THE REVIEW OF SYMBOLIC LOGIC
Volume 13, Number 4, December 2020

DE MORGAN INTERPRETATION OF THE LAMBEK–GRISHIN CALCULUS

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Abstract. We present an embedding of the Lambek–Grishin calculus into an extension of the nonassociative Lambek calculus with negation. The embedding is based on the De Morgan interpretation of the dual Grishin connectives.

§1. Introduction. In [6] Grishin introduced the extension ***LG*** of the nonassociative Lambek calculus ***NL*** [8] with dual connectives. The Lambek and Grishin connectives do not depend on each other, but the latter are defined by the rules of inference whose form is dual to the rules of ***NL***. Thus, the relationship between Grishin's connectives and Lambek's original connectives naturally resembles De Morgan's duality in classical logic. However, the former cannot be defined in such a way, because negation $\neg$ is not a part of the calculus. This naturally leads to the problem of extending ***NL*** with negation such that ***LG*** can be embedded into the extension.[1]

In this article we define the extension, $\boldsymbol{NL}_\neg$, of ***NL*** with *intuitionistic* negation and some additional *proper* classical negation axioms and interpret the Grishin connectives in that extension by means of De Morgan like laws. Namely, for the dual connective $\circledast$ of a Lambek connective $* \in \{/, \backslash, \cdot\}$, the formula $F \circledast G$ is (recursively) translated to $\neg(\neg F * \neg G)$. We show that our interpretation is strong, i.e., it preserves the consequence relation when passing from ***LG*** to $\boldsymbol{NL}_\neg$ and vice versa. In other words, a formula is provable (from assumptions) in ***LG*** if and only if its interpretation image is provable in $\boldsymbol{NL}_\neg$ from the interpretation images of the assumptions.

While the proof that the above translation preserves provability in ***LG*** is a routine inspection of the ***LG*** rules of inference, the proof of the converse direction is more involved. For that proof we introduce the sequent calculus $\boldsymbol{SNL}_\neg$ for $\boldsymbol{NL}_\neg$. We show that $\boldsymbol{SNL}_\neg$ extended with the rules of inference corresponding to the additional classical negation axioms and translations of proper ***LG*** axioms (assumptions) admits a restricted version of cut elimination. This will allow us to pass from the extension of $\boldsymbol{NL}_\neg$ to ***LG***.

The article is organized as follows. The next section deals with extensions of ***NL*** to ***LG*** and to $\boldsymbol{NL}_\neg$. It contains the statement of the interpretation theorem and the proof of its "only if" part. In §3 we present the sequent calculus $\boldsymbol{SNL}_\neg$ for $\boldsymbol{NL}_\neg$ with additional classical negation axioms and translations of proper ***LG*** axioms. Then, in §4, we interpret ***LG*** in $\boldsymbol{SNL}_\neg$, obtaining in such a way an interpretation of ***LG*** in $\boldsymbol{NL}_\neg$. Finally, we end

Received: January 24, 2018.
2010 *Mathematics Subject Classification*: 03B47, 03B47.
Key words and phrases: Lambek–Grishin calculus, duality, negation, De Morgan's laws, sequent calculus, cut elimination.

[1] In [4] De Groote embeds ***LG*** into a *bi-modal* version of ***NL*** extended with involutive negation, i.e., negation defined by De Morgan's laws.

doi:10.1017/S175502031900011X

the article with concluding remarks concerning ***LG*** and various extensions of ***NL*** with negation.

§2. The Lambek–Grishin calculus and its De Morgan interpretation. The language $\mathcal{L}$ of the (nonassociative) Lambek calculus ***NL*** [8] consists of propositional variables (atomic formulas) and the Lambek connectives $\cdot$,/, and \. Expressions of the form $F \to G$, where F and G are $\mathcal{L}$-formulas, are called $\mathcal{L}$-sequents.

The axioms of ***NL*** are $\mathcal{L}$-sequents of the form

$$F \to F \tag{1}$$

and the rules of inference are as follows.

$$(a)\ \frac{F\cdot G \to H}{F \to H/G} \qquad (b)\ \frac{F \to H/G}{F\cdot G \to H} \tag{2}$$

$$(a)\ \frac{F\cdot G \to H}{G \to F\backslash H} \qquad (b)\ \frac{G \to F\backslash H}{F\cdot G \to H} \tag{3}$$

and

$$\frac{F \to G \quad G \to H}{F \to H} \tag{4}$$

The language $\mathcal{L}^\circ$ of the (nonassociative) Lambek–Grishin calculus ***LG*** [6] is obtained from $\mathcal{L}$ by adding to it the *dual* Grishin connectives $\odot$, $\oslash$, and $\otimes$, and ***LG*** is obtained from ***NL*** by adding to it the dual rules of inference

$$(a)\ \frac{H \to F \odot G}{H \oslash G \to F} \qquad (b)\ \frac{H \oslash G \to F}{H \to F \odot G} \tag{5}$$

and

$$(a)\ \frac{H \to F \odot G}{F \otimes H \to G} \qquad (b)\ \frac{F \otimes H \to G}{H \to F \odot G} \tag{6}$$

Expressions of the form $F \to G$, where F and G are $\mathcal{L}^\circ$-formulas, are called $\mathcal{L}^\circ$-*sequents*.

