



RESEARCH ARTICLE

A McKay Correspondence in Positive Characteristic

Christian Liedtke 

TU München, Department of Mathematics, Boltzmannstr. 3, D-85748 Garching bei München, Germany;
E-mail: christian.liedtke@tum.de

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Abstract

We establish a McKay correspondence for finite and linearly reductive subgroup schemes of \mathbf{SL}_2 in positive characteristic. As an application, we obtain a McKay correspondence for *all* rational double point singularities in characteristic $p \geq 7$. We discuss linearly reductive quotient singularities and canonical lifts over the ring of Witt vectors. In dimension 2, we establish simultaneous resolutions of singularities of these canonical lifts via G -Hilbert schemes. In the appendix, we discuss several approaches towards the notion of *conjugacy classes* for finite group schemes: This is an ingredient in McKay correspondences, but also of independent interest.

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1. Introduction

1.1. Klein's classification and McKay's correspondence

Felix Klein [K184] classified finite subgroups G of $\mathbf{SL}_2(\mathbb{C})$: up to conjugation, there are two infinite series and three isolated cases.

1. The associated quotient singularity \mathbb{C}^2/G is called a *Klein singularity* and the singularities arising this way are precisely the *rational double point singularities*. Its minimal resolution of singularities is a union of \mathbb{P}^1 's, whose dual intersection graph Γ is a simply-laced Dynkin diagram of finite type, that is, of type A_n, D_n, E_6, E_7 or E_8 .
2. John McKay [Mc80] associated a finite graph $\widehat{\Gamma}$ to $G \subset \mathbf{SL}_2(\mathbb{C})$, whose vertices correspond to the isomorphism classes of the simple representations of G . This graph is a Dynkin diagram of affine type $\widehat{A}_n, \widehat{D}_n, \widehat{E}_6, \widehat{E}_7$ or \widehat{E}_8 .
3. After these preparations, the *classical McKay correspondence* consists of the following observations:
 - (a) The graph Γ is obtained from $\widehat{\Gamma}$ by removing the vertex corresponding to the trivial representation.
 - (b) There exists a bijection between conjugacy classes of G , vertices of $\widehat{\Gamma}$ and isomorphism classes of simple representations of G .
 - (c) There exists a bijection between finite subgroups of $\mathbf{SL}_2(\mathbb{C})$ up to conjugacy, the above Dynkin diagrams of affine type, Klein singularities and the above Dynkin diagrams of finite type.

By now, there are various approaches to and a vast literature on this subject, such as [Kn85, Ko85, Mc80, Mc81, St85] and many more. Also, there are now generalisations into very different directions:

higher dimensional algebraic geometry [Re02], K-theory [GV83], derived categories of coherent sheaves [BKR01, KV00], representations of quivers [Ki16], non-commutative geometry and Hopf algebras [CKWZ16] and string theory [Aetal09] - just to mention a few.

1.2. Positive characteristic

Now, let k be an algebraically closed field of characteristic $p > 0$.

1.2.1. Wild McKay correspondence

The classical McKay correspondence as sketched above is still partially available over k if G is assumed to be a finite subgroup of $\mathbf{SL}_2(k)$ of order prime to p - this is the *tame* case. If p divides the order of G - this is the *modular* or *wild* case - then this correspondence breaks down. We refer to Yasuda's surveys [Ya22, Ya23] about conjectures and partial results concerning such wild McKay correspondences.

1.2.2. Linearly reductive McKay correspondence

In this article, we show that if G is a finite and *linearly reductive* subgroup scheme of $\mathbf{SL}_{2,k}$, then there is a reasonable version of the classical McKay correspondence. For example, we obtain a McKay correspondence for *all* rational double point singularities if $p \geq 7$. Instead of considering groups, we allow non-reduced group schemes over k , but we require their categories of k -linear and finite-dimensional representations to be semi-simple. We refer to [TY20] about conjectures and partial results concerning a McKay correspondence for the group scheme α_p , which is not linearly reductive.

Thus, let G be a finite and linearly reductive subgroup scheme of $\mathbf{SL}_{2,k}$ with $p \geq 7$. (In this introduction, we will exclude small characteristics whenever this makes our discussion easier.) Let $x \in X := U/G$ with $U = \mathbb{A}_k^2$ or $U = \widehat{\mathbb{A}}_k^2$ be the associated Klein singularity, which is a rational double point. One goal of this article is to define a notion of conjugacy class for G , to construct graphs Γ and $\widehat{\Gamma}$ and to establish bijections as above.

Let us make three comments:

1. It is interesting in its own that a McKay correspondence can be extended from finite group schemes of length prime to p , that is, the tame case, to linearly reductive group schemes.
2. What makes this linearly reductive McKay correspondence really interesting is that the bijection in Theorem 1.1 is *not* true when considering finite groups of order prime to p only, see Example 1.2.
3. Probably, many more aspects of the classical McKay correspondence can be carried over to the linearly reductive setting, but rather than writing a whole monograph, we decided to establish only some basic bijections.

1.3. Linearly reductive group schemes

Let G be a *finite and linearly reductive group scheme* over an algebraically closed field k of characteristic $p \geq 0$. By definition, this means that the category $\text{Rep}_k(G)$ of k -linear and finite-dimensional representations of G is semi-simple.

If $p = 0$, then every finite group scheme over k is étale and linearly reductive. and in fact, it is the constant group scheme associated to a finite group. In fact, the functor $G \mapsto G_{\text{abs}} := G(k)$ induces an equivalence of categories between finite group schemes over k and finite groups.

If $p > 0$, then every linearly reductive group scheme admits a canonical semi-direct product decomposition

$$G \cong G^\circ \rtimes G^{\text{ét}},$$

where $G^{\text{ét}}$ is a group scheme of length prime to p (and thus, the constant group scheme associated to a finite group of order prime to p) and where G° is infinitesimal and diagonalisable. The latter implies

that G° is a product of group schemes of the form μ_{p^n} . Conversely, every such semi-direct product of a diagonalisable group scheme with the constant group scheme associated to a finite group of order prime to p is linearly reductive. This structure result is usually attributed to Nagata [Na61], but see also [AOV08], [Ch92] and [Ha15]. Linearly reductive group schemes over k strictly contains the class of constant group schemes associated to finite groups of order prime to p .

1.3.1. Abstract groups

Associated to G , there is an abstract finite group G_{abs} and we refer for the slightly technical definition to Section 2.2. The order of G_{abs} is equal to the length of G . For example, if G is étale over k , then we have $G_{\text{abs}} \cong G(k)$. The assignment $G \mapsto G_{\text{abs}}$ establishes an equivalence of categories

$$\left\{ \begin{array}{l} \text{finite linearly reductive} \\ \text{group schemes over } k \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{finite groups with a} \\ \text{unique } p\text{-Sylow subgroup} \end{array} \right\}, \tag{1.1}$$

see [LMM21] and Lemma 2.2.

1.3.2. Canonical lifts

In [LMM21], we showed that if G is a finite and linearly reductive group scheme, then there exists a lift of G over the ring of Witt vectors $W(k)$. We note that G° and $G^{\text{ét}}$ even lift uniquely to $W(k)$ and we define the *canonical lift* $G_{\text{can}} \rightarrow \text{Spec } K$ of G to be the unique lift that is a semi-direct product of the lifts of G° and $G^{\text{ét}}$. Any lift of G to some extension field of K becomes isomorphic to G_{can} after possibly passing to some further field extension. Moreover, the finite group $G_{\text{can}}(\bar{K})$ is isomorphic to G_{abs} .

1.3.3. Representation theory

By [LMM21] and Proposition 2.6, there exist canonical equivalences of representation categories

$$\text{Rep}_k(G) \rightarrow \text{Rep}_{\bar{K}}(G_{\text{can},\bar{K}}) \rightarrow \text{Rep}_{\mathbb{C}}(G_{\text{abs}}) \tag{1.2}$$

that are compatible with degrees, direct sums, tensor products, duals and simplicity. These equivalences induce isomorphisms of rings

$$K_k(G) \rightarrow K_{\bar{K}}(G_{\text{can},\bar{K}}) \rightarrow K_{\mathbb{C}}(G_{\text{abs}}),$$

see Corollary 2.8. Here, $K_F(G)$ denotes the K-group associated to F -linear and finite-dimensional G -representations.

1.3.4. Hopf algebras

If G is a finite group scheme over k , then the multiplication map turns $H^0(G, \mathcal{O}_G)$ into a finite-dimensional Hopf algebra over k . We discuss finite group schemes and among them the linearly reductive ones from the point of Hopf algebras in Appendix A.

1.4. Linearly reductive subgroup schemes of $\text{SL}_{2,k}$

Let k be an algebraically closed field of characteristic $p \geq 0$. Hashimoto [Ha15] extended Klein’s classification [Kl84] of finite subgroups of $\text{SL}_2(\mathbb{C})$ up to conjugation to the setting of finite and linearly reductive subgroup schemes of $\text{SL}_{2,k}$. If $p \geq 7$, then one obtains a list analogous to Klein’s classical list. If $p \in \{2, 3, 5\}$, then some classical cases are missing, but there are no new cases.

1.5. McKay graph and McKay correspondence

Let G be a finite and linearly reductive subgroup scheme of $\text{SL}_{2,k}$. As in McKay’s original construction [Mc80], we associate an affine Dynkin diagram $\hat{\Gamma}$ to G , its embedding into $\text{SL}_{2,k}$ and the set of

isomorphism classes of simple representations of G . This is the *McKay graph* associated to this data. In fact, we will see that it is compatible with the equivalences induced by (1.1) and (1.2) and we refer to Section 3.1 for details. We establish the following version of McKay’s theorem [Mc80] in positive characteristic.

Theorem 1.1 (Theorem 3.6). *Let k be an algebraically closed field of characteristic $p \geq 0$. There exists a bijection between non-trivial, finite and linearly reductive subgroup schemes of $\mathbf{SL}_{2,k}$ up to conjugation and affine Dynkin graphs of type*

$$\begin{aligned} \widehat{A}_n, \widehat{D}_n, \widehat{E}_6, \widehat{E}_7, \widehat{E}_8 & \text{ if } p = 0 \text{ or } p \geq 7, \\ \widehat{A}_n, \widehat{D}_n, \widehat{E}_6, \widehat{E}_7 & \text{ if } p = 5, \\ \widehat{A}_n, \widehat{D}_n & \text{ if } p = 3, \\ \widehat{A}_n & \text{ if } p = 2. \end{aligned}$$

By construction, this bijection is compatible with the classical McKay correspondence via the lifting results and the equivalences (1.1), (1.2).

Example 1.2. The linearly reductive group scheme corresponding to \widehat{A}_n is μ_{n+1} . This group scheme is reduced, that is, étale, if and only if p does not divide $n + 1$. In particular, it is crucial to allow non-reduced group schemes in order to obtain a bijection as in characteristic zero.

1.6. Linearly reductive quotient singularities

Consider $\mathbf{GL}_{2,k}$ with its usual linear action on $U = \mathbb{A}_k^2$ or $U = \widehat{\mathbb{A}}_k^2$. If G is a finite, linearly reductive and *very small* (see Definition 4.1) subgroup scheme of $\mathbf{GL}_{2,k}$, then the associated quotient singularity $x \in X := U/G$ is a two-dimensional *linearly reductive quotient singularity* in the sense of [LMM21]. By loc.cit., such a singularity determines G together with its embedding $G \rightarrow \mathbf{GL}_{2,k}$ up to isomorphism and conjugation, respectively.

In Section 4.3, we will see that a minimal resolution of singularities of a two-dimensional linearly reductive quotient singularity $x \in X = U/G$ is provided by the G -Hilbert scheme

$$\pi : G\text{-Hilb}(U) \rightarrow U/G, \tag{1.3}$$

which generalises work of Ishii, Ito and Nakamura [IN19, IN99].

Remark 1.3. In dimension two, a linearly reductive quotient singularity is the same as an F-regular singularity [LMM21]. Thus, every two-dimensional F-regular singularity can be resolved by a suitable G -Hilbert scheme, see also Remark 4.6.

If moreover G is a subgroup scheme of $\mathbf{SL}_{2,k}$, then $x \in X = U/G$ is called a *Klein singularity*. Klein singularities are rational double point singularities. If $p = 0$ or $p \geq 7$, then conversely every rational double point is a Klein singularity by Hashimoto [Ha15] and, independently, by [LS14]. If $p \in \{2, 3, 5\}$, then not every rational double point is a Klein singularity.

1.7. Canonical lifts and simultaneous resolutions

Let G be a very small, finite and linearly reductive subgroup scheme of $\mathbf{GL}_{2,k}$ and let $x \in X = U/G$ be the associated linearly reductive quotient singularity. Assume $p > 0$ and let $W(k)$ be the ring of Witt vectors of k . In Section 4.4, we will establish the existence of a *canonical lift*

$$\mathcal{X}_{\text{can}} \rightarrow \text{Spec } W(k)$$

of $x \in X = U/G$. Using G -Hilbert schemes in families, we will see in Section 4.5 that it admits a simultaneous and minimal resolution of singularities

$$\tilde{\pi} : \mathcal{Y} \rightarrow \mathcal{X}_{\text{can}} \rightarrow \text{Spec } W(k).$$

We will prove this resolution to be unique, see Theorem 4.10.

1.8. McKay correspondence for Klein singularities

Let G be a finite and linearly reductive subgroup scheme of $\mathbf{SL}_{2,k}$ and let $x \in X = U/G$ be the associated Klein singularity.

As discussed in Section 1.5, we have the McKay graph $\widehat{\Gamma}$ associated to G , its embedding into $\mathbf{SL}_{2,k}$ and the set of isomorphism classes of simple representations of G .

Let $\pi : Y \rightarrow X$ be a minimal resolution of singularities. Since $x \in X$ is a rational double point, the exceptional divisor $\text{Exc}(\pi)$ of π is a configuration of \mathbb{P}^1 's, whose dual intersection graph Γ is a simply-laced Dynkin diagram.

Theorem 1.4 (Theorem 5.3). *Let k be an algebraically closed field of characteristic $p \geq 0$. Let G be a finite and linearly reductive subgroup scheme of $\mathbf{SL}_{2,k}$ and let $x \in X = U/G$ be the associated Klein singularity. Then, there exists a natural bijection of the graph Γ with the graph obtained from $\widehat{\Gamma}$ by removing the vertex corresponding to the trivial representation.*

Since every rational double point in characteristic $p \geq 7$ is a Klein singularity, we obtain the following.

Corollary 1.5. *There exists a linearly reductive McKay correspondence for rational double point singularities in every characteristic $p \geq 7$.*

To establish this theorem, we use the Ishii-Ito-Nakamura resolution of singularities (1.3), as well as Hecke correspondences as in the work of Ito and Nakamura [IN99] and Nakajima [Na96, Na01].

1.9. Generalisations and variants

Let G be a very small, finite and linearly reductive subgroup scheme of $\mathbf{GL}_{2,k}$ and let $x \in X = U/G$ be the associated two-dimensional linearly reductive quotient singularity. Let $\pi : Y \rightarrow X$ be the minimal resolution of singularities and let $\text{Exc}(\pi)$ be the exceptional divisor of π .

