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Resonances for Slowly Varying Perturbations of a Periodic Schrödinger Operator

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Abstract. We study the resonances of the operator $P(h) = -\Delta_x + V(x) + \varphi(hx)$. Here V is a periodic potential, φ a decreasing perturbation and h a small positive constant. We prove the existence of shape resonances near the edges of the spectral bands of $P_0 = -\Delta_x + V(x)$, and we give its asymptotic expansions in powers of $h^{\frac{1}{2}}$.

0 Introduction

In this paper, we study the theory of resonances for periodic Schrödinger operator with decreasing perturbations. We consider Hamiltonians of the form:

$$P(h) = -\Delta_x + V(x) + \varphi(hx), \quad x \in \mathbf{R}^n, \ (h \searrow 0). \tag{0.1}$$

Hamiltonian (0.1) is one of the main models in the theory of solids. It describes a Bloch electron in a crystal placed in an external field. The function V represents the internal electric field of the crystal. It is real-valued, and periodic with respect to a lattice Γ in $\mathbf{R}^n$. $\varphi(hx)$ is an external potential with dimensionless scale parameter h, $h \ll 1$, which means that φ is slowly varying on the scale of the lattice. Usually, the external field can be considered as very regular. See [2], [5], [27], [28], [37].

First, let us consider the case $V = 0$. If one changes the variable x to $r = hx$, equation (0.1) becomes

$$\widehat{P}_0(h) = -h^2\Delta_r + \varphi(r). \tag{0.2}$$

Resonances of equation (0.2) have been studied quite extensively in the 30 last years. The Balslev-Combes theory of dilation analytic systems [3], [32], or one of its variants [1], [10], [22], [30] allows an elegant definition of the complex resonance energies for $\widetilde{P}(h)$. This theory identifies the resonances of a self-adjoint operator H with the complex eigenvalues of a closed operator $H(t)$, which is obtained from H by the method of spectral deformation.

If $V \neq 0$, the main difficulty encountered while trying to carry out the asymptotic analysis of equation (0.1) is to uncouple x, the fast variable and $r = hx$, the slow variable. V. Buslaev [9] has proposed an approach based on a two-scale expansion in which the electron coordinate x and the slowly variable $r = hx$ are regarded

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as independent variables. This method is based on the simple observation that if $u(\cdot, h) \in \mathcal{D}'(\mathbf{R}^{2n})$ is a solution Γ-periodic in x of

$$\big((hD_r + D_x)^2 + V(x) + \varphi(r) - \lambda\big)\, u(x, r, h) = 0, \tag{0.3}$$

then $v(x, h) = u(x, hx)$ satisfies

$$\big(P(h) - \lambda\big)\, v(x, h) = 0. \tag{0.4}$$

Buslaev has used this idea to construct asymptotic solutions of equation (0.4) by considering $(hD_r + D_x)^2 + V(x) + \varphi(r) - \lambda$ as an h-pseudodifferential operator in r with an operator valued symbol

$$p(r, k) := (k + D_x)^2 + V(x) + \varphi(r) - \lambda \in \mathcal{L}\big(H^2(\mathbf{T}^*), L^2(\mathbf{T}^*)\big). \tag{0.5}$$

Here, $\mathbf{T}^* = \mathbf{R}^n/\Gamma^*$ is the flat torus and Γ^* is the dual lattice of Γ. If the n-th band $\lambda_n(k)$ of the unperturbed periodic Schrödinger operator

$$P_0 = -\Delta + V(x)$$

is simple, and if $\lambda - \lambda_n(k) - \varphi(r)$ is the unique 0 eigenvalue of $p(r, k)$, Buslaev transformed the solvability of equation (0.4), modulo an error of order $\mathcal{O}(h)$, to the equation

$$\big(\varphi(hD_k) + \lambda_n(k) - \lambda\big)\, \widetilde{u}(k, h) = 0, \quad \widetilde{u}(\cdot, h) \in L^2(\mathbf{T}^*). \tag{0.6}$$

Using this idea, Gérard, Martinez and Sjöstrand [15] have showed that the spectral study of equation (0.1), near any fixed energy level z, can be reduced to the study of a finite system of h-pseudodifferential operator $E_{-+}(k, hD_k, z, h) = E^0_{-+}(k, hD_k, z) + hE^1_{-+}(k, hD_k, z) + \cdots$ acting on $L^2(\mathbf{T}^*\,; \mathbf{C}^N)$. The matrix $E^0_{-+}(k, r, z)$ satisfies:

$$\det E^0_{-+}(k, r, z) = 0 \iff \exists\, l \text{ such that } z = \varphi(r) + \lambda_l(k).$$

The articles [9] and [15] reduce the problem of one electron in a periodic lattice and additional perturbing potentials to a problem much like (0.2), and hence make the problem of electron in a periodic lattice not more complicated than free electron theory.

The goal of the present paper is to give a similar reduction for resonances, and applying it to prove the existence of shape resonances. We will give explicitly the leading terms of its asymptotic expansion in powers of h.

To our knowledge, the only known results on the existence of resonances for periodic Schrödinger operator perturbed by a decreasing potential were obtained for the exponentially decaying perturbations. See [13], [14] and [25]. In the one dimensional case ($n = 1$), N. E. Firsova [13] showed that in each gap of the Hill operator of sufficiently high energy, there exists an odd number of resonances after perturbations by an exponentially decaying potential. Under the same decay assumption, but without any restriction on the dimension, the meromorphic continuation of the resolvent of $P(1) = -\Delta + V(x) + \varphi(x)$ as a bounded operator between weighted Hilbert spaces was proven in [14]. In the semi-classical regime, F. Klopp [25] has studied the resonances of

$$H(h, \delta) = -h^2\Delta + V(x) + \delta W(x),$$

where $W(x)$ is a compactly supported potential, and δ is a small positive parameter depending on h. Using the same method as in [14], F. Klopp has proved the existence of one or more resonances near the edge of the first band when $n \neq 2$.

The method of [13], [14], [25] works only for the exponentially decaying perturbations. It excludes potentials of physical interest for which one expects resonances to exist.

In this work, we will use the Balslev-Combes theory of resonances [3]. So, we will identify the resonances of $P(h)$, near some fixed energy level λ, with the complex eigenvalues of a closed operator $P(t, h)$, which is obtained from $P(h)$ by the method of spectral deformation (t is the distortion parameter). In order to study the spectrum of the family $P(t, h)$, we will adapt a method similar to the one used in [15]. More precisely, for z in a small complex neighborhood Ω of λ, we construct an effective Hamiltonian $\widetilde{E}_{-+}(z, t, h)$ acting on $L^2(\mathbf{T}^*, \mathbf{C}^N)$, $N \in \mathbf{N}$ so that

$$z \in \sigma\big(P(t, h)\big) \iff 0 \in \sigma\big(\widetilde{E}_{-+}(z, t, h)\big).$$

Thus, the resonances of $P(h)$ near λ are the points z in the lower half plane for which $\widetilde{E}_{-+}(z, t, h)$ is not invertible for some t in $i\,]0, t_0[$, (t_0 is a small constant).

We are now going to briefly describe the main results of the paper.

Fix a point λ in the interior of some band Λ_l. We assume that the Fermi surface $\mathcal{F}(\lambda) := \bigcup_l \{k\ ;\ \lambda_l(k) = \lambda\}$ does not contain any critical points, *i.e.*, $\nabla\lambda_l(k) \neq 0$ for $k \in \mathcal{F}(\lambda)$. In Section 4, we will prove that, for all ϕ and ψ in a dense subset $\mathcal{A}$ of $L^2(\mathbf{R}^n)$,

$$\big((z - P_0)^{-1}\phi, \psi\big) \quad (\text{resp. } f_{\phi,\psi} := \Big(\big(z - P(h)\big)^{-1}\phi, \psi\Big))$$

has a holomorphic (resp. meromorphic) continuation from the upper half plane $\mathbf{C}^+$ to a complex disc around λ. Following [1] and [32, Sect. XII.6], the poles of $f_{\phi,\psi}$ are called resonances of $P(h)$.

Before stating the results concerning the existence of resonances, let us introduce some assumptions on the l-th band Hamiltonian:

$$W_l(k, r) = \lambda_l(k) + \varphi(r).$$

We suppose that $W_m^{-1}(\lambda) = \varnothing$ if $m \neq l$, $W_l^{-1}(\lambda) = \{(k_0, r_0)\} \cup \Sigma_\lambda$ (where Σ_λ is a connected component with $(k_0, r_0) \notin \Sigma_\lambda$), and W_l has a local nondegenerate extremum (local minimum or maximum) at (k_0, r_0). Finally, we assume that Σ_λ satisfies some nontrapping condition, see assumption (H5). Under these conditions, we prove in Section 5 that, for each $C_0 > 0$, $P(h)$ has a finite number of resonances in the disc $D(\lambda, C_0 h) = \{z \in \mathbf{C}\ ;\ |z - \lambda| < C_0 h\}$. Moreover, these resonances coincide, modulo $\mathcal{O}(h^{\frac{3}{2}})$, with the eigenvalues of the operator

$$\lambda - \frac{h^2}{2}\langle \varphi''(r_0)\nabla_k, \nabla_k\rangle + \frac{1}{2}\langle \lambda''(k_0)k, k\rangle.$$

Let us notice that our reduction can be used to study other types of resonances for $P(h)$, similar to those studied in [35] and [7].

The paper is organized as follows: In the next section, we introduce some notations and state the assumptions and the results precisely, which are proved in Sections 4 and 5. In Section 2, we define the distorted Hamiltonian. We also prove some h-pseudodifferential results on the torus which will be used in this paper. In Section 3, we recall the two-scale method of Buslaev and construct the effective Hamiltonian $\widetilde{E}_{-+}(z,t,h)$. Finally, some technical results on resolvent estimates and on pseudodifferential calculus with operator valued symbols are given in an appendix.

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1 Preliminaries and Main Result

Let $\Gamma = \bigoplus_{i=1}^{n} \mathbf{Z}a_i$ be the lattice generated by the basis $a_1, a_2, \dots, a_n$, $a_i \in \mathbf{R}^n$. The reciprocal lattice Γ^* is defined as the lattice generated by the dual basis $\{a_1^*, \dots, a_n^*\}$ determined by $a_i \cdot a_j^* = 2\pi\delta_{ij}$, $i, j = 1, \dots, n$. A fundamental domain of Γ is denoted by E, the one of Γ^* by E^*. If we identify opposite edges of E (resp. E^*) then it becomes a flat torus denoted by $\mathbf{T} = \mathbf{R}^n/\Gamma$ (resp. $\mathbf{T}^* = \mathbf{R}^n/\Gamma^*$).

Let V be a real-valued potential, C^∞ and Γ-periodic. For k in $\mathbf{R}^n$, we define on $L^2(\mathbf{T})$

$$P_k = (D_x + k)^2 + V(x). \tag{1.1}$$

P_k is a semi-bounded self-adjoint operator with k-independent domain $H^2(\mathbf{T})$. Since the resolvent of $(D_x+k)^2$ is compact, the resolvent of P_k is also compact, and therefore P_k has a complete set of (normalized) eigenfunctions $\phi_n(\cdot, k) \in H^2(\mathbf{T}^*)$, $n \in \mathbf{N}$, called Bloch functions. The corresponding eigenvalues accumulate at infinity and we enumerate them according to their multiplicities, $\lambda_1(k) \leq \lambda_2(k) \leq \cdots$. Since $e^{-ix\gamma^*} P_k e^{ix\gamma^*} = P_{\gamma^*+k}$, $\lambda_n(k)$ is periodic with respect to Γ^*. Ordinary perturbation theory, shows that $\lambda_j(k)$ are continuous functions in k for every fixed j, and $\lambda_j(k)$ is even an analytic function of k near every point $k_0 \in \mathbf{T}^*$ where $\lambda_j(k_0)$ is a simple eigenvalue of P_{k_0}. The function $\lambda_j(k)$ is called the band function and the closed intervals $\Lambda_l := \lambda_l(\mathbf{T}^*)$ are called bands.

Now, consider the self-adjoint operator with domain $H^2(\mathbf{R}^n)$

$$P(h) = P_0 + \varphi(hx), \quad (h \searrow 0), \tag{1.2}$$

where

$$P_0 = -\Delta + V(x). \tag{1.3}$$

By Bloch-Floquet theory, it is well known (see [26]) that

$$\sigma(P_0) = \sigma_{\text{ess}}(P_0) = \bigcup_{l=1}^{l=\infty} \Lambda_l. \tag{1.4}$$

Fix λ in $\sigma(P_0)$, and put

$$\mathcal{F}(\lambda) = \{k \in \mathbf{T}^* \,;\, \lambda \in \sigma(P_k)\}.$$

We assume:

(H1) There exist positive constants a and δ, such that φ extends analytically to $D(a) = \{z \in \mathbf{C}^n \,;\, |\Im z| \leq a\langle \Re z\rangle\}$, and

$$|\varphi(z)| \leq C\langle z\rangle^{-\delta}, \tag{1.5}$$

uniformly on z in $D(a)$. Here $\langle z\rangle = (1 + |z|^2)^{\frac{1}{2}}$.

(H2) For every $k \in \mathcal{F}(\lambda)$, λ is a simple eigenvalue of P_k.
In a small neighborhood of $\mathcal{F}(\lambda)$, we let $\lambda(k)$ be the simple eigenvalue which is close to λ. Then $\lambda(k)$ depends analytically on k, and is equal to λ when k belongs to $\mathcal{F}(\lambda)$.

(H3) $d\lambda(k) \neq 0$ for all $k \in \mathcal{F}(\lambda)$.

Let us introduce the set of analytic vectors (see [10] and [30]),

$$\mathcal{A} = \{u \in L^2(\mathbf{R}^n) \,;\, \forall c, s > 0, \exp(c\langle x\rangle)u(x) \in H^s(\mathbf{R}^n)\}.$$

Theorem 1.1 *Under the assumptions* (H1), (H2) *and* (H3), *there exist a neighborhood Ω of λ, and small constants η and $\epsilon > 0$, such that for every t in $I_\epsilon = i\,]0, \epsilon[$ and every $\psi, \phi \in \mathcal{A}$, we have:*

i) *The function*

$$f^0_{\phi,\psi}(z) := \left((P_0 - z)^{-1}\phi, \psi\right),$$

has an analytic continuation from the upper half plane $\mathbf{C}^+$ to $\Omega_{-\eta|t|}$. Here, $\Omega_s := \{z \in \Omega \,;\, \Im z > s\}$.

ii) *For h small enough,*

$$f_{\phi,\psi}(z) := \left((P(h) - z)^{-1}\phi, \psi\right),$$

has a meromorphic continuation, $f_{\phi,\psi,t}$, from $\mathbf{C}^+$ to $\Omega_{-\eta|t|}$.

Definition 1.2 Following [1] and [32, Sect. XII.6], any $z \in \Omega_{-\eta|t|}$ which is a pole of $f_{\phi,\psi}$ for some ϕ, ψ in $\mathcal{A}$ is called a resonance of $P(h)$. We do not consider here the resonances of $P(h)$ which are far from the real axis.

Theorem 1.3 **(Absence of Resonances)** *Under the assumptions* (H1), (H2) *and* (H3), *there exist a h-independent neighborhood Ω of λ and a small positive constant ϵ, such that if* $\operatorname{Sup}\{|\varphi(x)| \,;\, x \in D(a)\} \leq \epsilon$ *then for all ψ and $\phi \in \mathcal{A}$*

$$f_{\phi,\psi}(z) = \left(\left(P(h) - z\right)^{-1}\phi, \psi\right),$$

has a holomorphic continuation from $\mathbf{C}^+$ to Ω.

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Resonances We denote by $W_l(k,r) = \varphi(r) + \lambda_l(k)$ the l-th band Hamiltonian. For $m = 0, 1, 2, \dots$, set

$$\sum\nolimits_{\lambda,m} := \{(k,r) \in \mathbf{T}^* \times \mathbf{R}^n \,;\, W_m(k,r) = \lambda\}.$$

Assume

(H4) $\sum_{\lambda,m} = \varnothing$ for $m \neq l$, $\sum_{\lambda,l} = \{(k_0, r_0)\} \cup \Sigma_\lambda$, where Σ_λ is a connected component, $(k_0, r_0) \notin \Sigma_\lambda$ and W_l has a local non-degenerate extremum (local minimum or maximum) at (k_0, r_0). By a translation, we can assume that $(k_0, r_0) = (0, 0)$.

(H5) Near Σ_λ, $G_l(r,k) = r \cdot \lambda_l'(k)$ is an escape function, *i.e.*,

(1.6)
$$\frac{\partial W_l}{\partial k}\frac{\partial G_l}{\partial r} - \frac{\partial W_l}{\partial r}\frac{\partial G_l}{\partial k} = |\lambda_l'(k)|^2 - \langle \varphi'(r), \lambda_l''(k) r\rangle \geq c_0 > 0, \quad \forall (k,r) \in \Sigma_\lambda.$$

The non-degeneracy of W at $(0,0)$ means that the $2n \times 2n$ matrix,

$$W''(0,0) = \begin{pmatrix} \varphi''(0) & 0 \\ 0 & \lambda''(0) \end{pmatrix}$$

of second partial derivatives of W at $(0,0)$ is either positive or negative definite matrix. We define a reference Hamiltonian by

$$K = \pm\frac{1}{2}[-\langle \varphi''(0)\nabla, \nabla\rangle + \langle \lambda''(0)k, k\rangle]. \tag{1.7}$$

$+$ $(-)$ corresponds to a local minimum (maximum respectively).