Let $\mathcal{L}_\neg$ denote $\mathcal{L}$ augmented with negation $\neg$. Like in the case of $\mathcal{L}$ and $\mathcal{L}^\circ$, expressions of the form $A \to B$, where A and B are $\mathcal{L}_\neg$-formulas,[2] are called $\mathcal{L}_\neg$-sequents.

Now, following [3, equation (17)], we define the interpretation of ***LG***-formulas in $\mathcal{L}_\neg$ recursively as follows. The interpretation image, or just the *image*, of an ***LG***-formula F is denoted by $F^\sim$.

- If F is an atomic formula, then $F^\sim$ is F itself.
- For $* \in \{\cdot\, , /, \backslash\}$,
  - $(F * G)^\sim$ is $F^\sim * G^\sim$ and
  - $(F \circledast G)^\sim$ is $\neg(\neg F^\sim * \neg G^\sim)$.

Images $F^\sim * G^\sim$ and $\neg(\neg F^\sim * \neg G^\sim)$ will be referred to as $*$-images and $\circledast$-images, respectively.

2 In what follows, $\mathcal{L}_\neg$-formulas are denoted by A, B, and C, whereas $\mathcal{L}$ and $\mathcal{L}^\circ$-formulas are denoted by F, G, and H.

Next, for a set of $\mathcal{L}^{\circ}$-sequents Φ, we denote by $\Phi^{\sim}$ the set of $\mathcal{L}_{\neg}$-sequents

$$\Phi^{\sim} = \{F^{\sim} \to G^{\sim} : F \to G \in \Phi\}.$$

Finally, $\mathcal{L}_{\neg}$-sequents of the form $F^{\sim} \to G^{\sim}$ will be refereed to as *interpretations of **LG**-sequents*.

DEFINITION 2.1. *$\mathcal{L}_{\neg}$-formulas whose negations are images are called* prenegated *images.*

REMARK 2.2. *It follows from the definitions that the negations of prenegated images are $\circledast$-images and that subformulas of images are images or* (*pre-*) *negated images.*[3]

EXAMPLE 2.3. *Let* $* \in \{\cdot\,, /, \backslash\}$.

- *If A and B are images, then $A * B$ is also an image and vice versa*; *and*
- *if A and B are negated images, then $A * B$ is a prenegated image and vice versa.*

Finally, the extension $\boldsymbol{NL}_{\neg}$ of $\boldsymbol{NL}$ with negation $\neg$ is as follows. The language of $\boldsymbol{NL}_{\neg}$ is $\mathcal{L}_{\neg}$, the axioms of $\boldsymbol{NL}_{\neg}$ are axioms (1) of $\boldsymbol{NL}$ (over the extended language) added with

$$A \to \neg\neg A \tag{7}$$

and the set of all $\mathcal{L}_{\neg}$-sequents of the form

$$\neg\neg A \to A \tag{8}$$

where A is an atomic formula, a $*$-image, or a prenegated image,[4] and the rules of inference of $\boldsymbol{NL}_{\neg}$ are those of $\boldsymbol{NL}$ added with

$$\frac{A \to B}{\neg B \to \neg A} \tag{9}$$

cf. axioms (DN) and rule (TR) from [1].

REMARK 2.4. *The $\circledast$-image counterpart of* (*8*) *is*

$$\neg\neg\neg A \to \neg A \tag{10}$$

that is derivable from (*7*) *by* (*9*).

THEOREM 2.5. *Let Φ be a set of $\mathcal{L}^{\circ}$-sequents. Then, $\Phi \vdash_{\boldsymbol{LG}} F \to G$ if and only if $\Phi^{\sim} \vdash_{\boldsymbol{NL}_{\neg}} F^{\sim} \to G^{\sim}$.*

Proof of the "only if" part of Theorem 2.5. The proof is by an induction on the derivation length of $F \to G$ from Φ in $\boldsymbol{LG}$.

The basis is trivial, because the interpretations of the $\boldsymbol{LG}$ axioms are axioms of $\boldsymbol{NL}_{\neg}$ and the interpretations of the elements of Φ belong to $\Phi^{\sim}$.

For the induction step we consider the last step in the formula derivation. The cases of (2) and (3) immediately follow from the definition of interpretation of the Lambek connectives. The case of (4) is also immediate, because (4) is a rule of $\boldsymbol{NL}_{\neg}$.

Assume that the last step is by (5)(a). Then

[3] That is, prenegated or negated images.

[4] Because of this constraint on A, the $\boldsymbol{NL}_{\neg}$ negation is not classical.

https://doi.org/10.1017/S175502031900011X Published online by Cambridge University Press

1.	$H^{\sim} \to \neg(\neg F^{\sim} \cdot \neg G^{\sim})$	induction hypothesis
2.	$\neg\neg(\neg F^{\sim} \cdot \neg G^{\sim}) \to \neg H^{\sim}$	follows from 1 by (9)
3.	$\neg F^{\sim} \cdot \neg G^{\sim} \to \neg\neg(\neg F^{\sim} \cdot \neg G^{\sim})$	axiom (7)
4.	$\neg F^{\sim} \cdot \neg G^{\sim} \to \neg H^{\sim}$	follows from 3 and 2 by (4)
5.	$\neg F^{\sim} \to \neg H^{\sim} / \neg G^{\sim}$	follows from 4 by (2)(*a*)
6.	$\neg(\neg H^{\sim} / \neg G^{\sim}) \to \neg\neg F^{\sim}$	follows from 5 by (9)
7.	$\neg\neg F^{\sim} \to F^{\sim}$	(10), if $F^{\sim}$ is an $\circledast$-image, or axiom (8), otherwise
8.	$\neg(\neg H^{\sim} / \neg G^{\sim}) \to F^{\sim}$	follows from 6 and 7 by (4)

and, by definition, $\neg(\neg H^{\sim} / \neg G^{\sim}) \to F^{\sim}$ is $(H \oslash G \to F)^{\sim}$.

