ell)s}h(e^s)\Upsilon(\beta,\alpha)\right]\,\mathrm{d}s. \qquad (4.11)\end{aligned}$$

For $n = 10 + 4\ell$, we have by (4.10),

$$\psi(T) = C_1(T) + T\int_T^{+\infty}\left[e^{\lambda_1 s}G(\beta,\alpha) + e^{(\lambda_1-\ell)s}h(e^s)\Upsilon(\beta,\alpha)\right]\,\mathrm{d}s,$$

which implies that

$$\begin{aligned}d &= \psi(T) + \int_T^{+\infty} s\left[e^{\lambda_1 s}G(\beta,\alpha) + e^{(\lambda_1-\ell)s}h(e^s)\Upsilon(\beta,\alpha)\right]\,\mathrm{d}s\\ &\quad - T\int_T^{+\infty}\left[e^{\lambda_1 s}G(\beta,\alpha) + e^{(\lambda_1-\ell)s}h(e^s)\Upsilon(\beta,\alpha)\right]\,\mathrm{d}s. \qquad (4.12)\end{aligned}$$

5. Infinite multiplicity

In this section, we establish the existence of infinitely many entire solutions of (1.1) with the asymptotic behaviour (1.8).

5.1. Homogeneous equation

In order to find out how (1.4) under (1.12) possesses infinitely many entire solutions satisfying (1.8), we adopt similar arguments as in [**2**, **5**, **6**]. For our convenience, we fix a family $\{\overline{u}_\alpha\}$ of separated radial solutions of (1.6) indexed by $\alpha \in \mathbb{R}$ such that $\overline{u}_\alpha(0) = \alpha$, $\overline{u}_\alpha$ is monotonically increasing in α and

$$\lim_{r\to\infty}\left[\overline{u}_\alpha(r) - \log\frac{(2+\ell)(n-2)}{cr^{2+\ell}}\right] = 0. \tag{5.1}$$

It follows from proposition 4.2 that for each $\alpha \in \mathbb{R}$, there exists a super-solution $\overline{u}^+_\alpha > \overline{u}_\alpha$ of the equation $\Delta u + |x|^\ell e^u = 0$ satisfying

$$F_\alpha(r) := \overline{u}^+_\alpha(r) - \overline{u}_\alpha(r) = O(r^{-\lambda_2}) \quad \text{as} \quad r \to \infty. \tag{5.2}$$

and

$$\Delta F_\alpha \leqslant -|x|^\ell (e^{\overline{u}^+_\alpha} - e^{\overline{u}_\alpha}) \leqslant -|x|^\ell e^{\overline{u}_\alpha} F_\alpha.$$

Now, we prove infinite multiplicity of solutions in the radial case, and reveal the structure of partial separation.

PROPOSITION 5.1. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Suppose that K satisfies (KR), (1.12) for some $c > 0$. Then, there exists a constant $\alpha^* = \alpha^*(K)$ such that for each $\alpha \in (-\infty, \alpha^*)$, equation (1.4) possesses an entire solution u_α with $u_\alpha(0) = \alpha$ satisfying (1.8) and any two of them do not intersect.*

Proof. Let $c = 1$ for simplicity. For all β, (1.4) has a unique local solution u_β. First, we claim that for given β negatively large, there exists $\overline{\gamma} = \overline{\gamma}(\beta) < \beta$ such that $u_\gamma < \overline{u}_\beta$ for every $\gamma \leqslant \overline{\gamma}$.

Suppose to the contrary that for any $\gamma < \beta$, there exists $\tilde{\gamma} < \gamma$ such that $w_{\tilde{\gamma}}(r) := \overline{u}_\beta(r) - u_{\tilde{\gamma}}(r) > 0$ on $[0, R_{\tilde{\gamma}})$ but $w_{\tilde{\gamma}}(R_{\tilde{\gamma}}) = 0$ for some $R_{\tilde{\gamma}} > 0$. Then, $w_{\tilde{\gamma}}$ satisfies

$$\Delta w_{\tilde{\gamma}} = -|x|^\ell e^{\overline{u}_\beta} + K e^{u_{\tilde{\gamma}}}$$

in $\overline{B(R_{\tilde{\gamma}})}$. Fix $\alpha > \beta$. Applying Green's identity, we have

$$\begin{aligned} 0 &\leqslant \int_{\partial B(R_{\tilde{\gamma}})} \left(w_{\tilde{\gamma}} \frac{\partial F_\alpha}{\partial r} - F_\alpha \frac{\partial w_{\tilde{\gamma}}}{\partial r} \right) = \int_{B(R_{\tilde{\gamma}})} (w_{\tilde{\gamma}} \Delta F_\alpha - F_\alpha \Delta w_{\tilde{\gamma}}) \\ &\leqslant \int_{B(R_{\tilde{\gamma}})} \left\{ -|x|^\ell e^{\overline{u}_\alpha} w_{\tilde{\gamma}} F_\alpha + |x|^\ell e^{\overline{u}_\beta} F_\alpha - K e^{u_{\tilde{\gamma}}} F_\alpha \right\} \\ &\leqslant \int_{B(R_{\tilde{\gamma}})} \left\{ -|x|^\ell e^{\overline{u}_\alpha} w_{\tilde{\gamma}} F_\alpha + |x|^\ell e^{\overline{u}_\beta} w_{\tilde{\gamma}} F_\alpha + (|x|^\ell - K) e^{u_{\tilde{\gamma}}} F_\alpha \right\}, \end{aligned}$$

hence

$$\int_{B(R_{\tilde{\gamma}})} \left[e^{\overline{u}_\alpha} - e^{\overline{u}_\beta} \right] |x|^\ell w_{\tilde{\gamma}} F_\alpha \leqslant \int_{B(R_{\tilde{\gamma}})} \left(|x|^\ell - K \right) e^{u_{\tilde{\gamma}}} F_\alpha.$$

It follows from theorem 2.2 that for any $\tilde{\gamma}$ negatively large, $3/2\tilde{\gamma} \leqslant u_{\tilde{\gamma}} \leqslant 1/2\tilde{\gamma}$ on $[0, 1]$. Hence, for γ negatively large and thus, for negatively large $\tilde{\gamma} < \gamma$, we may assume that $R_{\tilde\gamma} > 1$ and $w_{\tilde\gamma} \geqslant \overline{u}_\beta(1) - 1/2\tilde\gamma$ in $B(1)$. Then, we have

$$\left(\overline{u}_\beta(1) - \frac{1}{2}\tilde\gamma\right)\int_{B(1.1)} |x|^\ell \left[e^{\overline{u}_\alpha} - e^{\overline{u}_\beta}\right] F_\alpha \leqslant \int_{B(R_{\tilde\gamma})} \left(|x|^\ell - K\right)_+ e^{u_{\tilde\gamma}} F_\alpha$$
$$\leqslant \int_{B(R_{\tilde\gamma})} \left(K - |x|^\ell\right)_- e^{\overline{u}_\beta} F_\alpha$$
$$\leqslant \int_{\mathbb{R}^n} \left(K - |x|^\ell\right)_- e^{\overline{u}_\beta} F_\alpha,$$

where $k_\pm = \max(\pm k, 0)$. Combining (5.1), (5.2) and (1.12), we observe that the last integral is finite. However, the left-hand side goes to ∞ as $\tilde\gamma \to -\infty$. This contradiction verifies the claim. Therefore, there exists $\overline{\gamma} = \overline{\gamma}(\beta) < \beta$ such that $u_\gamma < \overline{u}_\beta$ for all $\gamma \leqslant \overline{\gamma}$.

For $\beta \in \mathbb{R}$, let I_β be the set of $\gamma < \overline{\gamma}(\beta)$ satisfying

$$\frac{3}{4}(-\gamma)\int_{B(1)} |x|^\ell \left[e^{\overline{u}_\beta} - e^{u_\gamma}\right] F_\beta > \int_{B(R_\gamma)} \left(K - |x|^\ell\right)_+ e^{u_\gamma} F_\beta. \tag{5.3}$$

Note that $(K - |x|^\ell)_+ e^{u_\gamma} F_\beta$ converges pointwise to 0 as $\gamma \to -\infty$ and by (5.1), (5.2) and (1.12),

$$(K - |x|^\ell)_+ e^{u_\gamma} F_\beta \leqslant |K - |x|^\ell| e^{\overline{u}_\beta} F_\beta \in L^1(\mathbb{R}^n).$$

Hence, $I_\beta \supset (-\infty, \gamma_\beta)$ for some γ_β since the right-hand side of (5.3) goes to 0 as $\gamma \to -\infty$ by the Dominated Convergence theorem while the left-hand side is bounded below by a positive constant which is irrelevant to γ when γ is negatively large. It follows from lemma 2.4 that there exists $\hat\gamma \leqslant \gamma_\beta$ such that for all $\gamma < \hat\gamma$, and $u_\gamma(r) \geqslant 3/2\gamma$ on $[0,1]$. For negatively large $\gamma < \hat\gamma$ so that $u_\gamma(r) \geqslant 3/2\gamma$ for $0 \leqslant r \leqslant 1$, we claim that there exists $\eta < \gamma$ such that $u_\gamma > \overline{u}_\eta$ in $\mathbb{R}^n$. Suppose by contradiction that there exists $\hat\gamma_1 < \hat\gamma$ such that for each $\eta < \hat\gamma_1$, $\hat w_\eta(r) = u_{\hat\gamma_1}(r) - \overline{u}_\eta(r) > 0$ in $[0, r_\eta)$ for some $r_\eta > 0$ and $\hat w_\eta(r_\eta) = 0$. From Green's identity,

$$0 \leqslant \int_{B(r_\eta)} (\hat w_\eta \Delta F_\beta - F_\beta \Delta \hat w_\eta)$$
$$\leqslant \int_{B(r_\eta)} \left[-|x|^\ell \hat w_\eta e^{\overline{u}_\beta} F_\beta + K e^{u_{\hat\gamma_1}} F_\beta - |x|^\ell e^{\overline{u}_\eta} F_\beta\right]. \tag{5.4}$$