1. In Theorem 5.4, we associate a representation of G to each point of $\text{Exc}(\pi)$. This generalises results of Ishii and Nakamura [Is02, IN19].
2. In Theorem 5.6, we establish a bijection between the components of $\text{Exc}(\pi)$ and reflexive \mathcal{O}_X -modules. Probably, this result should be viewed as a theorem on two-dimensional F-regular singularities, see Remark 5.7. It generalises work of Artin and Verdier [AV85], Wunram [Wu88] and Ishii and Nakamura [IN19].
3. In Theorem 7.1, we establish an equivalence of derived categories of coherent sheaves $\mathcal{D}(Y)$ and $\mathcal{D}^G(U)$ (G -equivariant sheaves on U). This generalises work of Kapranov and Vasserot [KV00], Bridgeland, King and Reid [BKR01] and Ishii, Nakamura and Ueda [Is02, IN19, IU15]. Of course, these articles themselves generalise work of Gonzalez-Sprinberg and Verdier [GV83] from K-theory to derived categories of coherent sheaves.

1.10. Ito-Reid correspondence

If G is a finite subgroup of $\mathbf{SL}_2(\mathbb{C})$, then Ito and Reid [IR96] found a natural bijection, the *Ito-Reid correspondence*, between the conjugacy classes of G and the vertices of the McKay graph $\widehat{\Gamma}$.

Theorem 1.6. *Let k be an algebraically closed field of characteristic $p \geq 0$. There exists an Ito-Reid correspondence for finite and linearly reductive subgroup schemes of $\mathbf{SL}_{2,k}$.*

This can be proven using lifting results and the Ito-Reid correspondence over \mathbb{C} , see Section 6.3 for details.

The main difficulty is to define a notion of conjugacy class for finite and linearly reductive group schemes that makes such a correspondence work: Let us recall the ring $K_k(G)$ from Section 1.3.3. If G is a finite and linearly reductive group scheme over k , then we define the set of *conjugacy classes* of G to be

$$\mathrm{Spec}(\mathbb{C} \otimes K_k(G)).$$

At first sight, this might look rather artificial. One should think of it as defining conjugacy classes to be “dual” to simple representations. Moreover, our definition is compatible with lifting over $W(k)$ and it induces a bijection of the conjugacy classes of G with the conjugacy classes of G_{abs} . We refer to Section 6.1 and Appendix B.2 for details.

1.11. Conjugacy classes

In Theorem 1.6, we had to find a definition of conjugacy classes that makes this theorem true. This begs for the question whether there are other definitions or approaches, which is a question that is interesting in its own.

Let G be a finite group scheme (not necessarily linearly reductive) over an algebraically closed field k . In Appendix B, we study the following approaches to the notion of the set of conjugacy classes of G :

1. The set of conjugacy classes $G(k)/\sim$ of the group of k -rational points of G .
2. The spectrum of $F \otimes K_k(G)$, where F is a field of characteristic zero that contains “sufficiently many” roots of unity.
3. The scheme that represents the functor of conjugacy classes from schemes over k to sets defined by $S \mapsto G(S)/\sim$,
4. the isotypical component of the trivial representation of the adjoint representation of G .
5. The simple subrepresentations of the extended adjoint representations ${}^{\mathrm{Ad}}A$ and ${}^{\mathrm{Ad}}(A^*)$, which are defined using the quantum doubles of the Hopf algebra $A := H^0(G, \mathcal{O}_G)$ and its dual A^* .

All approaches lead essentially to the “same answer” in characteristic zero. However, they usually lead to very different notions in positive characteristic. On the other hand, all approaches have their merits and drawbacks. For linearly reductive group schemes, the approach (2) leads to a definition that is compatible with lifting and that leads to an Ito-Reid correspondence.

1.12. Organisation of this article

- In Section 2, we recall basic facts about finite and linearly reductive group schemes over algebraically closed fields.
- In Section 3, we construct the McKay graph $\widehat{\Gamma}$. The main result is Theorem 3.6, a McKay correspondence.
- In Section 4, we recall basic facts about linearly reductive quotient singularities. We establish an Ishii-Ito-Nakamura-type resolution of singularities, the canonical lift of such a singularity and a unique simultaneous resolution of singularities of the canonical lift.
- In Section 5, we revisit the Ishii-Ito-Nakamura-type resolution π of singularities of $x \in X = U/G$ and we introduce Hecke correspondences, which leads to Theorem 5.3, a bijection between simple and non-trivial representations of G and components of π . We also study generalisations of this result to two-dimensional linearly reductive quotient singularities.
- In Section 6, we establish an Ito-Reid correspondence between conjugacy classes of G and exceptional divisors in the minimal resolution of singularities of $x \in X = U/G$.
- In Section 7, we study derived categories of G -equivariant sheaves on U and on the minimal resolution $\pi : Y \rightarrow X = U/G$.

- In Appendix A, we recall results on finite group schemes from the point of view of Hopf algebras. We recall the adjoint representation, quantum doubles and the extended adjoint representation.
- In Appendix B, we study several approaches toward the notion of a conjugacy class for finite group schemes.

2. Linearly reductive group schemes

In this section, we recall a couple of general facts about finite and linearly reductive group schemes over algebraically closed fields. We discuss the close relationship between such a group scheme G and a certain abstracted group G_{abs} associated to it. For the relationship to Hopf algebras, we refer to Appendix A.

2.1. Group schemes

Let G be a finite group scheme over an algebraically closed field k of characteristic $p \geq 0$. Since k is perfect, there is a short exact sequence of finite group schemes over k

$$1 \rightarrow G^\circ \rightarrow G \rightarrow G^{\text{ét}} \rightarrow 1, \quad (2.1)$$

where G° is the connected component of the identity and where $G^{\text{ét}}$ is an étale group scheme over k . The reduction $G_{\text{red}} \rightarrow G$ provides a canonical splitting of (2.1) and we obtain a canonical semi-direct product decomposition $G \cong G^\circ \rtimes G^{\text{ét}}$. Since k is algebraically closed, $G^{\text{ét}}$ is the constant group scheme associated to the finite group $G(k) = G^{\text{ét}}(k)$ of k -rational points. Moreover, G° is an infinitesimal group scheme of length equal to some power of p . In particular, if $p = 0$ or if the length of G is prime to p , then G° is trivial and then, G is étale.

If M is a finitely generated abelian group, then the group algebra $k[M]$ carries a Hopf algebra structure and the associated commutative group scheme is denoted $D(M) := \text{Spec } k[M]$. By definition, such group schemes are called *diagonalisable*. For example, we have $D(\mathbf{C}_n) \cong \mu_n$, where \mathbf{C}_n denotes the cyclic group of order n . We have that μ_n is étale over k if and only if $p \nmid n$.

A finite group scheme G over k is said to be *linearly reductive* if every k -linear and finite-dimensional representation of G is semi-simple. If $p = 0$, then all finite group schemes over k are étale and linearly reductive. If $p > 0$, then, by a theorem that is often attributed to Nagata [Na61, Theorem 2] (but see also [AOV08, Proposition 2.10], [Ch92] and [Ha15, Section 2]), a finite group scheme over k is linearly reductive if and only if it is an extension of a finite and étale group scheme, whose length is prime to p , by a diagonalisable group scheme.

2.2. Abstract groups and canonical lifts

Let G be a finite and linearly reductive group scheme over an algebraically closed field k of characteristic $p > 0$. Following [LMM21, Section 2], we study the finite group

$$G_{\text{abs}} := \left(\underline{((G^\circ)^D(k))}_{\mathbf{C}} \right)^D(\mathbf{C}) \rtimes G^{\text{ét}}(k).$$

Here, $-^D$ denotes $\underline{\text{Hom}}(-, \mathbf{G}_m)$, the *Cartier dual*, of a commutative and finite group scheme. If G is étale, then $G_{\text{abs}} = G(k)$ and if $G = \mu_{p^n}$, then $G_{\text{abs}} = \mathbf{C}_{p^n}$. In any case, the order of G_{abs} is equal to the length of G .

Definition 2.1. The finite group G_{abs} is called the *abstract finite group* associated to G .

Lemma 2.2. *The functor*

$$G \mapsto G_{\text{abs}}$$

establishes an equivalence of categories between the category of finite and linearly reductive group schemes over k and the category of finite groups with a normal and abelian p -Sylow subgroup.

Proof. [LMM21, Lemma 2.1]. □

Remark 2.3. A finite group G with a normal and abelian p -Sylow subgroup P is the same as a finite group with a unique and abelian p -Sylow subgroup. In this case, the Schur-Zassenhaus theorem implies that G is isomorphic to a semi-direct product $P \rtimes G/P$.

Next, we study lifts to characteristic zero: let $W(k)$ be the ring of Witt vectors of k , let K be its field of fractions and let \bar{K} be an algebraic closure of K . By [LMM21, Proposition 2.4], there exist lifts of G as finite and flat group scheme over $W(k)$. More precisely, G° and $G^{\text{ét}}$ even lift uniquely to $W(k)$, but their extension class usually does not, see also [LMM21, Example 2.6]. However, there is a unique lift \mathcal{G}_{can} of G to $W(k)$ that is characterised by being a semi-direct product of the unique lift of G° with the unique lift of $G^{\text{ét}}$.

Definition 2.4. The lift $\mathcal{G}_{\text{can}} \rightarrow \text{Spec}W(k)$ is called the *canonical lift* of G . We set $G_{\text{can}} := \mathcal{G}_K \rightarrow \text{Spec}K$ and also call it canonical lift.

Every other lift of G to some extension of $R \supseteq W(k)$ differs from $\mathcal{G}_{\text{can},R}$ by a twist and thus, there is only one *geometric lift* of G to \bar{K} up to isomorphism, namely $G_{\text{can},\bar{K}}$, see [LMM21, Section 2.2].

Since \bar{K} is algebraically closed and of characteristic zero, $G_{\text{can},\bar{K}}$ is the constant group scheme associated to the finite group $G_{\text{can}}(\bar{K})$. In fact, we have $G^\circ \cong \prod_i \mu_{p^{n_i}}$ for some n_i 's and if we set $N := \max\{n_i\}_i$, fix a primitive p^N -th root of unity ζ_{p^N} and set $K_N := K(\zeta_{p^N})$, then G_{can,K_N} is the constant group scheme associated to the finite group $G_{\text{can}}(K_N)$ and we have $G_{\text{can}}(K_N) = G_{\text{can}}(\bar{K})$.

Lemma 2.5. *There exist isomorphisms of finite groups*

$$G_{\text{abs}} \cong G_{\text{can}}(K_N) = G_{\text{can}}(\bar{K}).$$

In particular, there exist isomorphisms

$$G_{\text{can},K_N} \cong \left(G_{\text{abs}} \right)_{K_N} \quad \text{and} \quad G_{\text{can},\bar{K}} \cong \left(G_{\text{abs}} \right)_{\bar{K}}$$

of finite group schemes over K_N and \bar{K} , respectively.

2.3. Representation theory

Let k be an algebraically closed field of characteristic $p > 0$, and let $W(k)$, K and \bar{K} be as in the previous section. The equivalence of Lemma 2.2 can be extended to representations and K -theory. If G is a finite group or a finite group scheme over k , we denote by $\text{Rep}_k(G)$ the category of its k -linear and finite-dimensional representations. Let us first recall [LMM21, Corollary 2.11].

Proposition 2.6. *Let G be a finite and linearly reductive group scheme over k . Then, there exist canonical equivalences of categories*

$$\text{Rep}_k(G) \rightarrow \text{Rep}_{K_N}(G_{\text{can},K_N}) \rightarrow \text{Rep}_{\bar{K}}(G_{\text{can},\bar{K}}) \rightarrow \text{Rep}_{\mathbb{C}}(G_{\text{abs}}),$$

which are compatible with degrees, direct sums, tensor products, duals and simplicity.

Remark 2.7. In particular, this allows us to define *characters* or even a *character table* of G via Proposition 2.6 and G_{abs} .

Let $K_k(G)$ be the K -group associated to $\text{Rep}_k(G)$. In fact, $K_k(G)$ has a natural structure of a commutative ring with one, where the sum (resp. product) structure comes from direct sums (resp. tensor products) of representations. A straightforward application of Proposition 2.6 is the following.

Corollary 2.8. *There exist isomorphisms of rings*

$$K_k(G) \rightarrow K_{K_N}(G_{\text{can},K_N}) \rightarrow K_{\overline{K}}(G_{\text{can},\overline{K}}) \rightarrow K_{\mathbb{C}}(G_{\text{abs}}).$$

3. McKay graph and McKay correspondence

In this section, we introduce the *McKay graph* associated to a finite and linearly reductive subgroup scheme G over an algebraically closed field k of characteristic $p \geq 0$ and a representation $\rho : G \rightarrow \mathbf{GL}_{n,k}$. This induces a bijection between certain affine Dynkin diagrams and finite and linearly reductive subgroup schemes of $\mathbf{SL}_{2,k}$. As an application, we establish a *linearly reductive McKay correspondence*.

3.1. McKay graph

Let G be a finite and linearly reductive group scheme over an algebraically closed field k of characteristic $p \geq 0$. Let $\{\rho_i\}_i$ be the finite set of isomorphism classes of k -linear and simple representations of G . Following tradition, we assume that ρ_0 is the trivial representation. We fix a representation $\rho : G \rightarrow \mathbf{GL}(V)$. If G is a subgroup scheme of $\mathbf{SL}_{n,k}$ or $\mathbf{GL}_{n,k}$, then ρ is usually the linear representation corresponding to the embedding of G into this linear algebraic group. By assumption, $\text{Rep}_k(G)$ is semi-simple. Therefore, there exist unique integers $a_{ij} \in \mathbb{Z}_{\geq 0}$ for each i , such that we have isomorphisms of k -linear representations

$$\rho \otimes \rho_i \cong \bigoplus_j \rho_j^{\oplus a_{ij}}.$$

Associated to this data, we define the *McKay graph*, denoted $\Gamma(G, \{\rho_i\}, \rho)$:

- The vertices are the $\{\rho_i\}_i$. (Some sources exclude the the trivial representation ρ_0 .)
- There are a_{ij} edges from the vertex corresponding to ρ_i to the vertex corresponding to ρ_j .

We now establish a couple of elementary properties of this graph, which are well-known in the classical case and which immediately carry over to the linearly reductive situation. We leave the proof of the first lemma to the reader.

Lemma 3.1. *We have*

$$a_{ij} = \dim_k \text{Hom}(\rho_i, \rho_j \otimes \rho).$$

In particular, if ρ is self-dual, that is, $\rho \cong \rho^\vee$, then $a_{ij} = a_{ji}$ for all i, j . In this case, we can consider $\Gamma(G, \{\rho_i\}, \rho)$ as an undirected graph.

Lemma 3.2. *Let $\rho : G \rightarrow \mathbf{SL}_{2,k}$ be a homomorphism of group schemes over k , considered as a 2-dimensional representation. Then, ρ is self-dual.*

Proof. Being a 2-dimensional representation, ρ^\vee is isomorphic to $\rho \otimes \det(\rho)$ and the lemma follows. □

Lemma 3.3. *Let $\rho : G \rightarrow \mathbf{GL}_{n,k}$ be a faithful representation. Then, every irreducible representation of G occurs as subrepresentation of $\rho^{\otimes m}$ for some suitable m .*

Proof. This is well-known for finite groups. Using Proposition 2.6, it carries over to finite and linearly reductive group schemes. □

Corollary 3.4. *If ρ is a faithful representation, then the graph $\Gamma(G, \{\rho_i\}, \rho)$ is connected.*

Proof. It suffices to note that the number of paths in Γ of length m that connect the vertices corresponding to ρ_i and ρ_j is equal to the multiplicity of ρ_i in $\rho_j \otimes \rho^{\otimes m}$, see, for example, the proof of [Ki16, Theorem 8.13]. □

Let k be an algebraically closed field of characteristic $p > 0$, and let $W(k)$, K and \bar{K} be as in Section 2.2. There, we also discussed the canonical lift G_{can} of G over K and we saw that there exists an isomorphism of finite groups $G_{\text{abs}} \cong G_{\text{can}}(\bar{K})$. Proposition 2.6 implies the following result.

