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Characterizing homomorphisms and derivations on C^*-algebras

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The main theorem states that a bounded linear operator h from a unital C^*-algebra A into a unital Banach algebra B must be a homomorphism provided that $h(\mathbf{1}) = \mathbf{1}$ and the following condition holds: if $x, y, z \in A$ are such that $xy = yz = 0$, then $h(x)h(y)h(z) = 0$. This theorem covers various known results; in particular it yields Johnson's theorem on local derivations.

1. Introduction

This paper is an analytic counterpart of the algebraic paper [1], in which we study characteristic properties of homomorphisms and derivations in rings containing non-trivial idempotents. Here we shall consider the same properties in C^*-algebras. Since C^*-algebras do not always contain non-trivial idempotents, a straightforward modification of the methods used in [1] cannot prove effective in this context.

Our main result, theorem 3.1, characterizes homomorphisms on C^*-algebras through their action on elements satisfying some special relations (specifically, elements x, y and z that satisfy $xy = yz = 0$). Our main motivation for treating the condition from theorem 3.1 is that, using a standard trick based on upper triangular 2×2 matrices, the results on this condition can be directly transformed into analogous results concerning a certain condition (see corollary 3.2) that is automatically satisfied by local derivations. As a corollary to our main result we shall thus obtain a new, short and self-contained proof of a theorem by Johnson [4] on local derivations on C^*-algebras (corollary 3.3). On the other hand, our method enables us to consider operators preserving zero products. Such operators were studied thoroughly in [2]. In § 4 we obtain generalizations and short proofs of some of their results.

For more details on the history and the background of the properties that are considered in §§ 3 and 4 we refer the reader to [1]. In § 2 we prove the key lemma that concerns bilinear maps on $C(I) \times C(I)$, where $C(I)$ is the C^*-algebra of continuous functions on an interval I.

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2. A lemma on bilinear maps

LEMMA 2.1. *Let X be a normed space, let I be a compact interval of $\mathbb{R}$ and let $\phi : C(I) \times C(I) \to X$ be a bounded bilinear map such that $\phi(f, g) = 0$ whenever $f, g \in C(I)$ are such that $fg = 0$. Then ϕ is symmetric (i.e. $\phi(f, g) = \phi(g, f)$ for all $f, g \in C(I)$).*

Proof. The map ϕ can be thought of as a bounded linear operator on the projective tensor product $V = C(I) \widehat{\otimes} C(I)$ defined through

$$f \otimes g \mapsto \phi(f, g)$$

for all $f, g \in C(I)$. On the other hand, the algebra V can be algebraically identified with a subalgebra of $C(I \times I)$ by defining

$$(f \otimes g)(s, t) = f(s)g(t)$$

for all $f, g \in C(I)$ and $s, t \in I$.

Let $F \in V$ be such that for some $\varepsilon > 0$ we have $F(s, t) = 0$ whenever $s, t \in I$ are such that $|s - t| < \varepsilon$. We claim that $\phi(F) = 0$. Of course, it is sufficient to prove that we can expand F as $F = \sum_{n=1}^{\infty} f_n \otimes g_n$ with $f_n, g_n \in C(I)$ and $f_n g_n = 0$ for each $n \in \mathbb{N}$. Write $F = \sum_{n=1}^{\infty} a_n \otimes b_n$ with $a_n, b_n \in C(I)$. For each $k \in \mathbb{Z}$, we define $\omega_k^{\varepsilon} \in C(I)$ by

$$\omega_k^{\varepsilon}(s) = \begin{cases} 5\varepsilon^{-1}s - 2k + 1, & s \in (\frac{1}{5}\varepsilon\,]2k-1, 2k]) \cap I; \\ 1, & s \in (\frac{1}{5}\varepsilon\,]2k, 2k+1]) \cap I; \\ 2k + 2 - 5\varepsilon^{-1}s, & s \in (\frac{1}{5}\varepsilon\,]2k+1, 2k+2]) \cap I; \\ 0, & \text{elsewhere.} \end{cases}$$

Note that $\omega_k^{\varepsilon} \neq 0$ only for finitely many k. It may easily be checked that $\omega_i^{\varepsilon}\omega_j^{\varepsilon} \neq 0$ if and only if $|i - j| \leqslant 1$ and that $1 = \sum_{i \in \mathbb{Z}} \omega_i^{\varepsilon}$, and therefore that

$$1 = \sum_{i,j \in \mathbb{Z}} \omega_i^{\varepsilon} \otimes \omega_j^{\varepsilon}.$$