Then, it follows from (5.4) that

$$\int_{B(r_\eta)} |x|^\ell \hat w_\eta \left[e^{\overline{u}_\beta} - e^{u_{\hat\gamma_1}}\right] F_\beta \leqslant \int_{B(r_\eta)} \left[|x|^\ell \hat w_\eta e^{\overline{u}_\beta} - |x|^\ell (e^{u_{\hat\gamma_1}} - e^{\overline{u}_\eta})\right] F_\beta$$
$$\leqslant \int_{B(r_\eta)} \left[K e^{u_{\hat\gamma_1}} - |x|^\ell e^{\overline{u}_\eta} - |x|^\ell (e^{u_{\hat\gamma_1}} - e^{\overline{u}_\eta})\right] F_\beta$$
$$\leqslant \int_{B(r_\eta)} \left(K - |x|^\ell\right)_+ e^{u_{\hat\gamma_1}} F_\beta.$$

Since $\overline{u}_\eta$ is monotonically decreasing to $-\infty$ as η decreases to $-\infty$ and thus $\overline{u}_\eta \to -\infty$ uniformly on $[0, R]$ for any fixed $R > 0$, we may assume that $r_\eta > 1$ and $\hat w_\eta(r) \geqslant$

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$3/2\hat{\gamma}_1 - \eta \geqslant 3/4(-\hat{\gamma}_1)$ in B_1 if $\eta \leqslant 9/4\hat{\gamma}_1$ and η is negatively large enough. Then, we have

$$\frac{3}{4}(-\hat{\gamma}_1)\int_{B(1)} |x|^\ell \left[e^{\overline{u}_\beta} - e^{u_{\hat{\gamma}_1}}\right] F_\beta \leqslant \int_{B(R_\gamma)} \left(K - |x|^\ell\right)_+ e^{u_{\hat{\gamma}_1}} F_\beta,$$

which is impossible because $\hat{\gamma}_1 \in I_\beta$. Therefore, for each β, there exist $\beta > \gamma > \eta$ satisfying $\overline{u}_\eta < u_\gamma < \overline{u}_\beta$ in $\mathbb{R}^n$. Repeating the arguments, we find a decreasing sequence $\{u_{\gamma_i}\}$ of entire solutions and a decreasing sequence $\{\alpha_i\}$ going to $-\infty$ as $i \to \infty$ such that $u_{\gamma_i} > \overline{u}_{\alpha_i} > u_{\gamma_{i+1}}$ in $\mathbb{R}^n$ for each $i \geqslant 1$.

In order to deal with the case that K has not non-negative near ∞, we consider the equation

$$u_{rr} + \frac{n-1}{r} u_r + K_+(r)e^u = 0, \tag{5.5}$$

and denote by $u_\alpha^+(r)$ the local solution of (5.5) with $u_\alpha^+(0) = \alpha \in \mathbb{R}$. From (1.12), we observe $K_+ \in MI(c)$ since

$$\int_1^\infty |r^{-\ell}K_+(r) - c|r^{-1+\lambda_1}\,\mathrm{d}r \leqslant \int_1^\infty |r^{-\ell}K(r) - c|r^{-1+\lambda_1}\,\mathrm{d}r < \infty.$$

Applying the previous arguments to (5.5), we obtain a decreasing sequence $\{u_{\xi_i}^+\}$ of entire solutions of (5.5) and a decreasing sequence $\{\beta_i\}$ going to $-\infty$ as $i \to \infty$ such that $u_{\xi_i}^+ > \overline{u}_{\beta_i} > u_{\xi_{i+1}}^+$ in $\mathbb{R}^n$ for each $i \geqslant 1$. By the separation property of solutions $\overline{u}_\alpha$ of (1.6), there exists γ_i such that $u_{\xi_1}^+ > u_{\xi_2}^+ > u_{\gamma_i}$ in $\mathbb{R}^n$. Then, it follows from lemma 3.1 that $u_\eta < u_\zeta$ in $\mathbb{R}^n$ for any $\eta < \zeta \leqslant \gamma_i$. Therefore, we conclude that there exists α^* such that u_α is monotone with respect to $\alpha \in (-\infty, \alpha^*)$, which completes the proof. □

In addition to the assumptions of proposition 5.1, if $K \geqslant 0$ and f satisfies (fR1) and $f \geqslant 0$, then (1.14) with $\mu > 0$ has the following structure of partial separation. If $\beta < \alpha < \alpha^*(K)$, then any two solutions $u_{\mu,\beta}$ and $u_{\mu,\alpha}$ do not intersect.

Now, we are ready to prove theorem 1.1.

Proof of theorem 1.1. Suppose (1.12) for some $c > 0$. Let u_α be an entire solution. Then, $u_\alpha \leqslant \overline{u}_\alpha$ by combining the separation property of $\overline{u}_\gamma$ and a comparison argument in [**5**, theorem 2.1]. Then, the second argument of proposition 5.1 implies that $u_\alpha > \overline{u}_\beta$ for some $\beta < \alpha$, and the existence of $D(\alpha)$ follows from lemma 4.3. Conversely, we assume that $D(\alpha)$ exists. Then, it follows from proposition 4.4 that the existence of $D(\alpha)$ is equivalent to

$$\int^{+\infty} e^{\lambda_1 s}\left[bg(W) + h(e^s)e^{-\ell s}e^W\right]\mathrm{d}s < \infty,$$

where W and g, h are defined in (4.3). Since $g(W) \geqslant 0$ near $+\infty$ and $h \geqslant 0$, we observe that

$$\int^{+\infty} he^{(\lambda_1 - \ell)s}\,\mathrm{d}s < \infty$$

which is a translation of condition (1.12), and the proof is complete. □

By combining proposition 5.1 with a comparison argument, we establish a more general result for (1.3). Let $n > N \geqslant 3$ and $x = (x_1, x_2) \in \mathbb{R}^{n-N} \times \mathbb{R}^N$. Assume that $K(x) = K(x_1, x_2) = K(x_1, r)$ is a function of variables x_1 and $r = |x_2|$. Moreover, we assume the following condition.

(K3) $\inf_{x_1 \in \mathbb{R}^{n-N}} K(x_1, r)$ and $\sup_{x_1 \in \mathbb{R}^{n-N}} K(x_1, r)$ are bounded below and above by locally Hölder continuous functions $K_1(r)$ and $K_2(r)$ on $(0, \infty)$, respectively, satisfying $\int_0 r|K_i(r)|\,\mathrm{d}r < \infty$ for $i = 1, 2$.

Hence, $K_1(|x_2|) \leqslant K(x) \leqslant K_2(|x_2|)$. Then, proposition 5.1 derives the following result.

THEOREM 5.2. *Let $n > N \geqslant 10 + 4\ell$ with $\ell > -2$. Assume that locally Hölder continuous function $K(x) = K(x_1, x_2) = K(x_1, |x_2|)$ in $\mathbb{R}^n \setminus \{0\}$ satisfies (K1), (K3), and for some constant $c > 0$,*

$$\int_1^\infty |r^{-\ell} K_i(r) - c| r^{-1+\lambda_1}\,\mathrm{d}r < \infty, \quad i = 1, 2.$$

Then, (1.3) possesses infinitely many entire solutions satisfying

$$\lim_{|x_2| \to \infty} \left[u(x_1, x_2) - \log \frac{(2+\ell)(n-2)}{c|x_2|^{2+\ell}} \right] = 0 \tag{5.6}$$

uniformly in $x_1 \in \mathbb{R}^{n-N}$ and any two of them do not intersect.

Proof. We first consider (1.4) with $K = K_i$ in the subspace $\mathbb{R}^N$, $N \geqslant 10 + 4\ell$. Then, it follows from proposition 5.1 that for each $i = 1, 2$, there exists a family $J_i = \{u_{\gamma,i}\}$ of ordered entire solutions satisfying (1.8). More precisely, there exists γ_i* such that solutions in J_i are indexed by $\gamma \in (-\infty, \gamma_i*)$. Moreover, (1.6) has a family $I_i = \{\overline{u}_{\alpha,i}\}$ of countable ordered entire solutions such that for each $\overline{u}_\alpha$ in I, there exist two solutions in J_i which are separated by $\overline{u}_\alpha$, and α can be chosen to be negatively large. By making use of the separation property of entire solutions of (1.6), we may choose $\gamma > \alpha > \beta > \xi > \eta > \rho > \delta$ such that

$$u_{\gamma,2} > \overline{u}_{\alpha,2} > \overline{u}_{\beta,1} > u_{\xi,1} > \overline{u}_{\eta,1} > \overline{u}_{\rho,2} > u_{\delta,2} \quad \text{in} \quad \mathbb{R}^N.$$

Setting $u_{\alpha,i}(x) = u_{\alpha,i}(x_1, x_2) = u_{\alpha,i}(|x_2|)$ and $K_i(x) = K_i(x_1, x_2) = K_i(|x_2|)$ in $\mathbb{R}^n$ for $i = 1, 2$, we have

$$u_{\gamma,2} > u_{\xi,1} > u_{\delta,2} \quad \text{in} \quad \mathbb{R}^n.$$

Obviously, $u_{\gamma,2}$ and $u_{\xi,1}$ are super- and sub-solution of (1.3) in $\mathbb{R}^n$, respectively. Then, the standard barrier method verifies the existence of an entire solution of (1.3). Repeating this process, we construct infinitely many ordered entire solutions satisfying (5.6). □

A direct consequence of theorem 5.2 is the following assertion.