Proposition 3.5. *Let G be a finite and linearly reductive group scheme over k . Let $\{\rho_i\}_i$ be the set of isomorphism classes of simple representations of G . Let ρ be a finite-dimensional representation of G .*

1. Proposition 2.6 yields sets of representations $\{\rho_{\text{can},i}\}_i$ and $\{\rho_{\text{abs},i}\}_i$ of G_{can} and G_{abs} , respectively, which are the sets of isomorphism classes of simple representations of G_{can} and G_{abs} , respectively.
2. Proposition 2.6 yields representations ρ_{can} and ρ_{abs} of G_{can} and G_{abs} , respectively.

This data leads to a bijection of the three McKay graphs

$$\Gamma(G, \{\rho_i\}_i, \rho), \quad \Gamma(G_{\text{can}}, \{\rho_{\text{can},i}\}_i, \rho_{\text{can}}), \quad \Gamma(G_{\text{abs}}, \{\rho_{\text{abs},i}\}_i, \rho_{\text{abs}}).$$

3.2. McKay correspondence

We now run through the classical McKay correspondence [Mc80] in our setting, where we follow [Ki16, Section 8.3]. Let k be an algebraically closed field of characteristic $p \geq 0$ and let G be a finite and linearly reductive subgroup scheme of $\mathbf{SL}_{2,k}$.

1. The K -group $K_k(G)$ carries a symmetric bilinear form

$$([V], [W])_0 := \dim_k \text{Hom}_G(V, W).$$

2. We consider the closed embedding $G \rightarrow \mathbf{SL}_{2,k}$ as a 2-dimensional representation $\rho : G \rightarrow \mathbf{SL}_{2,k} \rightarrow \mathbf{GL}_{2,k}$ and define an operator

$$A : K_k(G) \rightarrow K_k(G), \quad [V] \mapsto [V] \otimes \rho.$$

Since ρ is self-dual by Lemma 3.2, it follows that A is symmetric with respect to $(-, -)_0$, see [Ki16, Lemma 8.12].

3. Using ρ , we define a symmetric bilinear form on $K_k(G) \otimes_{\mathbb{Z}} \mathbb{R}$ by

$$([V], [W]) := ([V], (2 - A)[W])_0,$$

which is positive semi-definite. The class $\delta \in K_k(G)$ of the regular representation of G generates the radical of $(-, -)$.

4. Let $\{\rho_i\}$ be the set of isomorphism classes of simple representations of G .
 - (a) The McKay graph $\widehat{\Gamma} = \Gamma(G, \{\rho_i\}_i, \rho)$ is an affine Dynkin diagram of type $\widehat{A}_n, \widehat{D}_n$ with $n \geq 4$, $\widehat{E}_6, \widehat{E}_7$ or \widehat{E}_8 .
 - (b) After removing the vertex corresponding to the trivial representation ρ_0 from $\widehat{\Gamma}$, we obtain a finite Dynkin diagram of type A_n, D_n with $n \geq 4, E_6, E_7$ or E_8 , respectively.

If G is a finite subgroup of $\mathbf{SL}_2(\mathbb{C})$, then these statements are part of the classical McKay correspondence, see [Mc80] or [Ki16, Section 8.3]. In our setting of finite and linearly reductive subgroup schemes of $\mathbf{SL}_{2,k}$, the above claims immediately follow from the classical McKay correspondence together with the lifting results Proposition 2.6 and Proposition 3.5. We then obtain the following analog of McKay’s theorem [Mc80].

Theorem 3.6. *Let k be an algebraically closed field of characteristic $p \geq 0$. There exists a bijection between finite, non-trivial and linearly reductive subgroup schemes of $\mathbf{SL}_{2,k}$ up to conjugation and affine Dynkin diagrams of type*

$$\begin{aligned} \widehat{A}_n, \widehat{D}_n, \widehat{E}_6, \widehat{E}_7, \widehat{E}_8 & \text{ if } p = 0 \text{ or } p \geq 7, \\ \widehat{A}_n, \widehat{D}_n, \widehat{E}_6, \widehat{E}_7 & \text{ if } p = 5, \\ \widehat{A}_n, \widehat{D}_n & \text{ if } p = 3 \text{ and} \\ \widehat{A}_n & \text{ if } p = 2. \end{aligned}$$

Proof. If $p = 0$, then this is part of the classical McKay correspondence. If $p > 0$, then this follows from the linearly reductive McKay correspondence just discussed together with Hashimoto’s classification [Ha15, Theorem 3.8] of finite and linearly reductive subgroup schemes of $\mathbf{SL}_{2,k}$. \square

Remark 3.7. The linearly reductive group schemes corresponding to \widehat{E}_6 , \widehat{E}_7 and \widehat{E}_8 are étale and correspond to finite groups of order prime to p . The linearly reductive group scheme corresponding to \widehat{A}_n (resp. \widehat{D}_n) is étale if and only if $p \nmid (n + 1)$ (resp. $p \nmid (n - 2)$). We refer to [LS14, Proposition 4.2] for details. Thus, even if p is sufficiently large, then one does not obtain a bijection in Theorem 3.6 with finite groups of order to prime to p only.

Remark 3.8. Steinberg [St85] established many properties of the McKay graph and the McKay correspondence for finite subgroups of $\mathbf{SU}_2(\mathbb{C})$ without using classification lists. It seems plausible to obtain a proof of Theorem 3.6 along these lines without using lifting results or classification lists.

4. Linearly reductive quotient singularities

In this section, we recall *linearly reductive quotient singularities* in the sense of [LMM21] and some general results, including the *canonical lift* of such a singularity over the ring of Witt vectors. In dimension two, we establish a minimal resolution of singularities using G -Hilbert schemes as in the work of Ishii, Ito and Nakamura [Is02, IN19, IN99]. We also show that the canonical lift admits a unique minimal and simultaneous resolution of singularities. Finally, we discuss *Klein singularities* and their relation to rational double point singularities.

4.1. Quotient singularities

Let k be an algebraically closed field of characteristic $p \geq 0$. If G is a finite and linearly reductive group scheme over k , if V is a finite-dimensional k -vector space and if $\rho : G \rightarrow \mathbf{GL}(V)$ is a linear representation, then we define the λ -invariant of ρ as in [LMM21, Definition 2.7] to be

$$\lambda(\rho) := \max_{\{\text{id}\} \neq \mu_n \subseteq G} \dim V^{\mu_n},$$

where V^{μ_n} denotes the subspace of μ_n -invariants. As explained in [LMM21, Remark 2.8], the representation ρ is faithful if and only if $\lambda(\rho) \neq \dim V$. Moreover, ρ contains no pseudo-reflections if and only if $\lambda(\rho) \leq \dim V - 2$ and in this case, the representation ρ is said to be *small*.

Definition 4.1. The representation ρ is called *very small* if $\lambda(\rho) = 0$.

Note that in dimension two the notions of small and very small coincide. We refer to [LMM21, Section 3] for the classification of finite and linearly reductive group schemes that admit a very small representation. By [LMM21, Proposition 6.5], $\lambda(\rho)$ is equal to the dimension of the non-free locus of the induced G -action on the spectrum of the formal power series ring $k[[V^\vee]] = (k[V^\vee])^\wedge$. Following [LMM21, Definition 6.4] and using the linearisation result [LMM21, Proposition 6.3], we define:

Definition 4.2. A linearly reductive quotient singularity over k is an isolated singularity that is analytically isomorphic to $\text{Spec } k[[V^\vee]]^G$, where G is a finite and linearly reductive group scheme over k and where $\rho : G \rightarrow \mathbf{GL}(V)$ is a very small representation.

Remark 4.3. We set $U := \text{Spec } k[V^\vee]$ or $U := \text{Spec } k[[V^\vee]]$ and simply write $x \in X = U/G$ with the assumptions of Definition 4.2 implicitly understood.

Properties of these singularities have been studied in [LMM21]: for their local étale fundamental groups, class groups and F-signatures, we refer to [LMM21, Section 7].

By [LMM21, Theorem 8.1], a linearly reductive quotient singularity determines the finite and linearly reductive group scheme G together with the very small representation $\rho : G \rightarrow \mathbf{GL}(V)$ uniquely up to isomorphism and conjugacy, respectively. It thus makes sense to refer to $\rho(G) \subset \mathbf{GL}(V)$ as the *finite and linearly reductive subgroup scheme of $\mathbf{GL}(V)$ associated to the linearly reductive quotient singularity $x \in X = U/G$* . In particular, the classification of linearly reductive quotient singularities in dimension d is “the same” as the classification of very small, finite and linearly reductive subgroup schemes of $\mathbf{GL}_{d,k}$ up to conjugacy. We refer to [LMM21, Section 3] for details and this classification.

4.2. Minimal resolution of singularities

If $x \in X$ is a normal surface singularity, then it admits a unique minimal resolution of singularities

$$\pi : Y \rightarrow X.$$

If $x \in X$ is moreover a rational singularity, then the exceptional locus of π is a union of \mathbb{P}^1 's, whose dual intersection graph Γ contains no cycles. The graph Γ is called the *type* of $x \in X$. If the type determines the singularity up to analytic isomorphism, then the singularity is said to be *taut*.

Over \mathbb{C} , taut singularities have been classified by Laufer [La73]. For example, two-dimensional finite quotient singularities over \mathbb{C} are taut. Since two-dimensional linearly reductive quotient singularities over algebraically closed fields of positive characteristic are F-regular (see [LMM21, Proposition 7.1]), they are taut by Tanaka’s theorem [Ta15].

Remark 4.4. Very small, finite and linearly reductive subgroup schemes of $\mathbf{GL}_{2,k}$ have been classified in [LMM21, Theorem 3.4], extending Brieskorn’s classification [Br67, Satz 2.9] of small subgroups of $\mathbf{GL}_2(\mathbb{C})$. The types of the associated quotient singularities in terms of this classification are given by [Br67, Satz 2.11].

Finite and linearly reductive subgroup schemes of $\mathbf{SL}_{2,k}$ are automatically very small and they have been classified by Hashimoto [Ha15, Theorem 3.8], extending Klein’s classification [K184] of finite subgroups of $\mathbf{SL}_2(\mathbb{C})$ and we refer to [Ha15, LS14] for the types of the associated quotient singularities (see also Theorem 4.12 below).

4.3. The Ishii-Ito-Nakamura resolution

In [IN99], Ito and Nakamura showed that if G is a finite subgroup of $\mathbf{SL}_2(\mathbb{C})$, then a minimal resolution of singularities of \mathbb{C}^2/G is provided by the G -Hilbert scheme $G\text{-Hilb}(\mathbb{C}^2)$. Ishii [Is02] extended this to quotient singularities \mathbb{C}^2/G for G a finite and very small subgroup of $\mathbf{GL}_2(\mathbb{C})$ and Ishii and Nakamura [IN19] extended this to quotient singularities U/G for G a finite and very small subgroup of $\mathbf{GL}_2(k)$ of order prime to p .

Let k be an algebraically closed field of characteristic $p \geq 0$. Let G be a very small, finite and linearly reductive subgroup scheme of $\mathbf{GL}_{2,k}$. Set $U := \mathbb{A}_k^2$ or $\widehat{\mathbb{A}}_k^2$ and let $x \in X := U/G$ be the associated linearly reductive quotient singularity. By [BI11], there exists a G -Hilbert scheme $G\text{-Hilb}(U)$ over k that parametrises zero-dimensional G -invariant subschemes $Z \subset U$ (so-called *clusters*), such that the

G -representation on $H^0(Z, \mathcal{O}_Z)$ is the regular representation. Taking a cluster Z to its G -orbit (see, for example, [B111, Remark 3.3]) induces a morphism

$$\pi : Y := G\text{-Hilb}(U) \rightarrow U/G = X$$

over k .

Theorem 4.5. *The morphism π is a minimal resolution of singularities.*

Proof. If $p = 0$, then this is shown in [IN19], extending [IN99]. If $p > 0$ and G is of length prime to p , then this is shown in [IN19]. However, this proof also works if G is linearly reductive and p divides the length of G . □

Remark 4.6. By [LMM21, Theorem 11.5], every two-dimensional F-regular singularity $x \in X$ is a linearly reductive quotient singularity, say $x \in X = U/G$ for some finite and linearly reductive subgroup scheme G of $\mathbf{GL}_{2,k}$. By [LMM21, Theorem 8.1], G and its embedding into $\mathbf{GL}_{2,k}$ are unique up to isomorphism and conjugacy, respectively. By Theorem 4.5, $G\text{-Hilb}(U) \rightarrow U/G = X \ni x$ is the minimal resolution of singularities. In particular, every two-dimensional F-regular singularity can be resolved by a suitable G -Hilbert scheme.

4.4. Canonical lifts

Let k be an algebraically closed field of characteristic $p > 0$, let $W(k)$ be the ring of Witt vectors, let K be its field of fractions and let \bar{K} be an algebraic closure. Let $x \in X = U/G$ be a d -dimensional linearly reductive singularity, where $U = \mathbb{A}_k^d$ or $U = \widehat{\mathbb{A}}_k^d$. Next, we set $\mathcal{U} := \mathbb{A}_{W(k)}^d$ or $\widehat{\mathbb{A}}_{W(k)}^d$, respectively. In Section 2.2, we recalled the canonical lift \mathcal{G}_{can} of G over $W(k)$. By [LMM21, Proposition 2.9], there exists a unique lift of the G -action from U to \mathcal{U} . From this, we obtain a flat family

$$\mathcal{X}_{\text{can}} := \mathcal{U}/\mathcal{G}_{\text{can}} \rightarrow \text{Spec } W(k). \tag{4.1}$$

of linearly reductive quotient singularities over $W(k)$, whose special fibre over k is $x \in X = U/G$.

Definition 4.7. The family (4.1) is called the *canonical lift* of the linearly reductive quotient singularity $x \in X = U/G$.

By the Lefschetz principle, the geometric generic fibre of \mathcal{X}_{can} can be identified with a finite quotient singularity of the form $\mathbb{C}^d/G_{\text{abs}}$, where G_{abs} is the abstract group associated to G and the embedding of $G_{\text{abs}} \rightarrow \mathbf{GL}_d(\mathbb{C})$ corresponds to the embedding $G \rightarrow \mathbf{GL}_{d,k}$ provided by Proposition 2.6. The canonical lift is unique in the following sense:

Proposition 4.8. *We keep the notations and assumptions. Let $W(k) \subseteq R$ be a finite extension of complete DVRs and let $\mathcal{X} \rightarrow \text{Spec } R$ be a lift of $x \in X$ that is of the form $\mathcal{V}/\mathcal{G} \rightarrow \text{Spec } R$ for some flat lift \mathcal{G} of G to R and $\mathcal{V} \cong \mathcal{U} \times_R S$. Then, there exists a finite extension $R \subseteq S$ of complete DVRs, such that*

1. *There exists an isomorphism*

$$\mathcal{G}_{\text{can}} \times_{W(k)} S \cong \mathcal{G} \times_R S$$

of group schemes over S .