It is clear that $\sigma(K)$ is discrete and contained in $]0, \infty[$. Let e_n be the eigenvalues of K listed in increasing size, counting multiplicity, $e_1 < e_2 \leq e_3 \cdots$. Let $0 < C_0 \notin \{e_1, e_2, \dots\}$ and let N_0 be the number of e_j's in $[0, C_0]$, so that $e_{N_0} < C_0 < e_{N_0+1}$. Our main result is:

Theorem 1.4 *Fix C_0 as above. Under the assumptions* (H1), (H2), (H3), (H4) *and* (H5), *there exists $h_0 > 0$, such that for $h \in\,]0, h_0[$, $P(h)$ has precisely N_0 resonances $\big(e_i(h)\big)_{1\leq i\leq N_0}$, in $D(\lambda, C_0 h) = \{z \in \mathbf{C}\,;\, |z - \lambda| < C_0 h\}$ (counted with their algebraic multiplicities). Moreover,*

$$e_j(h) \sim \lambda \pm e_j h + \sum_{k\geq 1} \alpha_{j,k} h^{1+\frac{k}{2}}, \quad (\alpha_{j,k} \in \mathbf{R}),\ (h \searrow 0). \tag{1.8}$$

2 The Distorted Hamiltonian

2.1 Spectral Deformation Family

Let H be a Hilbert space. The scalar product in H will be denoted by $(\ ,\ )$. The set of linear bounded operators from H_1 to H_2 is denoted by $\mathcal{L}(H_1, H_2)$. We set $\mathcal{L}(H) = \mathcal{L}(H, H)$.

In this subsection, we define the spectral deformation family $\mathcal{U}_t$ which will be used in this paper. With the change of variable $r = hx$, $P(h)$ becomes

$$\widehat{P}(h) := -h^2\Delta_r + V\left(\frac{r}{h}\right) + \varphi(r) = \widehat{P}_0(h) + \varphi(r).$$

As indicated in the introduction, to study the spectrum of the distorted Hamiltinian, we will use the method of [15], which is based on Floquet theory. Then, we shall construct a family $\mathcal{U}_t$ such that $\mathcal{U}_t\widehat{P}_0(h)\mathcal{U}_t^{-1}$ commute with $\tau_{h\gamma}$, $\gamma \in \Gamma$. Here $\tau_{h\gamma}u(x) = u(x - h\gamma)$ is the translation operator. For that, we employ a technique of spectral deformation in the momentum space introduced by [10] (see also [30]).

Let $v = (v_1, \ldots, v_n) \in C^\infty(\mathbf{T}^* \,;\, \mathbf{R}^n)$ and let t_0 be a small positive constant. For $t \in D(t_0) = \{t \in \mathbf{C}\,;\, |t| < t_0\}$, set

$$v_t(k) = k - tv(k).$$

We denote by $J_t(k) := \det[Dv_t(k)]$ the Jacobian of $v_t(k)$. Since v is bounded with its derivatives, there exists a positive constant $t_0 > 0$ such that v_t is invertible for all t in $D(t_0)$.

For $t \in]-t_0, t_0[$, we define a map on $S(\mathbf{R}^n)$ by

$$\mathcal{U}_t u(r) = \mathcal{F}_h^{-1}\Big\{ J_t^{\frac{1}{2}}(k)(\mathcal{F}_h u)\big(v_t(k)\big) \Big\}, \tag{2.1}$$

where $\mathcal{F}_h$ is the semi-classical Fourier transform

$$\mathcal{F}_h u(k) := \int_{\mathbf{R}^n} e^{-irk/h} u(r)\, dr.$$

From now on we write $\mathcal{F}$ for $\mathcal{F}_h$. We adopt this notation henceforth.

***Lemma 2.1* [30]** *Let* $\mathcal{A} := \{u \in L^2(\mathbf{R}^n)\,;\, \forall\, c, s > 0, \exp(c\langle x\rangle)u(x) \in H^s(\mathbf{R}^n)\}$.

i) *For* $|t| < t_0$ *and real, the map* $\mathcal{U}_t$ *defined in (2.1) extends to a unitary operator on* $L^2(\mathbf{R}^n)$.
ii) *For any* $u \in \mathcal{A}$, $t \to \mathcal{U}_t u$ *can be extended to an* $L^2(\mathbf{R}^n)$*-valued analytic function on* $D(t_0)$*, and the range* $\mathcal{U}_t\mathcal{A}$ *is dense in* $L^2(\mathbf{R}^n)$.

Lemma 2.2 *For* $V \in C^\infty(\mathbf{T})$ *and* $t \in]-t_0, t_0[$*, the multiplication operator by* $V(\frac{\cdot}{h})$ *on* $L^2(\mathbf{R}^n)$ *is stable under decomposition with* $\mathcal{U}_t$*, i.e.,*

$$\mathcal{U}_t V\left(\frac{\cdot}{h}\right)\mathcal{U}_t^{-1}u(r) = V\left(\frac{r}{h}\right)u(r), \quad \forall\, u \in L^2(\mathbf{R}^n). \tag{2.2}$$

Proof By Lemma 2.1 i), (2.2) is equivalent to

$$\mathcal{U}_t\left(V\left(\frac{\cdot}{h}\right)u\right)(r) = V\left(\frac{r}{h}\right)\mathcal{U}_t u(r), \quad \forall\, u \in S(\mathbf{R}^n). \tag{2.3}$$

Let $u \in S(\mathbf{R}^n)$. Since $V \in C^\infty(\mathbf{T})$, the Fourier series $\sum_{\beta^* \in \Gamma^*} c_{\beta^*} e^{ir\beta^*}$ of V is uniformly convergent, which shows that

$$\sum_{|\beta^*| \le N} c_{\beta^*} e^{ir\beta^*} u \to Vu \quad \text{in } L^2(\mathbf{R}^n).$$

Then, it suffices to show (2.3) for $V(r) = e^{ir\beta^*}$, $\beta^* \in \Gamma^*$.

Using the fact that

$$J_t^{\frac{1}{2}}(k - \beta^*) = J_t^{\frac{1}{2}}(k), \quad \text{and} \quad \nu_t(k - \beta^*) = \nu_t(k) - \beta^*, \quad \forall \beta^* \in \Gamma^*, \tag{2.4}$$

we obtain

$$\begin{aligned} \mathcal{U}_t(e^{i\beta^* \cdot /h} u)(r) &= \mathcal{F}^{-1}\Big\{ J_t^{\frac{1}{2}}(k) \mathcal{F}\big(e^{i\beta^* \cdot /h} u(\cdot)\big)\big(\nu_t(k)\big) \Big\} \\ &= \mathcal{F}^{-1}\Big\{ J_t^{\frac{1}{2}}(k) \mathcal{F}u\big(\nu_t(k) - \beta^*\big) \Big\} \\ &= \mathcal{F}^{-1}\Big\{ J_t^{\frac{1}{2}}(k - \beta^*) \mathcal{F}u\big(\nu_t(k - \beta^*)\big) \Big\} \\ &= e^{ir\beta^*/h} \mathcal{U}_t u(r). \end{aligned}$$

This ends the proof of the lemma.

2.2 h-Pseudodifferential Operators

In this subsection, we prepare some results on h-pseudodifferential operator calculus which we use in the following sections.

Let $m(r,k)$ be an order function. For $l, \delta \in \mathbf{R}$, we define the class of semi-classical symbols on $T^*\mathbf{R}^n = \mathbf{R}^{2n}$:

$$\begin{aligned} S^l_\delta(\mathbf{R}^{2n}, m) = \{ & a(r,k\,;h) \in C^\infty(\mathbf{R}^{2n} \times]0,1]\,;\, \forall \alpha, \beta \in \mathbf{N}^n,\ \exists C_{\alpha,\beta}, \\ & |\partial_r^\alpha \partial_k^\beta a(r,k\,;h)| \le C_{\alpha,\beta} h^{-l-\delta(|\alpha|+|\beta|)} m(r,k) \}. \end{aligned} \tag{2.5}$$

We denote by $S^l_\delta(\mathbf{R}^{2n})$ (resp. $S^0(\mathbf{R}^{2n}, m)$), $S^l_\delta(\mathbf{R}^{2n}, 1)$ (resp. $S^0_0(\mathbf{R}^{2n}, m)$).

If $a = a(r,k\,;z,h)$ depends also on some parameter $z \in \Omega$, we say that $a \in S^l_\delta(\mathbf{R}^{2n}, m)$, if the constant $C_{\alpha,\beta}$ in (2.5) is independent of $z \in \Omega$.

Let $a(r,k\,;h) \in S^0(\mathbf{R}^{2n}, m)$. We say that $a(r,k\,;h)$ has an asymptotic expansion in powers of h in $S^0(\mathbf{R}^{2n}, m)$, and we write

$$a(r,k\,;h) \sim \sum_{j=0}^{\infty} a_j(r,k) h^j \quad \text{in } S^0(\mathbf{R}^{2n}, m),$$

if for every $N \in \mathbf{N}$, $h^{-(N+1)}(a - \sum_{j=0}^N a_j h^j) \in S^0(\mathbf{R}^{2n}, m)$.

For $a \in S^0(\mathbf{R}^{2n}, m)$, the h-Weyl operator $a^w(r, hD_r\,;h)$ is defined by

$$a^w(r, hD_r\,;h)u(r) = (2\pi h)^{-n} \iint e^{i(r-y)k/h} a\Big(\frac{r+y}{2}, k\,;h\Big) u(y)\,dy\,dk. \tag{2.6}$$

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Let $(r,k) \to p(r,k\,;\,h) \in S^0(\mathbf{R}^{2n})$ be Γ^*-periodic in k, and let a, δ be two positive constants. We assume that p extends holomorphically to the complex zone

$$S_{W,a} = \{(r,k) \in \mathbf{C}^n \times W\,;\, |\Im r| \leq a\langle \Re r\rangle\},$$

(where W is a complex neighborhood of the torus $\mathbf{T}^*$), and satisfies

$$|p(r,k)| \leq C\langle r\rangle^{-\delta}, \tag{2.7}$$

uniformly on $(r,k) \in S_{W,a}$.

Note that, by passing from a to $\tilde{a} = \frac{a}{2}$ we may assume that

$$\forall\, \alpha,\beta,\ \exists\, C_{\alpha,\beta}\,;\, |\partial_r^\alpha \partial_k^\beta p(r,k)| \leq C_{\alpha,\beta}\langle r\rangle^{-\delta-|\alpha|}, \tag{2.8}$$

uniformly on $S_{W,\tilde{a}}$. This is a simple consequence of (2.7) and Cauchy inequalities.

Theorem 2.3 *For $t_0 > 0$ small enough, the map*

$$]-t_0,t_0[\ \ni t \to \mathcal{U}_t p^w(r,hD_r)\mathcal{U}_t^{-1},$$

extends to $D(t_0)$ as a $\mathcal{L}\big(L^2(\mathbf{R}^n)\big)$-valued analytic function. Moreover, there exists $p_t \in S^0(\mathbf{R}^{2n},\langle r\rangle^{-\delta})$, Γ^*-periodic with respect to k such that

$$\mathcal{U}_t p^w(r,hD_r)\mathcal{U}_t^{-1} = p_t^w(r,hD_r\,;\,h), \tag{2.9}$$

and

$$p_t(r,k\,;\,h) \sim \sum_{j=0}^{\infty} p_{t,j}(r,k)h^j, \quad \text{in } S^0(\mathbf{R}^{2n},\langle r\rangle^{-\delta}).$$

Here

$$p_{t,0}(r,k) = p\Big(\big(1-tM(k)\big)^{-1}r, \nu_t(k)\Big), \quad M(k) = \left(\frac{\partial \nu_j}{\partial k_i}(k)\right)_{1\leq i,j\leq n}, \tag{2.10}$$

$$p_{t,1}(r,k) = 0. \tag{2.11}$$

Proof Let $u \in S(\mathbf{R}^n)$ and $t \in\ ]-t_0,t_0[$. Remembering the definition of $\mathcal{U}_t$ and using the well known formula

$$\mathcal{F}p^w(r,hD_r)\mathcal{F}^{-1} = p^w(-hD_r,r)$$

(see [21]), we obtain

$$\begin{aligned}
&\mathcal{F}\mathcal{U}_t p^w(r,hD_r)\mathcal{U}_t^{-1}\mathcal{F}^{-1}u(r)\\
&= J_t^{\frac12}(r)\Big(p^w(-hD_r,r)\Big(J_t^{-\frac12}(r)u\big(\nu_t^{-1}(r)\big)\Big)\Big)\big(\nu_t(r)\big)\\
&= (2\pi h)^{-n}\iint_{\mathbf{R}^n\times\mathbf{R}^n} e^{i(\nu_t(r)-y)k/h} J_t^{\frac12}(r)J_t^{-\frac12}(y)p\left(-k,\frac{\nu_t(r)+y}{2}\right)u\big(\nu_t^{-1}(y)\big)\,dy\,dk.
\end{aligned}$$

Next, we use a standard change of variables and we get:

(2.12)

$$\mathcal{F}\,\mathcal{U}_t p^w(r,hD_r)\mathcal{U}_t^{-1}\mathcal{F}^{-1}u(r)$$

$$= (2\pi h)^{-n}\iint_{\mathbf{R}^n\times\mathbf{R}^n} e^{i(r-y-t(\nu(r)-\nu(y))k/h}J_t^{\frac12}(r)J_t^{\frac12}(y)p\left(-k,\frac{\nu_t(r)+\nu_t(y)}{2}\right)u(y)\,dy\,dk$$

$$= (2\pi h)^{-n}\iint_{\mathbf{R}^n\times\mathbf{R}^n} e^{i(r-y)(1-t\kappa(r,y))k/h}J_t^{\frac12}(r)J_t^{\frac12}(y)p\left(-k,\frac{\nu_t(r)+\nu_t(y)}{2}\right)u(y)\,dy\,dk$$

$$= (2\pi h)^{-n}\iint_{\mathbf{R}^n\times\mathbf{R}^n} e^{i(r-y)k/h}G_t(r,y,k)u(y)\,dy\,dk,$$

where

(2.13)

$$G_t(r,y,k) = J_t^{\frac12}(r)J_t^{\frac12}(y)\Big(\det\big(1-t\kappa(r,y)\big)\Big)^{-1}p\left(\big(t\kappa(r,y)-1\big)^{-1}k,\frac{\nu_t(r)+\nu_t(y)}{2}\right).$$

$\kappa(r,y)$ is defined by

$$\nu(r)-\nu(y) = \int_0^1 \partial_y\nu\big(y+s(r-y)\big)\,ds(r-y) = \kappa(r,y)(r-y).$$

Since $\nu\in C^\infty(\mathbf{T}^*\,;\,\mathbf{R}^n)$, κ is bounded with all its derivatives. Combining this with (2.13) and using the analytic assumption on p, we deduce that for t_0 small enough $G_t(r,y,k)$ extends analytically on $t\in D(t_0)$. On the other hand, (2.8) and (2.13) show that

$$\forall\,\alpha,\beta,\gamma\in\mathbf{N}^n,\ \exists\, C_{\alpha,\beta,\gamma}\ ;\ |\partial_r^\alpha\partial_y^\beta\partial_k^\gamma G_t(r,y,k)|\le C_{\alpha,\beta,\gamma}\langle k\rangle^{-\delta}, \tag{2.14}$$

uniformly on t in $D(t_0)$.

By a classical result of h-pseudodifferential theory (see [33, Theorem II.27]), we deduce from (2.12), (2.13) and (2.14) that

$$\mathcal{F}\mathcal{U}_t p^w(r,hD_r)\mathcal{U}_t^{-1}\mathcal{F}^{-1} = b_t^w(r,hD_r\,;\,h), \tag{2.15}$$

with

$$b_t(r,k\,;\,h) = e^{-ihD_uD_k}G_t\left(r+\frac{u}{2},r-\frac{u}{2},k\right)_{|u=0}\sim\sum_{j=0}^\infty b_{t,j}(r,k)h^j, \tag{2.16}$$

in $S^0(\mathbf{R}^{2n},\langle k\rangle^{-\delta})$. Here

$$b_{t,j}(r,k) = \frac{i^j}{j!}(\langle\partial_u\partial_k\rangle)^j G_t\left(r+\frac{u}{2},r-\frac{u}{2},k\right)_{|u=0}. \tag{2.17}$$

In particular,

$$b_{t,0}(r,k) = G_t(r,r,k) = p\Big(\big(tM(r)-1\big)^{-1}k, \nu_t(r)\Big), \tag{2.18}$$

and

$$b_{t,1}(r,k) = \frac{1}{2}\big(\partial_r\partial_k G_t(r,r,k) - \partial_y\partial_k G_t(r,r,k)\big) = 0. \tag{2.19}$$

Clearly, (2.15) implies

$$\mathcal{U}_t p^w(r,hD_r)\mathcal{U}_t^{-1} = b_t^w(hD_r,-r\,;h) := p_t^w(r,hD_r\,;h),$$

which together with (2.16), (2.18) and (2.19) give (2.9), (2.10) and (2.11).

By assumption, $k \to p(r,k)$ is Γ^*-periodic. Hence,

$$e^{-ir\beta^*/h}p^w(r,hD_r)e^{ir\beta^*/h} = p^w(r,hD_r+\beta^*) = p^w(r,hD_r).$$

Combining this with the fact that $\mathcal{U}_t$ commutes with $e^{ir\beta^*/h}$ (see Lemma 2.2), we get

$$\begin{aligned} p_t^w(r,hD_r+\beta^*\,;h) &= e^{-ir\beta^*/h}p_t^w(r,hD_r\,;h)e^{ir\beta^*/h} \\ &= \mathcal{U}_t\big(e^{-ir\beta^*/h}p^w(r,hD_r)e^{ir\beta^*/h}\big)\mathcal{U}_t^{-1} = p_t^w(r,hD_r\,;h). \end{aligned}$$

Consequently, $p_t(r,k\,;h)$ is Γ^*-periodic with respect to k.