Assume now that the last step is by (5)(*b*). Then

1.	$\neg(\neg H^{\sim} / \neg G^{\sim}) \to F^{\sim}$	induction hypothesis
2.	$\neg F^{\sim} \to \neg\neg(\neg H^{\sim} / \neg G^{\sim})$	follows from 1 by (9)
3.	$\neg\neg(\neg H^{\sim} / \neg G^{\sim}) \to \neg H^{\sim} / \neg G^{\sim}$	axiom (8)
4.	$\neg F^{\sim} \to \neg H^{\sim} / \neg G^{\sim}$	follows from 2 and 3 by (4)
5.	$\neg F^{\sim} \cdot \neg G^{\sim} \to \neg H^{\sim}$	follows from 4 by (2)(*b*)
6.	$\neg\neg H^{\sim} \to \neg(\neg F^{\sim} \cdot \neg G^{\sim})$	follows from 5 by (9)
7.	$H^{\sim} \to \neg\neg H^{\sim}$	axiom (7)
8.	$H^{\sim} \to \neg(\neg F^{\sim} \cdot \neg G^{\sim})$	follows from 6 and 7 by (4)

and, by definition, $H^{\sim} \to \neg(\neg F^{\sim} \cdot \neg G^{\sim})$ is $(H \to F \odot G)^{\sim}$.

The case of (6) is similar. □

The rest of this article deals with the proof of the "if" part of Theorem 2.5. Namely, in the next sections, we introduce the sequent calculus ***SNL***$_{\neg}$ for ***NL***$_{\neg}$ and prove the cut elimination theorem. Then, in §4, we embed ***LG*** into ***SNL***$_{\neg}$.

§3. A sequent calculus for *NL*$_{\neg}$.

In this section we present a sequent calculus for (an extension of) ***NL***$_{\neg}$ that will be used for the proof of the "if" part of Theorem 2.5. For the definition of this calculus we need the notion of a *formula tree* (in this section, an $\mathcal{L}_{\neg}$-formula tree) that is as follows.

A *formula tree* is an ordered binary tree whose leaves are labeled with formulas. Such trees are denoted by Γ, possibly indexed, and for formula trees Γ_1 and Γ_2 we denote by (Γ_1, Γ_2) the formula tree obtained by joining Γ_1 and Γ_2 at a new root.

By $\Gamma[\Gamma_1]$ we denote a formula tree Γ with a designated formula subtree Γ_1 and, in this context, we denote by $\Gamma[\Gamma_2]$ the replacement of Γ_1 with Γ_2 in Γ for that particular occurrence of Γ_1.

We also identify *degenerate* one-node formula trees with their labels.[5]

Sequents are expressions of the form $\Gamma \to C$ or of the form $\Gamma_1; \Gamma_2 \to$, where Γ, Γ_1, and Γ_2 are formula trees and C is an $\mathcal{L}_{\neg}$-formula.[6]

Note. Pairs $\Gamma_1; \Gamma_2$ are *unordered*.

Let Φ be a set of $\mathcal{L}^{\circ}$-sequents. The axioms of the sequent calculus ***SNL***$_{\neg,\Phi}$ are sequents of the form $P \to P$, where P is an atomic formula and the rules of inference are as follows.

5 That is, a one-node formula tree is just a formula.

6 Even though, sequents with empty succeedent are disallowed in ***NL***, they are natural for ***NL***$_{\neg}$, because of negation, see the corresponding introduction rules below.

$(\cdot \rightarrow)$ (a) $\dfrac{\Gamma[(A,B)] \rightarrow C}{\Gamma[A \cdot B] \rightarrow C}$ (b) $\dfrac{\Gamma_1[(A,B)];\Gamma_2 \rightarrow}{\Gamma_1[A \cdot B];\Gamma_2 \rightarrow}$ $(\rightarrow \cdot)$ $\dfrac{\Gamma_1 \rightarrow A \quad \Gamma_2 \rightarrow B}{(\Gamma_1,\Gamma_2) \rightarrow A \cdot B}$

$(/ \rightarrow)$ (a) $\dfrac{\Gamma_1[B] \rightarrow C \quad \Gamma \rightarrow A}{\Gamma_1[(B/A,\Gamma)] \rightarrow C}$ (b) $\dfrac{\Gamma_1[B];\Gamma_2 \rightarrow \quad \Gamma \rightarrow A}{\Gamma_1[(B/A,\Gamma)];\Gamma_2 \rightarrow}$ $(\rightarrow /)$ $\dfrac{(\Gamma,A) \rightarrow B}{\Gamma \rightarrow B/A}$