2. *The very small representation $\rho : G \rightarrow \mathbf{GL}_{d,k}$ corresponding to the G -action on U lifts uniquely to \mathcal{G}_{can} and \mathcal{G} , respectively, and they become conjugate over S .*
3. *There is an isomorphism*

$$\mathcal{X}_{\text{can}} \times_{W(k)} S \cong \mathcal{X} \times_R S$$

of deformations of $x \in X$ over S .

Proof. Since \mathcal{G} and $\mathcal{G}_{\text{can},R}$ are lifts of G to R , they become isomorphic after passing to a finite extension $R \subseteq S$, see also the discussion in Section 2.2. There, we also saw that the linear representation $\rho : G \rightarrow \mathbf{GL}_{d,k}$ lifts uniquely to \mathcal{G} and \mathcal{G}_{can} , respectively, and that they become conjugate over S . From this, we deduce an isomorphism $\mathcal{X}_{\text{can}} \times_{W(k)} S \cong \mathcal{X} \times_R S$. \square

Remark 4.9. If $d \geq 3$, then [LMM21, Corollary 10.10] shows that a d -dimensional linearly reductive quotient singularity $x \in X$ admits precisely one lift over $W(k)$, namely the canonical lift. If $d = 2$, then linearly reductive quotient singularities usually have positive dimensional deformation spaces and admit many non-isomorphic lifts to $W(k)$, see [LMM21, Section 12].

4.5. Simultaneous resolution of singularities

If $\mathcal{X} \rightarrow S$ is a deformation of a rational double point singularity $x \in X$, then it admits a simultaneous resolution of singularities, but usually only after some finite base-change $S' \rightarrow S$, see [Ar74]. In the case where $\mathcal{X} \rightarrow S$ is a family of rational surface singularities, then such a finite $S' \rightarrow S$ exists if S maps to the so-called *Artin component* inside the versal deformation space of $x \in X$. Moreover, due to the existence of flops, these simultaneous resolutions (if they exist) are not unique in general.

In the special case where $x \in X$ is a two-dimensional linearly reductive quotient singularity over some algebraically closed field k of characteristic $p > 0$ and where $\mathcal{X}_{\text{can}} \rightarrow \text{Spec } W(k)$ is the canonical lift of $x \in X$, we will now show that there exists a simultaneous and minimal resolution of singularities over $W(k)$ and that it is unique. This simultaneous resolution can be most elegantly constructed using the Ishii-Ito-Nakamura resolution from Section 4.3 in families.

Let k be an algebraically closed field of characteristic $p > 0$. Let G be a finite and linearly reductive group scheme over k and let $\rho : G \rightarrow \mathbf{GL}_{2,k}$ be a very small representation. Let $\mathcal{G}_{\text{can}} \rightarrow \text{Spec } W(k)$ be the canonical lift of G and let $\tilde{\rho} : \mathcal{G}_{\text{can}} \rightarrow \mathbf{GL}_{2,W(k)}$ be the lift of ρ to $W(k)$. Let $\mathcal{U} := \mathbb{A}_{W(k)}^2$ and let

$$\mathcal{X}_{\text{can}} := \mathcal{U}/\mathcal{G}_{\text{can}} \rightarrow \text{Spec } W(k)$$

be the canonical lift of $x \in X$. By [B111], there exists a \mathcal{G}_{can} -Hilbert scheme

$$\mathcal{G}_{\text{can}}\text{-Hilb}(\mathcal{U}) \rightarrow \text{Spec } W(k)$$

that parametrises \mathcal{G}_{can} -invariant subschemes $Z \subset \mathcal{U}$ that are finite and flat over $W(k)$ (so-called \mathcal{G}_{can} -clusters) and such that the \mathcal{G}_{can} -representation on $H^0(Z, \mathcal{O}_Z)$ is the regular representation. Taking such a cluster Z to its \mathcal{G}_{can} -orbit (see, for example, [B111, Remark 3.3]) induces a morphism

$$\tilde{\pi} : \mathcal{Y} := \mathcal{G}_{\text{can}}\text{-Hilb}(\mathcal{U}) \rightarrow \mathcal{U}/\mathcal{G}_{\text{can}} = \mathcal{X}_{\text{can}} \tag{4.2}$$

over $W(k)$.

Theorem 4.10. *Keeping the assumptions and notations*

$$\tilde{\pi} : \mathcal{Y} \rightarrow \mathcal{X} \rightarrow \text{Spec } W(k)$$

is a simultaneous minimal resolution of singularities of the canonical lift $\mathcal{X} \rightarrow \text{Spec } W(k)$ of the linearly reductive quotient singularity $x \in X = U/G$.

1. *The simultaneous resolution $\tilde{\pi}$ is unique up to isomorphism.*
2. *The exceptional locus of $\tilde{\pi}$ is a union of $\mathbb{P}_{W(k)}^1$'s meeting transversally and we denote by Γ its dual intersection graph.*
3. *The special fibre and the generic fibre are linearly reductive quotient singularities of type Γ and $\tilde{\pi}$ identifies the components of the exceptional loci of $\tilde{\pi}_k$ and $\tilde{\pi}_K$.*

Proof. We recall that we defined $G_{\text{can}} := \mathcal{G}_{\text{can},K}$, which will simplify the notation in the following. The generic and special fibre of (4.2) over K and k are isomorphic to

$$\begin{aligned} G_{\text{can}}\text{-Hilb}(\mathcal{U}_K) &\rightarrow \mathcal{U}_K/G_{\text{can}} \rightarrow \text{Spec } K \\ \text{and } G\text{-Hilb}(U) &\rightarrow U/G \rightarrow \text{Spec } k, \end{aligned}$$

respectively, by [B11, Remark 3.1]. Now, $G_{\text{can},\bar{K}}\text{-Hilb}(\mathcal{U}_{\bar{K}}) \rightarrow \mathcal{U}_{\bar{K}}/G_{\text{can},\bar{K}}$ and $G\text{-Hilb}(U) \rightarrow U/G$, are minimal resolutions of singularities by Theorem 4.5. Thus, $G_{\text{can}}\text{-Hilb}(\mathcal{U}_K) \rightarrow \mathcal{U}_K/G_{\text{can}}$ is a resolution of singularities, which is minimal since it is minimal over \bar{K} . Thus, $\mathcal{Y} \rightarrow \mathcal{X} \rightarrow \text{Spec } W(k)$ is a simultaneous minimal resolution of singularities.

The exceptional fibres $\tilde{\pi}_K$ and $\tilde{\pi}_k$ of $\tilde{\pi}$ over K and k are unions of \mathbb{P}^1 's that intersect transversally. The types, that is dual resolution graphs, associated to $\tilde{\pi}_K$ and $\tilde{\pi}_k$ are the same (see also Remark 4.4) and we denote this graph by Γ . In particular, the exceptional fibres of $\tilde{\pi}_K$ and $\tilde{\pi}_k$ have the same numbers of irreducible components. In particular, the specialisation maps of Néron-Severi lattices

$$\text{NS}(\mathcal{Y}_K) \leftarrow \text{NS}(\mathcal{Y}) \rightarrow \text{NS}(\mathcal{Y}_k)$$

are isometries of lattices. These identify the components of the exceptional loci of $\tilde{\pi}_K$ and $\tilde{\pi}_k$. Moreover, given such a component of the exceptional locus of $\tilde{\pi}_k$, it is isomorphic to \mathbb{P}^1_k and it extends uniquely to a $\mathbb{P}^1_{W(k)}$ in the exceptional locus of $\tilde{\pi}$. This establishes claims (2) and (3).

It remains to prove claim (1): Let $\mathcal{Y}' \rightarrow \mathcal{X} \rightarrow \text{Spec } W(k)$ be a simultaneous resolution of singularities that coincides with the minimal resolution on special and generic fibres, respectively. Let $\alpha : \mathcal{Y}'_K \rightarrow \mathcal{Y}_K$ be an isomorphism over \mathcal{X}_K and choose a relatively (to \mathcal{X}) ample invertible sheaf \mathcal{L} on \mathcal{Y}_K . Then, $\alpha^*\mathcal{L}$ is relatively ample on \mathcal{Y}'_K . Since the types of the singularities of \mathcal{X}_K and \mathcal{X}_k are the same, the fibres of $\mathcal{Y}'_K \rightarrow \mathcal{X}_K$ and $\mathcal{Y}'_k \rightarrow \mathcal{Y}_k$ contain the same number of exceptional divisors. Thus, the specialisation map of Néron-Severi lattices $\text{NS}(\mathcal{Y}'_K) \rightarrow \text{NS}(\mathcal{Y}'_k)$ is an injective map between two lattices of the same rank and the same discriminant, whence an isometry. Similarly, the specialisation map of Néron-Severi lattices $\text{NS}(\mathcal{Y}_K) \rightarrow \text{NS}(\mathcal{Y}_k)$ is an isometry. In particular, we can identify the components of the exceptional loci of \mathcal{Y}_K and \mathcal{Y}_k . (If the types of \mathcal{X}_k and \mathcal{X}_K differ, then this specialisation map is usually only injective, there are usually more (-2) -curves in the special fibre and this is also the place where the difficulties with flops begin.) In particular, \mathcal{L} and $\alpha^*\mathcal{L}$ extend to relative ample invertible sheaves on \mathcal{Y} and \mathcal{Y}' , respectively. By [Ko09, Theorem 5.14], the isomorphism α extends to an isomorphism $\mathcal{Y}' \rightarrow \mathcal{Y}$ over \mathcal{X} and the claimed uniqueness follows. \square

4.6. Rational double point singularities and Klein singularities

We specialise Definition 4.2 to the following case.

Definition 4.11. A *Klein singularity* is a linearly reductive quotient singularity as in Definition 4.2 with $\dim V = 2$ and $\det \rho = 1$, that is, ρ is a homomorphism of G to $\mathbf{SL}_{2,k}$.

In particular, a Klein singularity is a two-dimensional and linearly reductive quotient singularity, it is a rational surface singularity, and since $\det(\rho) = 1$, it is Gorenstein. Quite generally, rational and Gorenstein surface singularities are precisely the *rational double point singularities* [Ar66]. We refer to [Du79] or [SSS] for more background on surface singularities and rational double point singularities.

If $x \in X$ is a rational double point, then its type Γ is a simply-laced finite Dynkin graph. In characteristic zero, every rational double point singularity is taut. In positive characteristic, a Klein singularity is F-regular and thus, taut. In the case of rational double points, this also follows from Artin's explicit classification [Ar77]. On the other hand, rational double point singularities that are not F-regular need not be taut. We have the following relation between rational double point singularities and Klein singularities in positive characteristic, see [LMM21, Theorem 11.2].

Theorem 4.12. *Let k be an algebraically closed field of characteristic $p \geq 0$. Let $x \in X$ be a normal surface singularity over k .*

1. If $p = 0$ or $p \geq 7$, then $x \in X$ is a Klein singularity if and only if $x \in X$ is a rational double point singularity.
2. If $p > 0$, then $x \in X$ is a Klein singularity if and only if $x \in X$ is a rational double point singularity and F -regular. The finite Dynkin graphs of these singularities are of type

$$\begin{array}{ll}
 A_n, D_n, E_6, E_7, E_8 & \text{if } p \geq 7, \\
 A_n, D_n, E_6, E_7 & \text{if } p = 5, \\
 A_n, D_n & \text{if } p = 3, \\
 A_n & \text{if } p = 2.
 \end{array}$$

A Klein singularity $x \in X = U/G$ is a linearly reductive quotient singularity and thus, determines G and $\rho : G \rightarrow \mathbf{SL}_{2,k}$ up to isomorphism and conjugacy, respectively. Thus, the classification of Klein singularities boils down to the classification of finite and linearly reductive subgroup schemes of $\mathbf{SL}_{2,k}$ and we refer to Remark 4.4 for this classification.

Remark 4.13. Theorem 3.6 and Theorem 4.12 rely on the classification of finite and linearly reductive subgroup schemes of $\mathbf{SL}_{2,k}$. It is therefore no surprise that the classification lists coincide.

5. Hecke correspondences

Let $x \in X = U/G$ be a Klein singularity over an algebraically closed field k of characteristic $p > 0$. In Section 2.2, we discussed the canonical lift $\mathcal{X} \rightarrow \text{Spec } W(k)$ of $x \in X$ and in Section 4.5 we established a simultaneous minimal resolution of singularities of the canonical lift. More precisely, this simultaneous resolution was constructed using G -Hilbert schemes extending previous work of Ishii, Ito and Nakamura [Is02, IN19, IN99].

In this section, we refine this resolution of singularities as in the work of Ito and Nakamura [IN99] and Nakajima [Na96, Na01]: we eventually obtain a bijection between the components of the minimal resolution of singularities of the Klein singularity $x \in X = U/G$ and the simple and non-trivial representations of G using special *Hecke correspondences*.

5.1. The Ito-Nakamura resolution revisited

We first slightly extend the setup of Section 4.3. Let k be an algebraically closed field of characteristic $p > 0$ and let $x \in X = U/G$ be a Klein singularity as in Definition 4.11, that is, we have $U = \mathbb{A}_k^2$ and G a very small, finite and linearly reductive subgroup scheme of $\mathbf{SL}_{2,k}$.

Let $\{\rho_i\}_{i \in I}$ be the set of isomorphism classes of simple representations of G . Given a finite-dimensional representation ρ of G , there exist non-negative integers $v_i \in \mathbb{Z}_+$, such that ρ is isomorphic to $\bigoplus_i \rho_i^{\oplus v_i}$ and we combine these into a multi-index $\nu = (\{v_i\}) \in \mathbb{Z}_{\geq 0}^I$. We set $\dim(\nu) := \sum_i v_i \dim \rho_i$, which is the dimension of the representation associated to ν .

For any integer $n \geq 1$, the G -action on U induces a G -action on the Hilbert scheme $\text{Hilb}^n(U)$, which parametrises zero-dimensional subschemes of length n of U . We consider the fixed point scheme

$$\text{Hilb}^{n,G}(U) := (\text{Hilb}^n(U))^G,$$

that is, the largest subscheme of $\text{Hilb}^n(U)$ on which G acts trivially. It parametrises G -invariant and zero-dimensional subschemes of length n of U . Given $\nu = (\{v_i\})_i \in \mathbb{Z}_{\geq 0}^I$ with $\dim(\nu) = n$, we define

$$H^\nu = \left\{ Z \in \text{Hilb}^{n,G}(U) \mid H^0(Z, \mathcal{O}_Z) \cong \bigoplus_i \rho_i^{\oplus v_i} \text{ as } G\text{-representation} \right\},$$

which defines a subscheme of $\text{Hilb}^{n,G}(U)$. Adapting [Ki16, Lemma 12.4], which follows [IN99, Section 9], to our situation, we have the following.

Lemma 5.1. *We keep the assumptions and notations and let $n \geq 1$ be an integer.*

1. *The scheme $\text{Hilb}^{n,G}(U)$ is smooth over k .*
2. *We have a decomposition*

$$\text{Hilb}^{n,G}(U) = \bigsqcup_{\nu} H^{\nu},$$

where the disjoint union runs over all multi-indices $\nu \in \mathbb{Z}_{\geq 0}^I$ of dimension n . Each H^{ν} is a smooth subscheme of $\text{Hilb}^{n,G}(U)$.