Now, it remains to show that $t \to \mathcal{U}_t p^w(r,hD_r)\mathcal{U}_t^{-1} \in \mathcal{L}(L^2)$ is analytic.

Let $u, \psi \in S(\mathbf{R}^n)$. By (2.12), one has

$$\langle \mathcal{U}_t p^w \mathcal{U}_t^{-1}\mathcal{F}^{-1}u, \mathcal{F}^{-1}\psi\rangle = (2\pi h)^{-n}\iiint e^{i(r-y)k/h}G_t(r,y,k)u(y)\overline{\psi}(r)\,dr\,dy\,dk.$$

Using repeated integration by parts with the help of the operator $(1+h^2\Delta_y)$, we get

$$\begin{aligned} &\langle \mathcal{U}_t p^w \mathcal{U}_t^{-1}\mathcal{F}^{-1}u, \mathcal{F}^{-1}\psi\rangle \\ &\quad = (2\pi h)^{-n}\iiint e^{i(r-y)k/h}\langle k\rangle^{-2N}(1+h^2\Delta_y)^N\big(G_t(r,y,k)u(y)\big)\overline{\psi}(r)\,dr\,dy\,dk. \end{aligned}$$

Clearly, for N large enough the right member of the above equality is analytic on t. Since $\|p_t^w(r,hD_r\ ;\ h)\|_{\mathcal{L}(L^2)}$ is uniformly bounded on $t \in D(t_0)$, by the Calderon-Vaillancourt theorem (see Theorem A.3), and since $\{\mathcal{F}^{-1}u\,;\,u \in S(\mathbf{R}^n)\}$ is dense in $L^2(\mathbf{R}^n)$, it follows from [23, p. 365] that $\mathcal{U}_t p^w(r,hD_r)\mathcal{U}_t^{-1}$ is analytic on t.

Remark 2.4 Note that, if $\nu(0) = 0$ and $p(r,k) = \mathcal{O}\big((r,k)^\alpha\big)$, for all $|\alpha| \le N$ then $p_{t,j}(r,k) = \mathcal{O}\big((r,k)^\beta\big)$ for all $|\beta| \le N-2j$. This is a simple consequence of (2.13) and (2.17).

Corollary 2.5 *Let $p(r,k)$ and $p_t(r,k\,;h)$ be as in Theorem 2.3. The family of operators $p_t^w(-hD_k,k\,;h)$, is well defined on $L^2(\mathbf{T}^*)$ and is unitarily equivalent to $p^w(-hD_k,k)$ for real t. Moreover,*

$$t \in D(t_0) \to p_t^w(-hD_r,r\,;h) = \mathcal{F}\mathcal{U}_t p^w(r,hD_r)\mathcal{U}_t^{-1}\mathcal{F}^{-1} \in \mathcal{L}\big(L^2(\mathbf{T}^*)\big)$$

is analytic.

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Proof Since p_t is Γ^*-periodic with respect to k, it follows from the Calderon-Vaillancourt theorem on the torus (see [16]) that $p_t^w(-hD_r, r\ ;\ h) \in \mathcal{L}\big(L^2(\mathbf{T}^*)\big)$. The analiticity on t can be proved as in Theorem 2.3.

Note that, for real t

$$u \in L^2(\mathbf{T}^*) \to \widehat{\mathcal{U}}_t u(k) := \mathcal{F}\,\mathcal{U}_t \mathcal{F}_t^{-1} u(k) = J_t^{\frac{1}{2}}(k) u\big(\nu_t(k)\big) \in L^2(\mathbf{T}^*),$$

is unitary. Hence, $\mathcal{F}\,\mathcal{U}_t p^w(r, hD_r)\mathcal{U}_t^{-1}\mathcal{F}^{-1} = \widehat{\mathcal{U}}_t \mathcal{F} p^w(r, hD_r)\mathcal{F}^{-1}\widehat{\mathcal{U}}_t^{-1}$ is unitarily equivalent to

$$\mathcal{F} p^w(r, hD_r)\mathcal{F}^{-1} = p^w(-hD_r, r). \qquad \blacksquare$$

Applying Theorem 2.3 to $p(r,k) = \varphi(r)$, we get:

Corollary 2.6 *Assume* (H1). *The map*

$$t \in D(t_0) \to \mathcal{U}_t \varphi \mathcal{U}_t^{-1} \in \mathcal{L}\big(L^2(\mathbf{R}^n)\big),$$

is analytic. Moreover, there exists $\varphi_t \in S^0(\mathbf{R}^{2n}, \langle r\rangle^{-\delta})$, Γ^*-periodic in k such that

$$\mathcal{U}_t \varphi \mathcal{U}_t^{-1} = \varphi_t^w(r, hD_r\ ;\ h), \tag{2.20}$$

with

$$\varphi_t(r,k\ ;\ h) \sim \sum_{j=0}^{\infty} \varphi_{t,j}(r,k) h^j, \quad \text{in } S^0(\mathbf{R}^{2n}, \langle r\rangle^{-\delta}).$$

In particular

$$\varphi_{t,0}(r,k) = \varphi\Big(\big(1 - tM(k)\big)^{-1} r\Big), \quad M(k) = \left(\frac{\partial \nu_j(k)}{\partial k_i}\right)_{1\le i,j\le n}, \tag{2.21}$$

and

$$\varphi_{t,1}(r,k) = 0. \tag{2.22}$$

$\blacksquare$

We end this section by a standard result on a weighted L^2-estimate [19]. Let f_1 and f_2 be two real-valued functions, bounded with all their derivatives. We assume that f_2 is Γ^*-periodic, and $\|\nabla f_i\|_\infty$ is small enough. Conjugating the left hand side of (2.12) by $e^{f_i/h}$, and using a standard change of variables (a complex version of Kuranishi trick) similar to the one used in the last equality of (2.12), we prove:

Proposition 2.7 *Let f_1 and f_2 be as above, and let $p_t^w(r, hD_r\ ;\ h)$ be given by (2.9). There exists $p_{f_i,t}(r,k\ ;\ h) \in S^0(\mathbf{R}^{2n}, \langle r\rangle^{-\delta})$, Γ^*-periodic with respect to k, such that*

$$e^{f_1(r)/h} p_t^w(r, hD_r\ ;\ h) e^{-f_1(r)/h} = p_{f_1,t}(r, hD_r\ ;\ h),$$

and

$$e^{f_2(hD_r)/h} p_t^w(r, hD_r\ ;\ h) e^{-f_2(hD_r)/h} = p_{f_2,t}(r, hD_r\ ;\ h).$$

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Moreover

$$p_{f_i,t}(r,k\,;h)\sim\sum_{j=0}^{\infty}p_{f_i,t,j}(r,k)h^j,\quad \text{in } S^0(\mathbf{R}^{2n},\langle r\rangle^{-\delta}),$$

with

$$p_{f_1,t,0}(r,k)=p_{t,0}\big(r,k+i\nabla f_1(r)\big),\tag{2.23}$$

and

$$p_{f_2,t,0}(r,k)=p_{t,0}\big(r-i\nabla f_2(k),k\big).\tag{2.24}$$

Here $p_{t,0}(r,k)$ *is given by (2.10).*

2.3 Distorted Hamiltonian

Now, we are ready to define the distorted Hamiltonian. Consider for $t\in\,]-t_0,t_0[$ the family of unitarily equivalent operators,

$$P(t,h)=\mathcal{U}_t\widehat{P}(h)\mathcal{U}_t^{-1}.$$

We recall that

$$\widehat{P}(h)=-h^2\Delta+V\left(\frac{r}{h}\right)+\varphi(r).$$

A simple calculus shows that

$$\mathcal{U}_t(-h^2\Delta)\mathcal{U}_t^{-1}=\big(\nu_t(hD_r)\big)^2,$$

which together with Lemma 2.2 and Corollary 2.6 yield

$$P(t,h)=\big(\nu_t(hD_r)\big)^2+V\left(\frac{r}{h}\right)+\varphi_t^w(r,hD_r\,;h).\tag{2.25}$$

Recall that $\nu_t(k)=k-t\nu(k)$, where $\nu(k)$ is bounded with all its derivatives. This ensures that the domain of $\big(\nu_t(hD_r)\big)^2$ is independent of t and $D\Big(\big(\nu_t(hD_r)\big)^2\Big)=D(-\Delta)=H^2(\mathbf{R}^n)$. Combining this with Corollary 2.6, we get:

Proposition 2.8 *Assume* (H1). *The self-adjoint operator* $P(t,h)$, *defined for* $t\in\,]-t_0,t_0[$, *extends to an analytic type-A family of operators on* $D(t_0)$ *with domain* $H^2(\mathbf{R}^n)$.

3 Effective Resonant Hamiltonians

When $V=0$, the spectrum of $P(t,h)$ was studied by Nakamura [30]. The main technique used in [30] is the calculus of h-pseudodifferential operators and the Fefferman-Phong inequalities.

The additional periodic, but rapidly oscillating potential modifies considerably the spectral study of $P(t,h)$. The main question is, how to uncouple x, the fast variable and $r=hx$, the slow variable.

As indicated in the introduction, one possible choice is to introduce a new operator $\mathbb{P}(t,h)$ in which x and r are regarded as independent variables. This is the two scale expansion method which we will describe in the next subsection.

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3.1 The Two Scale Expansion Method

Denote by T_Γ the distribution in $S'(\mathbf{R}^{2n})$ defined by

$$T_\Gamma(r,x) = \frac{1}{\mathrm{vol}(E)h^n} \sum_{\beta\in\Gamma^*} e^{i(r-hx)\beta/h}. \tag{3.1}$$

We recall that E is a fundamental domain of Γ.

Set

$$\mathbb{L} = \{v(r)T_\Gamma(r,x)\ ;\ v \in L^2(\mathbf{R}^n)\}.$$

By Poissons' formula, one has:

$$T_\Gamma(r,x) = \sum_{\gamma\in\Gamma} \delta(r - hx + h\gamma). \tag{3.2}$$

Then, for $\varphi \in \mathcal{S}(\mathbf{R}^n \times \mathbf{T}) := \{\varphi \in C^\infty(\mathbf{R}^{2n})\ ;\ \langle r\rangle^N \partial^\alpha_{r,x}\varphi \in L^2(\mathbf{R}^n \times \mathbf{T}), \forall N, \alpha\}$ and $u(r,x) = v(r)T_\Gamma(r,x)$

$$\begin{aligned}\langle u,\varphi\rangle &:= \iint_{\mathbf{R}^n\times E} u(r,x)\varphi(r,x)\,dx\,dr \\ &= \sum_{\gamma\in\Gamma} \int_E v\big(h(x-\gamma)\big)\,\varphi\big(h(x-\gamma),x\big)\ dx = \int_{\mathbf{R}^n} v(hx)\varphi(hx,x)\,dx.\end{aligned}$$

The last integral can be bounded by seminorms of φ in $\mathcal{S}(\mathbf{R}^n \times \mathbf{T})$. Hence, $\mathbb{L}$ can be viewed as a subspace of $\mathcal{S}'(\mathbf{R}^n \times \mathbf{T})$.

Moreover, the above equality shows that, $uT_\Gamma = 0$ in $\mathbb{L}$ implies that $u = 0$ in L^2. Therefore, $\mathbb{L}$ equipped with $(uT_\Gamma, vT_\Gamma) \to (u,v)_{L^2}$ has an Hilbert structure, and the map

$$U : L^2(\mathbf{R}^n) \ni v \to vT_\Gamma \in \mathbb{L} \tag{3.3}$$

is unitary.

Lemma 3.1 *Let $(r,k) \to p(r,k) \in S^0(\mathbf{R}^{2n})$ be Γ^* periodic with respect to k. One has*

$$U p^w(r,hD_r)U^{-1} = p^w(r,hD_r). \tag{3.4}$$

Proof (3.4) is equivalent to

$$\forall\, w = u(r)T_\Gamma(r,x) \in \mathbb{L}, \quad p^w(r,hD_r)w(r,x) = \big(p^w(r,hD_r)u\big)(r)T_\Gamma(r,x). \tag{3.5}$$

Since $p(r,k+\gamma^*) = p(r,k)$ for all $\gamma^* \in \Gamma^*$,

$$p^w(r,hD_r)e^{i(r-hx)\gamma^*/h} = e^{i(r-hx)\gamma^*/h}p^w(r,hD_r),$$

which yields (3.5).

Lemma 3.2 *Under assumption* (H1), *$P(t,h)$ acting on $L^2(\mathbf{R}^n)$ with domain $H^2(\mathbf{R}^n)$ is unitarily equivalent to*

$$\mathbb{P}(t,h) := \big(D_x + v_t(hD_r)\big)^2 + V(x) + \varphi^w_t(r,hD_r\ ;h) \tag{3.6}$$

acting on $\mathbb{L}$ with domain $\mathbb{L}^2 := \{u(r)T_\Gamma(r,x)\ ;\ \partial^\alpha_r u \in L^2(\mathbf{R}^n), \forall\,|\alpha| \leq 2\}$. In particular, $t \in D(t_0) \to \mathbb{P}(t,h)$ is analytic of type-A with domain $\mathbb{L}^2$.

Proof Recall that $\nu_t(hD_r+\gamma^*) = \nu_t(hD_r)+\gamma^*, \forall\gamma^* \in \Gamma^*$. Hence, for all $u(r)T_\Gamma(r,x) \in \mathbb{L}$

$$\begin{aligned}\big(\nu_t(hD_r)+D_x\big)^2u(r)T_\Gamma(r,x) &= \sum_{\gamma^*\in\Gamma^*} e^{i(r-hx)\gamma^*/h}\big(\nu_t(hD_r+\gamma^*)+D_x-\gamma^*\big)^2u(r)\\ &= \sum_{\gamma^*\in\Gamma^*} e^{i(r-hx)\gamma^*/h}\nu_t(hD_r)^2u(r)\\ &= T_\Gamma(r,x)\nu_t(hD_r)^2u(r),\end{aligned}$$

which yields

$$U\big(\nu_t(hD_r)\big)^2U^{-1} = \big(\nu_t(hD_r)+D_x\big)^2. \tag{3.7}$$

On the other hand, the periodicity of V and (3.2) give

$$V\left(\frac{r}{h}\right)T_\Gamma(r,x) = \sum_{\gamma\in\Gamma} V(x-\gamma)\delta(r-hx+h\gamma) = V(x)T_\Gamma(r,x),$$

which implies that

$$UV\left(\frac{r}{h}\right)U^{-1} = V(x). \tag{3.8}$$

Now, applying Lemma 3.1 to $\varphi_t(r,k\,;\,h)$ and using (3.7), (3.8) and Proposition 2.8 we get the lemma.

3.2 Grushin Problem for $\mathbb{P}(t,h)$

In this subsection, we will reduce the spectral study of $\mathbb{P}(t,h)$ to an h-pseudodifferential operator acting only on the r-variable. More precisely, we shall show that complete informations on the spectrum of $\mathbb{P}(t,h)$, near any fixed energy level z, is contained in a certain h-pseudodifferential operator $\widetilde{E}_{-+}(z,t,h) = E^w_{-+}(-hD_k,k\,;\,z,t,h)$, which is defined by constructing an inverse of an appropriate Grushin problem for $\mathbb{P}(t,h)$. See Theorem 3.8 below. Our method is quite similar to the one of [15], and that is why we omit sometimes the details of the proofs and we refer to [15].

We introduce the following Hilbert space with their natural norms

$$H_0 = L^2(\mathbf{T}),$$
$$H_{m,k} = \{u \in H_0\,;\,(D_x+k)^\alpha u \in H_0, \forall\,|\alpha| \le m\}.$$

We notice that only the norm on $H_{m,k}$ depends on k and not the space itself and we have:

$$\|u\|_{H_{m,k}} \le C\langle k-k'\rangle^m\|u\|_{H_{m,k'}}, \quad \forall\,u \in H_{m,0},\ k,k' \in \mathbf{R}^n.$$

Then, we can use the theory of h-pseudodifferential operator with operator valued symbol in $\mathcal{L}(H_{m,k}\,;\,H_{m',k})$. See Appendix A.

In the form (3.6), we can view $\mathbb{P}(t,h)$ as an h-pseudodifferential operator on r with operator valued symbols

$$P(r,k\,;t,h) := \big(D_x + \nu_t(k)\big)^2 + V(x) + \varphi_t(r,k,h) \in S^0\big(\mathbf{R}^{2n}_{r,k}\,;\mathcal{L}(H_{0,k},H_0)\big).$$

To construct a suitable Grushin problem for $\mathbb{P}(t,h)$, the first step is to construct a Grushin problem on the symbol level. This will be the object of the next lemma.

Set

$$P(r,k\,;t) := \big(D_x + \nu_t(k)\big)^2 + V(x) + \varphi_{t,0}(r,k), \tag{3.9}$$

$\varphi_{t,0}(r,k)$ is the principal term of $\varphi_t(r,k\,;h)$ given by (2.21).

