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Motion of bubbles towards the boundary for the Cahn–Hilliard equation

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In this work we describe some aspects of the dynamics of the Cahn–Hilliard equation. In particular, we consider the dynamics of 'bubble' solutions that is spherical interfaces which move superslowly towards the boundary without changing their shape. We show for the Cahn–Hilliard that the bubble drifts towards the closest point on the boundary provided it is sufficiently small. This is contrasted with the related mass conserving Allen–Cahn equation where size is not an issue.

1 Introduction

We are concerned with the Cahn–Hilliard equation

$$\begin{cases} u_t = -\Delta[\epsilon^2 \Delta u - W'(u)], & x \in \Omega \\ \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0, & x \in \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\frac{\partial}{\partial n}$ is the outward Neumann derivative, $0 < \epsilon \ll 1$ is a small parameter, and W a double well potential. This equation is widely accepted as a model that describes the space-time evolution of the concentration $u(x,t)$ of a binary alloy, which is originally homogeneous with concentration $\overline{u}$, and that separates into two coexisting phases with concentrations u_1, u_2, $u_1 < \bar{u} < u_2$. The separation phenomena begins after rapid quenching of the alloy below the curve of miscibility. Above this curve the homogeneous phase with concentration $\bar{u}$ is stable in thermodynamic equilibrium. Below the curve of miscibility the homogeneous phase becomes thermodynamically unstable and thermodynamical equilibrium corresponds to two equally favoured phases with concentration u_1, u_2. Therefore, after rapid quenching, a complicated separation phenomenon, which may include nucleation and spinodal decomposition, begins. We refer elsewhere [7, 8, 11, 16, 19, 20] for physical background and numerical studies. The Cahn–Hilliard equation can be viewed [12] as the gradient system corresponding to the free energy functional

$$\begin{cases} J_\epsilon(u) = \int_\Omega \left(\frac{\epsilon^2}{2}|\nabla u|^2 + W(u)\right) dx, \\ u \in \left\{v \in H^1(\Omega) : \frac{1}{|\Omega|}\int_\Omega v\, dx = \bar{u}\right\} \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain which represents the container of the alloy and $|\Omega|$ is the measure of Ω, u is the concentration, and W is a double well potential with two equal nondegenerate minima at $u = \pm 1$. A typical example is $W(u) = \frac{1}{4}(u^2 - 1)^2$. The Cahn–Hilliard preserves the average concentration

$$\frac{1}{|\Omega|}\int_{\Omega} u(x,t)\,dx = \overline{u} \tag{1.3}$$

where $\overline{u}$ is independent of t. The conservation of $\overline{u}$ expresses the fact that the amount of the two components of the alloy contained in the vessel does not change during separation. The gradient dynamics associated with the functional (1.2) under the constraint (1.3) depends on the choice of the Hilbert space H with respect to which the gradient is computed. For instance, if $H = L_0{}^2(\Omega) = \{\phi \in L^2(\Omega), \int_\Omega \phi = 0\}$ then, instead of (1.1), one obtains the nonlocal Allen–Cahn equation

$$\begin{cases} u_t = \epsilon^2 \Delta u - \left(W'(u) - \frac{1}{|\Omega|}\int_\Omega W'(u)\,dx\right), & in \quad \Omega \\ \frac{\partial u}{\partial n} = 0, & on \quad \partial\Omega \end{cases} \tag{1.4}$$

The functional $J_\epsilon(u)$ is the sum of the bulk free energy $\int_\Omega W(u)\,dx$ and of the term $\int_\Omega \frac{\epsilon^2}{2}|\nabla u|^2$, which models the contribution of the surface energy. The W term favors functions that take values close to its minima. We call such functions layered. We call interfaces the zero level sets of such a function, and we call states, the values close to ± 1, that u takes almost uniformly away from the interface. The parameter $\epsilon > 0$ is assumed to be very small $\epsilon \ll 1$, where ϵ is a measure of the relative importance of surface energy to bulk free energy. By direct calculation one can verify that along the solution to (1.1), we have

$$\frac{1}{|\Omega|}\int_\Omega u(t)\,dx = \overline{u}, \quad \frac{d}{dt}J_\epsilon(u(t)) \leqslant 0. \tag{1.5}$$

As mentioned by Fife [12], the Cahn–Hilliard equation is the gradient dynamic with respect to the Hilbert space $\mathscr{H}$ which is the completion of $L_0{}^2 = \{v \in L^2(\Omega)/\int_\Omega v\,dx = 0\}$, the subspace of $L^2(\Omega)$ of functions with zero average with respect to the inner product

$$(v_1, v_2)_{\mathscr{H}} = ((-\Delta)^{-1}v_1, (-\Delta)^{-1}v_2)_{L^2(\Omega)}. \tag{1.6}$$

Here $(-\Delta)^{-1}$ is the self-adjoint positive operator defined by $w = (-\Delta)^{-1}v$, and w is the unique solution of the problem

$$\begin{cases} -\Delta w = v \\ \frac{\partial w}{\partial n} = 0 \\ \int_\Omega w\,dx = \int_\Omega v\,dx = 0. \end{cases} \tag{1.7}$$

In Alikakos & Fusco [1], it was established that the Cahn–Hilliard equation in higher space dimensions supports superslow solutions called ‘bubble’ solutions. These solutions correspond to an approximately spherical interface drifting slowly towards the boundary, without changing its shape. Such solutions are typical of the final stages of evolution for general initial conditions. The order of magnitude of the speed of the 'bubble' is

$$\dot{\xi} = O\left(e^{-c\frac{d^{\xi}}{\epsilon}}\right) \tag{1.8}$$

where $\xi = \xi(t)$ represents the centre of the bubble, and $d^{\xi} = d(\xi, \partial\Omega) - \rho$ is the distance of the bubble from the boundary $\partial\Omega$. One of the remarkable features of this dependence of the time scale on ϵ is that changing for example ϵ by a factor of 100 slows down the process by a factor of e^{100}. Thus, the bubble is essentially stationary. The phenomenon of superslow motion in a related context, and in one spatial dimension, was first derived in Neu [18]. An explicit, and rigorous, characterization of metastability for the Allen–Cahn equation was done in the pioneering works of Carr & Pego [9] and Fusco & Hale [14]. For the Cahn–Hilliard equation, metastable motion was proved in Alikakos *et al.* [2], Bronsard & Hilhorst [6] and Grant [15]. Later, an explicit, rigorous analysis yielding ODE's was given in Bates & Xun [4]. A formal analysis comparing the ODEs for the Cahn–Hilliard equation, viscous Cahn–Hilliard, and constrained Allen–Cahn equation is given in Sun & Ward [22]. In Alikakos & Fusco [1], it was stated without proof that for the Cahn–Hilliard the bubble solution is drifting roughly towards the closest point on the boundary. Subsequently, in Ward [23] this statement for the bubble solutions of the related mass conserving Allen–Cahn equation, as introduced in Rubinstein & Sternberg [21], is established. The mass-conserving Allen–Cahn equation (see Rubinstein & Sternberg [21]), is

$$\begin{cases} u_t = \epsilon^2 \Delta u - \left(W'(u) - \frac{1}{|\Omega|}\int_\Omega W'(u)\,dx\right), & in \quad \Omega \\ \frac{\partial u}{\partial n} = 0, & on \quad \partial\Omega. \end{cases} \tag{1.9}$$

Slow Motion for (1.9) was established in Alikakos *et al.* [3]. For (1.9), Ward [23] established the following formula for the speed with careful formal asymptotics:

$$\dot{\xi} \sim \frac{\epsilon N \alpha^2 v^2}{\omega_N \beta} \int_{\partial\Omega} r^{1-N} e^{-2v_\epsilon \epsilon^{-1}(r-\rho)}\, \hat{r}[1 + \hat{r}\hat{n}]\hat{r}\hat{n}\, dS, \tag{1.10}$$

where $r = |x - \xi|$, $\hat{r} = \frac{x-\xi}{r}$, $\hat{n}$ denotes the unit vector on $\partial\Omega$, ω_N is the measure of the surface of the unit sphere in $\mathbb{R}^N$, and α, v, v_ϵ positive constants. Analysing then the integral term by Laplace's method the following expression is obtained:

$$\dot{\xi} \sim 2\frac{\epsilon N \alpha^2 v^2}{\omega_N \beta}\left(\frac{\pi\epsilon}{v_\epsilon r_m}\right)^{\frac{N-1}{2}} K(r_m) e^{-2v_\epsilon \epsilon^{-1}(r_m - \rho)}, \quad \text{as } \epsilon \to 0, \tag{1.11}$$

where $K(r_m) = (1 - \frac{r_m}{R_1})^{-\frac{1}{2}}(1 - \frac{r_m}{R_2})^{-\frac{1}{2}} \cdots (1 - \frac{r_m}{R_{N-1}})^{-\frac{1}{2}}$, $R_j \geqslant 0$, for $j = 1, \ldots, N-1$ are the principal radii of curvature and r_m denotes the minimum distance to the $\partial\Omega$. In the present work by building on Alikakos *et al.* [1] and Ward [23], we derive for the Cahn–Hilliard equation (1.1) the expression

$$\dot{\xi} \sim \left(\frac{\partial u^{\xi}}{\partial \xi_i}, \frac{\partial u^{\xi}}{\partial \xi_j}\right)^{-1}_{\mathscr{H}} \cdot \frac{\epsilon N \alpha^2 v^2}{\omega_N \beta} \int_{\partial\Omega} r^{1-N} e^{-2v_\epsilon \epsilon^{-1}(r-\rho)} \hat{r}[1 + \hat{r}\hat{n}]\hat{r}\hat{n}\, dS, \tag{1.12}$$

where u^{ξ} is a layered function that is near a step function, and is defined in §2. The equation above is also written as

$$\dot{\xi} \sim (\Gamma^{\xi})^{-1} \cdot \frac{\epsilon N \alpha^2 v^2}{\omega_N \beta} \int_{\partial\Omega} r^{1-N} e^{-2v_\epsilon \epsilon^{-1}(r-\rho)}\, \hat{r}[1+\hat{r}\hat{n}]\hat{r}\hat{n}\, dS. \tag{1.13}$$