COROLLARY 5.3. *Let $n > N \geqslant 10 + 4\ell$ with $\ell > -2$. Suppose that locally Hölder continuous function $K(x) = K(x_1, x_2) = K(x_1, |x_2|)$ in $\mathbb{R}^n \setminus \{0\}$ satisfies (K1), (K3) and there exists $c > 0$ such that*

$$K_i(r) = cr^{\ell} + O(r^{\ell-\lambda_1}(\log r)^{-\theta_i}) \quad at \quad \infty, \quad i = 1, 2,$$

for some constants $c > 0$ and $\theta_i > 1$, where $\lambda_1 = \lambda_1(N, \ell)$. Then, the same result as in theorem 5.2 holds.

When $n = N$, we observe the existence of $\mathcal{S}$ in theorem 1.3. However, the existence of a continuum $\mathcal{C}$ of solutions needs more detailed information about the asymptotic behaviour. The existence of $\mathcal{C}$ follows from theorem 1.5 with $\mu = 0$ in § 5.2.

5.2. Inhomogeneous equation

In this section, we study infinite multiplicity for the inhomogeneous equation (1.1). Here we argue similarly as in [**6**, § 4]. Under the assumptions on K as in proposition 5.1, equation (1.4) with $n \geqslant 10 + 4\ell$ and $\ell > -2$ has a family $\{u_\alpha\}$ of radial solutions indexed by $\alpha \in (-\infty, \alpha^*]$ for some $\alpha^* < \infty$ such that $u_\alpha(0) = \alpha$ and u_α is monotonically increasing with respect to α. For $\alpha \leqslant \alpha^*$, set $W(\alpha, t) := u_\alpha(r) - \log(((2 + \ell)(n - 2))/(cr^{2+\ell}))$, $t = \log r$ and

$$D(\alpha, t) := e^{\lambda_1 t} W(\alpha, t) = \mathcal{D}(\alpha, r) \quad \text{for } n > 10 + 4\ell,$$

$$D(\alpha, t) := t^{-1} e^{\lambda_1 t} W(\alpha, t) = \mathcal{D}(\alpha, r) \quad \text{for } n = 10 + 4\ell.$$

From the proof of proposition 5.1, we observe that for each $\alpha \in (-\infty, \alpha^*]$, there exist $\gamma < \alpha$ and $\beta > \alpha$ such that $\overline{u}_\gamma \leqslant u_\alpha \leqslant \overline{u}_\beta$ in $\mathbb{R}^n$ and thus, $u_\alpha(r) - \log(((2 + \ell)(n - 2))/(cr^{2+\ell})) \to 0$ as $r \to \infty$. Moreover, it follows from (4.2) and (1.13) that for fixed $-\infty < a < \alpha^*$, $D(\alpha, t)$ are uniformly bounded above and below near $+\infty$ on $[a, \alpha^*]$. For all $\alpha \in [a, \alpha^*]$, there exists $M = M(a)$ such that for $t \in [0, +\infty)$,

$$|W(\alpha, t)| \leqslant M e^{-\lambda_1 t} \quad \text{for } n > 10 + 4\ell \tag{5.7}$$

and

$$|W(\alpha, t)| \leqslant M t e^{-\lambda_1 t} \quad \text{for } n = 10 + 4\ell. \tag{5.8}$$

For fixed $-\infty < t < +\infty$, $D(\alpha, t)$ is continuous with respect to α. The next observation is that $D(\alpha, t)$ converges uniformly on $[a, \alpha^*]$ as $t \to +\infty$. We verify this under condition (1.12).

LEMMA 5.4. *Assume (5.7) and (5.8) for $n > 10 + 4\ell$ and $n = 10 + 4\ell$, respectively. For given $-\infty < a < \alpha^*$, $D(\alpha, t)$ converges uniformly on $[a, \alpha^*]$ as $t \to +\infty$.*

Proof. Setting $W(\alpha, t) := u_\alpha(r) - \log(((2 + \ell)(n - 2))/(cr^{2+\ell})), t = \log r$, we see that W satisfies (4.3).

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Case 1. Let $n > 10 + 4\ell$. Then, $D(\alpha, t) = e^{\lambda_1 t} W(\alpha, t)$ satisfies (4.5) and

$$(D_t e^{(\lambda_2-\lambda_1)t})_t = -e^{\lambda_2 t}\left[bg(W) + h(e^t)e^{-\ell t}e^W\right], \tag{5.9}$$

where $g(s) = 1/2s^2 + O(s^3)$ for s near 0. Integrating (5.9) over $[T, t]$ with $T \geqslant 0$, we have

$$D_t(\alpha, t) = e^{-(\lambda_2-\lambda_1)t}\left\{e^{(\lambda_2-\lambda_1)T} D_t(\alpha, T) - \int_T^t e^{\lambda_2 s}\left[bg(W) + h(e^s)e^{-\ell s}e^W\right] \mathrm{d}s\right\}. \tag{5.10}$$

It follows from (5.7) that for any $0 < \epsilon < \min\{\lambda_1, \lambda_2 - \lambda_1\}$ and for some $M_1 > 0$,

$$\begin{aligned} e^{(\lambda_1-\lambda_2)t}\int_T^t ce^{\lambda_2 s}|g(W(s))|\,\mathrm{d}s &\leqslant e^{(\lambda_1-\lambda_2)t}\int_T^t cM_1 e^{(\lambda_2-2\lambda_1)s}\,\mathrm{d}s \\ &\leqslant cM_1 e^{-\epsilon t}\int_T^t e^{-(\lambda_1-\epsilon)s}\,\mathrm{d}s \end{aligned} \tag{5.11}$$

which goes to 0 as $t \to +\infty$. From (1.12), we have

$$\begin{aligned} e^{(\lambda_1-\lambda_2)t}\int_T^t e^{(\lambda_2-\ell)s}|h(e^s)|\,\mathrm{d}s &= e^{(\lambda_1-\lambda_2)t}\int_T^t e^{(\lambda_2-\lambda_1)s}e^{(\lambda_1-\ell)s}|h(e^s)|\,\mathrm{d}s \\ &\leqslant \int_T^\infty e^{(\lambda_1-\ell)s}|h(e^s)|\,\mathrm{d}s < \infty. \end{aligned}$$

Hence, the function

$$F(t) := e^{(\lambda_1-\lambda_2)t}\int_T^t e^{(\lambda_2-\ell)s}|h(e^s)|\,\mathrm{d}s$$

is bounded and $F'(t) = (\lambda_1 - \lambda_2)F(t) + e^{(\lambda_1-\ell)t}|h(e^t)|$. Then,

$$\begin{aligned} (\lambda_2 - \lambda_1)\int_T^t F(s)\,\mathrm{d}s &= F(T) - F(t) + \int_T^t e^{(\lambda_1-\ell)s}|h(e^s)|\,\mathrm{d}s \\ &\leqslant F(T) + \int_T^{+\infty} e^{(\lambda_1-\ell)s}|h(e^s)|\,\mathrm{d}s < \infty. \end{aligned} \tag{5.12}$$

and thus, F is integrable near $+\infty$. Therefore, from (5.12), $F(t)$ converges as $t \to +\infty$, which combined with the integrability of F near $+\infty$ implies that

$$\lim_{t\to+\infty} F(t) = 0. \tag{5.13}$$

Hence, by (5.7), (5.10), (5.11) and (5.13), $D_t(\alpha, t)$ converges uniformly to 0 on $[a, \alpha^*]$ as $t \to +\infty$. Integrating (4.5) over $[T, t]$, we see that

$$\begin{aligned} (\lambda_2 - \lambda_1)(D(\alpha, t) - D(\alpha, T)) &= D_t(\alpha, T) - D_t(\alpha, t) \\ &\quad - \int_T^t e^{\lambda_1 s}\left[bg(W) + h(e^s)e^{-\ell s}e^W\right]\,\mathrm{d}s. \end{aligned}$$

Then, it follows immediately that $D(\alpha, t)$ converges uniformly on $[a, \alpha^*]$ as $t \to +\infty$.