Proof. Since U is two-dimensional, $\text{Hilb}^n(U)$ is smooth over k by [Fo68, Theorem 2.4]. To show Claim (1), it remains to show that the fixed point scheme for the G -action is also smooth, which follows from Lemma 5.2 below. Then, Claim (2) is obvious. □

Lemma 5.2. *Let k be an algebraically closed field, let X be a scheme that is smooth over k , let G be a finite and linearly reductive group scheme over k and assume that G acts on X . Then, the fixed point scheme $X^G \subseteq X$ is smooth over k .*

Proof. Let $x \in X^G$. Passing to the completion of the local ring $\mathcal{O}_{X,x}$, using that X is smooth over k and passing to coordinates such that the G -action is linear (this is always possible by [Sa12, proof of Corollary 1.8]), we may assume that

$$\widehat{\mathcal{O}}_{X,x} \cong k[[u_1, \dots, u_d]]$$

and that the G -action is linear. In this description it is easy to see that the G -invariant subscheme of $\text{Spec } \widehat{\mathcal{O}}_{X,x}$ is smooth (see also the proof of [IN99, Lemma 9.1]), which implies that X^G is smooth near x . □

An important special case is the *regular representation* of G , where we have $\nu_i = \dim \rho_i$ for all i and in this case, we will write δ for the corresponding multi-index. The dimension of δ is equal to the length of G . In this case, we have

$$\pi : H^{\delta} = G\text{-Hilb}(U) \rightarrow U/G = X,$$

which we already studied in Section 4.3. There, we saw that π is a minimal resolution of singularities of the Klein singularity $x \in X = U/G$.

5.2. Hecke correspondences

We keep the assumptions and notations of the previous section. We let n be the length of G and let δ be the multi-index corresponding to the regular representation. For $i \in I$, we set $\alpha_i := (0, \dots, 0, 1, 0, \dots)$ with the non-zero entry in the i .th position. We define

$$B_i := \{J_1 \in H^{\delta-\alpha_i}, J_2 \in H^{\delta} \mid J_2 \subseteq J_1\} \subseteq H^{\delta-\alpha_i} \times H^{\delta},$$

which is a *Hecke correspondence*, see [Na96], [Na01, Section 6.1] or [Ki16, Section 12.4]. We let

$$E_i := \text{pr}_2(B_i) \subseteq H^{\delta} = G\text{-Hilb}(U)$$

be the image under the projection pr_2 onto the second factor.

In characteristic zero, the following result is due to Nakajima [Na96] and independently to Ito and Nakamura [IN99], see also [Ki16, Section 12.4].

Theorem 5.3. *Let k be an algebraically closed field of characteristic $p \geq 0$. Let G be a finite and linearly reductive subgroup scheme of $\mathbf{SL}_{2,k}$, let $x \in X := U/G$ be the associated Klein singularity, and let $\pi : H^\delta \rightarrow X$ be the Ito-Nakamura resolution.*

1. *The assignment $i \mapsto E_i$ defines a bijection between the set $\{\rho_i\}_i$ of isomorphism classes of non-trivial simple representations of G and the set of irreducible components of the exceptional divisor $\text{Exc}(\pi)$ of π .*
2. *If $i \neq j$, then*

$$B_i \cdot B_j = A_{ij}$$

where the numbering of the $\{\rho_i\}_i$ is as in [IN99, Section 10].

We thus obtain a bijection between the dual resolution graph of π and the graph obtained by removing the vertex ρ_0 from the McKay graph of G with respect to the embedding $G \rightarrow \mathbf{SL}_{2,k} \rightarrow \mathbf{GL}_{2,k}$.

Proof. The case-by-case analysis of [IN99, Section 12] carries directly over from finite subgroups of $\mathbf{SL}_2(\mathbb{C})$ to finite and linearly reductive subgroup schemes of $\mathbf{SL}_{2,k}$. □

Here is a second proof of this theorem that uses our lifting results:

Proof. If $p = 0$, then the Lefschetz principle allows to reduce to $k = \mathbb{C}$, where the theorem is due to Ito, Nakajima and Nakamura [IN99, Na96].

We now assume $p > 0$. Let $\mathcal{X}_{\text{can}} = \mathcal{U}/\mathcal{G}_{\text{can}} \rightarrow \text{Spec } W(k)$ be the canonical lift of $x \in X = U/G$ (see Section 4.4) and let $\tilde{\pi} : \mathcal{Y} \rightarrow \mathcal{X}_{\text{can}} \rightarrow \text{Spec } W(k)$ be the simultaneous resolution of singularities (see Section 4.5). Moreover, we have the abstract group G_{abs} and the canonical lift G_{can} associated to G (see Section 2.2) and we have a bijection of simple representations of G_{abs} , G_{can} and G by Proposition 2.6.

The generic fibre \mathcal{X}_K is isomorphic to $\mathcal{U}_K/\mathcal{G}_{\text{can}}$ and using the Lefschetz principle, the geometric generic fibre can be identified with $\mathbb{C}^2/G_{\text{abs}}$. Over \mathbb{C} , we have Theorem 5.3 for G_{abs} $\mathbb{C}^2/G_{\text{abs}}$ and its minimal resolution of singularities by [IN99, Na96].

By the Lefschetz principle, we have it for G_{can} , $\mathcal{X}_K = \mathcal{U}_K/\mathcal{G}_{\text{can}}$ and $\tilde{\pi}_K$. Using the comparison results Proposition 2.6 and Theorem 4.10, we obtain it for $\mathcal{X}_k = \mathcal{U}_k/\mathcal{G}_{\text{can},k}$ and $\tilde{\pi}_k$, that is, for $X = U/G$, G and π . □

5.3. Local McKay correspondence

We now extend work of Ishii and Nakamura [Is02, IN19] to our linearly reductive setting. Let G be a very small, finite and linearly reductive subgroup scheme of $\mathbf{GL}_{2,k}$ and let $x \in X = U/G$ be the associated two-dimensional linearly reductive quotient singularity. Until the end of this section, we do not require it to be a Klein singularity. Note that in dimension two, linearly reductive quotient singularities are the same as F-regular singularities, see Remark 4.6.

Let $\{\rho_i\}_i$ be the set of simple representations of G . Let ρ_0 be the trivial representation and we choose our numbering of the ρ_i 's to be the one of [IN19, Theorem 3.6]. Let $\pi : Y \rightarrow X$ be its minimal resolution of singularities and let $\{E_i\}$ be the irreducible components of the exceptional divisor $\text{Exc}(\pi)$ of π . Then, there exists a connection between the $\{\rho_i\}$ and the exceptional divisors of π as follows - this is yet another version of the McKay correspondence.

Theorem 5.4. *Keeping assumptions and notations, let $\mathfrak{m} \subset \mathcal{O}_U$ be the maximal ideal corresponding to the origin, let $y \in Y$ be a closed point, let Z_y be the G -invariant cluster of U corresponding to y and let $\mathcal{I}_{Z_y} \subset \mathcal{O}_U$ be its ideal sheaf. Then, the G -representation on $\mathcal{I}_{Z_y}/\mathfrak{m}\mathcal{I}_{Z_y}$ is given by*

$$\begin{cases} \rho_i \oplus \rho_0 & \text{if } y \in E_i \setminus \bigcup_{j \neq i} E_j \\ \rho_i \oplus \rho_j \oplus \rho_0 & \text{if } y \in E_i \cap E_j \text{ with } i \neq j. \end{cases}$$

Proof. For $k = \mathbb{C}$, this is [Is02, Theorem 7.1]. For k algebraically closed of arbitrary characteristic and G a very small and finite subgroup of $\mathbf{GL}_2(k)$ of order prime to p , this is [IN19, Theorem 3.6]. However, these proofs also work if G is a very small and finite and linearly reductive subgroup scheme of $\mathbf{GL}_{2,k}$. \square

Remark 5.5. In [IN19], the G -quiver structure of $G\text{-Hilb}(U)$ was studied in the case where G is a very small and finite subgroup of $\mathbf{GL}_2(k)$. We leave the extension of these results to the linearly reductive case to the reader.

5.4. Reflexive sheaves on the minimal resolution

We end this section by shortly digressing on work of Wunram [Wu88], Ishii and Nakamura [IN19], which generalises work of Artin and Verdier [AV85]. We keep the assumptions and notations from Section 5.3. Let $F \subset Y$ be the fundamental divisor of π , see [Ar66].

Theorem 5.6. *Keeping assumptions and notations, there exists a bijection between*

1. *the set of irreducible components $\{E_i\}$ of $\text{Exc}(\pi)$ and*
2. *the set of non-trivial indecomposable full \mathcal{O}_Y -modules $\{\mathcal{M}_i\}$, special in the sense that $H^1(Y, \mathcal{M}_i^\vee) = 0$*

This correspondence $\mathcal{M}_i \mapsto E_i$ is defined by

$$c_1(\mathcal{M}_i) \cdot E_j = \delta_{i,j}.$$

The rank of \mathcal{M}_i is equal to $c_1(\mathcal{M}_i) \cdot F$, the multiplicity of E_i in F .

Proof. For $k = \mathbb{C}$, this is [Wu88]. For k algebraically closed of arbitrary characteristic and G a very small and finite subgroup of $\mathbf{GL}_{2,k}$ of order prime to p , this is [IN19, Theorem 2.8]. However, these proofs also work if G is a very small, finite and linearly reductive subgroup scheme of $\mathbf{GL}_{2,k}$. \square

Here, a full \mathcal{O}_Y -module is as defined in [IN19, Definition 2.4]. By [IN19, Corollary 2.5], the assignment $\mathcal{M} \mapsto \pi_*\mathcal{M}$ sets up a bijection between the set of (indecomposable) full \mathcal{O}_Y -modules with the set of (indecomposable) reflexive \mathcal{O}_X -modules. In particular, the above theorem yields a bijection between the set of irreducible components of $\text{Exc}(\pi)$ and the set of non-trivial, indecomposable and reflexive \mathcal{O}_X -modules.

Remark 5.7. The group scheme G plays no rôle in this theorem and the discussion thereafter. Probably, these results should be viewed as results on two-dimensional F-regular singularities (see Remark 4.6).

6. Conjugacy classes and Ito-Reid correspondence

For a finite group, the number of isomorphism classes of complex simple representations is equal to the number of conjugacy classes. Thus, one can *choose* a bijection between these two sets. In particular, for a finite subgroup of $\mathbf{SL}_2(\mathbb{C})$ one can *choose* a bijection between the vertices of its McKay graph and the conjugacy classes of G . In [IR96], Ito and Reid gave a *canonical* bijection between these two sets. In this section, we extend this *Ito-Reid correspondence* to finite and linearly reductive subgroup schemes of $\mathbf{SL}_{2,k}$. The main difficulty is to define a suitable notion of *conjugacy classes* for finite and linearly reductive group schemes.

6.1. Conjugacy classes

Given a finite group G_{abs} , a representation $\rho : G_{\text{abs}} \rightarrow \mathbf{GL}_n(\mathbb{C})$ and an element $g \in G_{\text{abs}}$, we have the trace $\text{tr}(\rho(g)) \in \mathbb{C}$. This pairing $(\rho, g) \mapsto \text{tr}(\rho(g))$ induces a non-degenerate pairing between isomorphism classes of simple representations of G_{abs} and conjugacy classes. Thus, conjugacy classes

can be thought of as being “dual” to semi-simple representations, which can be used to give an unusual definition of conjugacy classes. This definition generalises to finite and linearly reductive group schemes. More precisely, we make the following definition, which is inspired by a result of Serre [Se77, Section 11.4] and which we discuss in detail in Appendix B.2.

Definition 6.1. Let G be a finite and linearly reductive group scheme over an algebraically closed field k of characteristic $p \geq 0$. Then, the *set of conjugacy classes* is defined to be $\text{Spec } \mathbb{C} \otimes K_k(G)$.

We discuss several approaches to conjugacy classes for finite group schemes in Appendix B - there, we hope to convince the reader that Definition 6.1 is the best for the purposes of this article. For example, by Proposition B.2 it is compatible with canonical lifts and there is a natural bijection with the set of conjugacy classes of the abstract group G_{abs} associated to G . Concerning the choice of field \mathbb{C} in Definition 6.1, we refer to Remarks B.3.

6.2. An explicit bijection

For a finite and linearly reductive subgroup scheme G of $\mathbf{SL}_{2,k}$, we have an associated embedding of the finite group G_{abs} into $\mathbf{SL}_2(\mathbb{C})$. Let $\{\rho_i\}_i$ be the set of isomorphism classes of semi-simple representations of G and let $\{\rho_{\text{abs},i}\}_i$ be the corresponding set for G_{abs} obtained by Proposition 2.6. The associated McKay graphs $\Gamma(G, \rho, \{\rho_i\}_i)$ and $\Gamma(G_{\text{abs}}, \rho_{\text{abs}}, \{\rho_{\text{abs},i}\}_i)$ coincide by Proposition 3.5 and we denote both by $\widehat{\Gamma}$. Next, the group G_{abs} admits a presentation of the form

$$G_{\text{abs}} = \langle A, B, C \mid A^r = B^s = C^t = ABC \rangle$$

for suitable non-negative integers r, s, t . The non-trivial conjugacy classes of G_{abs} can be uniquely represented by the following elements

$$ABC, \quad A^i \text{ with } 1 \leq i \leq r - 1, \quad B^i \text{ with } 1 \leq i \leq s - 1, \quad C^i \text{ with } 1 \leq i \leq t - 1.$$

This allows to give an explicit bijection between the conjugacy classes of G_{abs} and the vertices of $\widehat{\Gamma}$. We refer to [Ki16, Section 2] for details. Using Proposition B.2, we obtain an explicit bijection between the conjugacy classes of G and the vertices of $\widehat{\Gamma}$.

6.3. The Ito-Reid correspondence

If G is a finite subgroup of $\mathbf{SL}_2(\mathbb{C})$, then Ito and Reid [IR96] (see also [Re02]) constructed a *canonical* bijection. Let us run through [Re02, Section 2] and show that this can be carried over to our linearly reductive setting: let k be an algebraically closed field of characteristic $p > 0$ and let G be a finite and linearly reductive subgroup scheme of $\mathbf{SL}_{2,k}$.

1. Associated to G , we have the abstract group G_{abs} . By Proposition 2.6, an embedding $G \rightarrow \mathbf{SL}_{n,k}$ yields an embedding $G_{\text{abs}} \rightarrow \mathbf{SL}_n(\mathbb{C})$. By Proposition B.2, we can identify conjugacy classes of G (in the sense of Definition 6.1) and conjugacy classes of G_{abs} , which allows us to define the *age* of a conjugacy class of G via the corresponding notion for G_{abs} as, for example, in [Re02, Section 2]. A conjugacy class of age 1 is called *junior* and if $n = 2$, then all conjugacy classes are junior.

Remark 6.2. Using the “toric mechanism” mentioned in [Re02, Section 2], one can define the age directly and without referring to lifts, but we will not pursue this here.

2. Let $x \in X := U/G$ be the associated Klein singularity. We have the canonical lift $\mathcal{X}_{\text{can}} \rightarrow \text{Spec } W(k)$ and the simultaneous resolution of singularities $\tilde{\pi} : \mathcal{Y} \rightarrow \mathcal{X}_{\text{can}} \rightarrow \text{Spec } W(k)$ by Theorem 4.10. Passing to geometric generic fibres and using the Lefschetz principle, we obtain the minimal resolution of singularities of $\mathbb{C}^2/G_{\text{abs}}$. The special fibre of $\tilde{\pi}$, the geometric generic fibre of $\tilde{\pi}$ and the minimal

resolution of $\mathbb{C}^2/G_{\text{abs}}$ are *crepant* in the sense of [Re87] and we can identify the exceptional divisors of these three resolutions with each other, see Theorem 4.10. This way, we obtain an Ito-Reid correspondence between junior conjugacy classes of G and crepant divisors of the resolution [Re02, Theorem 2.1].