Here Γ^{ξ} is the matrix, which for small bubble radius is given to leading order by

$$\Gamma^{\xi}_{ij} = \rho^{2N-2} \cdot M \int_{S^{N-1}} \int_{S^{N-1}} G(\rho u + \xi, \rho v + \xi)\langle u, e_i\rangle\langle v, e_j\rangle\, du\, dv. \tag{1.14}$$

Here $M = \frac{\omega_N}{N-2}[U(\infty) - U(-\infty)]^2$ is a constant, $u, v \in S^{N-1}$, $\{e_i\}_{i=1}^N$ is the standard basis of $I\!R^N$, ρ is the bubble radius, ξ is the centre of the bubble and $G(x,y)$ is the fundamental solution of the problem (3.6) below. The function $G(\rho u+\xi, \rho v+\xi)$ depends upon a global way from the shape of $\partial\Omega$ and (ρ,ξ). Therefore, $(\Gamma^{\xi})^{-1}$ is in general far from a multiple of the identity or even from a diagonal matrix and it cannot be computed explicitly. A consequence of this fact is that under the Cahn–Hilliard dynamics, the bubble drifts towards the boundary of Ω following a curved path and not a straight line as in the case of the conserved Allen–Cahn. Moreover, the point where the bubble hits the boundary is not in general the closest one. In the special case $\Omega = \{(x_1, x_2)/x_1 > 0, x_2 > 0\} \subset I\!R^2$, we can use the method of images to compute $G(\rho u+\xi, \rho v+\xi)$ and thus derive information on the dependence of $\dot{\xi}$ on $\xi = (\xi_1, \xi_2)$. In the following, we perform a rigorous asymptotic analysis of the matrix Γ^{ξ} under the assumption that the radius of the bubble is very small: $\rho \ll 1$ and show that for the Cahn–Hilliard equation the bubble drifts towards the closest point on the boundary, *provided it is sufficiently small* (Figure 1). Our analysis of Γ^{ξ} for $\rho \ll 1$ shows that to principal order in ρ, Γ^{ξ} is a multiple of the identity matrix. This is to be expected because if $N(|x-y|)$ is the fundamental solution of Δ we have

$$G(\rho u + \xi, \rho v + \xi) = N(\rho|u-v|) + \gamma(\rho u + \xi, \rho v + \xi)$$

for some smooth function $\gamma(x,y)$. This expression of G shows that G depends on the geometry of $\partial\Omega$ only through γ. Moreover, for $\rho \ll 1$ we have

$$\int_{S^{N-1}} \int_{S^{N-1}} \gamma(\rho u + \xi, \rho v + \xi)\langle u, e_i\rangle\langle u, e_j\rangle\, du\, dv \simeq \gamma(\xi,\xi) \int_{S^{N-1}} \int_{S^{N-1}} \langle u, e_i\rangle\langle v, e_j\rangle\, du\, dv = 0$$

for $i \neq j$. This should be contrasted with Ward's result [23] for (1.9), where size is not an issue. Besides the Cahn–Hilliard and the conserved Allen–Cahn equations, there are also other situations where slow motion occurs. For instance, in Chen & Kowalczyk [10] and Iron & Ward [17], it has been shown for the shadow Gierer–Meinhardt model that the spike shape is stable and the single interior spike moves superslowly towards the boundary. There are other systems where a spike and the boundary repel each other. Underlying all these results, there is the stability of a stationary radial solution in $I\!R^N$, something which is impossible for a second order scalar parabolic equation, but becomes possible for a system when conservation occurs.

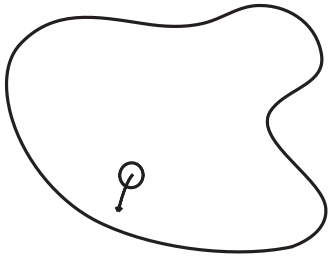

FIGURE 1. The bubble drifts towards the closest point on the boundary.

2 Preliminaries

2.1 Radial solutions

The following proposition, which was proved in Alikakos *et al.* [1], concerns the existence of radial solutions of

$$-\Delta[\Delta u - W'(u)] = 0, \quad on \quad I\!R^N, \tag{2.1}$$

or equivalently,

$$\Delta u - W'(u) = \sigma(\rho), \quad on \quad I\!R^N, \tag{2.2}$$

where ϵ is scaled out.

Proposition 2.1

(a) There exist a number $\bar{\rho} > 0$ and smooth functions $\sigma\colon (\bar{\rho}, \infty) \to I\!R$, $U^* : [0, \infty) \times (\bar{\rho}, \infty) \to I\!R$, such that for each $\rho \in (\bar{\rho}, \infty), \sigma(\rho)$ and $u(x, \rho) = U^*(|x|, \rho)$ satisfy equation (2.2). Moreover, $U^*(r, \rho)$ is increasing in r (Figure 2) and

(i) $\sigma(\rho) = \mathcal{X}\rho^{-1} + O(\rho^{-2})$,

(ii) $U^*(\rho, \rho) = O(\rho^{-1})$,

(iii) $1 + U^*(0, \rho) = O(\rho^{-1})$,

(iv) $lim_{r\to\infty} U^*(r, \rho) = \alpha(\rho)$, where $\mathcal{X} > 0$ is a constant and $\alpha(\rho)$ is the root near unity of the equation $W'(u) + \sigma(\rho) = 0$.

(v) $\alpha(\rho) - U^*(r, \rho) = O(e^{-\nu(\rho)(r-\rho)}), r > \rho, \nu(\rho) = (W''(\alpha(\rho)))^{\frac{1}{2}}$, and similar exponential estimates hold for the derivatives of U^* with respect to r. We expect that, as $\rho \to \infty$, $U^\rho(s) = U^*(s - \rho, \rho)$ tends to U, the unique bounded solution of $U'' - W'(U) = 0$, $\lim_{s\to\pm\infty} U(s) = \pm 1, U(0) = 0$.

(b) There is a number $C > 0$, independent of ρ, such that the functions σ, U^* satisfy the following estimates:

(vi) $\sigma'(\rho) = \mathcal{X}\rho^{-2} + O(\rho^{-3})$,

(vii) $$\begin{array}{ll} U^*(r, \rho) = U(r - \rho) + V(r - \rho, \rho) + O(\rho^{-2}), & r - \rho \in [-C\rho, \infty) \\ U^*{}_\rho(r, \rho) = -\dot{U}(r - \rho) + V_\rho(r - \rho, \rho) + O(\rho^{-3}), & r - \rho \in [-C\rho, \infty) \end{array}$$ where

$$V(r, \rho) = \mathcal{X}\rho^{-1}\int_{-\infty}^{\infty} G(r, s)\, ds, \quad \mathcal{X} = \int_{-\infty}^{\infty} \dot{U}^2 \Big/ \int_{-\infty}^{\infty} \dot{U}.$$

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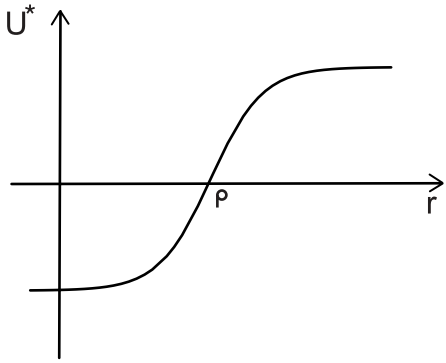

FIGURE 2. Radial solutions of (2.1).