Let $i, j \in \mathbb{Z}$ with $|i - j| \leqslant 1$ and let $p = \min\{i, j\}$. Then $\omega_i^{\varepsilon} \otimes \omega_j^{\varepsilon}$ vanishes on

$$(I \times I) \setminus (]\tfrac{1}{5}\varepsilon(2p - 1), \tfrac{1}{5}\varepsilon(2p + 4)[\times]\tfrac{1}{5}\varepsilon(2p - 1), \tfrac{1}{5}\varepsilon(2p + 4)[)$$

and, for $s, t \in]\frac{1}{5}\varepsilon(2p - 1), \frac{1}{5}\varepsilon(2p + 4)[$, we have $|s - t| < \varepsilon$. Therefore, the function

$$\sum_{|i-j| \leqslant 1} \omega_i^{\varepsilon} \otimes \omega_j^{\varepsilon}$$

vanishes on the set $\{(s, t) : I \times I : |s - t| \geqslant \varepsilon\}$. This implies that

$$F = F \sum_{i,j} \omega_i^{\varepsilon} \otimes \omega_j^{\varepsilon} = F \sum_{|i-j|>1} \omega_i^{\varepsilon} \otimes \omega_j^{\varepsilon} = \sum_{n=1}^{\infty} \sum_{|i-j|>1} a_n\omega_i^{\varepsilon} \otimes b_n\omega_j^{\varepsilon},$$

which gives the required decomposition, since $(a_n\omega_i^{\varepsilon})(b_n\omega_j^{\varepsilon}) = 0$ for all $n \in \mathbb{N}$ and $|i - j| > 1$.

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Let $0 < \varepsilon < \frac{1}{2}\pi$ and define $\sigma_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$\sigma_\varepsilon(s) = \begin{cases} 0, & -\pi < s \leqslant -2\varepsilon, \\ -2\varepsilon - s, & -2\varepsilon < s \leqslant -\varepsilon, \\ s, & -\varepsilon < s \leqslant \varepsilon, \\ 2\varepsilon - s, & \varepsilon < s \leqslant 2\varepsilon, \\ 0, & 2\varepsilon < s \leqslant \pi, \end{cases} \qquad \sigma_\varepsilon(s + 2\pi) = \sigma_\varepsilon(s), \quad s \in \mathbb{R}.$$

We also define $\varrho_\varepsilon \in C(I \times I)$ by $\varrho_\varepsilon(s,t) = \sigma_\varepsilon(s-t)$ for all $s,t \in I$. We see immediately that

$$\hat{\sigma}_\varepsilon(0) = 0, \quad \hat{\sigma}_\varepsilon(k) = \frac{\mathrm{i}}{\pi k^2}[\sin(2k\varepsilon) - 2\sin(k\varepsilon)] \quad (k \in \mathbb{Z} \setminus \{0\}),$$

where, as usual $\hat{\sigma}_\varepsilon$ stands for the Fourier transform of the 2π-periodic function σ_ε. Consequently,

$$\varrho_\varepsilon(s,t) = \sum_{k \neq 0} \hat{\sigma}_\varepsilon(k) \mathrm{e}^{\mathrm{i}ks} \mathrm{e}^{-\mathrm{i}kt}, \quad s,t \in I,$$

which clearly implies that $\varrho_\varepsilon \in V$ and $\|\varrho_\varepsilon\|_V \leqslant \mu(\varepsilon)$, where $\mu \in C(\mathbb{R})$ is defined by

$$\mu(s) = \sum_{k \neq 0} \frac{|\sin(2ks) - 2\sin(ks)|}{\pi k^2}, \quad s \in \mathbb{R}.$$

We now proceed to show that $\phi(f,g) = \phi(g,f)$ for all $f,g \in C(I)$. We need only consider the case where both f and g are polynomials. In such a case, we define $F = f \otimes g - g \otimes f \in V$ and we observe that

$$\begin{aligned} F(s,t) &= (f(s) - f(t))g(t) + (g(t) - g(s))f(t) \\ &= (s-t)[P(s,t)g(t) + Q(s,t)f(t)], \quad s,t \in I, \end{aligned}$$

for some polynomials P and Q. So $P, Q \in V$ and the function $R \in C(I \times I)$ defined by

$$R(s,t) = P(s,t)g(t) + Q(s,t)f(t), \quad s,t \in I,$$

also lies in V. For every $0 < \varepsilon < \frac{1}{2}\pi$ we set $F_\varepsilon = \varrho_\varepsilon R \in V$. Since $(F - F_\varepsilon)(s,t) = 0$ whenever $s,t \in I$ are such that $|s-t| < \varepsilon$, we conclude that $\phi(F) = \phi(F_\varepsilon)$. We finally observe that

$$\|F_\varepsilon\|_V \leqslant \|\varrho_\varepsilon\|_V \|R\|_V \leqslant \mu(\varepsilon)\|R\|_V$$

and so

$$\|\phi(F)\| \leqslant \|\phi\| \|R\|_V \mu(\varepsilon).$$

Hence $\|\phi(F)\| \leqslant \lim_{\varepsilon \to 0} \|\phi\| \|R\|_V \mu(\varepsilon) = \|\phi\| \|R\|_V \mu(0) = 0$ and therefore $\phi(f,g) - \phi(g,f) = 0$. □