$(\backslash \rightarrow)$ (a) $\dfrac{\Gamma_1[B] \rightarrow C \quad \Gamma \rightarrow A}{\Gamma_1[(\Gamma,A\backslash B)] \rightarrow C}$ (b) $\dfrac{\Gamma_1[B];\Gamma_2 \rightarrow \quad \Gamma \rightarrow A}{\Gamma_1[(\Gamma,A\backslash B)];\Gamma_2 \rightarrow}$ $(\rightarrow \backslash)$ $\dfrac{(A,\Gamma) \rightarrow B}{\Gamma \rightarrow A\backslash B}$

$(\neg \rightarrow)$ $\dfrac{\Gamma \rightarrow A}{\Gamma;\neg A \rightarrow}$ $(\rightarrow \neg)$ $\dfrac{\Gamma;A \rightarrow}{\Gamma \rightarrow \neg A}$

There are also two "ordinary" cut rules

$$(a)\ \frac{\Gamma_1 \rightarrow A \quad \Gamma_2[A] \rightarrow C}{\Gamma_2[\Gamma_1] \rightarrow C} \qquad (b)\ \frac{\Gamma_1 \rightarrow A \quad \Gamma_2[A];\Gamma \rightarrow}{\Gamma_2[\Gamma_1];\Gamma \rightarrow} \tag{11}$$

two resolution rules

$$(a)\ \frac{\Gamma_1;\neg A \rightarrow \quad \Gamma_2[A] \rightarrow C}{\Gamma_2[\Gamma_1] \rightarrow C} \qquad (b)\ \frac{\Gamma_1;\neg A \rightarrow \quad \Gamma_2[A];\Gamma \rightarrow}{\Gamma_2[\Gamma_1];\Gamma \rightarrow} \tag{12}$$

where A is an atomic formula, a $*$-image or, a prenegated image,
and two Φ-cut rules

$$(a)\ \frac{\Gamma_1 \rightarrow A \quad \Gamma_2[B] \rightarrow C}{\Gamma_2[\Gamma_1] \rightarrow C} \qquad (b)\ \frac{\Gamma_1 \rightarrow A \quad \Gamma_2[B];\Gamma \rightarrow}{\Gamma_2[\Gamma_1];\Gamma \rightarrow} \tag{13}$$

where $A \rightarrow B$ is a sequent from $\Phi^{\sim}$, cf. [2].

EXAMPLE 3.1. *Rule* (9) *is derivable in* $\mathbf{SNL}_{\neg,\Phi}$:

$$\cfrac{\cfrac{A \rightarrow B}{A;\neg B \rightarrow}\ (\neg \rightarrow)}{\neg B \rightarrow \neg A}\ (\rightarrow \neg)$$

REMARK 3.2. *A straightforward induction on the formula complexity shows that for all formulas* A, $\vdash_{\mathbf{SNL}_{\neg}} A \rightarrow A$.

EXAMPLE 3.3. *Axioms* (7) *are derivable in* $\mathbf{SNL}_{\neg,\Phi}$:

$$\cfrac{\cfrac{A \rightarrow A}{A;\neg A \rightarrow}\ (\neg \rightarrow)}{A \rightarrow \neg\neg A}\ (\rightarrow \neg)$$

EXAMPLE 3.4. *Axioms* (8) *are derivable in* $\mathbf{SNL}_{\neg,\Phi}$:

$$\cfrac{\cfrac{\neg A \rightarrow \neg A}{\neg\neg A;\neg A \rightarrow}\ (\neg \rightarrow) \qquad A \rightarrow A}{\neg\neg A \rightarrow A}\ \text{resolution}(a)$$

EXAMPLE 3.5. *All sequents from* $\Phi^{\sim}$ *are derivable in* $\mathbf{SNL}_{\neg,\Phi}$. *Let* $A \rightarrow B \in \Phi^{\sim}$. *Then*

$$\frac{A \rightarrow A \quad B \rightarrow B}{A \rightarrow B}\ \Phi\text{-cut}(a)$$

with Γ_1 *being* A *and* $\Gamma_2[B]$ *being* B.

https://doi.org/10.1017/S175502031900011X Published online by Cambridge University Press

PROPOSITION 3.6 (Inversion lemma).

(i) *If* $\vdash_{\mathbf{SNL}_{\neg,\Phi}} \Gamma \to B/A$, *then* $\vdash_{\mathbf{SNL}_{\neg,\Phi}} (\Gamma, A) \to B$.
(ii) *If* $\vdash_{\mathbf{SNL}_{\neg,\Phi}} \Gamma \to A\backslash B$, *then* $\vdash_{\mathbf{SNL}_{\neg,\Phi}} (A, \Gamma) \to B$.

For the proof of (*i*) see the proof of [7, Prop. 46(*ii*)] and the proof of (*ii*) is symmetric to that of (*i*).