Moreover,

(viii) $\int_{-\infty}^{\infty} W''(U)\dot{U}^2 V = 0.$

2.2 The fundamental block

The 'Bubble' $u^\xi(x)$

By means of Proposition 2.1, we can associate with each $\xi \in \Omega_{\rho+\delta} = \{\xi | d(\xi, \partial\Omega) - \rho > \delta\}$ a function $u^\xi : \Omega \to \mathbb{R}$ with the following properties:

(a) It is an almost stationary solution of the Cahn–Hilliard equation in the sense that it fails to satisfy the equation, or the boundary conditions, by terms which are of the order $O(e^{-c/\epsilon})$.

(b) It jumps from near -1 to near 1 in a thin layer of size of order ϵ around the circle of radius ρ and centre ξ. For $\epsilon \ll 1$ we set $u^\xi(x) = U^*(\frac{|x-\xi|}{\epsilon}, \frac{\rho - \alpha^\xi}{\epsilon}), x \in \Omega$, where the number a^ξ is chosen to be zero at some fixed $\xi_0 \in \Omega_{\rho+\delta}$ and is determined for generic $\xi \in \Omega_{\rho+\delta}$ by imposing that the 'mass' of u^ξ is constant on $\Omega_{\rho+\delta}$:

$$\int_\Omega u^\xi = \int_\Omega u^{\xi_0}, \quad \forall \xi \in \Omega_{\rho+\delta}. \tag{2.3}$$

We choose ξ_0 to be a point of maximal distance from $\partial\Omega$.

Lemma 2.2 [1]

The number a^ξ is uniquely determined by the condition (2.3) *and the assumption* $\alpha^{\xi_0} = 0$. *Moreover,* $0 \leqslant \alpha^\xi < Ce^{-(v_\epsilon/\epsilon)d^\xi}$, *where* $v_\epsilon = v(\frac{\rho - \alpha^\xi}{\epsilon})$; $d^\xi = d(\xi, \partial\Omega) - \rho$. *Similar estimates hold for derivatives of* α^ξ *with respect to* $\xi_i, i = 1, 2$.

2.3 The manifold

In this section, we quickly review the main geometric approach developed in Alikakos *et al.* [1] following the work in Fusco & Hale [14] and Carr & Pego [9], and building on Alikakos *et al.* [1] and Ward [23], we derive the asymptotics as given in (1.13). The

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motion of the bubble corresponds to the dynamics of the Cahn–Hilliard equation as an N-dimensional invariant manifold $\mathscr{M}^{\epsilon}_{\rho}$. This manifold turns out to be represented as a graph over the manifold of bubbles

$$M_{\rho}{}^{\epsilon} = \{u^{\xi}/\xi \in \Omega_{\rho+\delta}\} \tag{2.4}$$

where

$$\Omega_{\rho+\delta} = \{\xi \in \Omega/(\xi, \partial\Omega) > \rho + \delta\}, \quad u^{\xi} = u^{\xi}(x)$$
$$\mathscr{M}^{\epsilon}_{\rho} = \{u^{\xi} + v^{\xi}/\xi \in \Omega_{\rho+\delta}\}, \quad v^{\xi} \perp T_{u^{\xi}}{}^{\epsilon}_{\rho}.$$

Here $v^{\xi} = v^{\xi}(x)$ is small and orthogonal to the tangent space to M^{ϵ}_{ρ} in the Hilbert sense (1.6). Writing the Cahn–Hilliard equation in the abstract form

$$u_t = \mathscr{L}(u) \tag{2.5}$$

we can restate invariance equivalently as the condition of tangency of the vector field $\mathscr{L}$ to $\mathscr{M}^{\epsilon}_{\rho}$,

$$\begin{gathered} \mathscr{L}(u^{\xi} + v^{\xi}) = c_i(\xi) \cdot \frac{\partial}{\partial \xi_i}(u^{\xi} + v^{\xi}) \\ v^{\xi} \perp \frac{\partial u^{\xi}}{\partial \xi_1}, \dots, \frac{\partial u^{\xi}}{\partial \xi_n} \end{gathered} \tag{2.6}$$

where $c_i(\xi)$ is the ith component of the speed, $\dot{\xi} = c(\xi)$, and where the summation convention over repeated indices is employed. The Quasi-invariant manifold $\tilde{M}^{\epsilon}_{\rho}$ is an intermediate object between M^{ϵ}_{ρ} and $\mathscr{M}^{\epsilon}_{\rho}$, where $\hat{v}^{\xi} = \hat{v}^{\xi}(x)$ is defined via

$$\begin{gathered} \mathscr{L}(u^{\xi} + \hat{v}^{\xi}) = \hat{c}_i(\xi) \frac{\partial}{\partial \xi_i} u^{\xi} \\ \hat{v}^{\xi} \perp \frac{\partial u^{\xi}}{\partial \xi_1}, \dots, \frac{\partial u^{\xi}}{\partial \xi_n}, \quad \int_{\Omega} \hat{v}^{\xi} = 0. \end{gathered} \tag{2.7}$$

The system for $\hat{v}$ and $\hat{c}$ is analysed in Alikakos *et al.* [1].

2.4 The expression for the speed

It follows from theorems proved in Alikakos *et al.* [1] that $\|v^{\xi}\| = (e^{\frac{-v_{\epsilon} d_{\xi}}{\epsilon}}), d_{\xi} = d(\xi, \partial\Omega) - \rho$, $c(\xi) = O(e^{\frac{-2v_{\epsilon} d_{\xi}}{\epsilon}})$. Utilizing these estimates it follows that the product $c_i(\xi) \cdot \frac{\partial}{\partial \xi_i}(v^{\xi})$ is of the order $(e^{\frac{-3v_{\epsilon} d_{\xi}}{\epsilon}})$ and so it is insignificant with respect to $e^{\frac{-2v_{\epsilon} d_{\xi}}{\epsilon}}$. This argument suggests that $\hat{c}(\xi)$ is a good approximation of $c(\xi)$. We now proceed to derive an expression for $\hat{c}(\xi)$. We rewrite (2.7) in the form

$$\begin{cases} \Delta[\epsilon^2 \Delta(u^{\xi} + \hat{v}^{\xi}) - (W'(u^{\xi} + \hat{v}^{\xi})] = \hat{c}_i(\xi) \frac{\partial}{\partial \xi_i} u^{\xi}, & \textit{in } \Omega \\ \frac{\partial}{\partial n}(u^{\xi} + \hat{v}^{\xi}) = \frac{\partial}{\partial n} \Delta(u^{\xi} + \hat{v}^{\xi}) = 0, & \textit{on } \partial\Omega \\ \left(\frac{\partial u}{\partial_i}, \hat{v}\right)_{\mathscr{H}} = 0, & i = 1, \dots, N. \end{cases} \tag{2.8}$$

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Expanding the equation about u and making use of

$$-\Delta[\epsilon^2 \Delta u^\xi - W'(u^\xi)] = 0, \tag{2.9}$$

we can rewrite $(2.8)_1$ in the form

$$L^\xi \hat{v}^\xi + \mathscr{N}(\hat{v}^\xi) = \hat{c}_i(\xi) \frac{\partial}{\partial \xi_i} u^\xi, \tag{2.10}$$

where

$$L^\xi = -\Delta[\epsilon^2 \Delta - W''(u^\xi) I],$$

and $\mathscr{N}$ is the nonlinear term.

Taking the inner product of (2.10) with $\frac{\partial}{\partial \xi_j} u^\xi =: u^\xi_{\xi_j}$, we obtain

$$\left(u^\xi_{\xi_j}, L^\xi \hat{v}^\xi\right)_{\mathscr{H}} + \left(u^\xi_{\xi_j}, \mathscr{N}(\hat{v}^\xi)\right)_{\mathscr{H}} = \hat{c}_i \left(u^\xi_{\xi_i}, u^\xi_{\xi_j}\right)_{\mathscr{H}}. \tag{2.11}$$

Ignoring the nonlinear term in (2.11),

$$\left(u^\xi_{\xi_j}, L^\xi \hat{v}^\xi\right)_{\mathscr{H}} \simeq \hat{c}_i \left(u^\xi_{\xi_i}, u^\xi_{\xi_j}\right)_{\mathscr{H}}.$$

Utilizing the definition of Hilbert space in (1.6) we obtain

$$\left(u^\xi_{\xi_j}, \epsilon^2 \Delta \hat{v}^\xi - W''(u^\xi) \hat{v}^\xi\right)_{L^2} \simeq \hat{c}_i \left(u^\xi_{\xi_i}, u^\xi_{\xi_j}\right)_{\mathscr{H}}. \tag{2.12}$$

From

$$\epsilon^2 \Delta u^\xi - W'(u^\xi) = \sigma$$

we have

$$\epsilon^2 \Delta u^\xi_{\xi_j} - W''(u^\xi) u^\xi_{\xi_j} = \frac{\partial \sigma}{\partial \xi_j},$$

and so

$$\int_\Omega \epsilon^2 \Delta u^\xi_{\xi_j} \hat{v} \, dx - \int_\Omega W''(u^\xi) u^\xi_{\xi_j} \hat{v} \, dx = 0,$$

since $\int \hat{v}^\xi = 0$.