Remark 6.3. It seems reasonable that one can extend this correspondence to finite and linearly reductive subgroup schemes of $\mathbf{SL}_{n,k}$ with $n \geq 3$, but we will not pursue this here.

7. Derived categories

Let G be a very small, finite and linearly reductive subgroup scheme of $\mathbf{GL}_{2,k}$, let $x \in X := U/G$ be the associated linearly reductive quotient singularity and let $\pi : Y \rightarrow X$ be its minimal resolution of singularities. Gonzalez-Sprinberg and Verdier [GV83] gave an interpretation of the McKay correspondence as an isomorphism between the K-groups $K^G(U)$ and $K(Y)$. Kapranov and Vasserot [KV00] and Bridgeland, King and Reid [BKR01] generalised this to an equivalence of derived categories $\mathcal{D}^G(U)$ and $\mathcal{D}(Y)$. In this section, we extend this to our setting, following Ishii, Ito, Nakamura and Ueda [Is02, IN19, IU15].

We have a commutative diagram

$$\begin{CD} U \times_k Y @>\pi_U>> U \\ @V\pi_YVV @VV\varpi V \\ Y @>\pi>> X. \end{CD}$$

By Theorem 4.5, the minimal resolution π can be constructed by the Ishii-Ito-Nakamura resolution

$$G\text{-Hilb}(U) \rightarrow U/G$$

We let \mathcal{Z} be the universal cluster over $G\text{-Hilb}(U)$, we identify $G\text{-Hilb}(U)$ with Y and then, we have a commutative diagram

$$\begin{CD} \mathcal{Z} @>q>> U \\ @VpVV @VV\varpi V \\ Y @>\pi>> X. \end{CD}$$

Let $\mathcal{D}(Y)$ be the derived category of coherent sheaves on Y . Let $\mathcal{D}^G(U)$ be the derived category of G -equivariant coherent sheaves on U . Following [Is02] and [IN19], we define two functors

$$\begin{aligned} \Psi &: \mathcal{D}^G(U) \rightarrow \mathcal{D}(Y) \\ \Phi &: \mathcal{D}(Y) \rightarrow \mathcal{D}^G(U) \end{aligned}$$

by

$$\Psi(-) := [p_* \mathbf{L}q^*(-)]^G$$

and

$$\Phi(-) := \mathbf{R}\pi_{U,*} \left(\mathcal{O}_{\mathcal{Z}}^\vee \otimes^{\mathbf{L}} \pi_Y^*(- \otimes \rho_0) \otimes^{\mathbf{L}} \pi_U^* K_U \right) [2],$$

where $\mathcal{O}_{\mathcal{Z}}^\vee := \mathbf{R}Hom(\mathcal{O}_{\mathcal{Z}}, \mathcal{O}_{Y \times U})$ denotes the dual of $\mathcal{O}_{\mathcal{Z}}$, where $- \otimes \rho_0 : \mathcal{D}(Y) \rightarrow \mathcal{D}^G(Y)$ denotes the functor that attaches the trivial G -action and where K_U denotes the canonical sheaf of U . We refer to [IN19, Section 3.1] for details, conventions and notations.

Theorem 7.1. *Keeping assumptions and notations, Φ is fully faithful and Ψ is a left adjoint of Φ .*

Proof. For G is a very small subgroup of $\mathbf{GL}_2(\mathbb{C})$, this is [Is02, Section 6] and [IU15, Proposition 1.1 and Lemma 2.9]. For k algebraically closed of arbitrary characteristic and G a very small and finite subgroup of $\mathbf{GL}_2(k)$ of order prime to p , this is [IN19, Theorem 3.2]. However, these proofs also work if G is a very small, finite and linearly reductive subgroup scheme of $\mathbf{GL}_{2,k}$. \square

A. Hopf algebras

In the first section of the appendix, we study finite group schemes from the point of view of finite-dimensional Hopf algebras. We also recall the quantum double of a Hopf algebra, as well as the adjoint and the extended adjoint representation. Many results of this section should be well-known to the experts, but are somewhat scattered over the literature - especially, since many sources (sometimes implicitly) assume characteristic zero or work even over the complex numbers.

A.1. Generalities

If G is a finite group scheme over a field k , then the k -algebra $A := H^0(G, \mathcal{O}_G)$ is commutative and finite-dimensional as k -vector space. The multiplication $m : G \times G \rightarrow G$ the inverse $i : G \rightarrow G$ and the neutral element $e : \text{Spec } k \rightarrow G$ induce k -algebra homomorphisms $m^* : A \rightarrow A \otimes_k A$, $i^* : A \rightarrow A$ and $e^* : A \rightarrow k$, which turn A into a co-algebra over k with co-multiplication m^* and antipode $S := i^*$, and thus, into a Hopf algebra over k . Since i is the inverse of G , the antipode S satisfies $S^2 = \text{id}_A$, that is, A is an involutive Hopf algebra.

Conversely, if A is a finite-dimensional and commutative Hopf algebra over k , then it is involutive and $\text{Spec } A$ is a finite group scheme over k . Moreover, A is a co-commutative Hopf algebra if and only if G is a commutative group scheme.

Example A.1. Let G_{abs} be a finite group. The group algebra $k[G_{\text{abs}}]$ becomes a Hopf algebra by defining the co-multiplication to be $\Delta(g) = g \otimes g$ and the antipode to be $S(g) = g^{-1}$.

1. Clearly, $k[G_{\text{abs}}]$ is an involutive and co-commutative Hopf algebra. Moreover, $k[G_{\text{abs}}]$ is commutative if and only if G_{abs} is commutative.
2. There exist examples of non-isomorphic finite groups H_{abs} and G_{abs} , whose group rings $k[H_{\text{abs}}]$ and $k[G_{\text{abs}}]$ are isomorphic as k -algebras.

However, they are not isomorphic as Hopf algebras: Recall that an element $x \in B$ in a Hopf algebra B is called *group-like* if $\Delta(x) = x \otimes x$. The set of group-like elements of B form a group. If $B = k[G_{\text{abs}}]$, then the group of group-like elements of B is isomorphic to G_{abs} and thus, recovers the group.

3. There exists an isomorphism of finite group schemes over k

$$G \cong \text{Spec } k[G_{\text{abs}}]^*$$

where $k[G_{\text{abs}}]^*$ denotes the dual Hopf algebra and where G denotes the constant group scheme over k associated to G_{abs} .

A.2. Simplicity

A Hopf algebra A is *co-semi-simple* if its dual Hopf algebra A^* is semi-simple and A is *bi-semi-simple* if both A and A^* are semi-simple.

Let G be a finite group scheme over k and let $A := H^0(G, \mathcal{O}_G)$ be the associated Hopf algebra. To give a finite-dimensional and k -linear representation $\rho : G \rightarrow \text{GL}(V)$ is the same as to give an A -co-module $V \rightarrow V \otimes_k A$, see [Wa79, Section 3.2] for details. Using this equivalence, we see that G is

linearly reductive if and only if A is co-semi-simple. For the classification of linearly reductive group schemes (see Section 2) in the language of Hopf algebras we refer to [Ch92].

The following equivalences are probably well-known to the experts.

Proposition A.2. *Let G be a finite group scheme over an algebraically closed field k of characteristic $p \geq 0$. Let $A := H^0(G, \mathcal{O}_G)$ be the Hopf algebra associated to G .*

1. A is semi-simple if and only if G is étale.
2. A is co-semi-simple if and only if G is linearly reductive.
3. A is bi-semi-simple if and only if G is of length prime to p .

Proof. We already established Claim (2) above.

If G is étale, then $A \cong k[G_{\text{abs}}]^*$, where $G_{\text{abs}} := G(k)$ is the abstract group associated to G . Since group rings are co-semi-simple, A is semi-simple. Conversely, if A is semi-simple, then $\langle \varepsilon, \int_H^r \rangle \neq 0$ by Maschke’s theorem for Hopf algebras (see, for example, [Lo18, Section 12.3.1] for notation, statement and proof), which implies that A is a separable k -algebra, which implies that G is étale over k . This establishes Claim (1).

Of course, (3) follows immediately from (1) and (2), but we can also give an independent proof: Being an involutive Hopf algebra, A is bi-semi-simple if and only if p does not divide $\dim_k A$, see [EG98, Corollary 3.2] or [LR88, Corollary 2.6]. The latter is equivalent to G being of length prime to p . This establishes Claim (3). □

A.3. Adjoint representation

Let G_{abs} be a finite group and let k be an algebraically closed field of characteristic $p \geq 0$ (p may or may not divide the order of G_{abs}). Then, the action of G_{abs} on itself by conjugation is a permutation representation and we denote by

$$\rho_{\text{ad}} : G_{\text{abs}} \rightarrow \mathbf{GL}(V_{\text{ad}})$$

the associated k -linear representation.

Remark A.3. To be more precise: Depending on whether one considers $x \mapsto gxg^{-1}$ or $x \mapsto g^{-1}xg$, one should speak about *left* or *right* adjoint actions and representations. For our discussion, this is not important, as long as one chooses one of them and stays with it.

Remark A.4. Let us recall a couple of general results about ρ_{ad} , which are well-known (I thank Frank Himstedt for explaining them to me):

1. Let C be the set of conjugacy classes of G_{abs} and let $\{g_c\}_{c \in C}$ be a system of representatives. Then, we have an isomorphism of k -linear representations

$$\rho_{\text{ad}} \cong \bigoplus_{c \in C} \text{Ind}_{C(g_c)}^{G_{\text{abs}}} \mathbb{1},$$

where $C(g_c)$ denotes the centraliser of $g_c \in G_{\text{abs}}$, where $\mathbb{1}$ denotes the one-dimensional trivial representation and where Ind denotes induction from a subgroup. See, for example, [NT89, page 171 and Exercise 1.5].

2. Let $g \in G_{\text{abs}}$ and let

$$V := \text{Ind}_{C(g)}^{G_{\text{abs}}} \mathbb{1} \cong \bigoplus_i V_i$$

be a decomposition into indecomposable representations, which is unique up to isomorphism and numbering.

There is precisely one summand, say V_{i_1} , that has a subrepresentation that is isomorphic to $\mathbb{1}$. Moreover, this subrepresentation is unique. Also, there is precisely one summand, say V_{i_2} , that has a quotient representation that is isomorphic to $\mathbb{1}$. Moreover, this quotient representation is unique. Then, we have $i_1 = i_2$ and set

$$S(V) := V_{i_1} = V_{i_2},$$

which is called the *Scott representation* of V . See, for example, [NT89, page 296 and Theorem 8.4].

In particular, the dimension of the largest trivial subrepresentation (resp. quotient representation) of ρ_{ad} is equal to the number of conjugacy classes of G_{abs} .

If A is a finite dimensional Hopf algebra over k , then there is an *adjoint action* of A on itself (in fact, a *left adjoint action* and a *right adjoint action*), see, for example, [Mo93, Definition 3.4.1]. We will write

$$\text{ad} : A \rightarrow \text{End}(A)$$

or simply ${}^{\text{ad}}A$ for this representation.

Remarks A.5.

1. If G_{abs} is a finite group, then the adjoint representation of $H := k[G_{\text{abs}}]$ can be identified with the k -linear extension from G_{abs} to H of the conjugation action of G_{abs} on itself.
2. Let G be a finite group scheme over k and let $A := H^0(G, \mathcal{O}_G)$ be the $A := H^0(G, \mathcal{O}_G)$. There is an adjoint representation

$$\rho_{\text{ad}} : G \rightarrow \mathbf{GL}(V_{\text{ad}}),$$

which corresponds to the adjoint representation of the *dual Hopf algebra* A^* .

- (a) If G_{abs} is a finite group and G is the constant group scheme associated to it, then A is isomorphic to the dual of $k[G_{\text{abs}}]$ equipped with its usual Hopf algebra structure. This shows that ρ_{ad} should be defined via the adjoint representation of A^* rather than A .
- (b) The representation ρ_{ad} should not be confused with the adjoint representation of G on its Lie algebra, see for example, [Wa79, page 100, Exercise 13]. The latter can be identified with a subquotient of ρ_{ad} .

A.4. Quantum doubles

If A is a finite-dimensional Hopf algebra over a field k , then Drinfeld [Dr87] defined a Hopf algebra $D(A) := (A^{\text{op}})^* \bowtie A$, called the *quantum double* or *Drinfeld double*, where the bicrossed product structure is defined using the co-adjoint representation of A on A^* and the co-adjoint representation of A^* on A , see also [Mo93, Definition 10.3.1].

Remarks A.6. Let A be a finite-dimensional Hopf algebra over k and let $D(A) := (A^{\text{op}})^* \bowtie A$ be its quantum double.

1. As k -vector space, $D(A)$ is of dimension $(\dim_k A)^2$.
2. A is a Hopf subalgebra of $D(A)$ via $\varepsilon \bowtie A$ and $(A^{\text{op}})^*$ is a Hopf subalgebra of $D(A)$ via $(A^{\text{op}})^* \bowtie 1$.
3. If A is commutative and co-commutative, then we have $A^{\text{op}} = A$ and using the definition of the quantum double, we obtain isomorphisms

$$D(A) \cong A^* \otimes_k A \cong D(A^*)$$

of Hopf algebras over k , where the tensor product is the trivial tensor product of Hopf algebras. If we set $G := \text{Spec } A$, then we have $G^D \cong \text{Spec } A^*$, where $-^D$ denotes the Cartier dual group scheme,

and we have an isomorphism

$$\text{Spec } D(A) \cong G \times_{\text{Spec } k} G^D$$

of finite and commutative group schemes over k .

4. In particular, $D(A)$ is commutative if and only if A is commutative and co-commutative.

The assertions on simplicity in the next proposition extend results of Witherspoon [Wi96, Proposition 1.2] from groups to group schemes. They are also related to general semi-simplicity results of quasi-triangular Hopf algebras in positive characteristic due to Etingof and Gelaki [EG98] and in this form, they might be known to the experts.

Proposition A.7. *Let G be a finite group scheme over an algebraically closed field k of characteristic $p \geq 0$ and let $A := H^0(G, \mathcal{O}_G)$ be the associated Hopf algebra. Then, the following are equivalent*

- (1) $D(A)$ is semi-simple
- (2) $D(A)$ is co-semi-simple
- (3) A is bi-semi-simple
- (4) G is of length prime to p
- (1') $D(A^*)$ is semi-simple
- (2') $D(A^*)$ is co-semi-simple
- (3') A^* is bi-semi-simple

Proof. The equivalences (1) \Leftrightarrow (2) \Leftrightarrow (3) and (1') \Leftrightarrow (2') \Leftrightarrow (3') are shown in [Mo93, Corollary 10.3.13]. The equivalence (3) \Leftrightarrow (3') is trivial and finally, the equivalence (3) \Leftrightarrow (4) was shown in Proposition A.2. □

A.5. Extended adjoint representation

Given a Hopf algebra A , we defined the adjoint representation ${}^{\text{ad}}A$ in Section A.3 above. There is a way to extend this to a representation

$$\text{Ad} : D(A) \rightarrow \text{End}(A)$$

the *extended adjoint representation* of A , which is denoted by ${}^{\text{Ad}}A$. For its definition, we refer to Zhu's article [Zhu97], as well as [CW11] for subsequent work on this representation. We note that it can be described as an induced representation

$${}^{\text{Ad}}A \cong \text{Ind}_A^{D(A)} \mathbb{1}, \tag{A.1}$$

where $\mathbb{1}$ denotes the trivial one-dimensional representation and where we consider A as a subalgebra of $D(A)$ via $\varepsilon \bowtie A$, see [Bu06] or [Ja17, Section 4.3.4].