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Case 2. Let $n = 10 + 4\ell$. Then, $D(\alpha, t) = t^{-1}e^{\lambda_1 t}W(\alpha, t)$ satisfies (4.6) and

$$(t^2 D_t)_t = -te^{\lambda_1 t}\left[bg(W) + h(e^t)e^{-\ell t}e^W\right]. \tag{5.14}$$

Integrating (5.14) over $[T, t]$ with $T \geqslant 0$, we have

$$tD_t(\alpha, t) = t^{-1}\left\{T^2 D_t(\alpha, T) - \int_T^t se^{\lambda_1 s}[bg(W) + h(e^s)e^{-\ell s}e^W]\,\mathrm{d}s\right\}. \tag{5.15}$$

First, note that from (5.8),

$$t^{-1}\int_T^t bse^{\lambda_1 s}|g(W)|\,\mathrm{d}s \leqslant t^{-1}\int_T^t cM_2 se^{-\lambda_1 s}\,\mathrm{d}s \tag{5.16}$$

for some $M_2 > 0$. Second, letting

$$G(t) := t^{-1}\int_T^t se^{(\lambda_1 - \ell)s}|h(e^s)|\,\mathrm{d}s,$$

we have $G'(t) = -t^{-1}G(t) + e^{(\lambda_1-\ell)t}|h(e^t)|$. Then,

$$G(t) - G(T) = -\int_T^t s^{-1}G(s)\,\mathrm{d}s + \int_T^t e^{(\lambda_1-\ell)s}|h(e^s)|\,\mathrm{d}s. \tag{5.17}$$

Hence, we have

$$\int_T^{+\infty}\frac{G(s)}{s}\,\mathrm{d}s \leqslant G(T) + \int_T^{+\infty} e^{(\lambda_1-\ell)s}|h(e^s)|\,\mathrm{d}s < \infty, \tag{5.18}$$

which implies that by (5.17), $G(t)$ converges as $t \to +\infty$ and thus, to 0 by (5.18) again. Thus, from (5.8), (5.15) and (5.16), $tD_t(\alpha, t)$ converges uniformly to 0 on $[a, \alpha^*]$ as $t \to +\infty$. Multiplying (4.6) by t and integrating over $[T, t]$, we have

$$\begin{aligned} D(\alpha, t) = {} & D(\alpha, T) + TD_t(\alpha, T) - tD_t(\alpha, t) \\ & - \int_T^t e^{\lambda_1 s}\left[bg(W) + h(e^s)e^{-\ell s}e^W\right]\,\mathrm{d}s. \end{aligned}$$

Therefore, $D(\alpha, t)$ converges uniformly on $[a, \alpha^*]$ as $t \to +\infty$. □

It follows from lemma 5.4 that the limit $D(\alpha)$ of $D(\alpha, t)$ as $t \to +\infty$ is continuous in α.

Proposition 5.5. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Suppose the assumptions of proposition 5.1. Then, $D(\alpha) := \lim_{t\to+\infty} D(\alpha, t)$ is continuous for α negatively large. Moreover, $D(\alpha) \to -\infty$ as $\alpha \to -\infty$.*

Proof. Let $\alpha < \alpha^*$. Given $\varepsilon > 0$, we choose t sufficiently large so that $|D(\gamma) - D(\gamma, t)| < \varepsilon/3$ for all $\gamma \in [\alpha - 1, \alpha^*]$. In any fixed compact region, entire solutions

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u_β of (1.4) are close uniformly to u_α if β is sufficiently close to α. Hence, we may choose $0 < \delta < 1$ such that $|D(\alpha,t) - D(\beta,t)| < \varepsilon/3$ if $|\alpha - \beta| < \delta$. Then, we have

$$|D(\alpha) - D(\beta)| \leqslant |D(\alpha) - D(\alpha,t)| + |D(\alpha,t) - D(\beta,t)| + |D(\beta,t) - D(\beta)| < \varepsilon.$$

This implies the continuity of D. It follows from (1.13) for (1.6) that $D(\alpha) \to -\infty$ as $\alpha \to -\infty$. □

We are now ready to prove theorem 1.2.

Theorem 5.6. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Assume that K satisfies (Kr), (M) and (1.12) for some $c > 0$. Then, as $r \to \infty$, $\mathcal{D}(\alpha, r)$ converges uniformly on any compact subset of $(-\infty, \infty)$ to a continuous and strictly increasing negative function $D(\alpha)$. Moreover,*

$$\lim_{\alpha\to-\infty} D(\alpha) = -\infty \tag{5.19}$$

and

$$\lim_{\alpha\to\infty} D(\alpha) = D(\infty) \leqslant 0, \tag{5.20}$$

where

$$D(\infty) = \begin{cases} \lim_{r\to\infty} r^{\lambda_1}\left(U_K(r) - \log\dfrac{(2+\ell)(n-2)}{cr^{2+\ell}}\right) & \text{if} \quad n > 10 + 4\ell, \\ \lim_{r\to\infty} r^{\lambda_1}(\log r)^{-1}\left(U_K(r) - \log\dfrac{(2+\ell)(n-2)}{cr^{2+\ell}}\right) & \text{if} \quad n = 10 + 4\ell, \end{cases}$$

where U_K is the singular solution obtained in theorem C. If K_i, $K_1 \leqslant K_2$, satisfy the assumptions, then $U_{K_1} \geqslant U_{K_2}$.

Proof. Let $\beta > \alpha$. By (1.10), we observe that

$$\begin{aligned} r^2K[e^{u_\beta} - e^{u_\alpha}] - b[u_\beta - u_\alpha] &= \left(r^2K\frac{e^{u_\beta} - e^{u_\alpha}}{u_\beta - u_\alpha} - b\right)[u_\beta - u_\alpha] \\ &< (r^2Ke^{u_\beta} - b)[u_\beta - u_\alpha] \\ &< 0. \end{aligned}$$

The integral representation of D in proposition 4.4 shows that $D(\beta) > D(\alpha)$. Since $r^{-\ell}K(r) \geqslant c$, it follows from [**5**, theorem 2.1] that $u_\alpha \leqslant \overline{u}_\alpha$ for each α. Hence, (1.13) implies (5.19) and (5.20). From (1.13), we see that

$$U_K(r) - \log\frac{(2+\ell)(n-2)}{cr^{2+\ell}} \to 0$$

as $r \to \infty$. By lemma 4.3, $D(\infty)$ exists and D is continuous on $(-\infty, \infty]$. Let $u_{\alpha,K_i}(r)$ denote the solution of (1.4) with $K = K_i$ and $u_{\alpha,K_i}(0) = \alpha$. By similar arguments as in [**5**, theorem 2.1], we conclude that $u_{\alpha,K_1} \geqslant u_{\alpha,K_2}$, and thus, $U_{K_1} \geqslant U_{K_2}$. □

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Now, by the proofs in lemma 4.3 and proposition 4.4 under

$$\int_1^\infty |f(r)|r^{1+\lambda_1}\,\mathrm{d}r < \infty, \tag{5.21}$$

we derive the integral representation for $D_\mu(\alpha)$ for solution $u_{\mu,\alpha}$. Then, the arguments of theorem 5.6 implies that $D_\mu(\alpha)$ is strictly increasing as long as solutions are ordered and $r^2Ke^{u_{\mu,\alpha}} \leqslant b$ holds.

PROPOSITION 5.7. *Let $n \geqslant 10+4\ell$ with $\ell > -2$. Assume that (KR) and (1.12) for some $c > 0$, and (fR1), (5.21). If $u_{\mu,\alpha}$ is an entire solution of (1.14) satisfying (1.8), then $D_\mu(\alpha)$ exists and $u_{\mu,\alpha}$ has the integral representation*

$$D_\mu(\alpha) = \frac{-1}{\lambda_2-\lambda_1}\int_0^\infty \left[r^2Ke^{u_{\mu,\alpha}} + \mu r^2 f - b - b\left(u_{\mu,\alpha} - \log\frac{b}{cr^{2+\ell}}\right)\right] r^{-1+\lambda_1}\,\mathrm{d}r$$

for $n > 10+4\ell$,

$$D_\mu(\alpha) = -\int_0^\infty \left[r^2Ke^{u_{\mu,\alpha}} + \mu r^2 f - b - b\left(u_{\mu,\alpha} - \log\frac{b}{cr^{2+\ell}}\right)\right] r^{-1+\lambda_1}\,\mathrm{d}r$$

for $n = 10+4\ell$.

In addition to the existence of $D_\mu(\beta)$ and $D_\mu(\alpha)$ for some $\beta > \alpha$, if $u_{\mu,\beta} > u_{\mu,\alpha}$ and $r^2Ke^{u_{\mu,\beta}} \leqslant b$, then $D_\mu(\beta) > D_\mu(\alpha)$.

If $D_\mu(\alpha)$ exists for α in a range under (5.7) and (5.8) in the range for $n > 10+4\ell$ and $n = 10+4\ell$, respectively, where $D_\mu(\alpha,t)$ and $W_\mu(\alpha,t)$ are defined in the same manner with $u_{\mu,\alpha}$, then $D_\mu(\alpha)$ is continuous in the range.

We use β, α to denote two solutions of (1.14) near ∞. The following result is an extension of lemma 4.5 for (1.14).

LEMMA 5.8. *Let $n \geqslant 10+4\ell$ with $\ell > -2$. Assume (KR), (1.12) for some $c > 0$ and that K is non-negative near ∞, and (fR1), (5.21). If $u_{\mu,\beta}$ and $u_{\mu,\alpha}$ are two solutions of (1.14) near ∞ satisfying (1.8) such that $u_{\mu,\beta} > u_{\mu,\alpha}$ and $D_\mu(\beta) = D_\mu(\alpha)$, then*

$$\lim_{r\to\infty} r^{\lambda_2}(u_{\mu,\beta}(r) - u_{\mu,\alpha}(r)) = d$$

for some $d > 0$.