Integrating by parts

$$\int_{\partial\Omega} \left(\frac{\partial u^\xi_{\xi_j}}{\partial n} \hat{v} - \frac{\partial \hat{v}}{\partial n} u^\xi_{\xi_j} \right) dS \simeq \hat{c}_i \left(u^\xi_{\xi_i}, u^\xi_{\xi_j}\right)_{\mathscr{H}}. \tag{2.13}$$

Utilizing now the boundary conditions in (2.13), we obtain

$$\int_{\partial\Omega} \left(\frac{\partial u^\xi_{\xi_j}}{\partial n} \hat{v} + \frac{\partial u^\xi}{\partial n} u^\xi_{\xi_j} \right) dS = \hat{c}_i \left(u^\xi_{\xi_i}, u^\xi_{\xi_j}\right)_{\mathscr{H}}. \tag{2.14}$$

Finally, making use of

$$\frac{\partial u^\xi}{\partial \xi_i} = \frac{\partial u^\xi}{\partial x_i} + \frac{1}{\epsilon} U^*_\rho \frac{\partial u^\xi}{\partial \xi_i}, \tag{2.15}$$

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and of the estimates in Alikakos *et al.* [1], we can replace the ξ-derivatives in (2.14) with x-derivatives without affecting significantly $\hat{c}_i$:

$$\int_{\partial\Omega} \left(\frac{\partial u^{\xi}_{x_j}}{\partial n} \hat{v} + \frac{\partial u^{\xi}}{\partial n} u^{\xi}_{x_j} \right) dS \sim \hat{c}_i \left(u^{\xi}_{x_i}, u^{\xi}_{x_j} \right)_{\mathscr{H}}. \tag{2.16}$$

Next we utilize some of the asymptotics in Ward [23] for $x \in \partial\Omega$:

$$\begin{cases} u^{\xi}_{x_j} = v_\epsilon \alpha |x-\xi|^{-1} \epsilon^{-1} \left(\frac{|x-\xi|}{\rho} \right)^{\frac{1-N}{2}} e^{-v_\epsilon \epsilon^{-1}(|x-\xi|-\rho)} [(x_j - \xi_j) + O(\epsilon)] \\ \frac{\partial u^{\xi}_{x_j}}{\partial n} = -{v_\epsilon}^2 |x-\xi|^{-1} \epsilon^{-2} \left(\frac{|x-\xi|}{\rho} \right)^{\frac{1-N}{2}} e^{-v_\epsilon \epsilon^{-1}(|x-\xi|-\rho)} (x_j - \xi_j) \left[\frac{x-\xi}{|x-\xi|} \vec{n} + O(\epsilon) \right]. \end{cases} \tag{2.17}$$

To determine $\hat{v}$ to leading order, by following Ward [23], we argue that in (2.10) we can ignore the nonlinear term $\mathscr{N}$ and $\hat{c}$, since $\hat{c} \sim e^{\frac{-2v_\epsilon d}{\epsilon}}$. Therefore, (2.10) together with the boundary conditions (2.8) in x, y coordinates takes the form

$$\begin{cases} L^{\xi} \hat{v}^{\xi} = 0 \\ \frac{\partial}{\partial n} (u^{\xi} + \hat{v}^{\xi}) = \frac{\partial}{\partial n} \Delta (u^{\xi} + \hat{v}^{\xi}) = 0. \end{cases} \tag{2.18}$$

By utilizing canonical coordinates (s, d), where d denotes the distance from the boundary and s the projection on $\partial\Omega$, $\frac{d}{\epsilon} = \eta$, $\frac{s}{\epsilon} = \sigma$, (2.18) becomes

$$\begin{cases} \Delta(\sigma, \eta) \{ \Delta \tilde{v}^{\xi}(\sigma, \eta) - {v_\epsilon}^2 \tilde{v}^{\xi} \} = 0 \\ \frac{\partial \tilde{v}^{\xi}}{\partial n}(\sigma, 0) = \frac{\partial u^{\xi}}{\partial n}(\sigma, 0), \quad \frac{\partial^3 \tilde{v}^{\xi}}{\partial n^3}(\sigma, 0) = \frac{\partial^3 u^{\xi}}{\partial n^3}(\sigma, 0). \end{cases} \tag{2.19}$$

We seek a solution in the form $\tilde{v}^{\xi} = \tilde{v}^{\xi}(\eta)$ which decays exponentialy as $\eta \to \infty$. So, from (2.19) we have $\tilde{v}(\eta) = k e^{-v\epsilon\eta}$. After determining the constant k, we conclude that

$$\tilde{v}(\eta) = \left. \frac{\partial u^{\xi}}{\partial n} \right|_{n=0} \frac{1}{v_\epsilon} e^{-v_\epsilon \eta}. \tag{2.20}$$

Substituting $\tilde{v}$ from (2.20) and (2.17) into (2.16), we obtain the key formula (1.13). In the remainder of this paper we analyse the matrix (,) in (2.16) and show that for small bubbles this matrix is asymptotically a multiple of the identity.

2.5 Green's function

We call a fundamental solution $G(x, y)$ with pole y a Green's function (for the Neumann problem for the Laplace equation in the domain Ω), if

$$G(x, y) = N(x, y) + \gamma(x, y)$$

for $x \in \bar{\Omega}, y \in \Omega, x \neq y$, with $N(x, y)$ defined

$$N(x, y) = \psi(r) = \psi(|x-y|)$$

$$\psi(r) = \begin{cases} \frac{r^{2-N}}{(2-N)\omega_N} & \text{for } N > 2 \\ \frac{\log r}{2\pi} & \text{for } N = 2. \end{cases}$$

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Here, G is a modified Green's function for the Laplacian with Neumann boundary condition and satisfies (3.6) below and $\gamma(x,y)$ for $y \in \Omega$ is a solution of $\Delta\gamma = 0$, of class $C^2(\bar{\Omega})$ for which $G(x,y) = 0$ for $x \in \Omega, y \in \Omega$.

3 Analysis of the matrix $\left(\frac{\partial u^\xi}{\partial \xi_i}, \frac{\partial u^\xi}{\partial \xi_j}\right)_{\mathscr{H}}^{-1}$

As mentioned in the introduction, the purpose of this work is to establish that small bubbles drift towards the closest point on the boundary. So, the analysis of the matrix above is essential and the main ideas of its rigorous analysis can be described as follows. We are interested in obtaining the following estimate:

$$\alpha_{ij}{}^{\epsilon} = \left(\frac{\partial u^\xi}{\partial \xi_i}, \frac{\partial u^\xi}{\partial \xi_j}\right)_{\mathscr{H}}.$$

$$= C\rho^N \delta_{ij} + O(\rho^{2N-1}) + O\left(\frac{\epsilon}{\rho}\right) + O\left(e^{-\frac{cd_\xi}{\epsilon}}\right) \qquad (\star)$$

$$\text{as } \epsilon \to 0, \quad \text{and } \rho \text{ fixed.}$$

We shall establish $(\star)$ in three steps. First, we reduce u^ξ to the heteroclinic, then, using symmetry, we reduce Ω to the ring, and finally, we reduce the Green's function to the Newtonian potential. In this way, the desired result is obtained by calculation.

Step 1. Reduction to the heteroclinic

Set

$$a_{ij}{}^{\epsilon}(\xi) = -\int_\Omega \int_\Omega G(x,y) \frac{\partial u^\xi(x)}{\partial \xi_i} \cdot \frac{\partial u^\xi(y)}{\partial \xi_j}\, dx\, dy \qquad (3.1)$$

Lemma 3.1

$$a_{ij}{}^{\epsilon} = -\frac{1}{\epsilon^2} \int_\Omega \int_\Omega G(x,y) \dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right) \dot{U}\left(\frac{|y-\xi|-\rho}{\epsilon}\right) \frac{x_i - \xi_i}{|x-\xi|} \frac{y_j - \xi_j}{|y-\xi|}\, dx\, dy$$
$$+ O\left(\frac{\epsilon}{\rho}\right) + O\left(e^{\frac{-cd_\xi}{\epsilon}}\right) \qquad (3.2)$$

where $d_\xi = d(\xi, \partial\Omega) - \rho$.