Lemma 3.3 *Pick λ in $\mathbf{R}$. There exist $N \in \mathbf{N}$, a small constant $t_0 > 0$, a complex neighborhood ϑ of λ and functions ϕ_j in $C^\infty(\mathbf{R}^n_k\,;H_{2,k}) \cap C^\infty(\mathbf{R}^n_x \times \mathbf{R}^n_k)$, such that for each $k \in \mathbf{R}^n$, $t \in D(t_0)$ and each $z \in \vartheta$ the operator:*

$$\mathcal{P}(r,k\,;z,t) = \begin{pmatrix} P(r,k\,;t) - z & R_-\big(\nu_t(k)\big) \\ R_+\big(\nu_{\bar{t}}(k)\big) & 0 \end{pmatrix}, \tag{3.10}$$

is invertible from $H_{2,k} \times \mathbf{C}^N$ into $H_0 \times \mathbf{C}^N$, with an inverse $\mathcal{E}_0(r,k\,;z,t)$ uniformly bounded with respect to $(r,k\,;z,t)$ together with all its derivatives in $\mathcal{L}(H_0 \times \mathbf{C}^N, H_{2,k} \times \mathbf{C}^N)$. Here $\big(R_+(k)u\big)_j = \big(u,\phi_j(\cdot,k)\big)_{H_{0,0}}$ and $R_-(k)u^- = \sum_{j=1}^N u_j^-\phi_j(\cdot,k)$.

Moreover, $\phi_j\big(\cdot,\nu_t(k)\big)$ is analytic on t and satisfies

$$\begin{cases} \big\|\partial_k^\beta \phi_j\big(\cdot,\nu_t(k)\big)\big\|_{H_{2,k}} \le C_\beta, & \forall\,\beta \in \mathbf{N}^n,\ k \in \mathbf{R}^n,\ t \in D(t_0) \\ \phi_j\big(x,\nu_t(k+\gamma^*)\big) = e^{-ix\gamma^*}\phi_j\big(x,\nu_t(k)\big), & \forall\,\gamma^* \in \Gamma^*. \end{cases} \tag{3.11}$$

Proof Set

$$P_0(r,k) = (D_x + k)^2 + V(x) + \varphi(r).$$

Proposition 2.1 of [15], see also [18, Theorem 3.1.1], gives the existence of N functions $\phi_j(x,k)$ such that Lemma 3.3 holds when we replace $\mathcal{P}(r,k,z,t)$ by

$$\mathcal{P}_0(r,k,z) = \begin{pmatrix} P_0(r,k) - z & R_-(k) \\ R_+(k) & 0 \end{pmatrix}.$$

The functions ϕ_j constructed in [15], [18] are of the form

$$\phi_j(x,k) = \sum_{\gamma\in\Gamma} \psi_j(x-\gamma)e^{ik(\gamma-x)} = \sum_{\beta^*\in\Gamma^*} \widehat{\psi}_j(\beta^*-k)e^{i\beta^* x}, \tag{3.12}$$

with $\psi_j \in C_0^\infty(\mathbf{R}^n)$. By Paley-Wiener-Schwartz theorem, see for instance [21, Theorem 7.3.1], $\widehat{\psi}_j(\beta^*-k)$ extends analytically on k and satisfies,

$$\forall N \in \mathbf{N},\ \alpha \in \mathbf{N}^n,\ \partial_k^\alpha\Big(\widehat{\psi}_j\big(\beta^*-\nu_t(k)\big) - \widehat{\psi}_j(\beta^*-k)\Big) = \mathcal{O}_{N,\alpha}(|t_0|)(1+|\beta^*-k|)^{-N},$$

uniformly on $t \in D(t_0)$. We recall that $\nu_t(k) = k - t\nu(k)$ where $\nu \in C^\infty(\mathbf{T}^* ; \mathbf{R}^n)$.

From (3.12) and the above estimate, we deduce that $\phi_j\big(\cdot, \nu_t(k)\big)$ is analytic on t and

$$\big\| \partial_k^\alpha R_-\big(\nu_{\bar t}(k)\big) - \partial_k^\alpha R_-(k)\big\|_{\mathcal{L}(\mathbf{C}^N, H_{2,k})}, \big\| \partial_k^\alpha R_+\big(\nu_t(k)\big) - \partial_k^\alpha R_+(k)\big\|_{\mathcal{L}(H_0,\mathbf{C}^N)} = \mathcal{O}_\alpha(|t_0|).$$

Combining this with the following equality

$$P(r,k\,;t) - P_0(r,k) = -2t\nu(k)(D_x + k) + \nu(k)^2 t^2 + \varphi_{t,0}(r,k) - \varphi(r),$$

and using that $\partial_{r,k}^\beta\big(\varphi_{t,0}(r,k) - \varphi(r)\big) = \mathcal{O}_\beta(|t|)$, we obtain

$$\big\| \partial_{r,k}^\beta\big(\mathcal{P}(r,k\,;z,t) - \mathcal{P}_0(r,k\,;z)\big)\big\|_{\mathcal{L}(H_{2,k}\times\mathbf{C}^N; H_0\times\mathbf{C}^N)} = \mathcal{O}_\beta(|t_0|),$$

uniformly on $(r,k\,;t) \in \mathbf{R}^{2n} \times D(t_0)$.

Choosing t_0 small enough, and applying the results of [15] to $\mathcal{P}_0(r,k\,;z)$, we get Lemma 3.3. ∎

Now, we turn to the operator $\mathbb{P}(t,h)$. We denote by

$$\mathcal{E}_0(z,t) := \mathcal{E}_0^w(r, hD_r\,; z, t),$$

and

$$\mathcal{P}(z,t,h) := \begin{pmatrix} \mathbb{P}(t,h) - z & R_-^w\big(\nu_t(hD_r)\big) \\ R_+^w\big(\nu_{\bar t}(hD_r)\big) & 0 \end{pmatrix}$$

the Weyl quantization of $\mathcal{E}_0(r,k\,;z,t)$ and $\begin{pmatrix} P(r,k;t,h)-z & R_-(\nu_t(k)) \\ R_+\big(\nu_{\bar t}(k)\big) & 0 \end{pmatrix}$ respectively.

For $m \in \mathbf{N}$, set

$$\mathcal{K}_0 := L^2(\mathbf{R}_r^n \times \mathbf{T}),$$
$$\mathcal{K}_m := \{u \in \mathcal{K}_0\,;(D_x + hD_r)^\alpha u \in \mathcal{K}_0,\ \forall\,|\alpha| \le m\}.$$

Proposition 3.4 *The operator $\mathcal{E}_0(z,t)$ is continuous from $\mathcal{S}(\mathbf{R}^n\,; H_0 \times \mathbf{C}^N)$, (resp. $\mathcal{S}'(\mathbf{R}^n\,; H_0 \times \mathbf{C}^N)$) into $\mathcal{S}(\mathbf{R}^n\,; H_{2,0} \times \mathbf{C}^N)$ (resp. $\mathcal{S}'(\mathbf{R}^n\,; H_{2,0} \times \mathbf{C}^N)$) and uniformly bounded from $\mathcal{K}_0 \times L^2(\mathbf{R}_r^n\,;\mathbf{C}^N)$ into $\mathcal{K}_2 \times L^2(\mathbf{R}_r^n\,;\mathbf{C}^N)$. Moreover, we have*

$$\mathcal{P}(z,t,h)\circ\mathcal{E}_0(z,t) = 1 + h\mathcal{R}^w(r,hD_r\,;z,t,h), \tag{3.13}$$

where $\mathcal{R}(r,k\,;z,t,h) \sim \sum_{j=0}^\infty \mathcal{R}_j(r,k\,;z,t)h^j$ in $S^0\big(\mathbf{R}^{2n}\,;\mathcal{L}(H_0\times\mathbf{C}^N)\big)$ and $\mathcal{R},\mathcal{R}_j$ depend holomorphically on z.

Proof The continuity of $\mathcal{E}_0(z,t)$ in $\mathcal{S}$ and $\mathcal{S}'$ follows from Lemma 3.3 and Proposition A.1.

Let $P_\alpha(k) = \begin{pmatrix} (k+D_x)^\alpha & 0 \\ 0 & 1 \end{pmatrix}$ be the operator valued symbol in

$$S^0\big(\mathbf{R}^{2n}\,;\mathcal{L}(H_{2,k}\times\mathbf{C}^N, H_0\times\mathbf{C}^N)\big).$$

In view of the definition of $\mathcal{K}_2$, we have just to prove

(3.14) $$\|P_\alpha(hD_r)\circ\mathcal{E}_0(z,t)\|_{\mathcal{L}(\mathcal{K}_0\times L^2(\mathbf{R}^n_r;\mathbf{C}^N))}=\mathcal{O}(1),\quad\forall\,|\alpha|\le 2.$$

Theorem A.2 shows that:

$$P_\alpha(k)\circ\mathcal{E}_0(r,k\,;z,t)\in S^0\big(\mathbf{R}^{2n}\,;\mathcal{L}(H_0\times\mathbf{C}^N,H_0\times\mathbf{C}^N)\big),\quad\forall\,|\alpha|\le 2,$$

which together with Theorem A.3 give (3.14). Formula (3.13) is a simple consequence of Lemma 3.3 and Theorem A.2.

Proposition 3.5 *$t\in D(t_0)\to\mathcal{P}(z,t,h)\in\mathcal{L}\big(\mathcal{K}_2\times L^2(\mathbf{R}^n_r\,;\mathbf{C}^N)\,;\mathcal{K}_0\times L^2(\mathbf{R}^n_r\,;\mathbf{C}^N)\big)$ is analytic.*

Proof That $t\to\mathbb{P}(t,h)\in\mathcal{L}(\mathcal{K}_2,\mathcal{K}_0)$ is analytic follows from Corollary 2.6. On the other hand, by a proof similar to the one used in Theorem 2.3 we show, using the properties of $\phi\big(\cdot,\nu_t(k)\big)$ given in Lemma 3.3 that $R^w_+\big(\nu_{\bar t}(hD_r)\big)$ and $R^w_-\big(\nu_t(hD_r)\big)$ are analytic in t.

Proposition 3.6 *Fix $\lambda\in\mathbf{R}$. There exist a complex neighborhood ϑ of λ and small constants h_0, $t_0>0$, such that, for $(z,t,h)\in\vartheta\times D(t_0)\times\,]0,h_0[$, $\mathcal{P}(z,t,h)$ is bijective from $S'(\mathbf{R}^n\,;H_0\times\mathbf{C}^N)$ into $S'(\mathbf{R}^n,H_{2,0}\times\mathbf{C}^N)$, from $\mathcal{K}_2\times L^2(\mathbf{R}^n_r\,;\mathbf{C}^N)$ into $\mathcal{K}_0\times L^2(\mathbf{R}^n_r\,;\mathbf{C}^N)$ and has a uniformly bounded inverse of the form $\mathcal{E}(z,t,h):=\mathcal{E}^w(r,hD_r\,;z,t,h)$, where*

(3.15)
$$\mathcal{E}(r,k\,;z,t,h)\sim\sum_{j=0}^{\infty}\mathcal{E}_j(r,k\,;z,t)h^j\quad\textit{in } S^0\big(\mathbf{R}^{2n}\,;\mathcal{L}(H_0\times\mathbf{C}^N,H_{2,k}\times\mathbf{C}^N)\big).$$

The principal term $\mathcal{E}_0(r,k\,;z,t)$ is given by Lemma 3.3. The operator $\mathcal{E}(z,t,h)$ has the same continuity properties as $\mathcal{E}_0(z,t)$ in Proposition 3.4.

Proof Proposition 3.4 and Theorem A.3 imply that $\mathcal{R}^w(r,hD_r\,;z,t,h)$ is uniformly bounded on $\mathcal{L}\big(\mathcal{K}_0\times L^2(\mathbf{R}^n\,;\mathbf{C}^N)\big)$. Hence,

$$\|h\mathcal{R}^w(r,hD_r\,;z,t,h)\|_{\mathcal{L}(\mathcal{K}_0\times L^2(\mathbf{R}^n;\mathbf{C}^N))}\le 1/2,$$

for h small enough, and therefore $(1+h\mathcal{R}^w)^{-1}$ exists in $\mathcal{L}\big(\mathcal{K}_0\times L^2(\mathbf{R}^n\,;\mathbf{C}^N)\big)$. Using (3.13), we conclude that $\mathcal{P}(z,t,h)$ has a right inverse

$$\mathcal{E}(z,t,h)=\mathcal{E}_0(z,t)\circ(1+h\mathcal{R}^w)^{-1}.$$

Recalling that $\mathcal{P}(z,t,h)$ is self-adjoint for z and t real. Hence, $\mathcal{E}(z,t,h)$ is also a left inverse when $z\in\vartheta\cap\mathbf{R}$ and $t\in D(t_0)\cap\mathbf{R}$.

Since $t\to\mathcal{P}(z,t,h)$ is an analytic family of type A and $\mathcal{P}(\lambda,0,h)$ is bijective, it follows from Theorem XII.7 of [32] that $\mathcal{P}(z,t,h)$ is bijective for (z,t) in a small complex neighborhood of $(\lambda, 0)$. This shows that $\mathcal{E}(z,t,h)$ is also a left inverse of $\mathcal{P}(z,t,h)$ when $(z,t) \in \vartheta \times D(t_0)$.

Formula (3.15) is a consequence of Beals' result (see [15], [12, chapter 8]), and the fact that $\mathcal{R}$ has an asymptotic expansion in powers of h. This finishes the proof of Proposition 3.6. ∎

In the following, we denote

$$\begin{pmatrix} E(z,t,h) & E_+(z,t,h) \\ E_-(z,t,h) & E_{-+}(z,t,h) \end{pmatrix}$$

the matrix elements of $\mathcal{E}(z,t,h)$. By Proposition 3.6, $E_{-+}(z,t,h)$ has an asymptotic expansion in powers of h:

$$E_{-+}(r,k\,;z,t,h) \sim \sum_{j=0}^{\infty} E_{-+}^{j}(r,k\,;z,t)h^{j}, \quad \text{in } S^0\big(\mathbf{R}^{2n}\,;M(\mathbf{C}^N)\big). \tag{3.16}$$

Here $M(\mathbf{C}^N)$ is the space of square matrices with complex coefficients. The principal term $E_{-+}^{0}(r,k\ ;\ z,t)$ is the matrix which appears in the lower right corner of the inverse $\mathcal{E}_0(r,k\,;z,t)$ given in Lemma 3.3.

3.3 Effective Resonant Hamiltonians

Because of (3.11), we have:

$$\begin{cases} R_-\big(v_t(k+\gamma^*)\big) = e^{-ix\gamma^*}R_-\big(v_t(k)\big) \\ R_+\big(v_t(k+\gamma^*)\big) = R_+\big(v_t(k)\big)\,e^{ix\gamma^*}, \end{cases}$$

which implies

$$\begin{cases} e^{-ir\gamma^*/h}R_-^w\big(v_t(hD_r)\big)\,e^{ir\gamma^*/h} = e^{-ix\gamma^*}R_-^w\big(v_t(hD_r)\big) \\ e^{-ir\gamma^*/h}R_+^w\big(v_t(hD_r)\big)\,e^{ir\gamma^*/h} = R_+^w\big(v_t(hD_r)\big)\,e^{ix\gamma^*}. \end{cases}$$

Combining this with the fact that

$$P(r,k+\gamma^*,t,h) = e^{-ix\gamma^*}P(r,k,t,h)e^{ix\gamma^*},$$

we get

$$\left[\mathcal{P}(z,t,h), \begin{pmatrix} e^{i(r/h-x)\gamma^*} & 0 \\ 0 & e^{ir\gamma^*/h} \end{pmatrix}\right] = 0. \tag{3.17}$$

Obviously, (3.17) remains true if we replace $\mathcal{P}(z,t,h)$ by $\mathcal{E}(z,t,h)$. Hence,

$$E_{-+}(r,k+\gamma^*\,;z,t,h) = E_{-+}(r,k\,;z,t,h), \tag{3.18}$$

$$\begin{cases} E_+(r,k+\gamma^*\,;z,t,h) = e^{-ix\gamma^*}E_+(r,k\,;z,t,h) \\ E_-(r,k+\gamma^*\,;z,t,h) = E_-(r,k\,;z,t,h)e^{ix\gamma^*} \end{cases} \tag{3.19}$$

$$E(r,k+\gamma^*\,;z,t,h) = e^{-ix\gamma^*}E(r,k\,;z,t,h)e^{ix\gamma^*}, \quad \forall\gamma^* \in \Gamma^*. \tag{3.20}$$ ∎

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As in the proof of Theorem 3.7 in [15], (3.17)–(3.20) imply that

$$\mathcal{P}(z,t,h)\colon \mathbb{L}^2 \times V_0^N \to \mathbb{L} \times V_0^N, \tag{3.21}$$

and

$$\mathcal{E}(z,t,h)\colon \mathbb{L} \times V_0^N \to \mathbb{L}^2 \times V_0^N, \tag{3.22}$$

are bounded. Here $V_0 = \{\sum_{\gamma\in\Gamma} c_\gamma \delta(x - h\gamma) \in S'(\mathbf{R}^n)\ ;\ (c_\gamma)_{\gamma\in\Gamma} \in l^2\}$. Combining this with the fact that $\mathcal{P}(z,t,h)$ is bijective from $S'(\mathbf{R}^n\ ;\ H_0 \times \mathbf{C}^N)$ into $S'(\mathbf{R}^n, H_{2,0} \times \mathbf{C}^N)$ with inverse $\mathcal{E}(z,t,h)$ (see Proposition 2.6), as well as the fact that $\mathbb{L}^2 \times V_0^N \subset S'(\mathbf{R}^n, H_{2,0} \times \mathbf{C}^N)$, we deduce that the operator in (3.21) is bijective with inverse $\mathcal{E}(z,t,h)$.