3. Main results

THEOREM 3.1. *Let A be a unital C^*-algebra and let B be a unital Banach algebra. If $h : A \to B$ is a bounded linear operator such that $h(\mathbf{1}) = \mathbf{1}$ and, for all $x,y,z \in A$,*

$$xy = yz = 0 \quad \Longrightarrow \quad h(x)h(y)h(z) = 0,$$

then h is a homomorphism.

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Proof. Let a and b self-adjoint elements in A. Of course it is enough to prove that $h(ab) = h(a)h(b)$. Let I_1 (respectively, I_2) be a compact interval of $\mathbb{R}$ containing the spectrum of a (respectively, b). Pick $f_2, g_2 \in C(I_2)$ such that $f_2 g_2 = 0$, and define a bounded bilinear map $\phi_1 : C(I_1) \times C(I_1) \to B$ by

$$\phi_1(f_1, g_1) = h(f_1(a))h(g_1(a)f_2(b))h(g_2(b))$$

for all $f_1, g_1 \in C(I_1)$. If f_1 and g_1 are such that $f_1 g_1 = 0$, then

$$f_1(a)(g_1(a)f_2(b)) = (g_1(a)f_2(b))g_2(b) = 0,$$

and therefore the assumption on h implies that $\phi_1(f_1, g_1) = 0$. Using lemma 2.1 we thus get $\phi_1(f_1, g_1) = \phi_1(g_1, f_1)$ for all $f_1, g_1 \in C(I_1)$. In particular, by taking the functions $f_1(s) = 1$, $g_1(s) = s$ and using $h(\mathbf{1}) = \mathbf{1}$ we obtain

$$h(af_2(b))h(g_2(b)) = h(a)h(f_2(b))h(g_2(b)).$$

We have derived this identity under the assumption that f_2 and g_2 are any functions in $C(I_2)$ that satisfy $f_2 g_2 = 0$. Therefore, we may apply lemma 2.1 for $\phi_2 : C(I_2) \times C(I_2) \to B$ given by

$$\phi_2(f_2, g_2) = h(af_2(b))h(g_2(b)) - h(a)h(f_2(b))h(g_2(b)),$$

and hence we may conclude that $\phi_2(f_2, g_2) = \phi_2(g_2, f_2)$ for all $f_2, g_2 \in C(I_2)$. In particular, for $f_2(s) = 1$ and $g_2(s) = s$ we arrive at the desired conclusion $h(ab) = h(a)h(b)$. □

COROLLARY 3.2. *Let A be a unital C^*-algebra and let M be a unital Banach A-bimodule. If a bounded linear operator $d : A \to M$ is such that $d(\mathbf{1}) = 0$ and, for all $x, y, z \in A$,*

$$xy = yz = 0 \quad \Longrightarrow \quad xd(y)z = 0,$$

then d is a derivation.

Proof. The set B of all matrices of the form

$$\begin{pmatrix} x & m \\ 0 & x \end{pmatrix} \quad \text{with } x \in A \text{ and } m \in M,$$

becomes a Banach algebra under the usual matrix operations and the norm

$$\left\| \begin{pmatrix} x & m \\ 0 & x \end{pmatrix} \right\| = \|x\| + \|m\|.$$

Observe that $h : A \to B$, defined by

$$h(x) = \begin{pmatrix} x & d(x) \\ 0 & x \end{pmatrix},$$

satisfies the conditions of theorem 3.1. Therefore, h is a homomorphism, which implies that d is a derivation. □

Let M be an A-bimodule. Recall that a linear operator $d : A \to M$ is called a *local derivation* if for every $x \in A$ there exists a derivation $d_x : A \to M$ such that $d(x) = d_x(x)$. This concept was introduced by Kadison [5] and Larson and Sourour [6] in 1990, and since then has been studied by a number of authors (see, for example, the references in [1]). One of the most profound results in this area is the one by Johnson [4, theorem 5.3], which we now obtain as a corollary.