Proof We recall that

$$u^\xi(x) = U^*\left(\frac{|x-\xi|}{\epsilon}, \frac{a^\xi + \rho}{\epsilon}\right). \qquad (3.3)$$

By utilizing Proposition 2.1, we get

$$U^*(r,\rho) = U(r-\rho) + V(r-\rho,\rho) + O(\rho^{-2}), \quad r - \rho \in [-C\rho, \infty)$$
$$U^*{}_\rho(r,\rho) = -\dot{U}(r-\rho) + V_\rho(r-\rho,\rho) + O(\rho^{-3}), \quad r-\rho \in [-C\rho,\infty),$$

for a certain C independent of ρ, where

$$V(r,\rho) = \mathscr{X}\rho^{-1}\int_{-\infty}^{\infty} G(r,s)\, ds, \quad \mathscr{X} = \int_{-\infty}^{\infty} \dot{U}^2 \Big/ \int_{-\infty}^{\infty} \dot{U}, \qquad (3.4)$$

and $G(r,s)$ satisfies the estimate

$$\left| G(r,s) + \frac{1}{2\bar{v}} e^{-\bar{v}|s-\tau|} \right| \leqslant C e^{-|\tau|} e^{-|s|}. \tag{3.5}$$

By setting $r = |x - \xi|$, $\bar{\rho} = \rho + \alpha^{\xi}$, we then obtain from (3.3) that

$$\begin{aligned} \frac{\partial u^{\xi}}{\partial \xi_i} &= \frac{1}{\epsilon}\frac{\partial U^*}{\partial r}\frac{\partial r}{\partial \xi_i} + \frac{1}{\epsilon^2}\frac{\partial U^*}{\partial \rho}\frac{\partial \alpha^{\xi}}{\partial \xi_i} = \frac{1}{\epsilon}\frac{\partial U^*}{\partial r}\frac{\partial r}{\partial \xi_i} + O\left(e^{-\frac{cd_{\xi}}{\epsilon}}\right) \\ &= \left[\frac{1}{\epsilon}\dot{U}\left(\frac{r-\bar{\rho}}{\epsilon}\right) + \frac{1}{\epsilon}V_r\left(\frac{r-\bar{\rho}}{\epsilon}, \frac{\bar{\rho}}{\epsilon}\right) + O\left(\frac{\epsilon^2}{\bar{\rho}^2}\right)\right]\frac{\partial r}{\partial \xi_i} + O\left(e^{-\frac{cd_{\xi}}{\epsilon}}\right) \\ &= \left[\frac{1}{\epsilon}\dot{U}\left(\frac{r-\bar{\rho}}{\epsilon}\right) + \frac{1}{\rho}Q'\left(\frac{r-\bar{\rho}}{\epsilon}\right) + O\left(\frac{\epsilon^2}{\bar{\rho}^2}\right)\right]\frac{\partial r}{\partial \xi_i} + O\left(e^{-\frac{cd_{\xi}}{\epsilon}}\right) \end{aligned}$$

where $Q(r) = \int_{-\infty}^{\infty} G(r,s)\, ds$. Therefore,

$$\begin{aligned} a_{ij}{}^{\epsilon} = &-\frac{1}{\epsilon^2}\int_{\Omega}\int_{\Omega} G(x,y)\dot{U}\left(\frac{|x-\xi|-\bar{\rho}}{\epsilon}\right)\dot{U}\left(\frac{|y-\xi|-\bar{\rho}}{\epsilon}\right)\frac{x_i-\xi_i}{|x-\xi|}\frac{y_j-\xi_j}{|y-\xi|}\,dx\,dy \\ &+ O\left(\frac{1}{\epsilon\rho}\right)\int_{\Omega}\int_{\Omega} G(x,y)\dot{U}\left(\frac{|x-\xi|-\bar{\rho}}{\epsilon}\right)Q'\left(\frac{|y-\xi|-\bar{\rho}}{\epsilon}\right)\frac{x_i-\xi_i}{|x-\xi|}\frac{y_j-\xi_j}{|y-\xi|}\,dx\,dy \\ &+ O\left(\frac{1}{\rho^2}\right)\int_{\Omega}\int_{\Omega} G(x,y)Q'\left(\frac{|x-\xi|-\bar{\rho}}{\epsilon}\right)Q'\left(\frac{|y-\xi|-\bar{\rho}}{\epsilon}\right)\frac{x_i-\xi_i}{|x-\xi|}\frac{y_j-\xi_j}{|y-\xi|}\,dx\,dy \\ &+ O\left(\frac{\epsilon^2}{\rho^2}\right)\int_{\Omega}\int_{\Omega} G(x,y)\,dx\,dy + O\left(e^{-\frac{cd_{\xi}}{\epsilon}}\right) \\ = &\; I + II + III + IV + O\left(e^{-\frac{cd_{\xi}}{\epsilon}}\right). \end{aligned}$$

By converting to stretched coordinates

$$\frac{|x-\xi|-\bar{\rho}}{\epsilon} = \eta, \qquad \frac{|y-\xi|-\bar{\rho}}{\epsilon} = \bar{\eta},$$

we easily obtain that

$$|II| = O\left(\frac{1}{\epsilon\rho}\right)\epsilon^2 = O\left(\frac{\epsilon}{\rho}\right), \qquad |III| = O\left(\frac{1}{\rho^2}\right)\epsilon^2 = O\left(\frac{\epsilon^2}{\rho^2}\right).$$

□

Step 2. Reduction to the ring

Due to the radial geometry, and the fact that $\dot{U}$ localizes around the boundary of the bubble, we can have the following reduction to the ring, $\Omega_{\delta} = \{x / \| x - \xi| - \rho| \leqslant \delta\}$, where $\delta > 0$.

Lemma 3.2

Consider the problem

$$\begin{cases} \Delta_y G(x,y) = \delta_x(y) - \frac{1}{|\Omega|}, & x, y \in \Omega, \;\; \Omega \text{ bounded} \subset \mathbb{R}^N \\ \frac{\partial G}{\partial n_y} = 0, & \int_{\Omega} G_N(x,y) = 0. \end{cases} \tag{3.6}$$

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Then the following estimate holds true:

$$\|G(x,\cdot)\|_{W^{1,q}} < C, \tag{3.7}$$

where C *is independent of* x *and* $q < \frac{N}{N-1}$.

Proof We recall a result from Brezis & Strauss [5]. Let u be a weak solution of

$$\begin{cases} Lu = f & \text{in } \Omega \\ \frac{\partial u}{\partial n_L} = g & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Then, we have $u \in W^{1,q}(\Omega)$ for all $1 \leqslant q < \frac{N}{N-1}$ and

$$\|u\|_{1,q} \leqslant C_q\left(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Omega)}\right). \tag{3.9}$$

The estimate (3.9) is for functions. We would like to apply it to (3.6). For this purpose we introduce a δ – sequence. Let $f_n \geqslant 0$, $f_n \to \delta$, $\int_\Omega f_n = 1$, with

$$\begin{cases} \Delta G_n = f_n - \frac{1}{|\Omega|} \\ \frac{\partial G_n}{\partial n} = 0. \end{cases} \tag{3.10}$$

Applying the estimate (3.9) to (3.10) we take

$$\|G_n\|_{W^{1,q}} \leqslant C\left\|f_n - \frac{1}{|\Omega|}\right\|_{L^1} \leqslant C[\|f_n\|_{L^1} + 1]$$

$$\|\nabla G_n\|_{L^q} \leqslant C, \quad \|G_n\|_{L^q} \leqslant C. \tag{3.11}$$

Therefore, by using weak compactness we have

$$\nabla G_n \overset{L^q}{\rightharpoonup} \nabla w.$$

We pass to the limit in the weak formulation of (3.10) and obtain

$$-\int_\Omega \nabla G_n \nabla\phi\, dx = \int_\Omega \left(f_n - \frac{1}{|\Omega|}\right)\phi\, dx.$$

$$-\int_\Omega \nabla w \nabla\phi\, dx = \phi(0) - \int_\Omega \frac{1}{|\Omega|}\phi\, dx.$$

It follows that

$$G \equiv w.$$

By lower semicontinuity of the norm

$$G_n \overset{W^{1,q}}{\rightharpoonup} G$$

$$\liminf \|G_n\|_{W^{1,q}} \geqslant \|G\|_{W^{1,q}}.$$

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Therefore, by using (3.11) we conclude that

$$\|G\|_{W^{1,q}} \leqslant C$$

and the result is obtained. □

Note

It should be noted that estimate (3.9) is optimal, in the sense that $1 \leqslant q < \frac{N}{N-1}$ cannot be improved. We can easily check this by taking $N(x-y)$ which is in $W^{1,q}$, $q < \frac{N}{N-1}$ but not in $W^{1,\frac{N}{N-1}}$. Since $N(x,y)$ satisfies estimate (3.7) $\Rightarrow \gamma(x,y)$ also satisfies (3.7).