Let $\mathcal{F}$ be the semi-classical Fourier transform. Set

$$\widehat{\mathcal{P}}(z,t,h) = \begin{pmatrix} \mathbb{P}(t,h) - z & \widehat{R}_- \\ \widehat{R}_+ & 0 \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{F} \end{pmatrix} \mathcal{P}(z,t,h) \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{F}^{-1} \end{pmatrix}$$

and

$$\widehat{\mathcal{E}}(z,t,h) = \begin{pmatrix} \widetilde{E}(z,t,h) & \widetilde{E}_+(z,t,h) \\ \widetilde{E}_-(z,t,h) & \widetilde{E}_{-+}(z,t,h) \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{F} \end{pmatrix} \mathcal{E}(z,t,h) \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{F}^{-1} \end{pmatrix}.$$

Since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathcal{F} \end{smallmatrix}\right)$ is an isomorphism from $\mathbb{L}^2 \times V_0^N$ into $\mathbb{L}^2 \times L^2(\mathbf{T}^*\ ;\ \mathbf{C}^N)$, we have proved:

Theorem 3.7 *$\widehat{\mathcal{P}}(z,t,h)$ is uniformly bounded from $\mathbb{L}^2 \times L^2(\mathbf{T}^*\ ;\ \mathbf{C}^N)$ to $\mathbb{L} \times L^2(\mathbf{T}^*\ ;\ \mathbf{C}^N)$, and has the uniformly bounded two sided inverse $\widehat{\mathcal{E}}(z,t,h)$.*

The main result of this subsection is the following:

Theorem 3.8 *There exist a complex neighborhood ϑ of λ and small constants h_0, $t_0 > 0$, such that for $(z,t,h) \in \vartheta \times D(t_0) \times]0,h_0[$, one has:*

$$z \in \sigma\big(P(t,h)\big) \iff z \in \sigma\big(\mathbb{P}(t,h)\big) \iff 0 \in \sigma\big(\widetilde{E}_{-+}(z,t,h)\big). \tag{3.23}$$

Here, $\widetilde{E}_{-+}(z,t,h) = E^w_{-+}(-hD_k, k\ ;\ z,t,h)\colon L^2(\mathbf{T}^*\ ;\ \mathbf{C}^N) \to L^2(\mathbf{T}^*\ ;\ \mathbf{C}^N)$.

Proof The first equivalence is a consequence of Lemma 3.2, the second follows from the following standard identities (see [18]):

$$\big(z - \mathbb{P}(t,h)\big)^{-1} = -\widetilde{E}(z,t,h) + \widetilde{E}_+(z,t,h)\widetilde{E}_{-+}(z,t,h)^{-1}\widetilde{E}_-(z,t,h),$$
$$\widetilde{E}_{-+}^{-1}(z,t,h) = \widehat{R}_+\big(z - \mathbb{P}(t,h)\big)^{-1}\widehat{R}_-.$$

Remark 3.9 Recalling the definition of $\mathcal{P}(z,t,h)$ and using that $R\pm^w\big(\nu_t(hD_k)\big) = \mathcal{U}_t R^w_\pm(hD_k)\mathcal{U}_t^{-1}$ (which follows from Corollary 2.6), we see that $\mathcal{P}(z,t,h) = \mathcal{U}_t\mathcal{P}(z,0,h)\mathcal{U}_t^{-1}$. From this we deduce easily that $\widetilde{E}_{-+}(z,t,h) = \mathcal{U}_t\widetilde{E}_{-+}(z,0,h)\mathcal{U}_t^{-1}$ for real t.

4 Proofs of Theorem 1.1 and Theorem 1.3

From now on we assume (H1), (H2) and (H3). We choose $\nu(k)$ with $\nu(k) = \lambda'(k)$ near $\mathcal{F}(\lambda)$.

Let us begin with the non-perturbed case, $\varphi = 0$. We denote the distorted Hamiltonian of $P_0 = -\Delta + V(y)$ by $P(t)$. Let

$$\mathcal{E}(k,z,t) = \begin{pmatrix} E(k,z,t) & E_+(k,z,t) \\ E_-(k,z,t) & E_{-+}(k,z,t) \end{pmatrix},$$

be the inverse of

$$\mathcal{P}(k,z,t) = \begin{pmatrix} \big(D_x + \nu_t(k)\big)^2 + V(x) - z & R_-\big(\nu_t(k)\big) \\ R_+\big(\nu_{\bar{t}}(k)\big) & 0 \end{pmatrix}$$

given by Lemma 3.3. Since $\mathcal{P}(k,z,t)$ and $\mathcal{E}(k,z,t)$ are r-independent, we have:

$$\mathcal{P}^w(hD_r,z,t) \circ \mathcal{E}^w(hD_r,z,t) = I, \quad \text{and} \quad \mathcal{E}^w(hD_r,z,t) \circ \mathcal{P}^w(hD_r,z,t) = I,$$

which implies that, the effective Hamiltonian $\widetilde{E}_{-+}(z,t)$ (corresponding to the non-perturbed Hamiltonian P_0) given by Theorem 3.8 is the operator multiplication on $L^2(\mathbf{T}^* \,; \mathbf{C}^N)$ by the matrix $E_{-+}(k\,; z,t)$. Hence,

(4.1)
$$z \in \sigma\big(P(t)\big) \iff 0 \in \sigma\big(\widetilde{E}_{-+}(z,t)^{-1}\big) \iff \exists\, k \in \mathbf{T}^*, \quad \text{st } 0 \in \sigma\big(E_{-+}(k,z,t)\big).$$

On the other hand, using the fact that $\mathcal{E}(k\,; z,t)$ is a left and right inverse of $\mathcal{P}(k\,; z,t)$ we deduce as in the proof of Theorem 3.8 that

$$z \in \sigma\Big(\big(D_x + \nu_t(k)\big)^2 + V(x)\Big) \iff 0 \in \sigma\big(E_{-+}(k\,; z,t)\big). \tag{4.2}$$

Lemma 4.1 *Under the assumptions* (H1), (H2) *and* (H3), *there exist a neighborhood* Ω *of* λ *and a small positive constants* η, ϵ, *such that for every* $t \in i\,]0,\epsilon[$, $\widetilde{E}_{-+}(z,t)^{-1}$ *extends analytically from* $\Omega_+ = \{z \in \Omega\,; \Im z > 0\}$ *to* $\Omega_{-\eta|t|}$. *In particular*

$$\sigma\big(P(t)\big) \cap \Omega_{-\eta|t|} = \varnothing. \tag{4.3}$$

Proof Due to assumption (H2), there exists a small neighborhood Ω of λ such that

$$z \in \sigma\Big(\big(D_x + \nu_t(k)\big)^2 + V(x)\Big) \iff z = \lambda\big(\nu_t(k)\big). \tag{4.4}$$

By Taylor's formula, one has

$$z - \lambda\big(\nu_t(k)\big) = z - \lambda(k) + t\langle \lambda'(k).\nu(k)\rangle + \mathcal{O}(t^2). \tag{4.5}$$

Fix t in $i\,]0,\epsilon[$ with ϵ small enough. Since $\langle \lambda'(k).\nu(k)\rangle = |\lambda'(k)|^2$ near $\mathcal{F}(\lambda)$, assumption (H3) together with (4.5) yield

$$\exists\,\eta > 0 \text{ such that } \forall\, z \in \Omega, \quad \left| z - \lambda\big(\nu_t(k)\big) \right| \geq |\Im z + \eta \Im t|. \tag{4.6}$$

Clearly, Lemma 4.1 follows from equivalences (4.1), (4.2), (4.4) and (4.6). ■

We return now to the perturbed effective Hamiltonian $\widetilde{E}_{-+}(z,t,h)$ corresponding to the operator $\mathbb{P}(t,h)$.

Lemma 4.2 *Under assumption* (H1), $\widetilde{E}_{-+}(z,t,h) - \widetilde{E}_{-+}(z,t)$ *is a compact operator on* $L^2(\mathbf{T}^*\,;\mathbf{C}^N)$.

Proof The second resolvent equation gives

$$\begin{aligned}\widehat{\mathcal{E}}(z,t,h) - \widehat{\mathcal{E}}(z,t) &= \widehat{\mathcal{E}}(z,t)\big(\widehat{\mathcal{P}}(z,t,h) - \widehat{\mathcal{P}}(z,t)\big)\widehat{\mathcal{E}}(z,t,h)\\ &= \widehat{\mathcal{E}}(z,t)\begin{pmatrix}\varphi_t^w(r,hD_r\,;h) & 0\\ 0 & 0\end{pmatrix}\widehat{\mathcal{E}}(z,t,h),\end{aligned}$$

which yields

$$\widetilde{E}_{-+}(z,t,h) - \widetilde{E}_{-+}(z,t) = \mathcal{F}E_-(z,t)\varphi_t^w(r,hD_r\,;h)E_+(z,t,h)\mathcal{F}^{-1}. \tag{4.7}$$

Recall that $\varphi_t \in S^0(\mathbf{R}^{2n},\langle r\rangle^{-\delta})$, $E_+ \in S^0\big(\mathbf{R}^{2n}\,;\mathcal{L}(\mathbf{C}^N,H_0)\big)$ and $E_- \in S^0\big(\mathbf{R}^{2n}\,;\mathcal{L}(H_0,\mathbf{C}^N)\big)$. See Corollary 2.6 and Proposition 3.6. By a classical result of symbolic calculus [24] and [36, Section 4], we deduce that

$$\begin{aligned}K^w(r,hD_r\,;t,h) &:= E_-(z,t)\varphi_t(r,hD_r\,;h)E_+(z,t,h)\\ &\in \mathrm{Op}_h^w\Big(S^0\big(\mathbf{R}^{2n},\langle r\rangle^{-\delta}\,;M(\mathbf{C}^N)\big)\Big).\end{aligned}$$

Now Lemma 4.2 follows from the equality

$$\widetilde{E}_{-+}(z,t,h) - \widetilde{E}_{-+}(z,t) = K^w(-hD_k,k\,;t,h).$$

Note that if $p(k,r) \in S^0(\mathbf{T}^*\times\mathbf{R}^n_r,\langle r\rangle^{-\delta})$ with $\delta > 0$ then $p^w(k,hD_k)$ defines a compact operator on $L^2(\mathbf{T}^*)$ (see [16]).

4.1 Proof of Theorem 1.1

Fix $z_0 \in \Omega_+$. Since $\mathbb{P}(h,0)$ is self-adjoint and since $t \to \mathbb{P}(h,t)$ is an analytic family of type A, we may assume that $z_0 \notin \sigma\big(\mathbb{P}(t,h)\big)$ when $t \in i\,]0,\epsilon[$ (ϵ being small enough).

Set

$$K(z,t,h) = \widetilde{E}_{-+}(z,t,h) - \widetilde{E}_{-+}(z,t).$$

By Lemma 4.1 and Lemma 4.2, $z \to \widetilde{E}_{-+}(z,t)^{-1}K(z,t,h)$ is a compact, analytic, operator-valued function on $\Omega_{-\eta|t|}$. Since $\widetilde{E}_{-+}(z_0,t,h)$ is invertible by Theorem 3.8, the analytic Fredholm theorem [31, Theorem VI.14] implies that,

$$\widetilde{E}_{-+}(z,t,h) = \widetilde{E}_{-+}(z,t)\big(I + \widetilde{E}_{-+}(z,t)^{-1}K(z,t,h)\big), \tag{4.8}$$

is invertible on $\Omega_{-\eta|t|} \setminus S$, where S is a discrete subset of $\Omega_{-\eta|t|}$. Combining this with Theorem 3.8, we get

$$\sigma\big(P(t,h)\big) \cap \Omega_{-\eta|t|} \subset S. \tag{4.9}$$

Pick z in Ω_+. Let ϕ_0, ψ_0 be in $\mathcal{A}$, and set $\phi_h(x) = h^{-\frac{n}{2}}\phi_0(x/h)$, $\psi_h(x) = h^{-\frac{n}{2}}\psi_0(x/h)$. As long as t is real, $\mathcal{U}_t$ is unitary and

$$\begin{aligned} f_{\phi_0,\psi_0}(z) &= \Big(\big(P(h)-z\big)^{-1}\phi_0,\psi_0\Big) = \Big(\big(\widehat{P}(h)-z\big)^{-1}\phi_h,\psi_h\Big) \\ &= \Big(\big(P(t,h)-z\big)^{-1}\mathcal{U}_t\phi_h,\mathcal{U}_{\bar t}\psi_h\Big) := f_{\phi_0,\psi_0}(z,t), \end{aligned} \tag{4.10}$$

(4.11)

$$f^0_{\phi_0,\psi_0}(z) = \big((P_0-z)^{-1}\phi_0,\psi_0\big) = \Big(\big(P(t)-z\big)^{-1}\mathcal{U}_t\phi_h,\mathcal{U}_{\bar t}\psi_h\Big) := f^0_{\phi_0,\psi_0}(z,t).$$

By Lemma 2.1 and Proposition 2.8, $f_{\phi,\psi}(z,t)$ and $f^0_{\phi,\psi}(z,t)$ extend by analytic continuation in t to the disc $D(t_0)$.

Now for fixed t in $i\,]0,\epsilon[$, the right hand side of (4.10) (resp. (4.11)) is meromorphic (resp. holomorphic) on z in $\Omega_{-\eta|t|}$, due to (4.9) (resp. (4.3)). This ends the proof of Theorem 1.1.

Remark 4.3 Since $\{\mathcal{U}_t\phi\ ;\ \phi \in \mathcal{A}\}$ is dense in L^2, the right hand side of (4.10) has a pole $z \in \Omega_{-\eta|t|}$ if and only if $z \in \sigma\big(P(t,h)\big) \cap \Omega_{-\eta|t|}$.

4.2 Proof of Theorem 1.3

When $|\varphi| \le \epsilon$, (4.5) implies that $\|\widetilde{E}_{-+}(z,t,h) - \widetilde{E}_{-+}(z,t)\| \le C\epsilon$, which together with (4.8) and Theorem 1.1 i) give Theorem 1.3.

5 Proof of Theorem 1.4

5.1 Spectral Properties of the Effective Hamiltonian

For simplicity, we replace assumption (H4) by

$$W_l^{-1}(\mathbf{R}^n \times \mathbf{T}^*) \cap W_m^{-1}(\mathbf{R}^n \times \mathbf{T}^*) = \varnothing, \quad \forall\, m \neq l. \tag{$\tilde{\rm H}4$}$$

With a slight modification, our methods developed below work also under assumption (H4) (see Remark 5.10).

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Following Remark 4.3 and Theorem 3.8, the resonances of $P(h)$ in $\Omega_{-\eta|t|}$ are the 0 eigenvalues of the operator $\widetilde{E}_{-+}(z,t,h)$. In the next proposition, we will show that, $\widetilde{E}_{-+}(z,t,h)$ is an h-pseudodifferential operator with scalar valued symbol and we give explicitly the leading terms of its symbol.

We will first recall some well known facts about Bloch functions.

Let $\phi_l(\cdot,k) \in \ker\big(P_k - \lambda_l(k)\big)$ be the eigenfunction corresponding to the eigenvalue $\lambda_l(k)$. If $\lambda_l(k_0)$ is a simple eigenvalue of P_{k_0}, then the function ϕ_l can be chosen analytic on k in a neighborhood of $k_0 \in \mathbf{T}^*$. Moreover, if ($\tilde{\mathrm{H}}4$) is satisfied, we can choose ϕ_l analytic in a complex neighborhood W of $\mathbf{T}^*$ with

$$\Big(\big(D_x + \nu_t(k)\big)^2 + V(x)\Big)\phi_l\big(x,\nu_t(k)\big) = \lambda\big(\nu_t(k)\big)\phi_l\big(x,\nu_t(k)\big), \tag{5.0}$$

$$\int_{\mathbf{T}^*} \phi_l\big(x,\nu_t(k)\big)\overline{\phi}_l\big(x,\nu_{\bar t}(k)\big)\,dx = 1, \quad \forall\, t \in D(t_0)\cap \mathbf{R},\ k \in \mathbf{R}^n. \tag{5.1}$$

See [18] and [34, Lemma 4.1].

The left-hand sides of (5.0) and (5.1) have an analytic continuation on $t \in D(t_0)$. By uniqueness of analytic continuation, (5.0) and (5.1) remain true for all $t \in D(t_0)$. From now on we write ϕ for ϕ_l.

Proposition 5.1 *Fix $\lambda \in W_l$. Under assumption* ($\tilde{\mathrm{H}}4$)*, the matrix $E_{-+}(r,k\,;\,z,t,h)$ given by (3.16) can be chosen real-valued such that for all $m \in \mathbf{N}$:*

$$\begin{aligned} E_{-+}(r,k\,;z,t,h) &= z - E^0_{-+}(r,k\,;t) - hE^1_{-+}(r,k\,;t) \\ &\quad + \sum_{j=2}^{m} h^j E^j_{-+}(r,k\,;z,t) + h^{m+1}R(r,k\,;z,t,h), \end{aligned} \tag{5.2}$$

where $E^j_{-+}(r,k\,;z,t), R(r,k\,;z,t,h) \in S^0(\mathbf{R}^{2n})$, and

$$E^0_{-+}(r,k\,;t) = \varphi\Big(\big(1 - tM(k)\big)^{-1}r\Big) + \lambda\big(\nu_t(k)\big), \tag{5.3}$$

$$E^1_{-+}(r,k\,;t) = -i\big\langle \partial_k\phi\big(\cdot,\nu_t(k)\big)\,\phi\big(\cdot,\nu_{\bar t}(k)\big)\big\rangle_{H_0}\nabla_r\Big(\varphi\Big(\big(1-tM(k)\big)^{-1}r\Big)\Big). \tag{5.4}$$

Proof Following the procedure of Section 3, we have only to show that we can take $N = 1$ in Lemma 3.3, and prove (5.3), (5.4).