Example A.8. Let G_{abs} be a finite group and let $H := k[G_{\text{abs}}]$ be the group algebra equipped with its usual Hopf algebra structure. The elements $g \in G_{\text{abs}}$ form a basis of H as k -vector space and we denote by $\rho_g \in H^*$ the dual basis elements.

1. The multiplication of $D(H)$ is given by

$$(\rho_h \bowtie g) \cdot (\rho_k \bowtie \ell) = \delta_{h, gkg^{-1}} \cdot \rho_h \bowtie g\ell,$$

see, for example, [Ja17, Example 2.4.2].

(a) The extended adjoint representation ${}^{\text{Ad}}H$ is given by

$$(\rho_h \bowtie k) \mapsto (g \mapsto \delta_{h^{-1}, k g k^{-1}} \cdot k g k^{-1})$$

see, for example, [Ja17, Example 2.5.3].

- (b) The unit of H^* is $\varepsilon := \sum_{g \in G_{\text{abs}}} \rho_g$ and the restriction of ${}^{\text{Ad}}H$ to $\varepsilon \bowtie H$ is ${}^{\text{ad}}H$. Thus, the extended adjoint representation extends the adjoint representation from an A -representation to a $D(A)$ -representation, whence the name.
- (c) The extended adjoint representation ${}^{\text{Ad}}H$ is semi-simple. More precisely, the k -subvector space of H generated by the elements of a conjugacy class of G_{abs} is a simple $D(H)$ -subrepresentation of ${}^{\text{Ad}}H$. This gives a bijection between conjugacy classes of G_{abs} and simple subrepresentations of ${}^{\text{Ad}}H$. By [Ja17, Lemma 4.3.2], this is also true if p divides the order of G_{abs} .

We refer to [Bu06, DPR91, Go93, Wi96] for more results about the representation theory of $D(H)$ and ${}^{\text{Ad}}H$.

2. If we identify H^{**} with H , then the multiplication of $D(H^*)$ is given by

$$(g \bowtie \rho_h) \cdot (\ell \bowtie \rho_k) = \delta_{h,k} \cdot (g \cdot \ell) \bowtie \rho_h$$

and the extended adjoint representation ${}^{\text{Ad}}(H^*)$ is given by

$$(h \bowtie \rho_k) \mapsto (\rho_g \mapsto \rho_{gh^{-1}}),$$

which can be identified with the dual of the regular representation of G_{abs} . The restriction of ${}^{\text{Ad}}(H^*)$ to $1 \bowtie H^*$ is trivial, which is equal to the adjoint representation ${}^{\text{ad}}(H^*)$, which is also trivial.

Example A.9. Let A be a finite-dimensional Hopf algebra over k that is commutative and co-commutative.

1. The adjoint representations ${}^{\text{ad}}A$ and ${}^{\text{ad}}(A^*)$ are trivial.

By Remark A.6, we have isomorphisms

$$D(A) \cong A^* \otimes_k A \cong D(A^*)$$

of Hopf algebras over k , where the tensor product is the trivial tensor product of Hopf algebras. Thus, to give a representation $\rho : D(A) \rightarrow \text{End}(V)$ is equivalent to giving two representations $\rho_1 : A \rightarrow \text{End}(V)$ and $\rho_2 : A^* \rightarrow \text{End}(V)$, whose actions on V commute.

- 2. Using the description (A.1) of ${}^{\text{Ad}}A$ as induced representation, we obtain ${}^{\text{Ad}}A$ from the the trivial A -action on A (this is ρ_1) and the dual of the regular representation of A^* (this is ρ_2).
- 3. Similarly, we obtain ${}^{\text{Ad}}(A^*)$ from the trivial A^* -action on A^* and the dual of the regular representation of A .

B. Conjugacy classes for finite group schemes

Let G be a finite group scheme over an algebraically closed field k of characteristic $p \geq 0$. In the second section of the appendix, we discuss several approaches towards the notion of a *conjugacy class* for G . If $p = 0$, then all of them lead to the same notion, namely, the familiar one. All approaches look reasonable at first sight if $p > 0$ and they lead to essentially “the same” answer if G is of length prime to p . In general however, they lead to different notions, all of which have their merits and drawbacks. This appendix serves as a motivation for Definition 6.1, but the discussion and results may be interesting in themselves.

B.1. First approach: via rational points

Let $G(k)$ be the group of k -rational points. We obtain an equivalence relation \sim on this set by defining $g_1 \sim g_2$ if and only if there exists $h \in G(k)$ such that $g_1 = hg_2h^{-1}$. The quotient $G(k)/\sim$ is the first candidate for the set of conjugacy classes.

Example B.1. If G is étale over k , then it is the constant group scheme associated to the finite group $G_{\text{abs}} := G(k)$. In this case, $G(k)/\sim$ coincides with the set of conjugacy classes of G_{abs} .

If G is étale, which is automatic if $p = 0$, then this approach is satisfactory. However, if $p > 0$, then the connected-étale sequence (2.1) induces a bijection

$$G(k) \cong G^{\text{ét}}(k),$$

which is an isomorphism of finite groups. In particular, $G(k)/\sim$ depends on the maximal étale quotient $G^{\text{ét}}$ of G only. For example, in the extremal case where G is connected, we have $G(k) = \{1\}$ and then, $G(k)/\sim$ consists of one element, and we do not gain much information about G .

Concerning functoriality: if $\varphi : G \rightarrow H$ is a homomorphism of finite group schemes over k , then we have induced morphisms $G(k) \rightarrow H(k)$, $G^{\text{ét}} \rightarrow H^{\text{ét}}$ and $G(k)/\sim \rightarrow H(k)/\sim$.

B.2. Second approach: via representations and K-theory

If G_{abs} is a finite group of order prime to p , then the category $\text{Rep}_k(G)$ of k -linear and finite-dimensional G_{abs} -representations is semi-simple. In this case, the number of isomorphism classes of simple representations is equal to the number of conjugacy classes. More precisely, if ρ is a k -linear and finite-dimensional representation of G_{abs} and $g \in G_{\text{abs}}$ is an element, then

$$(\rho, g) \mapsto \text{Tr}(\rho(g)) \in k$$

induces a pairing between representations and conjugacy classes. Since the character table of a finite group is a quadratic and invertible matrix (see, for example, [Se77, Proposition I.7]), this pairing is non-degenerate and one can think of conjugacy classes as being “dual” to simple representations.

This idea can be made precise as follows: Let F be a field, let $\text{Cl}(G_{\text{abs}})$ be the set of conjugacy classes of G_{abs} and let $F^{\text{Cl}(G_{\text{abs}})}$ be the ring of class functions on G_{abs} with values in F . Let g be the order of G_{abs} , let $\zeta_g \in \mathbb{C}$ be a primitive g -th primitive root of unity and assume that F contains $\mathbb{Q}(\zeta_g)$. Since the characters of G_{abs} take values in F (here, we use $\mathbb{Q}(\zeta_g) \subseteq F$), we have injective ring homomorphisms

$$F \rightarrow F \otimes_{\mathbb{Z}} K_k(G_{\text{abs}}) \xrightarrow{\gamma} F^{\text{Cl}(G_{\text{abs}})}$$

and γ is an isomorphism, see [Se77, Section 11.4] and the first paragraph of [Se77, Section 9.1]. In particular,

$$\text{Spec } F \otimes_{\mathbb{Z}} K_k(G_{\text{abs}})$$

is a finite set consisting of maximal ideals only and the topology is discrete. Since γ is an isomorphism, the cardinality of this set is equal to that of $\text{Cl}(G_{\text{abs}})$. This carries over to finite and linearly reductive schemes as follows.

Proposition B.2. *Let G be a finite and linearly reductive group scheme over k and let G_{abs} be the associated abstract group. Let g be the length of G , fix a g -th root of unity $\zeta_g \in \mathbb{C}$ and let F be a field that contains $\mathbb{Q}(\zeta_g)$. Then, there exists a canonical bijection of sets*

$$\{ \text{conjugacy classes of } G_{\text{abs}} \} \rightarrow \text{Spec } F \otimes_{\mathbb{Z}} K_k(G_{\text{abs}})$$

and homeomorphisms

$$\text{Spec } F \otimes_{\mathbb{Z}} K_k(G_{\text{abs}}) \leftarrow \text{Spec } F \otimes_{\mathbb{Z}} K_{\overline{k}}(G_{\text{abs}}) \leftarrow \text{Spec } F \otimes_{\mathbb{Z}} K_k(G).$$

Proof. If c is a conjugacy class of G_{abs} , then $P_{0,c}$ (notation as in [Se77, Section 11.4, Proposition 30]) is an element of $\text{Spec } F \otimes K_k(G_{\text{abs}})$ and by loc. cit. this defines a bijection. The isomorphisms of Corollary 2.8 induce homeomorphisms and thus, bijections of spectra as stated. \square

Remarks B.3.

1. The maximal abelian extension $\mathbb{Q} \subset \mathbb{Q}^{\text{ab}}$ is generated by all roots of unity by the Kronecker-Weber theorem. Thus, if we have $\mathbb{Q}^{\text{ab}} \subseteq F$, then we have a field that works independent of the length of G . In Definition 6.1, we have chosen $F = \mathbb{C}$ as this field may be more familiar than \mathbb{Q}^{ab} .
2. In [Se77, Section 11.4], Serre described $\text{Spec } A \otimes_{\mathbb{Z}} K_k(G_{\text{abs}})$, where $A = \mathbb{Z}[\zeta_g]$. Using Corollary 2.8, we obtain a homeomorphism

$$\text{Spec } A \otimes_{\mathbb{Z}} K_k(G) \rightarrow \text{Spec } A \otimes_{\mathbb{Z}} K_k(G_{\text{abs}}).$$

In particular, Serre’s results from loc.cit. carry over to $A \otimes_{\mathbb{Z}} K_k(G)$. For the purposes of this article, we are only interested in the fibre over $0 \in \text{Spec } F$ with $F = \text{Frac}(A) = \mathbb{Q}(\zeta_g)$, that is, the Zariski open subset $\text{Spec } F \otimes_{\mathbb{Z}} K_k(G) \subset \text{Spec } A \otimes_{\mathbb{Z}} K_k(G)$.

Example B.4. Let G be the group scheme α_p or \mathbf{C}_p over the algebraically closed field k of characteristic $p > 0$. Then, $\text{Rep}_k(G)$ is not semi-simple: the only simple k -linear representation of G is the trivial one-dimensional representation $\mathbb{1}$. Thus, $\mathbb{1} \mapsto 1$ induces an isomorphism of rings $K_k(G) \cong \mathbb{Z}$ and $\text{Spec } F \otimes_{\mathbb{Z}} K_k(G)$ consists of one point only. The approach to conjugacy classes in this subsection may therefore lead to somewhat unexpected results if G is not linearly reductive.

Proposition B.5. Let $\varphi : G \rightarrow H$ be a morphism of finite and linearly reductive group schemes over k . Let $\varphi_{\text{abs}} : G_{\text{abs}} \rightarrow H_{\text{abs}}$ be the induced homomorphism of their associated abstract groups. Let F be a field that contains $\mathbb{Q}(\zeta_g, \zeta_h)$, where g (resp. h) denotes the length of G (resp. H).

1. The maps φ and φ_{abs} induce ring homomorphisms $K_k(H) \rightarrow K_k(G)$ and $K_{\overline{k}}(H_{\text{abs}}) \rightarrow K_{\overline{k}}(G_{\text{abs}})$, respectively. We obtain a commutative diagram of continuous maps

$$\begin{array}{ccc} \text{Spec } F \otimes_{\mathbb{Z}} K_k(G) & \longrightarrow & \text{Spec } F \otimes_{\mathbb{Z}} K_k(H) \\ \downarrow & & \downarrow \\ \text{Spec } F \otimes_{\mathbb{Z}} K_{\overline{k}}(G_{\text{abs}}) & \longrightarrow & \text{Spec } F \otimes_{\mathbb{Z}} K_{\overline{k}}(H_{\text{abs}}), \end{array}$$

whose vertical arrows are the homeomorphisms from Proposition B.2.

2. Let $G_{\text{abs}}/\sim \rightarrow H_{\text{abs}}/\sim$ be the map on conjugacy classes induced by φ_{abs} . We obtain a commutative diagram of maps of sets

$$\begin{array}{ccc} G_{\text{abs}}/\sim & \longrightarrow & H_{\text{abs}}/\sim \\ \downarrow & & \downarrow \\ \text{Spec } F \otimes_{\mathbb{Z}} K_{\overline{k}}(G_{\text{abs}}) & \longrightarrow & \text{Spec } F \otimes_{\mathbb{Z}} K_{\overline{k}}(H_{\text{abs}}), \end{array}$$

whose vertical maps are the bijections from Proposition B.2.

Proof. Clearly, φ induces a morphism $K_k(H) \rightarrow K_k(G)$ of rings since every H -representation becomes a G -representation via φ . Similarly, φ_{abs} induces a ring homomorphism $\varphi_{K, \text{abs}} : K_{\overline{k}}(H_{\text{abs}}) \rightarrow K_{\overline{k}}(G_{\text{abs}})$. We leave it to the reader to check the compatibility of these maps with the homeomorphisms of Proposition B.2.

To check commutativity of the second diagram, let $g \in G_{\text{abs}}$. With the notations and definitions of [Se77, Section 11.4], it is easy to see that we have

$$P_{0, \varphi_{\text{abs}}(g)} = \varphi_{K, \text{abs}}^{-1}(P_{0, g}) = \varphi_{K, \text{abs}}^{\#}(P_{0, g}),$$

where

$$\varphi_{K, \text{abs}}^{\#} : \text{Spec } F \otimes K_{\overline{K}}(G_{\text{abs}}) \rightarrow \text{Spec } F \otimes K_{\overline{K}}(H_{\text{abs}})$$

is the induced map on spectra. Since the image of the conjugacy class $[g]$ of G_{abs} is the conjugacy class $[\varphi_{\text{abs}}(g)]$ of H_{abs} , the assertion follows. \square

B.3. Third approach: the scheme of conjugacy classes

Just as group schemes generalise the notion of a group, one could try to replace the *set* of conjugacy classes by a suitable notion of *scheme* of conjugacy classes. More precisely, let G be a finite group scheme over k and let $\underline{\text{Aut}}(G)$ be the automorphism group scheme of G . For every scheme $T \rightarrow \text{Spec } k$, we have the set $G(T)$ of T -valued points of G and a conjugation action of $G(T)$ on $G(T)$. This induces a morphism of schemes $G \rightarrow \underline{\text{Aut}}(G)$. We will say that two elements of $G(T)$ are *equivalent* if they differ by such an automorphism and we denote the resulting equivalence relation by \sim .

We obtain a functor from the category of schemes over k to sets

$$\begin{aligned} \underline{\text{Conj}}_G & : (\text{Schemes}/k) \rightarrow (\text{Sets}) \\ T & \quad \mapsto G(T)/\sim . \end{aligned}$$

This functor should somehow represent the conjugacy classes of G in the sense of schemes.