Lemma 3.3

$$a_{ij} = -\frac{1}{\epsilon^2}\int_{\Omega_\delta}\int_{\Omega_\delta} G(x,y)\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)\dot{U}\left(\frac{|y-\xi|-\rho}{\epsilon}\right)\cos\theta_i(x)\cos\theta_j(y)\,dx\,dy$$
$$+O\left(e^{-\frac{c\delta}{\epsilon}}\right), \quad c>0 \tag{3.12}$$

where

$$cos\theta_i(x) = \frac{x_i-\xi_i}{|x-\xi|} = \frac{((x-\xi),e_i)}{|x-\xi|}, \quad cos\theta_j(y) = \frac{y_i-\xi_i}{|y-\xi|} = \frac{((y-\xi),e_i)}{|y-\xi|}.$$

Proof By utilizing Lemma 3.2 and $|\dot{U}(\eta)| \leqslant ce^{-c|\eta|}$, we compute

$$\int_\Omega\int_\Omega G(x,y)\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)\dot{U}\left(\frac{|y-\xi|-\rho}{\epsilon}\right)\cos\theta_i(x)\cos\theta_j(y)\,dx\,dy$$
$$= \int_{\Omega_\delta}\left(\int_{\Omega_\delta} G(x,y)\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)\dot{U}\left(\frac{|y-\xi|-\rho}{\epsilon}\right)\cos\theta_i(x)\cos\theta_j(y)\,dx\right)dy$$
$$+ \int_{\Omega_\delta}\left(\int_{\Omega\setminus\Omega_\delta} G(x,y)\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)\dot{U}\left(\frac{|y-\xi|-\rho}{\epsilon}\right)\cos\theta_i(x)\cos\theta_j(y)\,dx\right)dy$$
$$+ \int_{\Omega\setminus\Omega_\delta}\left(\int_{\Omega_\delta} G(x,y)\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)\dot{U}\left(\frac{|y-\xi|-\rho}{\epsilon}\right)\cos\theta_i(x)\cos\theta_j(y)\,dx\right)dy$$
$$+ \int_{\Omega\setminus\Omega_\delta}\left(\int_{\Omega\setminus\Omega_\delta} G(x,y)\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)\dot{U}\left(\frac{|y-\xi|-\rho}{\epsilon}\right)\cos\theta_i(x)\cos\theta_j(y)\,dx\right)dy$$
$$= I + II + III + IV.$$

(a)

$$\left|\int_{\Omega\setminus\Omega_\delta} G(x,y)\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)\cos\theta_i(x)\,dx\right| \leqslant e^{-c\frac{\delta}{\epsilon}}\int_{\Omega\setminus\Omega_\delta}|G(x,y)|\,dx \leqslant e^{-c\frac{\delta}{\epsilon}}C$$

by Lemma 3.2 and the symmetry of G_N.

Therefore,

$$|II|, |IV| < Ce^{-c\frac{\delta}{\epsilon}}. \tag{3.13}$$

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(b)

$$\left|\int_{\Omega\setminus\Omega_\delta}\left(\int_{\Omega_\delta} G(x,y)\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)\dot{U}\left(\frac{|y-\xi|-\rho}{\epsilon}\right)\cos\theta_i(x)\cos\theta_j(y)\,dx\right)dy\right|$$
$$\leqslant \int_{\Omega\setminus\Omega_\delta} e^{-\frac{c\delta}{\epsilon}}\left(\int_{\Omega_\delta}|G(x,y)|\left|\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)\right|dx\right)dy$$
$$\leqslant Ce^{-c\frac{\delta}{\epsilon}},$$

as before. Therefore,

$$|III| < Ce^{-c\frac{\delta}{\epsilon}}. \tag{3.14}$$

□

Step 3. Reduction to the Newtonian Potential
By Lemma 3.2 and the fact that (3.7) holds for the Newtonian Potential (as can be checked by explicit calculation), it follows that it also holds for $\gamma(x,y)$, where

$$G(x,y) = N(x,y) + \gamma(x,y).$$

Here $N(x,y)$ is the Newtonian Potential, and γ satisfies

$$\begin{cases} \Delta_y\gamma(x,y) = -\frac{1}{|\Omega|}, & in\ \Omega \\ \frac{\partial\gamma(x,y)}{\partial n_y} = -\frac{\partial N(x,y)}{\partial y}, & y\in\partial\Omega. \end{cases} \tag{3.15}$$

By interior elliptic estimates we obtain

$$\left|\partial_x^\alpha\partial_y^\beta\gamma(x,y)\right| < C \tag{3.16}$$

$(\alpha+\beta=2)$, for $(x,y)\in\Omega_\delta\times\Omega_\delta$.

Lemma 3.4

$$\left|\frac{1}{\epsilon^2}\int_{\Omega_\delta}\int_{\Omega_\delta}\gamma(x,y)\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)\dot{U}\left(\frac{|y-\xi|-\rho}{\epsilon}\right)\cos\theta_i(x)\cos\theta_j(y)\,dx\,dy\right| \leqslant C\rho^{2N-1}. \tag{3.17}$$

Proof

$$\begin{aligned}\gamma(x,y) &= \gamma(x,y)-\gamma(\xi,\xi)+\gamma(\xi,\xi)\\ &= \nabla_x\gamma(\xi,\xi)(x-\xi)+\nabla_y\gamma(\xi,\xi)(y-\xi)+O(|x-\xi|^2)+\gamma(\xi,\xi).\end{aligned} \tag{3.18}$$

We note that

$$\int_{\Omega_\delta}\int_{\Omega_\delta}\gamma(x,y)\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)\dot{U}\left(\frac{|y-\xi|-\rho}{\epsilon}\right)\cos\theta_i(x)\cos\theta_j(y)\,dx\,dy = 0, \tag{3.19}$$

and by (3.16)

$$|\gamma(x,y)-\gamma(\xi,\xi)| \leqslant C\rho. \tag{3.20}$$

Hence,

$$\frac{1}{\epsilon^2}\int_{\Omega_\delta}\int_{\Omega_\delta}|\gamma(x,y)-\gamma(\xi,\xi)|\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)\dot{U}\left(\frac{|y-\xi|-\rho}{\epsilon}\right)dx\,dy$$
$$\leqslant C\rho\int_{|\eta|\leqslant\frac{\delta}{\epsilon}}\int_{|\overline{\eta}|\leqslant\frac{\delta}{\epsilon}}\dot{U}(\eta)\dot{U}(\overline{\eta})(\epsilon\eta+\rho)^{N-1}(\epsilon\overline{\eta}+\rho)^{N-1}\,d\eta\,d\overline{\eta}\leqslant C\rho^{2N-1},$$

and the proof of the lemma is complete. □

Lemma 3.5
Set

$$I_{ij}=-\frac{1}{\epsilon^2}\int_{\Omega_\delta}\int_{\Omega_\delta}N(|x-y|)\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)\dot{U}\left(\frac{|y-\xi|-\rho}{\epsilon}\right)\cos\theta_i(x)\cos\theta_j(y)\,dx\,dy, \tag{3.21}$$

where $N(|x-y|)=N(x,y)$. *Then,*

$$\lim_{\epsilon\to0}I_{ij}=C\rho^N\delta_{ij}. \tag{3.22}$$

Proof We give now the proof of the lemma for $N=2$ while the proof for $N>2$ can be found in the appendix. We have

$$I_{ij}=-\frac{1}{\epsilon^2}\int_{\Omega_\delta}\int_{\Omega_\delta}N(|x-y|)\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)\dot{U}\left(\frac{|y-\xi|-\rho}{\epsilon}\right)\cos\theta_i(x)\cos\theta_j(y)\,dx\,dy.$$

Claim 1

$$\lim_{\epsilon\to0}I_{ij}=-4\pi^2\rho^2[U(\infty)-U(-\infty)]^2\int_0^{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\ln[\rho|e^{i\theta}-e^{i\overline{\theta}}|]Q_{ij}(\theta,\overline{\theta})\,d\theta\,d\overline{\theta} \tag{3.23}$$

where

$$Q_{11}(\theta,\overline{\theta})=\cos\theta\cos\overline{\theta},\quad Q_{12}(\theta,\overline{\theta})=\cos\theta\sin\overline{\theta}$$
$$Q_{21}(\theta,\overline{\theta})=\sin\theta\cos\overline{\theta},\quad Q_{22}(\theta,\overline{\theta})=\sin\theta\sin\overline{\theta}.$$

Proof
We show that

$$\frac{1}{U(\infty)-U(-\infty)}\cdot\frac{1}{2\pi\rho}\cdot\frac{1}{\epsilon}\cdot\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)\to\delta_\rho(|x-\xi|)$$
$$\frac{1}{U(\infty)-U(-\infty)}\cdot\frac{1}{2\pi\rho}\cdot\frac{1}{\epsilon}\cdot\dot{U}\left(\frac{|y-\xi|-\rho}{\epsilon}\right)\to\delta_\rho(|y-\xi|),$$

where δ_ρ is a δ-sequence as $\epsilon\to0$. Indeed,

(a)

$$\frac{1}{2\pi\rho}\frac{1}{\epsilon}\int_{\mathbb{R}^2}\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)dx=\frac{1}{2\pi\rho}\int_0^{2\pi}\int_0^{\infty}\frac{1}{\epsilon}\dot{U}\left(\frac{r-\rho}{\epsilon}\right)r\,dr\,d\theta$$
$$=\frac{1}{2\pi\rho}\int_0^{2\pi}\int_{\frac{-\rho}{\epsilon}}^{\infty}\dot{U}(\eta)(\epsilon\eta+\rho)\,d\eta\,d\theta\to U(\infty)-U(-\infty)$$

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(by making the change of variables $r = |x - \xi|, \eta = \frac{r-\rho}{\epsilon}$).