REMARK 5.9. Suppose that $\tilde{K} \in M(c)$ for some $c > 0$. If $K \leqslant \tilde{K}$, the proof of proposition 5.1 shows that there exists an entire solution $\tilde{u}$ of (1.4) with $\tilde{K}$ such that $\tilde{u} > u_\alpha$ in $\mathbb{R}^n$ for all α negatively large, where solutions are obtained in proposition 5.1. Since $r^2Ke^{u_\alpha} \leqslant r^2\tilde{K}e^{\tilde{u}} \leqslant b$, arguing as in the proof of theorem 5.6 we see that $D(\alpha)$ is strictly increasing for α negatively large. For equation (1.14), this is valid for all $\alpha \in I_\mu$, where I_μ is chosen properly. For example, $\tilde{K}(r) = cr^\ell$ if $r^{-\ell}K(r) \leqslant c$. More generally, if $r^{-\ell}K(r)$ is bounded above and $\sup_{r<s<\infty} s^{-\ell}K(s) \to c$ as $r \to \infty$, then we may set $\tilde{K}(r) = r^\ell \sup_{r<s<\infty} s^{-\ell}K(s)$ for $r > 0$.

The continuity of $D(\alpha)$ plays a crucial role in establishing the following multiplicity result. The characteristic function χ_E of E is defined by $\chi_E(x) = 1$ if $x \in E$, and 0 if $x \notin E$.

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THEOREM 5.10. *Let $n \geqslant 10+4\ell$ with $\ell > -2$. Assume that K and f satisfy (K1) and (f1) respectively. Suppose that*

$$K^+ + (\chi_{[0,1]} + r^{2+\ell}\chi_{[1,\infty)})F_+ \geqslant 0 \tag{5.22}$$

and

$$\int_1^\infty |r^{-\ell}K^\pm \pm r^2 F_\pm - c|r^{-1+\lambda_1}\,\mathrm{d}r < \infty \tag{5.23}$$

for some $c > 0$, where $K^-(r) := \min_{|x|=r} K(x)$, $K^+(r) := \max_{|x|=r} K(x)$, $\lambda_1 = \lambda_1(n,\ell)$, $F_\pm(r) = \max_{|x|=r}\{\pm f(x), 0\}$. Then, there exists $\mu_* > 0$ such that for every $\mu \in [0,\mu_*)$, (1.1) has a continuum of entire solutions with the asymptotic behaviour

$$\lim_{|x|\to\infty}\left[u(x) - \log\frac{(2+\ell)(n-2)}{c|x|^{2+\ell}}\right] = 0. \tag{5.24}$$

Proof. We consider the following homogeneous problems

$$v' + \frac{n-1}{r}v' + (K^\pm \pm H^\pm)e^v \;=\; 0 \quad \text{in } (0,\infty), \quad v(0) = \alpha, \tag{5.25}$$

where $H^\pm(|x|) = F_\pm(|x|)$ in $B(1)$ and $H^\pm(|x|) = |x|^{2+\ell}F_\pm(|x|)$ in $\mathbb{R}^n \setminus B(1)$. We may consider only the case that $K^- - H^- \not\equiv cr^\ell \not\equiv K^+ + H^+$ and $f \not\equiv 0$ because the other cases can be handled similarly. By v_α^+ and v_α^-, denote the solutions of (5.25) with $K^+ + H^+$ and $K^- - H^-$, respectively. From proposition 5.1, there exists α^* such that for each $\alpha \in (-\infty, \alpha^*]$, there exist entire solutions $v_\alpha^\pm$ of (5.25) which increase as α increases and are located below $\overline{u}_\theta$ for some $\theta > \alpha^*$. Moreover, for given $\alpha \in (-\infty, \alpha^*]$, there exist $\eta < \gamma < \xi < \alpha$ such that $\overline{u}_\eta < v_\gamma^- < \overline{u}_\xi < v_\alpha^+$ in $\mathbb{R}^n$. Define $\gamma_\alpha = \sup\{\,\beta \in (\eta,\alpha) \mid v_\beta^- < v_\alpha^+ \text{ in } \mathbb{R}^n\,\}$. Obviously, $v_{\gamma_\alpha}^- \leqslant v_\alpha^+$. Then, the strong maximum principle with (5.22) implies that $v_{\gamma_\alpha}^- < v_\alpha^+$ in $\mathbb{R}^n$. By lemma 5.4, we may set

$$D^-(\gamma_\alpha) := \lim_{r\to\infty} r^{\lambda_1}\left[v_{\gamma_\alpha}^-(r) - \log\frac{(2+\ell)(n-2)}{cr^{2+\ell}}\right]$$

and

$$D^+(\alpha) := \lim_{r\to\infty} r^{\lambda_1}\left[v_\alpha^+(r) - \log\frac{(2+\ell)(n-2)}{cr^{2+\ell}}\right]$$

if $n > 10+4\ell$, and

$$D^-(\gamma_\alpha) := \lim_{r\to\infty} \frac{r^{\lambda_1}}{\log r}\left[v_{\gamma_\alpha}^-(r) - \log\frac{(2+\ell)(n-2)}{cr^{2+\ell}}\right]$$

and

$$D^+(\alpha) := \lim_{r\to\infty} \frac{r^{\lambda_1}}{\log r}\left[v_\alpha^+(r) - \log\frac{(2+\ell)(n-2)}{cr^{2+\ell}}\right]$$

if $n = 10+4\ell$. Then, it follows from proposition 5.5 that $D^-(\gamma_\alpha) = D^+(\alpha)$. Indeed, if $D^-(\gamma_\alpha) < D^+(\alpha)$, then $v_{\gamma_\alpha}^- < v_\alpha^+$ near ∞. Hence, the continuity of D^- implies that there exist $R > 0$ and $\delta > 0$ such that if $0 < \beta - \gamma_\alpha < \delta$ and $\beta < \alpha$, then

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$v_\beta^-(r) < v_\alpha^+(r)$ for $r \in [R, \infty)$. Since v_β^- is monotonically decreasing to $v_{\gamma_\alpha}^-$ as β decreases to γ_α and $v_\beta^- \to v_{\gamma_\alpha}^-$ uniformly on $[0, R]$, there exists $\gamma_\alpha < \gamma_1 < \beta$ such that $v_{\gamma_1}^- < v_\alpha^+$ in $\mathbb{R}^n$ which contradicts the definition of γ_α.

Fix $\alpha_1 \in (-\infty, \alpha^*]$. For simplicity, assume $\alpha^* < 0$. By proposition 5.1, there exist $\eta_1 < \gamma_{\alpha_1}$ and $\eta_2 < \alpha_2 < 2\eta_1$ such that $\overline{u}_{\eta_2} < v_{\gamma_{\alpha_2}}^- < v_{\alpha_2}^+ < \overline{u}_{2\eta_1} < \overline{u}_{\eta_1} < v_{\gamma_{\alpha_1}}^-$ in $\mathbb{R}^n$. Since $D(\alpha)$ in (1.13) for $\overline{u}_\alpha$ is strictly increasing as α increases, we have $D^-(\gamma_{\alpha_2}) = D^+(\alpha_2) < D^-(\gamma_{\alpha_1}) = D^+(\alpha_1)$. The continuity of D^+ implies that $D^+([\alpha_2, \alpha_1]) \supset [D^+(\alpha_2), D^+(\alpha_1)]$. We apply (1.8) to find $\mu^\pm$ satisfying $\mu^+ f_+ \leqslant H^+ \exp(v_{\alpha_1}^+)$, $\mu^- f_- \leqslant H^- \exp(v_{\gamma_{\alpha_1}}^-)$. For each $0 \leqslant \mu \leqslant \min\{\mu^+, \mu^-\}$, we conclude by the super- and sub-solution method that for every $\alpha \in [\alpha_2, \alpha_1]$, equation (1.1) possesses an entire solution $u_{(\mu,\alpha)}$ satisfying $v_{\gamma_\alpha}^- < u_{(\mu,\alpha)} < v_\alpha^+$ in $\mathbb{R}^n$, and moreover,

$$\lim_{|x|\to\infty} \left[u_{(\mu,\alpha)}(x) - \log \frac{(2+\ell)(n-2)}{c|x|^{2+\ell}} \right] = 0.$$

Every $u_{(\mu,\alpha)}$ is characterized by the asymptotic behaviour

$$D_\mu(\alpha) := \lim_{|x|\to\infty} |x|^{\lambda_1} \left[u_{(\mu,\alpha)}(x) - \log \frac{(2+\ell)(n-2)}{c|x|^{2+\ell}} \right] = D^+(\alpha)$$

if $n > 10 + 4\ell$ and

$$D_\mu(\alpha) := \lim_{|x|\to\infty} \frac{|x|^{\lambda_1}}{\log|x|} \left[u_{(\mu,\alpha)}(x) - \log \frac{(2+\ell)(n-2)}{c|x|^{2+\ell}} \right] = D^+(\alpha)$$

if $n = 10 + 4\ell$. The continuity of $D_\mu(\alpha)$ follows from the continuities of $D^\pm(\alpha)$. □

If f satisfies (f2), then (5.23) is equivalent to

$$\int_1^\infty |r^{-\ell} K^\pm - c| r^{-1+\lambda_1} \, \mathrm{d}r < \infty. \tag{5.26}$$

Moreover, (K2) implies (5.26). Then, (5.22) is redundant since K^+ in (5.22) can be replaced by K_+^+. Hence, theorem 1.5 is a special case of theorem 5.10.