Proposition B.6. *Let G be a finite group scheme over k . Then, the functor $\underline{\text{Conj}}_G$ is representable by a scheme Conj_G , which is finite over $\text{Spec } k$.*

Proof. Set $V := H^0(G, \mathcal{O}_G)$ and let $\rho_{\text{ad}} : G \rightarrow \mathbf{GL}(V)$ be the adjoint representation of G (see Appendix A.3). Then, the functor $\underline{\text{Conj}}_G$ can be rephrased as the functor that associates to each $T \rightarrow \text{Spec } k$ the quotient of $V \times_{\mathcal{O}_T} \underline{\text{Conj}}_G$ modulo $G(T)$. This amounts to representing the quotient V/G by a scheme. Since V is a vector space and thus, can be identified with an affine scheme, and since G is a finite group scheme, this quotient is representable by a scheme. In fact, it is representable by the spectrum of the invariant ring $\text{Spec } V^G$. \square

Definition B.7. Conj_G is called the *scheme of conjugacy classes* of G .

Remark B.8. If $\rho_{\text{ad}} : G \rightarrow \mathbf{GL}(H^0(G, \mathcal{O}_G))$ is the adjoint representation, then the previous proof shows that the length of Conj_G over $\text{Spec } k$ is equal to the dimension of the maximal trivial subrepresentation of ρ_{ad} . If G_{abs} is a finite group or if G is a finite and étale group scheme over $\text{Spec } k$, then we gave an explicit description of this maximal trivial subrepresentation of ρ_{ad} in Remark A.4.

Examples B.9.

1. If G is étale over k , then G is isomorphic to the constant group scheme associated to $G_{\text{abs}} := G(k)$. In this case, Conj_G is a disjoint union of copies of $\text{Spec } k$, one copy for each conjugacy class of the abstract group G_{abs} .
2. If G is commutative, then the adjoint representation is trivial and thus, Conj_G is isomorphic to the scheme underlying G . For example, if $G = \alpha_p$ or $G = \mu_p$, then Conj_G is a non-reduced scheme of length p with reduction $(\text{Conj}_G)_{\text{red}} \cong \text{Spec } k$.

In characteristic zero, all finite group schemes are étale and Case (1) applies. However, in characteristic $p > 0$, the second example shows that for non-reduced group schemes Conj_G may also be non-reduced. The reduction $(\text{Conj}_G)_{\text{red}}$ is related to our first approach to conjugacy classes:

Proposition B.10.

1. A morphism $G \rightarrow H$ of finite group schemes over k induces a morphism $\text{Conj}_G \rightarrow \text{Conj}_H$ of schemes over k .
2. Assume $p > 0$ and let $G \rightarrow G^{\text{ét}}$ be the maximal étale quotient of the finite group scheme G over k . Then, reduction identifies G_{red} with $G^{\text{ét}}$ and the induced natural inclusion $G^{\text{ét}} \rightarrow G$ induces an isomorphism

$$\text{Conj}_{G^{\text{ét}}} \rightarrow (\text{Conj}_G)_{\text{red}}$$

of schemes over k . In particular, $(\text{Conj}_G)_{\text{red}}$ is a disjoint union of copies of $\text{Spec } k$ with one copy for each conjugacy class of the abstract group $G_{\text{abs}} := G(k) = G^{\text{ét}}(k)$.

Proof. We first prove Claim (1). If $G_{\text{abs}} \rightarrow H_{\text{abs}}$ is a homomorphism of groups, then we get a well-defined induced map of conjugacy classes. Thus, if $G \rightarrow H$ is as in (1) and if T is a scheme over k , then we get a well-defined map $(G(T)/\sim) \rightarrow (H(T)/\sim)$. This induces a morphism of functors $\underline{\text{Conj}}_G \rightarrow \underline{\text{Conj}}_H$ and thus, a morphism of schemes $\text{Conj}_G \rightarrow \text{Conj}_H$.

Let G be as in Claim (2). Let $G_{\text{red}} \rightarrow G$ be the reduction, which is a morphism of group schemes. We thus obtain canonical homomorphisms $G^{\text{ét}} \rightarrow G \rightarrow G^{\text{ét}}$ of group schemes over k , which (by Claim (1)) induce morphisms of their associated schemes of conjugacy classes

$$\text{Conj}_{G^{\text{ét}}} \rightarrow \text{Conj}_G \rightarrow \text{Conj}_{G^{\text{ét}}}.$$

The composition is the identity. Since $\text{Conj}_{G^{\text{ét}}}$ is reduced, we obtain a factorisation

$$\text{Conj}_{G^{\text{ét}}} \rightarrow (\text{Conj}_G)_{\text{red}} \rightarrow \text{Conj}_{G^{\text{ét}}}. \tag{B.1}$$

All these schemes are reduced and finite over the algebraically closed field k . Thus, all of them are finite disjoint unions of copies of $\text{Spec } k$. To prove that the morphisms in (B.1) are isomorphisms, it suffices to check that the induced maps on k -rational points are bijections. This follows easily from the fact that the maps $G_{\text{red}}(k) \rightarrow G(k) \rightarrow G^{\text{ét}}(k)$ are bijections. \square

If G is moreover linearly reductive, then the length of Conj_G is related to the approach to conjugacy classes from Section B.2.

Proposition B.11. *Let G be a finite and linearly reductive group scheme over k , let G_{abs} be the abstract group associated to G and let F be a field as in Proposition B.2. Then,*

$$\text{length}_k \text{Conj}_G = |\text{Spec } F \otimes_{\mathbb{Z}} K_k(G)|$$

and this length agrees with the number of conjugacy classes of G_{abs} .

Proof. By Remark B.8, the length of $\text{Conj}_k(G)$ is equal to the dimension of the largest trivial subrepresentation of ρ_{ad} . Since $\text{Rep}_k(G)$ is semi-simple, this is the same as the multiplicity of $\mathbb{1}$ in ρ_{ad} . Using the isomorphism from Prop 2.6, this multiplicity is the same as the multiplicity of $\mathbb{1}$ in the adjoint representation of G_{abs} . This latter multiplicity is equal to the number of conjugacy classes of G_{abs} , see, for example, Remark A.4. By Proposition B.2, this number is equal to the cardinality of $\text{Spec } F \otimes_{\mathbb{Z}} K_k(G)$. \square

B.4. Fourth approach: via lifting to characteristic zero

Another idea might be to use lifting to characteristic zero: Let k be an algebraically closed field of characteristic $p > 0$, let $W(k)$ be the ring of Witt vectors of k and let K be field of fractions of $W(k)$.

Example B.12 (Mumford–Oort). Let $\varphi : \mu_p \rightarrow \underline{\text{Aut}}(\alpha_p) \cong \mathbb{G}_m$ be a non-trivial homomorphism and let $G := \alpha_p \rtimes_{\varphi} \mu_p$ be the corresponding semi-direct product group scheme. Then, G is a non-commutative

group scheme of length p^2 over k . Oort and Mumford [MO68, Introduction, Example (-B)] (but see also [Oo71, page 266]) showed that there does *not* exist a lift of G to any extension of $W(k)$.

In particular, one *cannot* define conjugacy classes by first lifting G over some possibly ramified extension of $W(k)$, then passing to the geometric generic fibre of the lift, which would be the constant group associated to an abstract finite group G_{abs} , and finally use the conjugacy classes of G_{abs} as a replacement for the conjugacy classes of G . The reason is simply that lifts may not exist to start with.

In the following two cases, lifts do exist and we leave the straightforward proofs of our assertions to the reader:

Remark B.13. Assume that G is étale over k . Then, there exists a unique flat lift of G over $W(k)$, which is the constant group scheme associated to $G_{\text{abs}} := G(k)$ over $W(k)$. The constant scheme associated to the set of conjugacy classes of G_{abs} is the unique a flat lift of the scheme Conj_G over $W(k)$, which is étale over $W(k)$. More precisely, it is the constant scheme associated to the set of conjugacy classes of G_{abs} over $W(k)$.

Remark B.14. Assume that G is linearly reductive. Let G_{abs} abstract group associated to G and recall that their representation categories are equivalent by Proposition 2.6. Proposition B.2 shows that Definition 6.1 is compatible with G_{abs} .

If $\mathcal{G} \rightarrow \text{Spec } W(k)$ is a flat lift of G over $W(k)$, for example, the canonical lift \mathcal{G}_{can} , then

$$\text{Spec } H^0(\mathcal{G}, \mathcal{O}_{\mathcal{G}})^{\mathcal{G}} \rightarrow \text{Spec } W(k),$$

where the invariants are taken with respect to the adjoint representation, is a flat lift of Conj_G over $W(k)$, whose geometric generic fibre is the constant scheme associated to the set of conjugacy classes of G_{abs} . (Flatness follows from the fact that the special and the geometric generic fibre have the same length by Remark B.8.)

The length of Conj_G is equal to the number of conjugacy classes of G_{abs} . The length of Conj_G is at least the number of k -rational points $\text{Conj}_G(k)$ and in general not equal, since Conj_G may not be étale over k .

If G is of length prime to p , then it is étale and linearly reductive and in this case, all previous approaches yield essentially the same notion of conjugacy class. On the other hand, the examples of the above subsections show that if G is étale of length divisible by p or if G is linearly reductive but not étale, then the various approaches of the above subsections usually lead to different notions of conjugacy classes.

B.5. Fifth approach: via adjoint representation

Let G_{abs} be a finite group and let $\rho_{\text{ad}} : G_{\text{abs}} \rightarrow \mathbf{GL}(V_{\text{ad}})$ be its adjoint representation over k . By Remark A.4, the dimension of the largest trivial subrepresentation of ρ_{ad} is equal to the dimension of largest trivial quotient representation of ρ_{ad} (even if p divides the order of G_{abs}) and these dimensions are equal to the number of conjugacy classes of G_{abs} . Unfortunately, these trivial sub- or quotient representations do not admit *canonical* decompositions into one-dimensional subspaces. Thus, there is no canonical bijection between the conjugacy classes of G_{abs} with trivial sub- or quotient representations of ρ_{ad} .

Remark B.15. Let $H := k[G_{\text{abs}}]$ be the group algebra with its usual Hopf algebra structure. The set of group-like elements of H recovers the group G_{abs} , see Example A.1. This suggests to use the adjoint representation ${}^{\text{ad}}H$ together with the Hopf algebra structure of H - in particular, its co-algebra structure - to define a useful notion of conjugacy class.

As we have seen (somewhat implicitly) in Section B.3 above, this works: Let $G := \text{Spec } H^*$ be the constant group scheme associated to G_{abs} . Then, the topological space underlying Conj_G is the set of

group-like elements of the Hopf algebra H modulo the adjoint representation. This set can be identified with the set of conjugacy classes of G_{abs} .

Unfortunately, the previous remark does not carry over to finite group schemes that are not étale, as the following example shows.

Example B.16. Let $G = \mu_p$ or $G = \alpha_p$ over the algebraically closed field k of characteristic $p > 0$.

1. The adjoint representation ρ_{ad} of G is trivial of dimension p , but there is no canonical way to decompose it into one-dimensional subspaces. This would suggest to have p conjugacy classes, but without being able to distinguish them.
2. Moreover, there is only one group-like element in the Hopf algebra $H^0(G, \mathcal{O}_G)^*$ and the adjoint representation on this element is trivial as well. This would suggest to have one conjugacy class only, see also Example B.9.(2).

B.6. Sixth approach: via extended adjoint representation

Given a finite-dimensional Hopf algebra A over a field k , we have the adjoint representation ${}^{\text{ad}}A$, see Appendix A.3. In Appendix A.4, we recalled the quantum double $D(A) = (A^{\text{op}})^* \bowtie A$. In Appendix A.5, we recalled that ${}^{\text{ad}}A$ can be extended to a representation of $D(A)$ on A , the extended adjoint representation ${}^{\text{Ad}}A$.

Generalising work of Witherspoon [Wi99], Cohen and Westreich [CW11] defined the set of conjugacy classes of A for a finite-dimensional semi-simple Hopf algebra over \mathbb{C} to be the set of simple subrepresentations of ${}^{\text{Ad}}A$. In this context, we also refer to the work of Jacoby [Ja17] and Zhu [Zhu97].

If G is a finite group scheme over k with Hopf algebra $A := H^0(G, \mathcal{O}_G)$, then $D(A)$ is semi-simple if and only if $D(A^*)$ is semi-simple if and only if G is of length prime to p , see Proposition A.7. In particular, the (extended) adjoint representations of A or A^* may not be semi-simple. We will now study these representations and their relation to conjugacy classes.

Example B.17. Let G be étale over k . Then it is the constant group scheme associated to $G_{\text{abs}} := G(k)$. If $A := H^0(G, \mathcal{O}_G)$ is the associated Hopf algebra, then we have $A^* \cong k[G_{\text{abs}}]$. As seen in Example A.8, the extended adjoint representation ${}^{\text{Ad}}(A^*)$ is semi-simple and there is a natural bijection between conjugacy classes of G_{abs} and simple subrepresentations of ${}^{\text{Ad}}(A^*)$. We stress that this is also true if p divides the order of G_{abs} .

Example B.18. Let G be a finite and linearly reductive group scheme over k , let G_{abs} be the associated abstract finite group and let $A := H^0(G, \mathcal{O}_G)$ be the associated Hopf algebra. Since A is commutative, ${}^{\text{ad}}A$ is trivial. Using the description (A.1) of ${}^{\text{Ad}}A$ as an induced representation, we can identify it with the dual of the regular representation

$$\rho_{\text{reg}}^\vee : A^* \rightarrow \text{End}(A)$$

together with the trivial representation of A . By Proposition 2.6, representations of A^* can be identified with representations of $\mathbb{C}[G_{\text{abs}}]$ and thus, representations of G_{abs} . Thus, we can decompose ${}^{\text{Ad}}A$ like the dual of the regular representation of G_{abs} : the number of isotypical components of this representation is equal to the number of conjugacy classes of G_{abs} . Thus, one can think of the conjugacy classes of G_{abs} as being “dual” to these isotypical components similarly to Appendix B.2.

The upshot of this discussion is the following: Let G be a finite group scheme over k and let $A := H^0(G, \mathcal{O}_G)$ be the associated Hopf algebra.

1. If G is étale over k , then simple subrepresentations of ${}^{\text{Ad}}(A^*)$ are in bijection with conjugacy classes of $G_{\text{abs}} := G(k)$.
2. If G is linearly reductive, then the isotypical components of ${}^{\text{Ad}}A$ give a reasonably good definition for the “dual” of a conjugacy class.

The following example shows that in general, neither ${}^{\text{Ad}}A$ nor ${}^{\text{Ad}}(A^*)$ leads to a good approach toward the notion of a conjugacy class - at least none that is better than the ones already discussed.

Example B.19. Let G be a finite and commutative group scheme over k and let $A := H^0(G, \mathcal{O}_G)$ be the associated Hopf algebra. As a consequence of Example A.9, we have the following.

1. If $G = \mu_p$, then we have $A = k[\mathbf{C}_p]$. Thus, ${}^{\text{Ad}}A$ splits into the direct sum of p pairwise non-isomorphic one-dimensional representations, which correspond to the characters of G . On the other hand, ${}^{\text{Ad}}(A^*)$ is a non-trivial successive p -fold and non-split extension of the trivial representation $\mathbb{1}$, whose semi-simplification is trivial of dimension p .
2. If $G = \mathbf{C}_p$, then we obtain the same as before with the rôles of A and A^* interchanged.
3. If $G = \alpha_p$, then $A \cong A^*$ and ${}^{\text{Ad}}A \cong {}^{\text{Ad}}(A^*)$ is a successive p -fold and non-split extension of the trivial representation $\mathbb{1}$.

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