(b)

$$\frac{1}{U(\infty) - U(-\infty)} \frac{1}{2\pi\rho} \frac{1}{\epsilon} \int_{I\!R^2} \dot{U}\left(\frac{|x-\xi| - \rho}{\epsilon}\right) g(x)\, dx \to \int_{|y-\xi|=\rho} g(y)\, dy.$$

Claim 2

$$\int_0^{2\pi} \int_0^{2\pi} (\ln \rho) Q_{ij}(\theta, \overline{\theta})\, d\theta\, d\overline{\theta} = 0.$$

Proof

By periodicity.

Claim 3

$$\lim_{\epsilon \to 0} I_{ij} = \begin{cases} -2\pi\rho^2 [U(\infty) - U(-\infty)]^2 \int_0^{2\pi} \int_0^{2\pi} \ln |e^{i\theta} - e^{i\overline{\theta}}| Q_{11}(\theta, \overline{\theta})\, d\theta\, d\overline{\theta} \\ 0, \quad i \neq j. \end{cases}$$

$$\lim_{\epsilon \to 0} I_{ij} = \begin{cases} -2\pi\rho^2 [U(\infty) - U(-\infty)]^2 \int_0^{2\pi} \int_0^{2\pi} \ln |e^{i\theta} - e^{i\overline{\theta}}| Q_{22}(\theta, \overline{\theta})\, d\theta\, d\overline{\theta} \\ 0, \quad i \neq j. \end{cases}$$

Proof

We first show that for $i \neq j \Rightarrow \lim_{\epsilon \to 0} I_{ij} = 0$.

(a) Set

$$u(\theta) = \int_0^{2\pi} \ln |e^{i\theta} - e^{i\overline{\theta}}| \cos \overline{\theta}\, d\overline{\theta}. \tag{3.24}$$

We show that $u(\theta) = u(-\theta)$. From this it follows by cancellation that

$$\int_0^{2\pi} \int_0^{2\pi} \ln |e^{i\theta} - e^{i\overline{\theta}}| Q_{ij}(\theta, \overline{\theta})\, d\theta\, d\overline{\theta} = 0.$$

Indeed, we first observe that (3.24) defines a 2π-periodic function. We calculate

$$\begin{aligned} u(-\theta) &= \int_0^{2\pi} \ln |e^{-i\theta} - e^{i\overline{\theta}}| \cos \overline{\theta}\, d\overline{\theta} = \int_0^{2\pi} \ln |e^{-i\theta} - e^{-i\overline{\overline{\theta}}}| \cos \overline{\overline{\theta}}\, d\overline{\overline{\theta}} \\ &= \int_0^{2\pi} \ln |e^{i\theta} - e^{i\overline{\overline{\theta}}}| \cos \overline{\overline{\theta}}\, d\overline{\overline{\theta}} = u(\theta) \end{aligned}$$

and the result is obtained.

(b) Our aim now is to show that for $i = j \Rightarrow \lim_{\epsilon \to 0} I_{11} = \lim_{\epsilon \to 0} I_{22}$.

$$\begin{aligned} \int_0^{2\pi} \int_0^{2\pi} \ln |e^{i\theta} - e^{i\overline{\theta}}| \cos \theta \cos \overline{\theta}\, d\theta\, d\overline{\theta} &= \int_0^{2\pi} \int_0^{2\pi} \ln \left| e^{i(\theta^* + \frac{\pi}{2})} - e^{i(\overline{\theta}^* + \frac{\pi}{2})} \right| \sin \theta^* \sin \overline{\theta}^*\, d\theta^*\, d\overline{\theta}^* \\ &\left(\theta^* = \theta - \frac{\pi}{2}, \quad \overline{\theta}^* = \overline{\theta} - \frac{\pi}{2} \text{ by utilizing } 2\pi\text{-periodicity} \right) \\ &= \int_0^{2\pi} \int_0^{2\pi} \ln |e^{i\theta^*} - e^{i\overline{\theta}^*}| \sin \theta^* \sin \overline{\theta}^*\, d\theta^*\, d\overline{\theta}^*, \end{aligned}$$

and hence

$$\lim_{\epsilon\to 0} I_{11} = \lim_{\epsilon\to 0} I_{22}.$$

□

Theorem 3.6
Let $a_{ij}{}^{\epsilon}$ *be as in Lemma* 3.1. *Then the following estimate holds:*

$$a_{ij}{}^{\epsilon} = C\rho^N \delta_{ij} + O(\rho^{2N-1}) + O\left(\frac{\epsilon}{\rho}\right) + O\left(e^{-\frac{cd_\xi}{\epsilon}}\right)$$

as $\epsilon \to 0$, for fixed ρ.

Proof By a combination of Lemmas 3.1–3.5, where

$$a_{ij}{}^{\epsilon}(\xi) = -\int_\Omega \int_\Omega G(x,y)\frac{\partial u^\xi(x)}{\partial \xi_i} \cdot \frac{\partial u^\xi(y)}{\partial \xi_j}\, dx\, dy, \quad i,j = 1,\ldots,N.$$

□

The proof of Theorem 3.6 implies from (1.12) the desired result that small bubbles for the Cahn–Hilliard are directed towards the closest point on the boundary.

4 Conclusion

Both the conserved Allen–Cahn and the Cahn–Hilliard equation exhibit superslow motion of bubble solutions. They have the same set of equilibria with the same stability property. In both cases the bubble is attracted to the boundary. This happens because the whole evolution takes place so that the free energy $J_\epsilon(u(t))$ is monotone in t, and that for small ϵ, J_ϵ registers the perimeter of the interface lying inside Ω. Therefore, spheres are the favored intermediate states, while interfaces intersecting the boundary are the favored asymptotic states (Figure 3).

It is worth mentioning that the path of the bubble towards the boundary is different in the two cases. In the case of the conserved Allen–Cahn equation, the bubble sees only the closest point on the boundary and moves towards it by following the segment of minimum distance. In the Cahn–Hilliard case, the bubble interacts with the full boundary and moves towards it by following a path which depends globally on the whole boundary and changes drastically with the size of the bubble. Only in the limit of bubbles with a very small size does a bubble under Cahn–Hilliard dynamics move along the segment of the minimal distance, as is the case for the Allen-Cahn equation (Figure 4).

Appendix A Proof of Lemma 3.5 for $N > 2$

In this appendix, we generalize the proof of Lemma 3.5 to N dimensions. We have

$$I_{ij} = -\frac{1}{\epsilon^2}\int_{\Omega_\delta}\int_{\Omega_\delta} N(|x-y|)\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right)\dot{U}\left(\frac{|y-\xi|-\rho}{\epsilon}\right)\cos\theta_i(x)\cos\theta_j(y)\, dx\, dy.$$

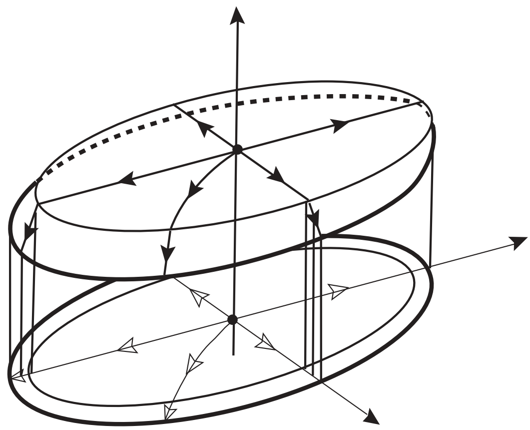

FIGURE 3. The dynamic of the energy of the bubble and the half bubble with the same volume.

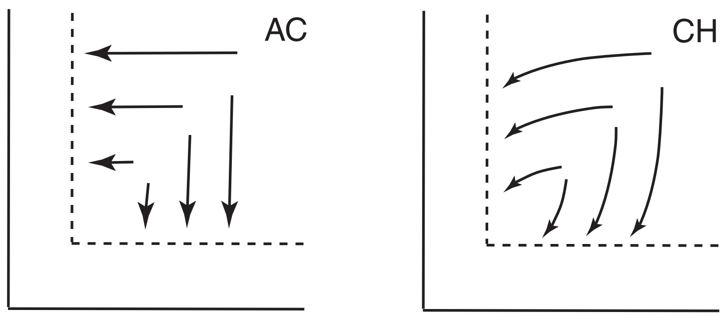

FIGURE 4. Bubble paths for the Allen–Cahn and Cahn–Hilliard equations.