The radial version of theorem 5.10 is as follows.

THEOREM 5.11. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Assume that K and f satisfy (KR) and (fR1) respectively. Suppose that*

$$K + (\chi_{[0,1]} + r^{2+\ell}\chi_{[1,\infty)}) f_+ \geqslant 0 \tag{5.27}$$

and

$$\int_1^\infty |r^{-\ell} K \pm r^2 f_\pm - c| r^{-1+\lambda_1} \, \mathrm{d}r < \infty \tag{5.28}$$

for some $c > 0$, where $f_\pm = \max(\pm f, 0)$ and $\lambda_1 = \lambda_1(n, \ell)$. Then, there exists $\mu^* > 0$ with the property that for fixed $0 < \mu < \mu^*$, there exists an interval $I_\mu = (\alpha_\mu, \beta_\mu)$, $-\infty \leqslant \alpha_\mu < \beta_\mu \leqslant +\infty$, such that for each $\xi \in I_\mu$, (1.14) has an entire solution $u_{\mu,\xi}$ satisfying (1.8), and two solutions among them are separated. Moreover, the limit

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$D_\mu(\alpha)$ is continuous and increasing in $\alpha \in I_\mu$. If K is bounded above by a function in $M(c)$, then $D_\mu(\alpha)$ is strictly increasing in $\alpha \in I_\mu$.

REMARK 5.12. In theorems 5.10 and 5.11, we utilize (5.22) and (5.27) to employ the strong maximum principle. In order to verify the existence in theorem 5.11, we combine comparison arguments in [**5**, theorem 2.1] with proposition 5.7 rather than the barrier method. The above proof even without (5.22) shows that there exist finitely many ordered entire solutions of (1.1) with (5.24) as long as $\mu > 0$ sufficiently small. Then, the number of solutions for (1.1) is arbitrary large as $\mu \to 0$ while (1.3) possesses countably many ordered entire solutions with (5.24). The separation property in theorem 5.11 follows from lemma 3.1 with $\tilde{K} = K + (\chi_{[0,1]} + r^{2+\ell}\chi_{[1,\infty)})f_+ \in MI(c)$. When K is non-negative near ∞, we may choose $\tilde{K} = K_+ + (\chi_{[0,1]} + r^{2+\ell}\chi_{[1,\infty)})f_+ \in MI(c)$. On the other hand, if f satisfies (fR2), then this choice also works since (1.12) implies (5.28), which is the assertion of theorem 1.4. Hence, for these two cases, we can remove (5.27) in the assumptions of theorem 5.11.

THEOREM 5.13. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Assume that K and f satisfy (KR) and (fR1) respectively. Suppose that K is non-negative near ∞ and satisfies (5.28) for some $c > 0$. Then, the result of theorem 5.11 holds.*

6. Stability

Now, we apply our results to the Cauchy problem

$$\begin{cases} u_t = \Delta u + K(x)e^u + \mu f(x) & \text{in} \quad \mathbb{R}^n \times (0,T), \\ u(x,0) = \phi & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{6.1}$$

where $T > 0$ and $\phi \not\equiv 0$ is a bounded continuous function in $\mathbb{R}^n$. It is known that there exists $T = T[\phi] > 0$ such that (6.1) has a *unique* solution $u(x,t;\phi)$ in $C^{2,1}_{loc}(\mathbb{R}^n \setminus \{0\} \times (0,T)) \cap C(\mathbb{R}^n \times [0,T))$ which is bounded in $\mathbb{R}^n \times [0,T']$ for all $T' < T[\phi]$. Define weighted L^∞ norms as follows: For $\lambda > 0$, $\theta > 0$, let

$$||\psi||_{(\lambda,\theta)} = \sup_{x\in\mathbb{R}^n} \left| \frac{(1+|x|)^\lambda}{[\log(2+|x|)]^\theta} \psi(x) \right|,$$

where ψ is a continuous function in $\mathbb{R}^n$. We say that a regular steady state u_α of (6.1) is *stable* with respect to the norm $||\cdot||_{(\lambda,\theta)}$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that for ϕ satisfying $||\phi - u_\alpha||_{(\lambda,\theta)} < \delta$,

$$||u(\cdot,t;\phi) - u_\alpha||_{(\lambda,\theta)} < \epsilon$$

for all $t > 0$; u_α is *weakly asymptotically stable* with respect to $||\cdot||_{(\lambda,\theta)}$ if u_α is stable with respect to $||\cdot||_{(\lambda,\theta)}$ and there exists $\delta > 0$ such that for ϕ satisfying $||\phi - u_\alpha||_{(\lambda,\theta)} < \delta$,

$$\lim_{t\to\infty} ||u(\cdot,t;\phi) - u_\alpha||_{(\lambda',\theta)} = 0$$

for all $\lambda' < \lambda$. A special family of super- and sub-solutions surrounding a given solution may derive the weak asymptotic stability. For example, assuming theorem

B(ii) and (1.12), we construct a sequence of pairs of super-solutions $u^+_{\alpha,j} > u_\alpha$ and sub-solutions $u^-_{\alpha,j} < u_\alpha$ of (1.4) surrounding u_α such that all limits $D^{\pm,j}(\alpha)$ of the functions defined similarly as in (1.11) have the same limit $D(\alpha)$ as u_α. Then, the topology is induced by the following asymptotic behaviour.

PROPOSITION 6.1. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Assume that K satisfies (Kr), (M) and (1.12) for some $c > 0$. For each $\alpha \in \mathbb{R}$, there exist a super-solution $u^+_\alpha(r)$ and a sub-solution $u^-_\alpha(r)$ of (1.4) such that $u^+_\alpha > u_\alpha > u^-_\alpha$ on $[0, \infty)$ and $u^+_\alpha(r) - u^-_\alpha(r) = O(r^{-\lambda_2})$ as $r \to \infty$.*

See lemmas 4.3 and 4.5 for the proof of the asymptotic behaviour. Here, we explain how to construct super- and sub-solutions in a general setting. For our convenience, we use u_α instead of $u_{\mu,\alpha}$ to denote solutions of (1.14) with μ fixed.

PROPOSITION 6.2. *Suppose the hypotheses of theorem 5.11 or theorem 1.4 with $0 \not\equiv K \geqslant 0$ and (fR2). For $\alpha_\mu < \eta < \alpha < \xi < \beta_\mu$, there exist a super-solution u^+_α and a sub-solution u^-_α of (1.14) such that $u_\eta < u^-_\alpha < u_\alpha < u^+_\alpha < u_\xi$ in $\mathbb{R}^n$.*

Proof. Let H be a smooth radial function with compact support such that $0 \not\equiv H \not\equiv K$, $0 \leqslant H \leqslant K$ and $\operatorname{supp}(H) \subset \operatorname{supp}(K)$. Consider the problem

$$u_{rr} + \frac{n-1}{r}u_r + (K(r) + H(r))e^u + \mu f(r) = 0, \quad u(0) = \alpha. \tag{6.2}$$

Let $\check{u}_\alpha$ be the solution of (6.2). Let $\alpha_\mu < \alpha < \xi < \beta_\mu$. Suppose that for $\alpha < \beta < \xi$, there exists $R > 0$ such that $\check{u}_\beta > u_\alpha$ in $[0, R)$ and $\check{u}_\beta(R) = u_\alpha(R)$. Then, $\operatorname{supp}(H) \cap B(R) \neq \emptyset$. Since H is small enough, R is arbitrarily large. Hence, we may assume that $\operatorname{supp}(H) \subset B(R)$. Let $R_1 = \sup\{r > 0 \mid \check{u}_\beta(r) - u_\alpha(r) \geqslant (\beta - \alpha)/(2^i))\}$ for i large. For H_1 and H_2, $0 \leqslant H_1 \leqslant H_2$, satisfying the above conditions, let $\check{u}_{\zeta,H_1}$ and $\check{u}_{\beta,H_2}$ for $\beta < \zeta < \xi$ be solutions of (6.2) with $H = H_1$ and H_2 respectively. By the same way as in [**1**, lemma 2.3] with $\check{w}_2 := u_\xi - u_\zeta$, we have $u_{\zeta,H_1} \leqslant u_\zeta$ and $\check{w}_1 := \check{u}_{\zeta,H_1} - \check{u}_{\beta,H_2} > 0$. See [**5**, theorem 2.1] for the ideas of the arguments. Hence, we derive $\check{u}_{\beta,H_1} \geqslant \check{u}_{\beta,H_2}$ by taking $\zeta \to \beta$. Therefore, we may assume that $\operatorname{supp}(H) \subset B(R_1)$ by choosing i sufficiently large and H sufficiently small. Then, $\operatorname{supp}(H) \subset B(R_1) \subset B(R)$. Now, setting $w_3 := \check{u}_\beta - u_\alpha$, we have

$$\begin{cases} \Delta w_3 + k_3 w_3 = 0 \quad \text{in } B(R), \\ w_3 > 0 \text{ in } B(R) \text{ and } w_3(R) = 0, \end{cases}$$

where

$$k_3 := K\frac{e^{\check{u}_\beta} - e^{u_\alpha}}{\check{u}_\beta - u_\alpha} + H\frac{e^{\check{u}_\beta}}{\check{u}_\beta - u_\alpha} \leqslant Ke^{u_\beta} + \frac{2^i H}{\beta - \alpha}e^{u_\beta}$$

in $B(R)$, and $w_3'(R) \leqslant 0$. On the other hand, we have $w_4 := u_\gamma - u_\xi > 0$ in $[0, \infty)$ for any $\xi < \gamma < \beta_\mu$, and

$$\Delta w_4 + k_4 w_4 = 0$$

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in $\mathbb{R}^n$, where

$$k_4 := K\frac{e^{u_\gamma} - e^{u_\xi}}{u_\gamma - u_\xi} \geqslant Ke^{u_\xi}.$$