COROLLARY 3.7. *If* $\Phi^{\sim} \vdash_{\mathbf{NL}_{\neg}} A \to B$, *then* $\vdash_{\mathbf{SNL}_{\neg,\Phi}} A \to B$.[7]

Proof. The proof is by a straightforward induction on the length of an $\mathbf{NL}_{\neg}$-derivation of $A \to B$ from Φ. The basis is Remark 3.2 and Examples 3.3–3.5 and the induction step follows from (11)(*a*), Example 3.1, and Proposition 3.6. □

THEOREM 3.8. *If a sequent is derivable in* $\mathbf{SNL}_{\neg,\Phi}$, *then it is derivable without cuts* (11).[8]

Proof. The proof is a straightforward combination of the proofs in [8, Sec. 9] and [9, Sec. 4.1]. Namely, by the outer induction on the derivation length up to the first cut and the inner induction on the complexity of the cut formula, we eliminate the first cut in the derivation.

All cases of the outer induction, including resolutions (12) and Φ-cuts (13), are standard switchings of the order of applications of the rules of inference and, in view of [8], for the inner induction it suffices to consider the case of the principal connective $\neg$. In this case we replace the derivation

$$\dfrac{\dfrac{\Gamma_1; A \to}{\Gamma_1 \to \neg A}\,(\to \neg) \qquad \dfrac{\Gamma_2 \to A}{\Gamma_2; \neg A \to}\,(\neg \to)}{\Gamma_1; \Gamma_2 \to}\,\mathrm{cut}(b)$$

with

$$\dfrac{\Gamma_2 \to A \qquad \Gamma_1; A \to}{\Gamma_1; \Gamma_2 \to}\,\mathrm{cut}(b)$$

□

The following corollary to Theorem 3.8 is similar to [2, Lemma 1].

COROLLARY 3.9 (Cf. [2], Lemma 1). *If a sequent S is derivable in* $\mathbf{SNL}_{\neg,\Phi}$, *then there exists a derivation of S in* $\mathbf{SNL}_{\neg,\Phi}$ *such that all formulas appearing in it are subformulas of formulas occurring in S, or subformulas of formulas occurring in the sequents from* $\Phi^{\sim}$, *or subformulas of "the resolution formulas" from* (12).

Proof. By Theorem 3.8, there is a derivation of S without cuts (11). In such a derivation, all formulas are like in the statement of the corollary.[9] □

[7] The converse of the corollary is also true, but is not required for the proof of the "if" part of Theorem 2.5.
[8] Of course, resolutions (12) and Φ-cuts (13) cannot be eliminated in general.
[9] Actually, here we need a trivial induction on the derivation length that we leave to the reader.

COROLLARY 3.10. *Let S be a sequent derivable in* $\boldsymbol{SNL}_{\neg,\Phi}$ *such that all formulas occurring in S are images or (pre) negated images. Then there exists a derivation of S in* $\boldsymbol{SNL}_{\neg,\Phi}$ *such that all formulas appearing in it are images or (pre) negated images.*[10]

Proof. The proof follows from Corollary 3.9, because all subformulas of the antecedents or the succeedents of the sequents from $\Phi^{\sim}$ or "the resolution formulas" are images or (pre) negated images, see Remark 2.2. □

§4. Proof of the "if" part of Theorem 2.5. By Corollary 3.7, for the proof of the "if" part of Theorem 2.5 it suffices to show that $\vdash_{\boldsymbol{SNL}_{\neg,\Phi}} F^{\sim} \to G^{\sim}$ implies $\Phi \vdash_{\boldsymbol{LG}} F \to G$. For this we shall need the following notations, definitions, and auxiliary results.

For an image A, we denote by A° the $\boldsymbol{LG}$-formula F whose image is A. That is, $(F^{\sim})^{\circ}$ is F and $(A^{\circ})^{\sim}$ is A.

For a pre (respectively, negated) image A, we denote by A° the $\boldsymbol{LG}$-formula F such that $F^{\sim}$ is $\neg A$ (respectively, A is $\neg F^{\sim}$).

EXAMPLE 4.1. *If A and B are negated images and* $* \in \{\cdot, /, \backslash\}$, *then* $(A * B)^{\circ}$ *is* $A^{\circ} \circledast B^{\circ}$.[11]

Formula trees whose all leaves are labelled with images are called *positive* and formula trees whose all leaves are labelled with (pre) negated images are called *negative*.

Next, for a formula tree Γ whose leaves are labelled with images or (pre) negated images we denote by Γ° the $\mathcal{L}^{\circ}$-formula tree obtained from Γ by replacing each leaf label A with A°.

Finally, let Γ be an $\mathcal{L}^{\circ}$-formula tree. The $\mathcal{L}^{\circ}$-formulas $\Gamma^{\bullet}$ and $\Gamma^{\odot}$ are defined by the following recursion.

- If Γ is a single node tree F, then both $\Gamma^{\bullet}$ and $\Gamma^{\odot}$ are F itself,
- $(\Gamma_1, \Gamma_2)^{\bullet}$ is $\Gamma_1^{\bullet} \cdot \Gamma_2^{\bullet}$, and $(\Gamma_1, \Gamma_2)^{\odot}$ is $\Gamma_1^{\odot} \odot \Gamma_2^{\odot}$.

EXAMPLE 4.2. *If* Γ *is a negative formula tree, then* $(\Gamma[(A, B)])^{\circ\odot}$ *is* $(\Gamma^{\circ}[(A^{\circ} \odot B^{\circ})])^{\odot}$.

EXAMPLE 4.3. *If* Γ_1 *and* Γ_2 *are negative formula trees, then* $(\Gamma_1, \Gamma_2)^{\circ\odot}$ *is* $\Gamma_1^{\circ\odot} \odot \Gamma_2^{\circ\odot}$.