Claim

$$\lim_{\epsilon\to 0} I_{ij} = \frac{\omega_N}{N-2}\rho^{2N-2}[U(\infty)-U(-\infty)]^2 \int_{S^{N-1}}\int_{S^{N-1}} \frac{1}{|x-y|^{N-2}} \cos\theta_i(x)\cos\theta_j(y)\, dS_x\, dS_y, \tag{A.1}$$

where ω_N denotes the surface area of the unit sphere in $I\!R^N$

$$\cos\theta_i(x) = \frac{x_i-\xi_i}{|x-\xi|} = \frac{((x-\xi), e_i)}{|x-\xi|}, \qquad \cos\theta_j(y) = \frac{y_i-\xi_i}{|y-\xi|} = \frac{((y-\xi), e_i)}{|y-\xi|}.$$

Verification
We show that

$$\frac{1}{\epsilon}\dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right) \to \omega_N\rho^{N-1}[U\left(\infty\right)-U(-\infty)]\delta_\rho(|x-\xi|)$$
$$\frac{1}{\epsilon}\dot{U}\left(\frac{|y-\xi|-\rho}{\epsilon}\right) \to \omega_N\rho^{N-1}[U(\infty)-U(-\infty)]\delta_\rho(|y-\xi|).$$

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Indeed,

(1)

$$\frac{1}{\epsilon}\int_{\mathbb{R}^N} \dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right) dx = \frac{1}{\epsilon}\int_{S^{N-1}}\int_0^\infty \dot{U}\left(\frac{r-\rho}{\epsilon}\right) r^{N-1}\, dr\, dS$$
$$= \frac{1}{\epsilon}\int_{S^{N-1}}\int_{\frac{-\rho}{\epsilon}}^{\infty} \dot{U}(\eta)(\epsilon\eta+\rho)^{N-1}\epsilon\, d\eta\, ds \to \omega_N \int_{\frac{-\rho}{\epsilon}}^{\infty} \dot{U}(\eta)\rho^{N-1}\, d\eta$$
$$\to \omega_N \rho^{N-1}[U(\infty)-U(-\infty)].$$

(2)

$$\frac{1}{\epsilon}\int_{\mathbb{R}^N} \dot{U}\left(\frac{|x-\xi|-\rho}{\epsilon}\right) g(x)\, dx \to \omega_N \rho^{N-1}[U(\infty)-U(-\infty)]\int_{|y-\xi|=\rho} g(y)\, dy.$$

We would like to establish the following:

$$\lim_{\epsilon\to 0} I_{ij} = C\rho^N \delta_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ C\rho^N, & \text{when } i = j. \end{cases}$$

$i \neq j$

From the Claim it is sufficient to show that

$$\int_{S^{N-1}}\int_{S^{N-1}} \frac{1}{|x-y|^{N-2}} \cos\theta_i(x)\cos\theta_j(y)\, dS_x\, dS_y = 0.$$

- We first show that the above integral is well defined. This is true because the singularity exists only when $x \to y$. So, we fix $\delta > 0$ so small that $|x-y| < \delta$, $x, y \in \mathbb{R}^{N-1}$ and by the following calculation we conclude that the singularity is integrable:

$$\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}^{N-1}} \frac{1}{|x-y|^{N-2}}\, dx\, dy = \int_{\mathbb{R}^{N-1}}\left(\int_{|x-y|<\delta} \frac{1}{|x-y|^{N-2}}\, dx\right) dy$$
$$= \int_{\mathbb{R}^{N-1}}\left(\int \frac{1}{r^{N-2}} r^{N-2}\, dr\, d\theta\right) dy = \int\left(\int dr\, d\theta\right) dy$$

The calculation is similar for $N = 2$.

From this equation we have

$$\int_{S^{N-1}}\int_{S^{N-1}} \frac{1}{|x-y|^{N-2}} \frac{x_i-\xi_i}{|x-\xi|}\frac{y_j-\xi_j}{|y-\xi|}\, dS_x\, dS_y.$$

Set

$$g(y) = \int_{S^{N-1}} \frac{1}{|x-y|^{N-2}} \frac{(x_i-\xi_i)}{|x-\xi|}\, dS_x.$$

By making the transformation $y \to y^*$ where $y^* = (y_1, \ldots, y_i, \ldots, y_{j-1}, -y_j, \ldots)$ is the image of $y = (y_1, \ldots, y_i, \ldots, y_j, \ldots)$ with respect to the x_i axis, it is easy to prove that $g(y^*) = g(y)$.

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Then, by setting

$$Q(y) = \int_{S^{N-1}} g(y) \frac{(y_j - \xi_j)}{|y - \xi|} \, dS_y,$$

and calculating $Q(y^*)$, we take

$$Q(y^*) = \int_{S^{N-1}} g(y^*) \frac{(-y_j + \xi_j)}{|y - \xi|} \, dS_y = -\int_{S^{N-1}} g(y) \frac{(y_j - \xi_j)}{|y - \xi|} \, dS_y = -Q(y).$$

Therefore,

$$\int_{S^{N-1}} \int_{S^{N-1}} \frac{1}{|x - y|^{N-2}} \frac{(x_i - \xi_i)}{|x - \xi|} \frac{(y_j - \xi_j)}{|y - \xi|} \, dS_x \, dS_y = 0 \Rightarrow \lim_{\epsilon \to 0} I_{ij} = 0.$$

$i = j$

We would like to show first that

$$R_{11} = R_{22} = \ldots = R_{NN},$$

where

$$R_{ii} = \int_{S^{N-1}} \int_{S^{N-1}} \frac{1}{|x - y|^{N-2}} \frac{(x_i - \xi_i)}{|x - \xi|} \frac{(y_i - \xi_i)}{|y - \xi|} \, dS_x \, dS_y, \quad i = 1, \ldots, N.$$

By applying the transformation

$$u - \xi \to (x_i - \xi_i, \ldots, x_1 - \xi_1, \ldots, x_N - \xi_N),$$
$$v - \xi \to (y_i - \xi_i, \ldots, y_1 - \xi_1, \ldots, y_N - \xi_N),$$

we have

$$\begin{aligned} R_{11} &= \int_{S^{N-1}} \int_{S^{N-1}} \frac{1}{|x - y|^{N-2}} \frac{(x_1 - \xi_1)}{|x - \xi|} \frac{(y_1 - \xi_1)}{|y - \xi|} \, dS_x \, dS_y \\ &= \int_{S^{N-1}} \int_{S^{N-1}} \frac{1}{|u - v|^{N-2}} \frac{(u_i - \xi_i)}{|u - \xi|} \frac{(v_i - \xi_i)}{|v - \xi|} \, dS_u \, dS_v = R_{ii}. \end{aligned}$$

Here we used the fact that f is an orthogonal transformation. By utilizing $R_{11} = R_{22} = \ldots = R_{NN}$, it is obvious that

$$R_{ii} = \frac{1}{N} \sum_{i=1}^{N} R_{ii}$$

$$\int_{S^{N-1}} \int_{S^{N-1}} \frac{1}{|x - y|^{N-2}} \frac{x_i - \xi_i}{|x - \xi|} \frac{y_j - \xi_j}{|y - \xi|} \, dS_x \, dS_y = \frac{1}{\rho^2} R_{ii}.$$

We sum up all the R_{ii}'s and calculate

$$\begin{aligned} \frac{1}{\rho^2} \sum_{i=1}^{N} R_{ii} &= \frac{1}{\rho^2} \int_{S^{N-1}} \int_{S^{N-1}} \frac{1}{|x - y|^{N-2}} \langle x - \xi, y - \xi \rangle \, dS_x \, dS_y \\ &= \frac{1}{\rho^2} \int_{S^{N-1}} \int_{S^{N-1}} \frac{1}{|x - y|^{N-2}} \|x - \xi\| \cdot \|y - \xi\| \cos\theta \, dS_x \, dS_y \end{aligned}$$

$$= \frac{1}{\rho^2}\int_{S^{N-1}}\int_{S^{N-1}} \frac{1}{|x-y|^{N-2}}\rho^2 \cos\theta \, dS_x \, dS_y$$
$$= \int_{S^{N-1}}\int_{S^{N-1}} \frac{1}{\rho^{N-2}} \frac{1}{(2-2\cos\theta)^{\frac{N-2}{2}}} \cos\theta \, dS_x \, dS_y$$
$$= \frac{1}{\rho^{N-2}}\int_{S^{N-1}}\int_{S^{N-1}} \frac{\cos\theta}{(2-2\cos\theta)^{\frac{N-2}{2}}} \, dS_x \, dS_y \neq 0.$$

By the Claim, we conclude

$$\lim_{\epsilon\to 0} I_{ij} = \frac{\omega_N}{N-2}\rho^{2N-2}\frac{1}{\rho^{N-2}}\frac{1}{N}[U(\infty)-U(-\infty)]^2 \int_{S^{N-1}}\int_{S^{N-1}} \frac{\cos\theta}{(2-2\cos\theta)^{\frac{N-2}{2}}} \, dS_x \, dS_y$$
$$= \frac{\omega_N}{N-2}\rho^{N}\frac{1}{N}[U(\infty)-U(-\infty)]^2 \int_{S^{N-1}}\int_{S^{N-1}} \frac{\cos\theta}{(2-2\cos\theta)^{\frac{N-2}{2}}} \, dS_x \, dS_y$$

where $\int_{S^{N-1}} \frac{\cos\theta}{(2-2\cos\theta)^{\frac{N-2}{2}}} \, dS_x \neq 0$.

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