Furthermore, we may choose H satisfying $H < ((\beta-\alpha)/(2^i))K[e^{u_\xi - u_\beta} - 1]$ in $B(R_1) \cap \operatorname{supp}(K)^\circ$. Hence, $k_3 \leqslant k_4$ but $k_3 \not\equiv k_4$ in $B(R)$. From Green's identity, it follows that

$$\begin{aligned} \omega_n R^{n-1} w_3'(R) w_4(R) &= \int_{B(R)} (w_4 \Delta w_3 - w_3 \Delta w_4) \\ &\geqslant \int_{B(R)} (k_4 - k_3) w_3 w_4 \ > 0, \end{aligned}$$

which in turn implies that $w_3'(R) > 0$, a contradiction. Therefore, for given $\alpha < \beta < \xi$, $\check{u}_\beta > u_\alpha$ in $\mathbb{R}^n$ if H is chosen suitably. Set $u_\alpha^+ = \check{u}_\beta$.

In order to find a sub-solution of (1.14), consider the problem,

$$u_{rr} + \frac{n-1}{r}u_r + (K(r) - H(r))e^u + \mu f(r) = 0, \quad u(0) = \alpha. \tag{6.3}$$

Let $\hat{u}_\alpha$ be the solution of (6.3). Suppose that for some $\alpha_\mu < \eta < \delta < \alpha < \beta_\mu$, there exists $R > 0$ such that $\hat{u}_\delta < u_\alpha$ in $[0, R)$ and $\hat{u}_\delta(R) = u_\alpha(R)$. Let $R_2 = \sup\{r > 0 \mid u_\alpha(r) - \hat{u}_\delta(r) \geqslant ((\alpha-\delta)/(2^i))\}$. Then, we see that $\operatorname{supp}(H) \subset B(R_2)$ for i sufficiently large and H sufficiently small. Setting $w_5 := u_\alpha - \hat{u}_\delta$, we have

$$\begin{cases} \Delta w_5 + k_5 w_5 = 0 \quad \text{in } B(R), \\ w_5 > 0 \text{ in } B(R) \text{ and } w_5(R) = 0, \end{cases}$$

where

$$k_5 := K\frac{e^{u_\alpha} - e^{\hat{u}_\delta}}{u_\alpha - \hat{u}_\delta} + H\frac{e^{\hat{u}_\delta}}{u_\alpha - \hat{u}_\delta} \leqslant Ke^{u_\alpha} + \frac{2^i H}{\alpha - \delta}e^{u_\alpha}$$

in B_R, and $w_5'(R) \leqslant 0$. We may choose H satisfying $H < ((\alpha-\delta)/(2^i))K[e^{u_\xi - u_\alpha} - 1]$ in $B(R_2) \cap \operatorname{supp}(K)^\circ$. Hence, $k_5 \leqslant k_4$ but $k_5 \not\equiv k_4$ in $B(R)$. From Green's identity, it follows that

$$\begin{aligned} \omega_n R^{n-1} w_5'(R) w_4(R) &= \int_{B(R)} (w_4 \Delta w_5 - w_5 \Delta w_4) \\ &\geqslant \int_{B(R)} (k_4 - k_5)\, w_5 w_4 \ > 0 \end{aligned}$$

which implies a contradiction, $w_5'(R) > 0$. Therefore, we have $u_\alpha > \hat{u}_\delta$ for $\eta < \delta < \alpha$. Then, it follows from lemma 3.1 that $\hat{u}_\eta < \hat{u}_\delta$ in $\mathbb{R}^n$ and by the same argument as in [**1**, lemma 2.3], we have $u_\eta \leqslant \hat{u}_\eta$. Set $u_\alpha^- = \hat{u}_\delta$. □

We are ready to improve proposition 6.1 by constructing the following family which explains the topology for the weak asymptotic stability with respect to $|| \cdot ||_{(\lambda_2, 0)}$.

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THEOREM 6.3. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Assume (Kr), (1.12) for some $c > 0$ and (fR1,2). If K is bounded above by a function in $M(c)$, then for each fixed solution u_α of (1.14) obtained in theorem 5.11 or theorem 1.4, there exist a sequence $\{u_\alpha^{+,j}\}$ of radial super-solutions, $u_\alpha^{+,1} > u_\alpha^{+,2} > \cdots > u_\alpha$, and a sequence $\{u_\alpha^{-,j}\}$ of radial sub-solutions, $u_\alpha^{-,1} < u_\alpha^{-,2} < \cdots < u_\alpha$, such that u_α is the only solution of (1.14) in the ordered interval $u_\alpha^{-,j} < u_\alpha < u_\alpha^{+,j}$ for every j, and*

$$\lim_{j\to\infty} u_\alpha^{\pm,j} = u_\alpha.$$

Moreover, there exist two sequences $\{d_\alpha^{+,j}\}$ and $\{d_\alpha^{-,j}\}$ defined by

$$d_\alpha^{\pm,j} := \lim_{r\to\infty} r^{\lambda_2}\left(u_\alpha^{\pm,j}(r) - u_\alpha(r)\right) \tag{6.4}$$

such that $d_\alpha^{+,j}$ is strictly decreasing to 0 and $d_\alpha^{-,j}$ is strictly increasing to 0 as $j \to \infty$.

Proof. Fix $0 < \mu < \mu^*$ and $\alpha_\mu < \alpha < \beta_\mu$. From proposition 6.2, (1.14) has a super-solution $\check{u}_\beta$ by a suitable choice of H such that $\check{u}_\beta > u_\alpha$ in $\mathbb{R}^n$. Define

$$\overline{\beta} = \min\{\alpha < \beta < \xi \mid \check{u}_\beta \geqslant u_\alpha\}.$$

By the maximum principle, $u_\alpha^{+,1} := \check{u}_{\overline{\beta}} > u_\alpha$. We may construct a function in $M(c)$ which dominates $K + H$. Hence, there exists a continuous function $D_+^{(1)}(\beta)$ given by $\check{u}_\beta$. Since $D_+^{(1)}$ is non-decreasing, $D_\alpha^{+,1} := D_+^{(1)}(\overline{\beta}) = D(\alpha)$. Indeed, if $D_+^{(1)}(\overline{\beta}) > D(\alpha)$, then by using the continuity of $D_+^{(1)}(\beta)$, we may choose $\alpha < \tilde{\beta} < \overline{\beta}$ and $R > 0$ such that $\check{u}_\beta > u_\alpha$ on (R, ∞) for every $\tilde{\beta} < \beta < \overline{\beta}$. Since $\check{u}_\beta$ is monotonically increasing to $\check{u}_{\overline{\beta}}$ as β increases to $\overline{\beta}$, there exists $\tilde{\beta} < \beta < \overline{\beta}$ such that $\check{u}_{\overline{\beta}} > \check{u}_\beta > u_\alpha$ in $\mathbb{R}^n$, which contradicts the definition of $\overline{\beta}$.

It follows from proposition 6.2 that (1.14) has a sub-solution $\hat{u}_\delta$ for a suitable choice of H in (6.3) such that $\hat{u}_\delta < u_\alpha$ in $\mathbb{R}^n$. Define

$$\underline{\delta} = \max\{\delta < \alpha \mid \hat{u}_\delta \leqslant u_\alpha\}.$$

By the maximum principle, $u_\alpha^{-,1} := \hat{u}_{\underline{\delta}} < u_\alpha$. Similarly, we define $D_-^{(1)}(\delta)$ and $D_\alpha^{-,1} := D_-^{(1)}(\underline{\delta})$. Then, $D_\alpha^{-,1} = D(\alpha)$.