The "if" part of Theorem 2.5 follows from Theorem 4.4$(i)(a)$ with Γ and C being $F^{\sim}$ and $G^{\sim}$, respectively.

THEOREM 4.4. *Let* Φ *be a set of* $\boldsymbol{LG}$*-sequents and let S be a sequent appearing in a cut-free* $\boldsymbol{SNL}_{\neg,\Phi}$*-derivation of the interpretation of an* $\boldsymbol{LG}$*-sequent.*

(i) *If S is of the form* $\Gamma \to C$, *then either*

(a) Γ *is positive, C is an image, and*

$$\Phi \vdash_{\boldsymbol{LG}} \Gamma^{\circ\bullet} \to C^{\circ}$$

or

(b) Γ *is negative, C is a (pre) negated image, and*

$$\Phi \vdash_{\boldsymbol{LG}} C^{\circ} \to \Gamma^{\circ\odot}$$

[10] Since resolutions and Φ-cuts are not eliminable in $\boldsymbol{SNL}_{\neg}$, some formulas appearing in a derivation might not occur in S.

[11] Note that $A * B$ is a prenegated image.

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(ii) *If S is of the form $\Gamma_1; \Gamma_2 \rightarrow$, then one of Γ_1, Γ_2 is positive and the other is negative, say, Γ_1 is positive and Γ_2 is negative, and*

$$\Phi \vdash_{LG} \Gamma_1^{\circ \bullet} \rightarrow \Gamma_2^{\circ \odot}.$$

For the proof of Theorem 4.4 we need the *derivable* rules of inference of ***LG*** given by Proposition 4.5 below.

PROPOSITION 4.5. *The following rules of inference are deraivable in* ***LG***.

$$\frac{F \rightarrow G \quad (\Gamma[G])^{\bullet} \rightarrow H}{(\Gamma[F])^{\bullet} \rightarrow H} \tag{14}$$

$$\frac{F \rightarrow G \quad F' \rightarrow G'}{F \cdot F' \rightarrow G \cdot G'} \tag{15}$$

$$\frac{(\Gamma[G])^{\bullet} \rightarrow H \quad F' \rightarrow F}{(\Gamma[G/F, F'])^{\bullet} \rightarrow H} \tag{16}$$

$$\frac{(\Gamma[G])^{\bullet} \rightarrow H \quad F' \rightarrow F}{(\Gamma[F', F\backslash G])^{\bullet} \rightarrow H} \tag{17}$$

$$\frac{F \rightarrow (\Gamma[G])^{\odot} \quad G \rightarrow H}{F \rightarrow (\Gamma[H])^{\odot}} \tag{18}$$

$$\frac{F \rightarrow G \quad F' \rightarrow G'}{F \odot F' \rightarrow G \odot G'} \tag{19}$$

$$\frac{H \rightarrow (\Gamma[G])^{\odot} \quad F \rightarrow F'}{H \rightarrow (\Gamma[G \oslash F, F'])^{\odot}} \tag{20}$$

and

$$\frac{H \rightarrow (\Gamma[G])^{\odot} \quad F \rightarrow F'}{H \rightarrow (\Gamma[F', F \,\overline{\oslash}\, G])^{\odot}} \tag{21}$$

The proof of (14)–(17) is a verbatim of the corresponding proof in [8, Sec. 9] and the proof of (18)–(21) is dual to the above. We omit the proofs.

Proof of Theorem 4.4. The proof is by induction on the length of a cut-free derivation (Theorem 3.8) of the corresponding sequent. The basis (i.e., the derivation is of length one) is immediate, because in this case the sequent is of the form $P \rightarrow P$, where P is an atomic formula. The induction step cases are similar each to other. We consider only some of them. Namely, for each rule of inference, but Φ-cut (13), we consider either the positive or the negative case of the formula tree, only, and we consider only case (a) of Φ-cut.

Before moving to the induction step cases, we observe that, by the induction hypothesis, formula trees occurring in the premise(s) of the last rule of inference in the derivation are either positive or negative.

- Assume that the last rule of inference in the derivation is

$$\frac{\Gamma[(A, B)] \rightarrow C}{\Gamma[A \cdot B] \rightarrow C} \ (\cdot \ \rightarrow)(a)$$

and assume that $\Gamma[(A, B)]$ is positive. By the induction hypothesis for $(i)(a)$, C is an image and

$$\Phi \vdash_{LG} (\Gamma[A, B])^{\circ \bullet} \rightarrow C^{\circ}.$$

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By definition, $(\Gamma[A, B])^{\circ\bullet}$ is $(\Gamma[A \cdot B])^{\circ\bullet}$. Obviously, $\Gamma[A \cdot B]$ is positive.