Let $\Pi_k \colon L^2(\mathbf{T}) \to L^2(\mathbf{T})$ be the projection defined by

$$(\Pi_k u)(x,k) = \Big(u, \phi\big(\cdot,\nu_{\bar t}(k)\big)\Big)_{H_0}\phi\big(x,\nu_t(k)\big). \tag{5.5}$$

Under assumption ($\tilde{\mathrm{H}}4$), $(1-\Pi_k)\big(P(r,k,t) - z\big)^{-1}(1-\Pi_k)$ is well defined for z in a small complex neighborhood of λ.

Let $\mathcal{P}(r,k\,;z,t)$ be the operator constructed in Lemma 3.3 with

$$R_+\big(\nu_{\bar t}(k)\big)\,u = \Big(u, \phi\big(\cdot, \nu_{\bar t}(k)\big)\Big)_{H_0}.$$

Using (5.0) and (5.1), we see easily that

(5.6)
$$\mathcal{E}_0(r,k\,;z,t) = \begin{pmatrix} (1-\Pi)\big(P(r,k\,;t)-z\big)^{-1}(1-\Pi) & R_-\big(\nu_t(k)\big) \\ R_+\big(\nu_{\bar t}(k)\big) & z-\lambda\big(\nu_t(k)\big)-\varphi_{t,0}(r,k)\end{pmatrix}$$

is the inverse of $\mathcal{P}(r,k,z,t)$.

Recalling that $E_{-+}(r,k\,;z,t,h)$ is the lower right corner of the matrix

$$\begin{aligned}\mathcal{E}(z,t,h) &= \mathcal{E}_0^w(r,hD_r\,;z,t)\circ(1+h\mathcal{R}^w)^{-1}\\ &= \mathcal{E}_0^w(r,hD_r\,;z,t) - h\mathcal{E}_0^w(r,hD_r\,;z,t)\circ\begin{pmatrix} a_1 & a_3\\ a_2 & a_4\end{pmatrix} + \mathcal{O}(h^2).\end{aligned} \tag{5.7}$$

Hence,

(5.8)
$$\begin{aligned}E_{-+}(r,k\,;z,t,h) &= z-\lambda\big(\nu_t(k)\big)-\varphi_{t,0}(r,k)\\ &\quad - h\Big[\Big(z-\lambda\big(\nu_t(k)\big)-\varphi_{t,0}(r,k)\Big)a_4 + R_+\big(\nu_{\bar t}(k)\big)a_3\Big] + \mathcal{O}(h^2).\end{aligned}$$

Here $\left(\begin{smallmatrix} a_1 & a_3\\ a_2 & a_4\end{smallmatrix}\right)$ denotes the principal term of $\mathcal{R}$. Formula (3.13) and Theorem A.2 show that

$$\begin{pmatrix} a_1 & a_3\\ a_2 & a_4\end{pmatrix} = \frac{1}{2i}\{\mathcal{P}(r,k\,;z,t),\mathcal{E}_0(r,k\,;z,t)\},$$

where $\{\cdot,\cdot\}$ is the Poisson bracket. Consequently,

$$a_4 = \frac{1}{2i}\Big\{R_+\big(\nu_{\bar t}(k)\big), R_-\big(\nu_t(k)\big)\Big\},\quad a_3 = \frac{1}{i}\Big\{\varphi_{t,0}(r,k), R_-\big(\nu_t(k)\big)\Big\}. \tag{5.9}$$

Since R_+ and R_- depend only on k, $a_4 = 0$. Combining this with (2.21), (5.8) and (5.9) we get (5.3) and (5.4). ■

Remark The purpose of this remark is to provide a broad outline of the proof. Some of ideas presented here come from older work of [29], [38].

Set

$$B(h) = \mathrm{Op}_h^w\big(\varphi(r)+\lambda(k)+hE^1_{-+}(r,k,0)\big),\quad \widehat{B}(h) := \mathcal{F}B(h)\mathcal{F}^{-1} \tag{5.10}$$

where $E^1_{-+}(r,k,0)$ is the right-hand side of (5.4) token at $t=0$. By Corollary 2.5,

$$D(t_0)\ni t\to \mathcal{F}\mathcal{U}_tB(h)\mathcal{U}_t^{-1}\mathcal{F}^{-1} := B_t(h)\in\mathcal{L}\big(L^2(\mathbf{T}^*)\big)$$

is analytic and unitarily equivalent to $\widehat{B}(h)$ for real t.

On the other hand, Theorem 2.3 and Corollary 2.6 show that $B_t(h) - \lambda\big(\nu_t(k)\big)$ is a compact operator on $L^2(\mathbf{T}^*)$, and

$$z - B_t(h) = \widetilde{E}_{-+}(z,t,h) + \mathcal{O}(h^2). \tag{5.11}$$

Therefore, modulo $\big(\mathcal{O}(h^2)\big)$, we are led to study the spectrum of $B_t(h)$ near λ.

It results from Lemma 3 of [32, p. 111] that,

$$\sigma_{\text{ess}}\big(B_t(h)\big) = \big\{\lambda\big(\nu_t(k)\big)\ ; k \in \mathbf{T}^*\big\}.$$

Fix t small with $\Im t > 0$, and let $\Omega = \Omega_t$ be a small complex neighborhood of λ. Under assumption (H3), (4.5) and (4.6) imply that

$$\sigma_{\text{ess}}\big(B_t(h)\big) \cap \Omega_t = \varnothing.$$

Therefore the spectrum of $B_t(h)$ in Ω_t consists of discrete eigenvalues of finite multiplicity. Moreover, since $B_t(h)$ is obtained from $\widehat{B}(h)$ by a spectral deformation, these eigenvalues are t-independent, and therefore should be considered as resonances of $\widehat{B}(h)$.

Under assumption (H5), we will show that $\big(\lambda - B_t(h)\big)$ is elliptic except for $(k,r) = (0,0)$. Using this, we conclude by some weighted L^2 estimate that, only a microlocal version of $B_t(h)$ near $(0,0)$ is needed to study the spectrum of $B_t(h)$ near λ. Hence, constructing an analytic family $\widetilde{B}_t(h)$ such that $\widetilde{B}_0(h)$ coincide with $K = \frac{h^2}{2}\langle\varphi''(r_0)\nabla_k, \nabla_k\rangle + \frac{1}{2}\langle\lambda''(k_0)k,k\rangle$, near $(0,0)$, we deduce that

$$\sigma\big(B_t(h)\big) \cap \Omega = \big(\lambda + \sigma(K) + \mathcal{O}(h^2)\big) \cap \Omega.$$

Finally we give the complete asymptotic expansion by constructing an asymptotic solution of the equation $\widetilde{E}_{-+}(z,t,h)u = 0$.

5.2 Spectral Properties of $B_t(h)$

Without any loss of generality, we can assume that the band Hamiltonian $W_l(k,r)$ has a non-degenerate minimum. Otherwise we consider $-B_t(h)$ near $-\lambda$.

We denote by $B(0,\epsilon)$ a ball in $\mathbf{T}^* \times \mathbf{R}^n$ of center $(0,0)$ and radius $\epsilon > 0$. By $S^0(\mathbf{T}^* \times \mathbf{R}^n)$, we denote the space of symbols $p(k,r)$ in $S^0(\mathbf{R}^{2n})$ which are Γ^*-periodic with respect to k.

In the following, $t = i\Im t$ is fixed with $\Im t > 0$. The next lemma shows that $B_t(h) - \lambda$ is elliptic except at $(k,r) = (0,0)$.

Lemma 5.2 *Under the assumptions* (H1), (H2), (H3) *and* (H5), *for sufficiently small positive* ϵ, *there exists* $C > 0$ *such that*

$$|p(k,r\,;t) - \lambda| \geq \frac{|t|}{C}, \quad \forall (k,r) \notin B(0,\epsilon) \tag{5.12}$$

$$\Re\big(p(k,r\,;t) - \lambda\big) \geq \frac{|r|^2 + |k|^2}{C}, \quad \forall (k,r) \in B(0,\epsilon). \tag{5.13}$$

Here

$$p(k,r\,;t) := E^0_{-+}(-r,k\,;t) = \varphi\Big(\big(tM(k)-1\big)^{-1}r\Big) + \lambda\big(\nu_t(k)\big)$$

denotes the principal term of $B_t(h)$.

Proof By Taylor's formula, one has:

(5.14)
$$p(k,-r\,;t) - \lambda = \varphi(r) + \lambda(k) - \lambda - t\big(|\lambda'(k)|^2 - \lambda''(k)r\cdot\varphi'(r)\big) + \mathcal{O}_{r,k}(t^2),$$

where $\mathcal{O}_{r,k}(t^2) = \mathcal{O}(t^2)$ uniformly on $(r,k) \in \mathbf{R}^{2n}$ and $\mathcal{O}_{r,k}(t^2) = \mathcal{O}\big(t^2(r,k)^2\big)$ near $(0,0)$. Assumption (H2) implies that

$$|\lambda(k) - \lambda| + |\lambda'(k)| \geq c_0, \tag{5.15}$$

for some $c_0 > 0$. Since $\varphi(r)$ and $r\varphi'(r)$ tends to 0 when r tends to infinity, (5.14) and (5.15) give (5.12) when $|r| \geq R$ with R large enough.

On the other hand, due to assumption (H5), we have: for $(k,r) \notin B(0,\epsilon)$ with $|\lambda(k) + \varphi(r) - \lambda| \leq \delta$ and $|r| < R$:

$$|\lambda'(k)|^2 - \lambda''(k)r\cdot\varphi'(r) \geq \frac{1}{C}, \quad \text{for some } C > 0.$$

This together with (5.14) ends the proof of (5.12).

Recalling that $(0,0)$ is a nondegenerate minimum of $\varphi(r) + \lambda(k)$. Combining this with (5.14), we obtain:

$$p(k,r\,;t) - \lambda = \varphi''(0)\frac{r^2}{2} + \lambda''(0)\frac{k^2}{2} + t\mathcal{O}\big((r,k)^2\big),$$

for all (k,r) in $B(0,\epsilon)$. Choosing $|t|$ small enough, we get (5.13). ∎

Remark 5.3 The estimates of Lemma 5.2 remain true if we replace r (resp. k) in the right hand side of (5.12) and (5.13) by $r + i\nabla f(k)$ (resp. $k + i\nabla g(r)$), where $f(k) \in C^\infty(\mathbf{T}^*)$ (resp. $g(r) \in C^\infty(\mathbf{R}^n)$) is a non-negative function such that $f(k), g(r) = \mathcal{O}(|t|)$, $g(k), f(k) \sim c_0k^2$ near a neighborhood Ω of 0 in $\mathbf{T}^*$ (resp. $\mathbf{R}^n$) and $g(k), f(k) > c|t|$, for $k \notin \Omega$.

Theorem 5.4 *Let* $E(h) \in D(\lambda, C_0h)$ *be an eigenvalue of* $B_t(h)$. *Let* u_h *be a normalized associated eigenfunction,*

$$B_t(h)u_h = E(h)u_h, \quad \|u_h\|_{L^2(\mathbf{T}^*)} = 1.$$

Then, there exists $C > 0$ *which does not depend on* $h \in]0, h_0[$, *such that*

$$\|e^{f/h}u_h\|_{L^2(\mathbf{T}^*)} \leq C. \tag{5.16}$$

Here f *is a function satisfying the properties of Remark 5.3.*

Proof Put $B_t^f(h) := e^{f/h} B_t(h) e^{-f/h}$ and $u_{f,h} = e^{f/h} u_h$. We have

$$\big(B_t^f(h) - E(h)\big)\, u_{f,h} = 0. \tag{5.17}$$

Proposition 2.7 and Corollary 2.5 imply that

$$B_t^f(h) = p^w\big(k, hD_k + i\partial_k f(k), t\big) + \mathcal{O}(h), \quad \text{in } \mathcal{L}\big(L^2(\mathbf{T}^*)\big). \tag{5.18}$$

Let η be a small positive constant. Let $\chi_j \in C^\infty(\mathbf{T}^* \times \mathbf{R}^n\ ;\ ]0,1[)$, $j = 1,2$ be smooth functions, such that supp $\chi_1 \subset B(0,\eta)$, $\chi_1 = 1$ near $(0,0)$ and $\chi_1 + \chi_2 = 1$. Lemma 5.2 and Remark 5.3 imply that

$$(k,r) \to \widetilde{p}_t(k,r\,;z,t) := \chi_2(k,r)\Big(z - p\big(k, r + i\partial_k f(k), t\big)\Big)^{-1} \in S^0(\mathbf{T}^* \times \mathbf{R}^n).$$

Set, $\widehat{B}_t(z,h) = \widetilde{p}_t^w(k, hD_k\,; z, t)$. One has,

$$\widehat{B}_t(z,h)\big(z - B_t^f(h)\big) = \chi_2^w(k, hD_k) + \mathcal{O}(h). \tag{5.19}$$

Taking $z = E(h)$ in (5.19), we get

$$\chi_2^w(k, hD_k) u_{f,h} = \mathcal{O}(h) u_{f,h}, \tag{5.20}$$

due to (5.17). Since $\chi_1 + \chi_2 = 1$, then

$$u^1_{f,h} := \chi_1^w(k, hD_k) u_{f,h} = u_{f,h} - \chi_2^w(k, hD_k) u_{f,h} = \big(1 + \mathcal{O}(h)\big)\, u_{f,h}. \tag{5.21}$$

Combining this with (5.17), we obtain

$$\big(B_t^f(h) - E(h)\big)\, u^1_{f,h} = \mathcal{O}(h) u_{f,h}. \tag{5.22}$$

In view of (5.13), we can apply the semi-classical sharp Gårding inequality (see for instance [12, Theorem 7.12]) to the operator $\chi_1^w(k, hD_k)\big(B_t^f(h) - E(h)\big)\, \chi_1^w(k, hD_k)$, and get:

$$\Re\Big(\big(B_t^f(h) - E(h)\big)\, u_f^1(h), u_f^1(h)\Big) \geq c\big((h^2D_k^2 + k^2 - Ch) u^1_{f,h}, u^1_{f,h}\big), \tag{5.23}$$

for some c and $C > 0$.

Let R be a large positive constant. Write

(5.24)
$$\big((h^2D_k^2 + k^2 - Ch) u^1_{f,h}, u^1_{f,h}\big) = \big((h^2D_k^2 + k^2 - Ch) u^1_{f,h}, u^1_{f,h}\big)_{L^2(|k| > (Rh)^{\frac12})} + \big((h^2D_k^2 + k^2 - Ch) u^1_{f,h}, u^1_{f,h}\big)_{L^2(|k| < (Rh)^{\frac12})}.$$

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The first term of the right member is bounded from below by $(R-C)h\|u^1_{f,h}\|^2$, since $h^2D_k^2 + k^2 - Ch \geq k^2 - Ch$ in the sense of self-adjoint operators. On the other hand, by construction of f (see Remark 5.3), $e^{f/h} = \mathcal{O}_R(1)$ for $|k|^2 \leq Rh$. Consequently,

$$\big((h^2D_k^2 + k^2 - Ch)u^1_{f,h}, u^1_{f,h}\big)_{L^2(|k|<(Rh)^{\frac12})} \geq -Ch\int_{\{|k|<(Rh)^{\frac12}\}} |u^1_{f,h}(k)|^2\,dk \geq -C_Rh$$

where C_R depends only on R. Here we have used the fact that $\|u_h\| = 1$. Combining this with (5.23) and (5.24), we get:

$$\big\|\big(B_t^f(h) - E(h)\big)u^1_{f,h}\big\|\,\|u^1_{f,h}\| \geq (R-C)h\|u^1_{f,h}\|^2 - C_Rh. \tag{5.25}$$

Using (5.22), (5.25) and the fact that $ab \leq \frac{R}{4}a^2 + \frac{1}{R}b^2$, we obtain

$$\|u^1_{f,h}\|^2 \leq \frac{\widetilde{C}}{R}\|u_{f,h}\|^2 + \widetilde{C}_R. \tag{5.26}$$

If we substitute (5.21) in the right hand side of (5.26), we get (5.16). This finishes the proof of the theorem. ∎

Remark 5.5 Let g be a function satisfying the properties of Remark 5.3. Using the above arguments and Proposition 2.7 we obtain

$$\|e^{g(hD_k)/h}u_h\|_{L^2(\mathbf{T}^*)} \leq C, \tag{5.27}$$

uniformly on $h \in\,]0, h_0[$.

Remark 5.6 Theorem 5.4 and Remark 5.5 show that the energy of eigenfunctions of $B_t(h)$ associated to an eigenvalue $E(h) \in D(\lambda, C_0h)$ are microlocally exponentially concentrated near $(0,0)$. So, to compute the eigenvalues of $B_t(h)$ in $D(\lambda, C_0h)$, we need just to study a microlocal version of $B_t(h)$ near $(0,0)$.

Let C_0, $(e_i)_{1\leq i\leq N_0+1}$ as in Theorem 1.4. We let Υ_j (resp. $\widehat{\Upsilon}_j$) be the complex circle (resp. disk) of center $\lambda + he_j$ and radius δh. We choose δ small enough such that $\widehat{\Upsilon}_j \cap \widehat{\Upsilon}_k = \varnothing$ when $e_k \neq e_j$. Set

$$\widehat{D}_\lambda(h) := D(\lambda, C_0h) \setminus \bigcup_{k=1}^{N_0} \widehat{\Upsilon}_k. \tag{5.28}$$

Now, we construct a microlocally version $\widetilde{B}_t(h)$ of $B_t(h)$ as indicated in Remark 5.6. Fix $N \geq 5$. Let $\widetilde{\lambda} \in C^\infty(\mathbf{T}^*)$ and $\widetilde{\varphi} \in C^\infty(\mathbf{R}^n)$ be two positive valued functions satisfying:

i) $\widetilde{\lambda}(k)$ extends analytically in a small complex neighborhood of the torus $\mathbf{T}^*$.
ii) $\widetilde{\lambda}^{-1}(0) = \Gamma^*$ and $\widetilde{\lambda}(k) = \sum_{|\alpha|\leq N} \frac{1}{\alpha!}\lambda^{(\alpha)}(0)k^\alpha + \mathcal{O}(|k|^{N+1})$ near zero.
iii) $\widetilde{\varphi}(r)$ extends analytically in D_a, for some $a > 0$, and satisfies (1.5) with $\delta = 0$.