For each j, we consider the equations

$$u' + \frac{n-1}{r}u' + \left(K(r) \pm \frac{H(r)}{j}\right)e^u + \mu f(r) = 0, \quad u(0) = \alpha > 0. \tag{6.5}$$

Applying the standard barriers method to (6.5), we obtain a sequence of radial strict super-solutions of (1.14) $u_\alpha^{+,1} > u_\alpha^{+,2} > \cdots > u_\alpha$ and a sequence of radial strict

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sub-solutions of (1.14) $u_\alpha^{-,1} < u_\alpha^{-,2} < \cdots < u_\alpha$. For each j, setting

$$D_\alpha^{\pm,j} := \lim_{r\to\infty} \begin{cases} r^{\lambda_1}\left(u_\alpha^{\pm,j}(r) - \log\dfrac{(2+\ell)(n-2)}{cr^{2+\ell}}\right) & \text{if} \quad n > 10+4\ell, \\ r^{\lambda_1}(\log r)^{-1}\left(u_\alpha^{\pm,j}(r) - \log\dfrac{(2+\ell)(n-2)}{cr^{2+\ell}}\right) & \text{if} \quad n = 10+4\ell, \end{cases}$$

we have $D_\alpha^{+,j} = D(\alpha) = D_\alpha^{-,j} < 0$ for all j. Since $D(\alpha)$ is strictly increasing as α increases, u_α is the unique solution of (1.14) satisfying $u_\alpha^{-,j} < u_\alpha < u_\alpha^{+,j}$. Now, we see that there exists a decreasing sequence $\{\beta_j\}$ such that $\beta_j = u_\alpha^{+,j}(0) > \alpha$ and

$$u_\alpha^{+,j}(r) = \beta_j - \frac{1}{n-2}\int_0^r s\left\{1 - \left(\frac{s}{r}\right)^{n-2}\right\}\left(\left[K(s) + \frac{H(s)}{j}\right]e^{u_\alpha^{+,j}} + \mu f(s)\right)\mathrm{d}s.$$

Since $u_\alpha^{+,j}$ is monotonically decreasing as $j\to\infty$ and thus $u_\alpha^{+,j}$ converges uniformly in any compact subset of $[0,\infty)$ to a continuous function $\tilde{u}$ which satisfies that $\tilde{u} \geqslant u_\alpha$ and for any $r > 0$,

$$\tilde{u}(r) = \beta - \frac{1}{n-2}\int_0^r s\left\{1 - \left(\frac{s}{r}\right)^{n-2}\right\}\left(K(s)e^{\tilde{u}} + \mu f(s)\right)\mathrm{d}s$$

for some $\beta \geqslant \alpha$. Hence, $\tilde{u} = u_\beta$. The uniqueness of u_α implies that $\beta = \alpha$. Similarly, we observe that $u_\alpha = \lim_{j\to\infty} u_\alpha^{-,j}$. On the other hand, lemma 5.8 implies that there exist two sequences $\{d_\alpha^{+,j}\}$ and $\{d_\alpha^{-,j}\}$ defined by (6.4). It follows from lemma 5.8 again that $d_\alpha^{+,j}$, $d_\alpha^{-,j}$ are strictly decreasing and strictly increasing as $j\to\infty$, respectively. Set

$$W^{+,j}(\alpha,t) := u_\alpha^{+,j}(r) - \log\frac{(2+\ell)(n-2)}{cr^{2+\ell}}, \quad t = \log r$$

and $\psi^{+,j}(t) := r^{\lambda_2}(u_\alpha^{+,j}(r) - u_\alpha(r))$.

Case 1. Let $n > 10+4\ell$. From (4.11) with sufficiently large T, we have

$$\begin{aligned} d_\alpha^{+,j} &= \psi^{+,j}(T) + \frac{1}{\lambda_2-\lambda_1}\int_T^{+\infty} e^{\lambda_2 s}\left[bg(W_\alpha^{+,j}) - bg(W_\alpha)\right]\mathrm{d}s \\ &\quad + \frac{1}{\lambda_2-\lambda_1}\int_T^{+\infty} e^{(\lambda_2-\ell)s}h(e^s)\left(e^{W_\alpha^{+,j}} - e^{W_\alpha}\right)\mathrm{d}s \\ &\quad - \frac{1}{\lambda_2-\lambda_1}e^{(\lambda_2-\lambda_1)T}\int_T^{+\infty} e^{\lambda_1 s}\left[bg(W_\alpha^{+,j}) - bg(W_\alpha)\right]\mathrm{d}s \\ &\quad - \frac{1}{\lambda_2-\lambda_1}e^{(\lambda_2-\lambda_1)T}\int_T^{+\infty} e^{(\lambda_1-\ell)s}h(e^s)\left(e^{W_\alpha^{+,j}} - e^{W_\alpha}\right)\mathrm{d}s. \end{aligned}$$

Since $u_\alpha^{+,j}$ converges uniformly on any compact set of $[0,\infty)$, we see that $\psi^{+,j}(T)\to 0$ as $j\to\infty$. The remaining parts are going to 0 also by the Dominated Convergence theorem. Therefore, $d_\alpha^{+,j}\to 0$ as $j\to\infty$. Similarly, $d_\alpha^{-,j}\to 0$ as $j\to\infty$.

Case 2. Let $n = 10+4\ell$. Similar arguments with (4.12) lead to the conclusion. □

For similar results to theorem 6.3, see [**1**, theorem 4.2] and [**3**, theorem 3.1] which is motivated by [**8**, theorem 4.1]. In [**8**], one of the key ingredients is [**8**, proposition 2.27] to study a weighted L^∞ stability of positive radial solutions of (1.9). The contents of [**8**, proposition 2.27] are a comparison principle, monotone decrease (increase) in time of super-(sub-)solutions and radial symmetry in space of solutions with radially symmetric initial data. In fact, [**8**, proposition 2.27] adopts some results in [**11**]. See [**11**, lemma 1.3, lemma 2.6(ii), lemma 2.6(i)] for classical case and [**11**, theorem 2.4(i), (ii), theorem 2.3] for more general case. More precisely, the former and the latter cover the case of $K(r) = |x|^\ell$ for $\ell \geqslant 0$ and $-2 < \ell < 0$, respectively. The arguments can be applied to (1.4) under (Kr), and even to (1.14) under (fR1) as in [**7**, proposition 4.1] for (1.2). In order to employ the arguments without substantial changes, we consider only the case that K is non-negative.

THEOREM 6.4. *Let $n \geqslant 10 + 4\ell$ with $\ell > -2$. Assume (Kr), (1.12) for some $c > 0$ and (fR1,2) and K is bounded above by a function in $M(c)$. Let $\alpha \in I_\mu$ and $u_{\mu,\alpha}$ be one of the radially symmetric steady states of (6.1) obtained by theorem 5.11 or theorem 1.4.*

(i) *If $n = 10 + 4\ell$, then $u_{\mu,\alpha}$ is stable with respect to the norm $||\cdot||_{(\lambda_1,1)}$ and is weakly asymptotically stable with respect to the norms $||\cdot||_{(\lambda_2,0)}$.*

(ii) *For $n > 10 + 4\ell$, $u_{\mu,\alpha}$ is stable with respect to the norm $||\cdot||_{(\lambda_1,0)}$ and is weakly asymptotically stable with respect to the norm $||\cdot||_{(\lambda_2,0)}$.*

In particular, theorem 6.3 is essential in verifying the weak asymptotic stability. See [**3**, **7**, **8**, **10**, **11**] for the arguments of the proof. We refer the reader to [**7**, **12**] for further related works for stability questions. The proofs of theorems 5.10 and 1.5 show the following stability even when K and f are not supposed to be radially symmetric.

THEOREM 6.5. *Assume that $K \geqslant 0$ and the hypotheses of theorem 5.10 or theorem 1.5. Let u be one of the steady states of (6.1) obtained in theorem 5.10 or theorem 1.5.*

(i) *For $n = 10 + 4\ell$, u is stable with respect to the norm $||\cdot||_{(\lambda_1,1)}$.*

(ii) *For $n > 10 + 4\ell$, u is stable with respect to the norm $||\cdot||_{(\lambda_1,0)}$.*

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References

1 S. Bae. Separation structure of positive radial solutions of a semilinear elliptic equation in $\mathbb{R}^n$. *J. Diff. Equ.* **194** (2003), 460–499.

2 S. Bae. On positive entire solutions of indefinte semilinear elliptic equations. *J. Diff. Equ.* **247** (2009), 1616–1635.

https://doi.org/10.1017/prm.2018.98 Published online by Cambridge University Press

3 S. Bae. Weighted L^∞ stability of positive steady states of semilinear heat equations in $\mathbb{R}^n$. *Discrete and Conti. Dyn. Syst. Ser. A* **26** (2010), 823–837.

4 S. Bae. Entire solutions with asymptotic self-similarity for elliptic equations with exponential nonlinearity. *J. Math. Anal. Appl.* **428** (2015), 1085–1116.

5 S. Bae and W.-M. Ni. Existence and infinite multiplicity for an inhomogeneous semilinear elliptic equation on $\mathbb{R}^n$. *Math. Ann.* **320** (2001), 191–210.

6 S. Bae, T. K. Chang and D. H. Pahk. Infinite multiplicity of positive entire solutions for a semilinear elliptic equation. *J. Diff. Equ.* **181** (2002), 367–387.

7 Y. Deng, Y. Li and F. Yang. On the stability of the positive steady states for a nonhomogeneous semilinear Cauchy problem. *J. Diff. Equ.* **228** (2006), 507–529.

8 C. Gui, W.-M. Ni and X. Wang. On the stability and instability of positive steady states of a semilinear heat equation in $\mathbb{R}^n$. *Comm. Pure Appl. Math.* **45** (1992), 1153–1181.

9 D. D. Joseph and T. S. Lundgren. Quasilinear Dirichlet problems driven by positive sources. *Arch. Rational Mech. Anal.* **49** (1973), 241–269.

10 J. I. Tello. Stability of steady states of the Cauchy problem for the exponential reaction-diffusion equation. *J. Math. Anal. Appl.* **324** (2006), 381–396.

11 X. Wang. On Cauchy problems for reaction-diffusion equations. *Trans. Amer. Math. Soc.* **337** (1993), 549–590.

12 F. Yang and Z. Zhang. On the stability of the positive radial steady states for a semilinear Cauchy problem. *Nonlinear Anal. TMA* **80** (2013), 109–121.