- Assume that the last rule of inference in the derivation is

$$\frac{\Gamma_1[(A, B)]; \Gamma_2 \rightarrow}{\Gamma_1[A \cdot B]; \Gamma_2 \rightarrow} \ (\cdot \ \rightarrow)(b)$$

and assume that $\Gamma_1[(A, B)]$ is negative. Then $A \cdot B$ is a prenegated image, implying that both A and B are negated images. By the induction hypothesis for (ii),

$$\Phi \vdash_{LG} \Gamma_2^{\circ\bullet} \rightarrow (\Gamma_1[A, B])^{\circ\odot}.$$

By definition, $(\Gamma_1[A, B])^{\circ\odot}$ is $(\Gamma_1^{\circ}[A^{\circ} \odot B^{\circ}])^{\odot}$ and, by Example 4.1, $A^{\circ} \odot B^{\circ}$ is $(A \cdot B)^{\circ}$. Obviously, $\Gamma_1[A \cdot B]$ is negative and Γ_2 remains positive.

- Assume that the last rule of inference in the derivation is

$$\frac{\Gamma_1 \rightarrow A \qquad \Gamma_2 \rightarrow B}{(\Gamma_1, \Gamma_2) \rightarrow A \cdot B} \ (\rightarrow \cdot)$$

and assume that Γ_1 is positive. Then, by the induction hypothesis for $(i)(a)$, A is an image and,

$$\Phi \vdash_{LG} \Gamma_1^{\circ\bullet} \rightarrow A^{\circ}. \tag{22}$$

By Corollary 3.10, $A \cdot B$ is either an image or a prenegated image. Thus, since A is an image, B must be an image as well. Therefore, by the induction hypothesis for $(i)(a)$, Γ_2 is positive and,

$$\Phi \vdash_{LG} \Gamma_2^{\circ\bullet} \rightarrow B^{\circ}. \tag{23}$$

It follows from (22) and (23), by (15), that

$$\Phi \vdash_{LG} \Gamma_1^{\circ\bullet} \cdot \Gamma_2^{\circ\bullet} \rightarrow A^{\circ} \cdot B^{\circ}.$$

By definition, $\Gamma_1^{\circ\bullet} \cdot \Gamma_2^{\circ\bullet}$ is $(\Gamma_1, \Gamma_2)^{\circ\bullet}$ and $A^{\circ} \cdot B^{\circ}$ is $(A \cdot B)^{\circ}$. Since both Γ_1 and Γ_2 are positive, (Γ_1, Γ_2) is positive and, since both A and B are images, $A \cdot B$ is also an image.

- Assume that the last rule of inference in the derivation is

$$\frac{\Gamma_1[B] \rightarrow C \qquad \Gamma \rightarrow A}{\Gamma_1[(B/A, \Gamma)] \rightarrow C} \ (/ \rightarrow)(a)$$

and assume that $\Gamma_1[B]$ is negative. By the induction hypothesis for $(i)(b)$,

$$\Phi \vdash_{LG} C^{\circ} \rightarrow \left(\Gamma_1^{\circ}[B^{\circ}]\right)^{\odot}. \tag{24}$$

Also, since $\Gamma_1[B]$ is negative, B is a (pre) negated image and, since B/A is either an image or a (pre) negated image, by Corollary 3.10, both A and B must be negated images. Thus, by the induction hypothesis for $(i)(b)$, Γ is negative and

$$\Phi \vdash_{LG} A^{\circ} \rightarrow \Gamma^{\circ\odot}. \tag{25}$$

It follows from (24) and (25), by (20), that

$$\Phi \vdash_{LG} C^{\circ} \rightarrow \left(\Gamma_1^{\circ}[B^{\circ}/A^{\circ}, \Gamma^{\circ\odot}]\right)^{\odot}$$

and, by definition, $\left(\Gamma_1^{\circ}[B^{\circ}/A^{\circ}, \Gamma^{\circ\odot}]\right)^{\odot}$ is $(\Gamma_1[B/A, \Gamma])^{\circ\odot}$. Finally, since A and B are negated images and Γ_1 and Γ are negative, $\Gamma_1[(B/A, \Gamma)]$ is negative.

- Assume that the last rule of inference in the derivation is

$$\frac{\Gamma_1[B]; \Gamma_2 \rightarrow \qquad \Gamma \rightarrow A}{\Gamma_1[(B/A, \Gamma)]; \Gamma_2 \rightarrow} \ (/ \rightarrow)(b)$$

https://doi.org/10.1017/S175502031900011X Published online by Cambridge University Press

and assume that $\Gamma_1[B]$ is positive. By the induction hypothesis for (ii),

$$\Phi \vdash_{LG} \left(\Gamma_1^{\circ}[B^{\circ}]\right)^{\bullet} \to \Gamma_2^{\circ\odot}. \tag{26}$$

Since $\Gamma_1[B]$ is positive, B is an image and, since, by Corollary 3.10, B/A is either an image or a negated image, A must be an image. Thus, by the induction hypothesis for $(i)(a)$, Γ is positive and

$$\Phi \vdash_{LG} \Gamma^{\circ\bullet} \to A^{\circ}. \tag{27}$$

It follows from (26) and (27), by (16), that

$$\Phi \vdash_{LG} \left(\Gamma_1^{\circ}[B^{\circ}/A^{\circ}, \Gamma^{\circ}]\right)^{\bullet} \to \Gamma_2^{\circ\odot}$$

and, by definition, $\left(\Gamma_1^{\circ}[B^{\circ}/A^{\circ}, \Gamma^{\circ}]\right)^{\bullet}$ is $(\Gamma_1[B/A, \Gamma])^{\circ\bullet}$. Since A and B are images, B/A is an image. Therefore, $\Gamma_1[(B/A, \Gamma)]$ is positive, because both Γ_1 and Γ are positive,