Corollary 3.3 (Johnson [4]). *A bounded local derivation d from a C^*-algebra A into a Banach A-bimodule is a derivation.*

Proof. Without loss of generality we may assume that A is a unital algebra and M is a unital A-bimodule. If this was not true, then we would adjoin a unity $\mathbf{1}$ to A, set $\mathbf{1}m = m\mathbf{1} = m$ for every $m \in M$, and extend d by setting $d(\mathbf{1}) = 0$. (Incidentally, it is easy to see that every local derivation sends $\mathbf{1}$ into 0 if A and M are unital.)

Let $x, y, z \in A$ be such that $xy = yz = 0$. Then

$$xd(y)z = xd_y(y)z = (d_y(xy) - d_y(x)y)z = 0.$$

Thus, d satisfies the conditions of corollary 3.2 and so d is a derivation. □

We remark that the proof just given is entirely different from Johnson's. Moreover, it is short and, unlike the one in [4], it avoids using another deep theorem of Johnson on Jordan derivations [3]. On the other hand, Johnson proved that local derivations from A into M are automatically continuous [4, theorem 7.5] and so the assumption of boundedness can be removed from corollary 3.3. This, of course, does not follow from our arguments.

4. Operators preserving zero products

Theorem 3.1 also gives new information about operators preserving zero products, i.e. operators $h : A \to B$ such that, for all $x, y \in A$, $xy = 0$ implies $h(x)h(y) = 0$. Such operators were recently studied in [2] (see also references therein). Theorem 3.1 certainly yields the definitive conclusion about zero-product preservers, but only under the assumption that was avoided in [2], namely, $h(\mathbf{1}) = \mathbf{1}$. Our proof can easily be modified so that without this assumption it gives the following result.

Theorem 4.1. *Let A be a unital C^*-algebra and let B be a Banach algebra. If $h : A \to B$ is a bounded linear operator preserving zero products, then $h(\mathbf{1})h(xy) = h(x)h(y)$ for all $x, y \in A$.*

Proof. Let $a, y \in A$ with a self-adjoint and let I be a compact interval of $\mathbb{R}$ containing the spectrum of a. Define $\phi : C(I) \times C(I) \to B$ by

$$\phi(f, g) = h(f(a))h(g(a)y).$$

If $f, g \in C(I)$ are such that $fg = 0$, then $f(a)(g(a)y) = 0$ and so $\phi(f, g) = 0$. On account of lemma 2.1, we have $\phi(f, g) = \phi(g, f)$ for all $f, g \in C(I)$. In particular, for $f(s) = 1$ and $g(s) = s$ we obtain $h(\mathbf{1})h(ay) = h(a)h(y)$, which readily implies the desired conclusion. □

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In [2] there are several results giving the same conclusion as theorem 4.1. In particular, [2, theorem 4.1] establishes theorem 4.1 for the case where A and B are von Neumann algebras. For general C^*-algebras, however, our theorem seems to be new.

The condition $h(\mathbf{1})h(xy) = h(x)h(y)$ obviously characterizes maps preserving zero products. However, a more desirable characterization is that $h(x) = \lambda\varphi(x)$, where φ is an algebra homomorphism and λ is a central element in the algebra B' generated by the range of h. The assumptions of theorem 4.1 do not allow us to conclude this; for instance, B can be an algebra with trivial multiplication and hence every map preserves zero products. Anyhow, under rather mild additional assumptions, for example,

(i) B' is a unital algebra, or

(ii) h is onto and $B^2 = B$,

it follows that h is of the form $h(x) = \lambda\varphi(x)$. Indeed, this can be easily checked; for (i) one merely has to follow the proof of [2, theorem 2.2], and for (ii) one has to make an obvious modification in the proof of [2, theorem 4.6]. In particular, we now see that [2, theorem 4.11] holds true not only for a C^*-algebra B, but also for every Banach algebra satisfying $B^2 = B$.

We conclude this paper with an analogue of theorem 4.1 for derivations.

Corollary 4.2. *Let A be a unital C^*-algebra and let M be a unital Banach A-bimodule. If a bounded linear operator $d : A \to M$ is such that, for all $x, y \in A$,*

$$xy = 0 \quad \Longrightarrow \quad xd(y) + d(x)y = 0,$$

then $\lambda = d(\mathbf{1})$ lies in the centre of M and there is a derivation $\delta : A \to M$ such that $d(x) = \lambda x + \delta(x)$ for all $x \in A$.

Proof. Following the same method as in the proof of corollary 3.2 we find from theorem 4.1 that $d(xy) + d(\mathbf{1})xy = xd(y) + d(x)y$ for all $x, y \in A$. Setting $y = \mathbf{1}$, we see that $\lambda = d(\mathbf{1})$ lies in the centre of M. Consequently, $\delta(x) = d(x) - \lambda x$ is a derivation. □

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