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iv) $\liminf \widetilde{\varphi} > 0$, $\widetilde{\varphi}^{-1}(0) = \{0\}$ and $\widetilde{\varphi}(r) = \sum_{|\alpha|\leq N} \frac{1}{\alpha!}\varphi^{(\alpha)}(0)r^\alpha + \mathcal{O}(|r|^{N+1})$ near zero.

We define the operators $\widetilde{B}(h)$, $\widetilde{B}_t(h)$ as $B(h)$, $B_t(h)$, but with $\widetilde{\lambda}$, $\widetilde{\varphi}$ instead of λ, φ. Evidently, the spectrum of $\mathcal{F}\widetilde{B}(h)\mathcal{F}^{-1}$ near $D(\lambda, C_0h)$ is discrete and

$$\sigma\big(\mathcal{F}\widetilde{B}(h)\mathcal{F}^{-1}\big) \cap D(\lambda, C_0h) = \{\widetilde{e}_1(h), \dots, \widetilde{e}_{N_0}(h)\},$$

with

$$\widetilde{e}_k(h) = \lambda + e_kh + \mathcal{O}(h^{\frac{3}{2}}). \tag{5.29}$$

For further information on the spectrum of an h-pseudodifferential operator near a nondegenerate minimum we refer to [12, Chapter 4].

Since $\widetilde{B}_t(h)$ is an analytic family of type A by Corollary 2.5, it follows from the stability theorem of discrete spectrum (see [32]) that

$$\sigma\big(\widetilde{B}_t(h)\big) \cap D(\lambda, C_0h) = \{\widetilde{e}_1(t,h), \dots, \widetilde{e}_N(t,h)\},$$

where $\widetilde{e}_i(t,h)$ are analytic on t. Combining this with the fact that $\widetilde{B}_t(h)$ is unitarily equivalent to $\mathcal{F}\widetilde{B}(h)\mathcal{F}^{-1}$ for real t, we get: $\widetilde{e}_i(t,h) = \widetilde{e}_i(h)$ for all $t \in D(t_0)$. Consequently,

$$\sigma\big(\widetilde{B}_t(h)\big) \cap D(\lambda, C_0h) = \{\widetilde{e}_1(h), \dots, \widetilde{e}_{N_0}(h)\}. \tag{5.30}$$

The following result shows that there is no spectrum of $B_t(h)$ in $\widehat{D}_\lambda(h)$.

Proposition 5.7 *For $h_0 > 0$ small enough, there exists a constant $c > 0$, such that*

$$\sigma\big(B_t(h)\big) \cap \widehat{D}_\lambda(h) = \varnothing$$

for all $h \in \,]0, h_0[$, and

$$\big\|\big(B_t(h) - z\big)^{-1}\big\| \leq ch^{-1}, \tag{5.31}$$

uniformly on $z \in \widehat{D}_\lambda(h)$.

Proof Let $\epsilon > 0$ be a small constant which will be fixed later. Let χ_j, $j = 1, 2$ be a partition of unity on $\mathbf{T}^* \times \mathbf{R}^n$ with $\chi_1 = 1$ near $B(0, \epsilon)$ and equals 0 outside $B(0, 2\epsilon)$. Lemma 5.2 implies that $B_t(h)$ is elliptic except for $(k, r) = (0, 0)$. Hence, on supp χ_2 we can apply the proof of (5.19) and show that

$$\widehat{B}_t(z,h)\big(z - B_t(h)\big) = \chi_2^w(k, hD_k) + \mathcal{O}(h), \tag{5.32}$$

uniformly on $z \in D$, where $\widehat{B}_t(z,h) \in \mathrm{Op}_h^w\big(S^0(\mathbf{T}^* \times \mathbf{R}^n)\big)$.

On supp χ_1, we compare $\big(z-B_t(h)\big)$ with $\big(z-\widetilde{B}_t(h)\big)^{-1}$ via the obvious equality

$$\begin{aligned}\chi_1^w(k,hD_k)\big(z-\widetilde{B}_t(h)\big)^{-1}\big(z-B_t(h)\big)\\ = \chi_1^w(k,hD_k)+\chi_1^w(k,hD_k)\big(z-\widetilde{B}_t(h)\big)^{-1}\big(\widetilde{B}_t(h)-B_t(h)\big).\end{aligned} \tag{5.33}$$

Let χ_3 be equal to 1 near support of χ_1 and 0 outside $B(0,3\epsilon)$. Lemma B.4 shows that

$$\chi_1\big(z-\widetilde{B}_t(h)\big)^{-1}(1-\chi_3)=\mathcal{O}(h^\infty). \tag{5.34}$$

By construction, $\widetilde{B}_0(h)-B_0(h)=\mathcal{O}\big((r,k)^\beta\big)$ for all $|\beta|\le N+1$, which together with Remark 2.4 give

$$\big(B_t(h)-\widetilde{B}_t(h)\big)=\mathcal{O}\big(|(k,r)|^{N+1}\big)+\mathcal{O}\big(h^2|(k,r)|^{N-3}\big)+\mathcal{O}(h^3). \tag{5.35}$$

Since $|k|+|r|\le 3\epsilon$ on supp χ_3, it follows from (5.35) and Lemma B.3 that

$$\big\|\chi_1^w(k,hD_k)\big(z-\widetilde{B}_t(h)\big)^{-1}\chi_3\big(B_t(h)-\widetilde{B}_t(h)\big)\big\|\le C(\epsilon+h), \tag{5.36}$$

where $C>0$ is independent of ϵ and h. Now, (5.32), (5.33), (5.34) and (5.36) give

$$Ch^{-1}\big\|\big(B_t(h)-z\big)u\big\|\ge(1-C\epsilon)\|u\|. \tag{5.37}$$

This shows that $\ker\big(B_t(h)-z\big)=\{0\}$ for $z\in\widehat{D}_\lambda(h)$. Since the spectrum of $B_t(h)$ in D is discrete, this also shows that $\sigma\big(B_t(h)\big)\cap\widehat{D}_\lambda(h)=\varnothing$. Formula (5.31) is a consequence of (5.37).

Theorem 5.8 *For h small enough, one has:*

$$\sigma\big(B_t(h)\big)\cap D(\lambda,C_0h)=\bigcup_{j=1}^{N_0}\{E_j(h)\}, \tag{5.38}$$

with

$$E_j(h)=\lambda+he_j+\mathcal{O}(h^{3/2}). \tag{5.39}$$

Proof Fix j in $\{1,\dots,N_0\}$, and define

$$\pi(h)=\frac{1}{2\pi i}\int_{\Upsilon_j}\big(z-B_t(h)\big)^{-1}dz \tag{5.40}$$

$$\pi_0(h)=\frac{1}{2\pi i}\int_{\Upsilon_j}\big(z-\widetilde{B}_t(h)\big)^{-1}dz. \tag{5.41}$$

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We recall that Υ_j is the circle of center $(\lambda + he_j)$ and radius δh (δ is small so that $(\lambda + he_k)$ is outside Υ_j when $e_k \neq e_j$). In view of (5.32) and (5.33), we have:

(5.42)
$$\begin{aligned}&\big(z - B_t(h)\big)^{-1} - \chi_1^w(k, hD_k)\big(z - \widetilde{B}_t(h)\big)^{-1}\\ &\quad = \widehat{B}_t(z,h) - \Big[\mathcal{O}(h) + \chi_1^w(k,hD_k)\big(z-\widetilde{B}_t(h)\big)^{-1}\big(B_t(h) - \widetilde{B}_t(h)\big)\Big]\big(z - B_t(h)\big)^{-1},\end{aligned}$$

uniformly on $z \in \widehat{D}_\lambda(h)$.

Since $\widehat{B}_t(z,h)$ is holomorphic inside Υ_j, it results from (5.36), (5.42), Proposition 5.7 and the fact that $\operatorname{diam}(\Upsilon_j) = \mathcal{O}(h)$,

$$\|\pi(h) - \chi_1^w(k, hD_k)\pi_0\| \le C(\epsilon + h).$$

Moreover, Lemma B.3 shows that $\chi_2^w(k, hD_k)\pi_0(h) = \mathcal{O}(h)$. We recall that $\chi_1 + \chi_2 = 1$ and $\chi_2 = 0$ near 0. Consequently, for h small enough

$$\|\pi(h) - \pi_0(h)\| \le \widetilde{C}\epsilon.$$

Next we choose ϵ so small that $\widetilde{C}\epsilon < 1$. Since $\pi(h)$ and $\pi_0(h)$ are projectors, it follows from Lemma 1.23 of [23, p. 438] that,

$$\dim\Big(rg\big(\pi_0(h)\big)\Big) = \dim\Big(rg\big(\pi(h)\big)\Big).$$

This gives (5.38).

Let us prove (5.39). By construction, $\widetilde{B}(h)$ is elliptic except for $(k,r) = (0,0)$ and is analytic in a complex neighborhood of $\mathbf{R}^{2n}$. Hence, we can apply Theorem 5.4 and Remark 5.5 to $\widetilde{B}_t(h)$ and deduce that, the eigenfunctions corresponding to an eigenvalue $\lambda \in D$ of $\widetilde{B}_t(h)$ are microlocally concentrated near $(k,r) = (0,0)$. In particular, if $\varphi_1, \ldots, \varphi_{\widetilde{N}}$ is an orthonormal basis of $\operatorname{rang}\big(\pi_0(h)\big)$ and if $\chi \in S^0(\mathbf{T}^* \times \mathbf{R}^n)$ with $\chi(k,r) = \big((k,r)^\alpha\big)$, then

$$\|\chi^w(k, hD_k)\varphi_i\| = \mathcal{O}(h^{|\alpha|/2}), \quad i = 1, \ldots, \widetilde{N}.$$

Combining this with (5.35), we get

$$\big(B_t(h) - \widetilde{B}_t(h)\big)\varphi_i = \mathcal{O}(h^3). \tag{5.43}$$

Consequently,

$$\big(B_t(h) - \widetilde{e}_j(h)\big)\varphi_i = \big(B_t(h) - \widetilde{B}_t(h)\big)\varphi_i = \mathcal{O}(h^3). \tag{5.44}$$

Put $\psi_i = \pi(h)\varphi_i$. (5.43) and (5.44) yield

$$\begin{aligned}\psi_i - \varphi_i &= \big(\pi(h) - \pi_0(h)\big)\varphi_i\\ &= \frac{1}{2\pi i}\int_{\Upsilon_j}\big(z - B_t(h)\big)^{-1}\big(B_t(h) - \widetilde{B}_t(h)\big)\big(z - \widetilde{B}_t(h)\big)^{-1}\varphi_i\,dz\\ &= \mathcal{O}(h^2).\end{aligned} \tag{5.45}$$

This shows that $\psi_1, \dots, \psi_{\widetilde{N}}$ is a basis of rang$\big(\pi(h)\big)$ when h is small enough. Moreover, (5.44) gives

$$\big(B_t(h) - \widetilde{e}_j(h)\big)\,\psi_j = \pi(h)\big(B_t(h) - \widetilde{e}_j(h)\big)\,\varphi_j = \mathcal{O}(h^3), \tag{5.46}$$

which means that the matrix M of $\pi(h)B_t(h)\pi(h)$ in the basis $(\psi_1, \dots, \psi_{\widetilde{N}})$ is of the form:

$$M = \widetilde{e}_j(h)I + \mathcal{O}(h^3). \tag{5.47}$$

Now, it is clear that (5.39) follows from (5.47) and (5.29).

5.3 End of the Proof of Theorem 1.4

Since $\widetilde{E}_{-+}(z,t,h) = z - B_t(h) + \mathcal{O}(h^2)$, it follows from Proposition 5.7 that $\widetilde{E}_{-+}(z,t,h)$ is invertible for $z \in \widehat{D}_\lambda(h)$. Then it suffices to study the invertibility of $\widetilde{E}_{-+}(z,t,h)$ for z in a fixed $\widehat{\Upsilon}_j$.

Let $\widetilde{B}_t^*(h)$ (resp. $B_t^*(h)$) be the adjoint of $\widetilde{B}_t(h)$ (resp. $B_t(h)$). Let $(\varphi_1^*, \dots, \varphi_N^*)$ be a basis of $\big(\widetilde{B}_t(h) - \widetilde{e}_j(h)\big)^*$ satisfying

$$(\varphi_i, \varphi_j^*) = \delta_{ij}. \tag{5.48}$$

As in the proof of (5.45), we can construct a basis $(\psi_1^*, \dots, \psi_N^*)$ of $\ker\big(B_t(h) - E_j(h)\big)^*$ such that

$$\psi_i^* = \varphi_i^* + \mathcal{O}(h^2),$$

which together with (5.45) and (5.48) imply

$$(\psi_i, \psi_j^*) = \delta_{ij} + \mathcal{O}(h^2). \tag{5.49}$$

Let Π be the spectral projector defined by

$$\Pi u = \sum_{i=1}^{N} (u, \psi_i^*)\psi_i.$$

Set

$$\widehat{\Pi} = 1 - \Pi, \quad \text{and} \quad \widehat{B}_t(h) = \widehat{\Pi} B_t(h) \widehat{\Pi}.$$

Then, the reduced resolvent $\widehat{R}(z,t,h) = \big(z - \widehat{B}_t(h)\big)^{-1}$ of $\widehat{B}_t(h)$ is well defined on the range of $\widehat{\Pi}$. Moreover, the arguments used in the proof of Proposition 5.7 show that

$$\|\widehat{R}(z,t,h)\widehat{\Pi}\| \le Ch^{-1}, \tag{5.50}$$

uniformly on z in $\widehat{\Upsilon}_j$.

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Consider the Grushin problem for $\widetilde{E}_{-+}(z,t,h)$,

$$\mathcal{G}(z,t,h) := \begin{pmatrix} \widetilde{E}_{-+}(z,t,h) & r_- \\ r_+ & 0 \end{pmatrix} : L^2(\mathbf{T}^*) \oplus \mathbf{C}^N \to L^2(\mathbf{T}^*) \oplus \mathbf{C}^N, \tag{5.51}$$

where $r_- : \mathbf{C}^N \to L^2(\mathbf{T}^*)$ and $r_+ : L^2(\mathbf{T}^*) \to \mathbf{C}^N$ are defined by

$$r_-(\alpha_1, \dots, \alpha_N) = \sum_{i=1}^{N} \alpha_i \psi_i, \quad \text{and} \quad r_+ u = \big((u, \psi_1^*), \dots, (u, \psi_N^*)\big).$$

In view of (5.48) and (5.49), one has

$$r_+ r_- = I + \mathcal{O}(h^2), \quad r_- r_+ = \Pi. \tag{5.52}$$

Set

$$\mathcal{R}(z,t,h) := \begin{pmatrix} \widehat{R}(z,t,h)\widehat{\Pi} & r_- \\ r_+ & \big(E_j(h) - z\big) I_{\mathbf{C}^N} \end{pmatrix}.$$

Using (5.50), (5.52) and the fact that $\widetilde{E}_{-+}(z,t,h) = z - B_t(h) + \mathcal{O}(h^2)$, we check easily

$$\mathcal{G}(z,t,h)\mathcal{R}(z,t,h) = I + \begin{pmatrix} \mathcal{O}(h) & \mathcal{O}(h^2) \\ \mathcal{O}(h) & \mathcal{O}(h^2) \end{pmatrix}, \tag{5.53}$$

and

$$\mathcal{R}(z,t,h)\mathcal{G}(z,t,h) = I + \begin{pmatrix} \mathcal{O}(h) & \mathcal{O}(h) \\ \mathcal{O}(h^2) & \mathcal{O}(h^2) \end{pmatrix}. \tag{5.54}$$

Hence, $\mathcal{G}(z,t,h)$ is invertible with inverse

$$\mathcal{F}(z,t,h) := \begin{pmatrix} F(z,t,h) & F_+(z,t,h) \\ F_-(z,t,h) & F_{-+}(z,t,h) \end{pmatrix}.$$

On the other hand, the right hand side of (5.53) and (5.54) show that

$$F_{-+}(z,t,h) = \big(E_j(h) - z\big) I_{\mathbf{C}^N} + \mathcal{O}(h^2).$$

Therefore,

$$\det\big(F_{-+}(z,t,h)\big) = \big(E_j(h) - z\big)^N + \sum_{k=1}^{N} a_k(z,t,h)\big(E_j(h) - z\big)^{N-k}, \tag{5.55}$$

where $a_i(z,t,h)$ depends holomorphically on z in $\widehat{\Upsilon}_j$ and $a_k(z,t,h) = \mathcal{O}(h^{2k})$. Now, by a Rouché's theorem, we deduce from (5.55) that $\det\big(F_{-+}(z,t,h)\big) = 0$ has N roots $z_k(h)_{1\le k\le N}$ in $\widehat{\Upsilon}_j$ with

$$z_k(h) = E_j(h) + \mathcal{O}(h^2) = \lambda + he_j + \mathcal{O}(h^{3/2}).$$

We recall that $E_j(h) = \lambda + he_j + \mathcal{O}(h^{3/2})$.