- Assume that the last rule of inference in the derivation is

$$\frac{(\Gamma, A) \to B}{\Gamma \to B/A}\ (\to /)$$

and assume that (Γ, A) is positive. By the induction hypothesis for $(i)(a)$, B is an image and

$$\Phi \vdash_{LG} (\Gamma, A)^{\circ\bullet} \to B^{\circ}. \tag{28}$$

Since (Γ, A) is positive, Γ is positive and A is an image. Thus, B/A is also an image. Since, by definition, $(\Gamma, A)^{\circ\bullet}$ is $\Gamma^{\circ\bullet} \cdot A^{\circ}$ and $(B/A)^{\circ}$ is B°/A°,

$$\Phi \vdash_{LG} \Gamma^{\circ\bullet} \to (B/A)^{\circ}$$

follows from (28) by $(2)(a)$.

- Assume that the last rule of inference in the derivation is

$$\frac{\Gamma \to C}{\Gamma; \neg C \to}\ (\neg \to)$$

and assume that Γ is negative. By the induction hypothesis for $(i)(b)$, C is a (pre) negated image and

$$\Phi \vdash_{LG} C^{\circ} \to \Gamma^{\circ\odot}.$$

- Assume that the last rule of inference in the derivation is

$$\frac{\Gamma_1; \neg A \to \qquad \Gamma_2[A] \to C}{\Gamma_2[\Gamma_1] \to C}\ \text{resolution}(a)$$

and assume that Γ_1 is positive. By the induction hypothesis for (ii), A is an image and

$$\Phi \vdash_{LG} \Gamma_1^{\circ\bullet} \to A^{\circ}. \tag{29}$$

Since A is an image, $\Gamma_2[A])$ is also positive and, by the induction hypothesis for $(i)(a)$,

$$\Phi \vdash_{LG} (\Gamma_2[A])^{\circ\bullet} \to C^{\circ}. \tag{30}$$

Since, by definition, $(\Gamma_2[A])^{\circ\bullet}$ is $\left(\Gamma_2^{\circ}[A^{\circ}]\right)^{\bullet}$ and $(\Gamma_2[\Gamma_1])^{\circ\bullet}$ is $\left(\Gamma_2^{\circ}[\Gamma_1^{\circ\bullet}]\right)^{\bullet}$,

$$\Phi \vdash_{LG} (\Gamma_2[\Gamma_1])^{\circ\bullet} \to C^{\circ}$$

follows from (29) and (30), by (14). Since both Γ_1 and $\Gamma_2[A]$ are positive, $\Gamma_2[\Gamma_1]$ is also positive.

- Assume that the last rule of inference in the derivation is

$$\frac{\Gamma_1 \to A \qquad \Gamma_2[B] \to C}{\Gamma_2[\Gamma_1] \to C}\ \Phi\text{-cut}(a).$$

Then both A and B are images.

Since A is an image, by the induction hypothesis for $(i)(a)$, Γ_1 is positive and,

$$\Phi \vdash_{LG} \Gamma_1^{\circ\bullet} \to A^\circ. \tag{31}$$

Similarly, since B is an image, by the induction hypothesis for $(i)(a)$, $\Gamma_2[B])$ is also positive and,

$$\Phi \vdash_{LG} (\Gamma_2[B])^{\circ\bullet} \to C^\circ. \tag{32}$$

Since, by definition, $(\Gamma_2[B])^{\circ\bullet}$ is $\left(\Gamma_2^\circ[B^\circ]\right)^\bullet$ and $(\Gamma_2[\Gamma_1])^{\circ\bullet}$ is $\left(\Gamma_2^\circ[\Gamma_1^{\circ\bullet}]\right)^\bullet$,

$$\Phi \vdash_{LG} (\Gamma_2[\Gamma_1])^{\circ\bullet} \to C^\circ$$

follows from (31), $A^\circ \to B^\circ \in \Phi$, and (32), by two applications of (14). Since both Γ_1 and Γ_2 are positive, $\Gamma_2[\Gamma_1]$ is also positive. □

§5. Concluding remarks. So, in this article, we (strongly) embedded ***LG*** into $\boldsymbol{NL}_\neg$. The latter calculus is the weakest extension of the intuitionistic counterpart of Boolean negation from [1] that suits our purpose. Actually, $\boldsymbol{NL}_\neg$ is merely an *ad hoc* technical tool. In particular, the restriction on the formula A in the axioms (8) implies that $\boldsymbol{NL}_\neg$ is not closed under substitution. It is closed under substitution of images only.

It seems to be of interest to find a stronger negation that allows the De Morgan embedding. We do not know whether ***LG*** embeds into ***NL*** extended with full Boolean negation ([1]), i.e., the calculus obtained by allowing A in (8) to be any $\mathcal{L}_\neg$-formula. Obviously, classical nonassociative Lambek calculus ***CNL*** from [5] is too strong, because the sequents $\neg A/B \to A\backslash\neg B$ and $A\backslash\neg B \to \neg A/B$ are derivable in this calculus.

§6. Acknowledgment. The authors are grateful to Wojciech Buszkowski for his comments on the first version of this article.

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https://doi.org/10.1017/S175502031900011X Published online by Cambridge University Press