Summing up, we have proved:

Lemma 5.9 *There exists a matrix $F_{-+}(z,t,h)\colon \mathbf{C}^N \to \mathbf{C}^N$, depending holomorphically on $z \in \widehat{\Upsilon}_j$, such that the following properties hold.*

i) *$z = z(h) \in \widehat{\Upsilon}_j$ is a root of multiplicity m of* $\det\big(F_{-+}(z,t,h)\big) = 0$*, if and only if* 0 *is an eigenvalue of multiplicity m of $\widetilde{E}_{-+}(z,t,h)$.*

ii) $\det\big(F_{-+}(z,t,h)\big) = 0$ *has N roots in $\widehat{\Upsilon}_j$, $z_1(h),\dots,z_N(h)$ (counted with their multiplicities) and*

$$z_i(h) = \lambda + he_j + \mathcal{O}(h^{3/2}). \tag{5.56}$$

It follows from Theorem 3.8 and Lemma 5.9 that the resonances of $P(h)$ in $D(\lambda, C_0h)$ are all given by (5.56) with $e_j < C_0$. It remains to prove that $z_i(h)$ has an asymptotic expansion in powers of $h^{\frac{1}{2}}$, *i.e.*,

$$z_i(h) \sim \lambda + he_j + \sum_{l\geq 1} \alpha_{i,l} h^{1+\frac{l}{2}}, \quad (\alpha_{i,l} \in \mathbf{R}),\ (h \searrow 0). \tag{5.57}$$

The most essential step in the proof of (5.57) is to construct asymptotic solutions of the equation

$$\widetilde{E}_{-+}(z,t,h)u = 0.$$

A similar problem was studied by [24], [29], [38] and that is why we omit the details.

Using Proposition 5.1 and Remark 3.9 we deduce as in the proof of (5.11) that, for all $m \in \mathbf{N}$, there exists $B^m(h,z) = z - B(h) + h^2 b_2(z) + \cdots + h^{m+2} b_{m+2}(z)$ where $b_i(z) \in \mathrm{Op}_h\big(S^0(\mathbf{R}^{2n})\big)$ such that

$$B_t^m(h,z) := \widehat{\mathcal{U}}_t \widehat{B}^m(h,z) \widehat{\mathcal{U}}_t^{-1} = \widetilde{E}_{-+}(z,t,h) + \mathcal{O}(h^{m+2}). \tag{5.58}$$

Here $B(h)$ is given by (5.10). We recall that $\widehat{B}^m(h,z) = \mathcal{F} B^m(h,z) \mathcal{F}^{-1}$.

Fix $t = is$ with $0 < s$ small enough. The arguments used in Section 5.2 show that $B_t^m(h,z)$ is elliptic uniformly on $z \in D(\lambda, C_0h)$ except for $(k,r) = (0,0)$. In particular, the normalized solutions of the equation $B_t^m(h,z)u = 0$ are microlocally concentrated near $(0,0)$.

Since $\varphi(r) + \lambda(k)$ has a non-degenerate minimum near $(0,0)$, the construction of Helffer-Sjöstrand in [20] (see also [24], [29], [38]) gives N asymptotic solutions in the form

$$u_i(k,h) = e^{-S(k)/h} \sum_{l=0}^{2m+2} c_{i,l}(k) h^{l/2 - m_j}, \quad (m_j \in \mathbf{R}),$$

associated with

$$\widetilde{z}_i(h) = \lambda + e_j h + \sum_{l=2}^{2m} \alpha_{i,l} h^{1+\frac{l}{2}},$$

such that near $(r,k) = (0,0)$

$$\widehat{B}^m\big(h, \widetilde{z}_i(h)\big)\, u_i = \mathcal{O}(h^{m+1}).$$

Here $S(k)$ satisfies

$$\varphi(-\nabla_k S) + \lambda(k) = 0,$$

with $S(0) = 0$, $\nabla S(0) = 0$ and $\Re S > 0$ for $k \neq 0$.

Let $\chi \in C_0^\infty(\mathbf{T}^*)$ be supported in a small neighborhood of zero and equals one near 0. Set $v_i^t(k,h) := \chi(k)\widehat{\mathcal{U}}_t u_i(k,h)$. One has

$$\widehat{B}_t^m\big(h, \widetilde{z}_i(h)\big)\, v_i^t(k,h) = \mathcal{O}(h^{m+1}),$$

which together with (5.58) give

$$\widetilde{E}_{-+}\big(\widetilde{z}_i(h), t, h\big)\, v_i^t(k,h) = \mathcal{O}(h^{m+1}).$$

Now to finish the proof of (5.57), we just need to study a Grushin problem for $\widetilde{E}_{-+}(z,t,h)$, similar to (5.51), but with $\psi_i = v_i^t(k,h)$.

Remark 5.10 The purpose of this remark is to explain why our methods work under assumption (H4).

Let us denote by $E_{-+}^0(r,k\,;z,t)$ the lower right corner of the matrix $\mathcal{E}^0(r,k\,;z,t)$ given in Lemma 3.3. Under assumption (H4), we can construct $E_{-+}^0(r,k\,;z,t)$ scalar valued with

$$E_{-+}^0(r,k\,;z,t) = \Big(z - \lambda\big(v_t(k)\big) - \varphi_{t,0}(r,k)\Big)\, g(r,k\,;z,t),$$

near $\Sigma := (E_{-+}^0)^{-1}\{0\}$ and $|g| \geq C > 0$. See [18, Theorem 3.5]. Thus, Lemma 5.2 remains true. In particular, to study the spectrum of $\widetilde{E}_{-+}(z,t,h)$ near 0, one only needs a microlocal version of $\widetilde{E}_{-+}(z,t,h)$ near $(0,0)$.

A Operator-Valued Symbols

We recall some basic results about operator-valued symbols. Our main reference is [15]. We shall consider a family of Hilbert spaces $\mathcal{A}_X$, $X = (x,\xi) \in \mathbf{R}^{2n}$ satisfying:

(A.1) $$\mathcal{A}_X = \mathcal{A}_Y \quad \text{as vector spaces for all } X, Y \in \mathbf{R}^{2n},$$

(A.2)
$$\exists\, N_0 > 0, C > 0, \text{ tq } \|u\|_{\mathcal{A}_X} \leq C\langle X - Y\rangle^{N_0}\|u\|_{\mathcal{A}_Y} \quad \text{for all } u \in \mathcal{A}_0, X, Y \in \mathbf{R}^{2n}.$$

Let $\mathcal{B}_X$ and $\mathcal{C}_X$ satisfy (A.1) and (A.2). We say that $p \in C^\infty\big(\mathbf{R}^{2n}\,;\mathcal{L}(\mathcal{A}_0, \mathcal{B}_0)\big)$ belongs to $S^0\big(\mathbf{R}^{2n}\,;\mathcal{L}(\mathcal{A}_X, \mathcal{B}_X)\big)$ if for every $\alpha \in \mathbf{N}^{2n}$ there is a constant C_α such that

(A.3) $$\|\partial_X^\alpha p\|_{\mathcal{L}(\mathcal{A}_X;\mathcal{B}_X)} \leq C_\alpha, \quad \text{for all } X \in \mathbf{R}^{2n}.$$

We can then associate with p the operator $p^w(x, hD_x)$. As in the scalar case, one has:

Proposition A.1 **[15]** *Let* $p \in S^0\big(\mathbf{R}^{2n}\,;\mathcal{L}(\mathcal{A}_X, \mathcal{B}_X)\big)$. *Then* $p^w = p^w(x,hD_x)$ *is uniformly continuous* $\mathcal{S}(\mathbf{R}^n\,;\mathcal{A}_0) \to \mathcal{S}(\mathbf{R}^n\,;\mathcal{B}_0)$.

Theorem A.2 **[15]** *Let* $p \in S^0\big(\mathbf{R}^{2n}\,;\mathcal{L}(\mathcal{B}_X,\mathcal{C}_X)\big)$, $q \in S^0\big(\mathbf{R}^{2n}\,;\mathcal{L}(\mathcal{A}_X,\mathcal{B}_X)\big)$. *Then* $p^w \circ q^w = r^w$, *where* $r \in S^0\big(\mathbf{R}^{2n}\,;\mathcal{L}(\mathcal{A}_X,\mathcal{C}_X)\big)$ *is given by*

$$r = \exp\left(\frac{ih}{2}\sigma(D_x, D_\xi\,; D_y, D_\eta)\right)\big(p(x,\xi)q(y,\eta)\big)_{|_{x=y,\xi=\eta}},$$

where σ *is the usual symplectic 2 form. We have the asymptotic formula:*

$$r \sim \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{ih}{2}\sigma(D_x, D_\xi\,; D_y, D_\eta)\right)^k p(x,\xi)q(y,\eta)_{|_{x=y,\xi=\eta}}.$$

Theorem A.3 **[15]** *Assume* $\mathcal{A}_X = \mathcal{A}_0$, $\mathcal{B}_X = \mathcal{B}_0$, $\forall X \in \mathbf{R}^{2n}$. *If* $p \in S^0\big(\mathbf{R}^{2n}\,;\mathcal{L}(\mathcal{A}_0,\mathcal{B}_0)\big)$ *then* $\mathrm{Op}_h^w(p)$ *is uniformly bounded*

$$L^2(\mathbf{R}^n\,;\mathcal{A}_0) \to L^2(\mathbf{R}^n\,;\mathcal{B}_0).$$

B Some Resolvent Estimates

Lemma B.1 *Fix* $m \in \{0,1,2\}$. *Let* $p, \chi \in S^0(\mathbf{R}^{2n})$ *be real valued satisfying:*

i) $p(x,\xi) \geq 0$, $\liminf p(x,\xi)_{(x,\xi)\to\infty} > 0$ *and* $p^{-1}(0) = \{(0,0)\}$.
ii) p *has a non-degenerate minimum at* $(0,0)$.
iii) $\chi(x,\xi) = \mathcal{O}(x^\alpha \xi^\beta)$ *for all* $|\alpha| + |\beta| \leq m$.

Then, there are $c, C, h_0 > 0$ *such that,* $\big(p^w(x,hD_x) + Ch\big)$ *is invertible and*

$$\big\|\chi^w(x,hD_x)\big(p^w(x,hD_x)+Ch\big)^{-1}\big\| \leq ch^{\frac{m}{2}-1}, \tag{B.1}$$

uniformly on $h \in\,]0,h_0[$.

Proof For $\lambda > 0$, set

$$Q_\lambda(x,\xi) = \big(p(x,\xi)+\lambda\big)^{-1}.$$

The following properties were shown in [12, proof of Theorem 7.12],

1) $\forall \alpha, \beta, \exists C_{\alpha,\beta}$ (independent of λ) such that

$$|\partial_x^\alpha \partial_\xi^\beta Q_\lambda| \leq C_{\alpha,\beta} Q_\lambda(x,\xi)\lambda^{-\frac{|\alpha|+|\beta|}{2}}. \tag{B.2}$$

2) There are $C, h_0 > 0$ such that $\big(p^w(x,hD_x) + Ch\big)$ is invertible and

$$\big(p^w(x,hD_x)+Ch\big)^{-1} = Q^w_{Ch}(x,hD_x)\circ A, \tag{B.3}$$

where $\|A\| = \mathcal{O}(1)$, uniformly on $h \in\,]0,h_0[$.

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On the other hand, assumptions ii) and iii) imply that

$$\left|\chi(x,\xi)\big(p(x,\xi)+\lambda\big)^{-1}\right| \le C_0\lambda^{\frac{m}{2}-1}.$$

Combining this with (B.2), we get

$$|\partial_x^\alpha\partial_\xi^\beta(\chi Q_\lambda)| \le C_{\alpha,\beta}\lambda^{\frac{m}{2}-1}\lambda^{-\frac{|\alpha|+|\beta|}{2}}.$$

This shows that $\chi Q_{Ch} \in S_{\frac{1}{2}}^{\frac{m}{2}-1}(\mathbf{R}^{2n})$. Now, Lemma B.1 follows from Proposition 7.7 of [12].

Remark B.2 Note that Lemma B.1 remains true if we replace $\lambda = Ch$ by $\lambda = z$ with $|z - Ch| \le C_1 h$ and $\operatorname{dist}\big(z,\sigma(P)\big) \ge C_2 h$ for some $C_1, C_2 > 0$. This follows easily from Lemma B.1 and the first resolvent equation

$$(P-z)^{-1} = (P+Ch)^{-1} + (P+Ch)^{-1}(z+Ch)(P-z)^{-1}.$$

We recall that $\operatorname{dist}\big(z,\sigma(P)\big) \ge C_2 h$ and $P = P^*$ imply $\|(z-P)^{-1}\| \le (C_2 h)^{-1}$.

Lemma B.3 *Let $\chi \in S^0(\mathbf{T}^* \times \mathbf{R}^n)$ satisfy $\chi(k,r) = \mathcal{O}(k^\alpha r^\beta)$ for all $|\alpha| + |\beta| \le m$ $(m \in \{0,1,2\})$. There exists $C > 0$ (independent of h and $z \in D_0$), such that*

$$\left\|\chi^w(k,hD_k)\big(z-\widetilde{B}_t(h)\big)^{-1}\right\| \le Ch^{\frac{m}{2}-1} \tag{B.4}$$

$$\left\|\big(z-\widetilde{B}_t(h)\big)^{-1}\chi^w(k,hD_k)\right\| \le Ch^{\frac{m}{2}-1}. \tag{B.5}$$

In particular for $m = 0$, we get:

$$\left\|\big(z-\widetilde{B}_t(h)\big)^{-1}\right\| \le Ch^{-1}, \quad \forall z \in D_0.$$

Proof We only prove (B.4). The proof of (B.5) is similar. By construction, $\widetilde{B}(h) = \mathcal{O}\big((r,k)^\alpha\big) + \mathcal{O}(h)$ for all $|\alpha| \le N+1$. Remembering the expression of $\widetilde{B}_t(h)$ and using Proposition 2.6, we get

$$\widetilde{B}_t(h) - \widetilde{B}(h) = t\big(r_0(x,\xi\,;t) + hr_1(x,\xi,t)\big) + h^2 R(x,\xi\,;t,h),$$

where $r_0, r_1, R \in S^0(\mathbf{R}^{2n})$ and $r_0(x,\xi\,;t) = \mathcal{O}(x^\alpha\xi^\beta)$, for all $|\alpha| + |\beta| \le N+1$.

Applying Lemma B.1 and Remark B.2 to $p = \widetilde{B}(h)$ and $\chi(x,\xi) = r_0(x,\xi\,;t)$, we obtain

$$\left\|\big(\widetilde{B}_t(h) - \widetilde{B}(h)\big)\big(z-\widetilde{B}(h)\big)^{-1}\right\| = \mathcal{O}(t).$$

Consequently, for t_0 small enough, $\Big(I - \big(\widetilde{B}_t(h) - \widetilde{B}(h)\big)\big(z-\widetilde{B}(h)\big)^{-1}\Big)$ is invertible and

$$\left\|\Big(I - \big(\widetilde{B}_t(h) - \widetilde{B}(h)\big)\big(z-\widetilde{B}(h)\big)^{-1}\Big)^{-1}\right\| = \mathcal{O}(1).$$

Now, (B.4) follows from Lemma B.1 and the following formula

$$\left(z-\widetilde{B}_t(h)\right)^{-1}=\left(z-\widetilde{B}(h)\right)^{-1}\left(I-\left(\widetilde{B}_t(h)-\widetilde{B}(h)\right)\left(z-\widetilde{B}(h)\right)^{-1}\right)^{-1}.$$

Lemma B.4 *Let $\chi_i\in S^0(\mathbf{T}^*\times\mathbf{R}^n)$, $i=1,2$. We assume that* $\operatorname{supp}(\chi_1)$ *is a compact set, $\chi_1=1$ near $(0,0)$ and $d(\operatorname{supp}\chi_1,\operatorname{supp}\chi_2)>\epsilon>0$. Then, for all $M\in\mathbf{N}$,*

$$\text{(B.6)}\qquad \left\|\chi_2^w(k,hD_k)\left(z-\widetilde{B}_t(h)\right)^{-1}\chi_1^w(k,hD_k)\right\|=\mathcal{O}_M(h^M).$$

Proof For any M we can find $\psi_1,\dots,\psi_M\in C_0^\infty(\mathbf{R}^{2n})$, constant in a neighborhood of supp χ_1 and such that

$$\chi_1\psi_1=\chi_1,\quad \psi_{i-1}\psi_i=\psi_{i-1},\quad 1\le i\le M,\ \psi_M\chi_2=0.$$

Then

(B.7)

$$\begin{aligned}
&\chi_2\left(\widetilde{B}_t(h)-z\right)^{-1}\chi_1\\
&\quad=\chi_2\left(\widetilde{B}_t(h)-z\right)^{-1}\psi_1\cdots\psi_M\chi_1\\
&\quad=\chi_2\left(\widetilde{B}_t(h)-z\right)^{-1}[\psi_1,\widetilde{B}_t(h)]\left(\widetilde{B}_t(h)-z\right)^{-1}[\psi_2,\widetilde{B}_t(h)]\left(\widetilde{B}_t(h)-z\right)^{-1}\\
&\qquad\cdots\left(\widetilde{B}_t(h)-z\right)^{-1}[\psi_M,\widetilde{B}_t(h)]\left(\widetilde{B}_t(h)-z\right)^{-1}\chi_1.
\end{aligned}$$

Symbolic calculus shows that $[\psi_k,\widetilde{B}_t(h)]=h\mathcal{O}\left((x,\xi)^2\right)$. Combining this with Lemma B.3, we get

$$\left\|\left(\widetilde{B}_t(h)-z\right)^{-1}[\psi_k,\widetilde{B}_t(h)]\right\|=\mathcal{O}(h),$$

which together with (B.7) give (B.6).

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