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RENEWAL THEORY WITH EXPONENTIAL AND HYPERBOLIC DISCOUNTING

J. A. M. van der Weide
Faculty of Electrical Engineering, Mathematics and Computer Science
Delft University of Technology
NL-2600 GA Delft, The Netherlands
E-mail: j.m.vanderweide@tudelft.nl

Suyono
Jurusan Matematika FMIPA Universitas Negeri Jakarta
Jakarta Timur 13200, Indonesia
E-mail: synjkt@yahoo.com

J. M. van Noortwijk
HKV Consultants
NL-8203 AC Lelystad, The Netherlands
and
Faculty of Electrical Engineering, Mathematics and Computer Science
Delft University of Technology
NL-2600 GA Delft, The Netherlands
E-mail: j.m.van.noortwijk@hkv.nl

To determine optimal investment and maintenance decisions, the total costs should be minimized over the whole life of a system or structure. In minimizing life-cycle costs, it is important to account for the time value of money by discounting and to consider the uncertainties involved. This article presents new results in renewal theory with costs that can be discounted according to any discount function that is nonincreasing and monotonic over time (such as exponential, hyperbolic, generalized hyperbolic, and no discounting). The main results include expressions for the first and second moment of the discounted costs over a bounded and unbounded time horizon as well as asymptotic expansions for nondiscounted costs.

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1. INTRODUCTION

In determining optimal investment and maintenance decisions, the total costs should ideally be computed over the whole life of a system or structure. In optimizing life-cycle costs, it is important to account for the time value of money and to consider the uncertainties involved. Examples of life-cycle costing are balancing the initial cost of investment against the future cost of maintenance and balancing the cost of preventive maintenance against the cost of corrective maintenance.

Investment and maintenance optimization can be computationally expensive. To reduce the computational effort, renewal theory can be used (Tijms [20, Chaps. 2 and 8]). Maintenance can be modeled as a renewal process if we can identify independent renewals that bring a system or structure back into its original condition or "good as new state." Although renewal theory attracted a huge amount of applications in the fields of mechanical and electrical engineering, it penetrated the field of structural engineering just very recently.

In finding an optimal balance between the initial cost of investment and the future cost of maintenance, it is essential to take the time value of money into account by applying a discount function (van Noortwijk [21]). Although there is a huge amount of literature on renewal theory, the bulk of this literature does not consider cost discounting. Mathematical derivations of analytic life-cycle models on the basis of continuous-time and discrete-time renewal processes with discounting can be found in Rackwitz [13,14] and van Noortwijk [21], respectively. As a discount function, they used discounting with a constant discount rate (exponential discounting).

This article was inspired by the work of van Noortwijk [21] on discrete-time renewal processes with exponential discounting. Van Noortwijk's work presented analytic expressions for the expected value and the variance of the discounted cost over a bounded and unbounded time horizons. This article extends van Noortwijk's model in several aspects. First, other types of discounting are considered, such as hyperbolic and generalized hyperbolic discounting. Second, expressions are derived for the first and second moments of the discounted cost over a bounded and unbounded horizon for any discount function that is nonincreasing and monotonic over time. Third, following the steps of the proofs in Tijms [20, Chap. 8], asymptotic expansions are derived for the first and second moments of nondiscounted cost for a time horizon tending to infinity.

The new findings in this article contain the following formerly derived results as special cases. Feller [5] and Smith [17,18] derived asymptotic expansions for the first and second moments of the number of renewals for discrete-time and continuous-time renewal processes, respectively (see also Tijms [20, Chap. 8]). Dall'Aglio [3] studied renewal processes with exponential discounting and unit cost. Léveillé and Garrido [8] considered compound renewal processes with exponential discounting and independence of renewal interoccurrence time and associated cost. Washburn [23] regarded a similar renewal model, but with possible dependence between the renewal interoccurrence time and cost. Wolff [26, Chap. 2] presented a central limit theorem for the renewal-reward process with corresponding asymptotic mean and variance (resulting in the first term of the corresponding asymptotic expansions). For discrete-time renewal processes, van Noortwijk [21] derived expressions for the first and second moments of the discounted cost over an unbounded time horizon for renewal cost being a function of the renewal interoccurrence time. For continuous-time renewal processes, Rackwitz [13,14] obtained the expected discounted costs over an unbounded time horizon.

The outline of this article is as follows. A brief overview on discount functions is given in Section 2. The mathematical model and notation can be found in Section 3. Only assuming the monotonicity of the discount function, explicit formulas for the first two moments of the discounted costs over both bounded and unbounded time horizons are presented in Section 4. The following types of discounting are studied: exponential discounting in Section 5, hyperbolic discounting in Section 6, and no discounting in Section 7. Finally, conclusions are formulated in Section 8.

2. DISCOUNT FUNCTIONS

Investment and maintenance decisions involve trade-offs among costs occurring at different times. In economic models, discount functions describe the weights placed on costs that occur at different points in time. In the psychological and economic literature, several mathematical functions that give more weight to present cost than to future cost have been proposed (for an extensive literature review on time discounting, see Frederick, Loewenstein, and O'Donoghue [6]). The most well-known discount function is the discounted-utility model proposed by Samuelson [16] in 1937. According to this model, the discount factor at time t is given by

$$D(t) = e^{-rt}, \qquad r > 0, \tag{1}$$

where r is the discount rate (rate of time preference; in finance, usually defined as the nominal interest rate adjusted for the inflation). This type of discounting is called exponential discounting. In economic analyses of intertemporal choice, exponential discounting is currently regarded as a normative standard for comparing public policies using costs–benefit analyses. However, in the last two decades, experimental psychology showed that intertemporal preference often cannot be characterized by a single constant discount rate. Weitzman [24] showed that the "lowest possible" discount rate should be used for discounting far-distant future costs and benefits.

As an alternative to exponential discounting, several hyperbolic functional forms for the discount function have been proposed: Herrnstein [7] and Mazur [10] suggested the function

$$D(t) = (1 + \beta t)^{-1}, \qquad \beta > 0, \tag{2}$$

and Loewenstein and Prelec [9] generalized this form to

$$D(t) = (1 + \alpha t)^{-\beta/\alpha}, \qquad \alpha, \beta > 0. \tag{3}$$

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Equations (2) and (3) are called hyperbolic discounting and generalized hyperbolic discounting, respectively. For hyperbolic discounting, future cost is attached more weight than for exponential discounting, and a person's discount rate is declining over time rather than being a constant. A hyperbolic discount function often fits empirical data better than the exponential discount function. Exponential discounting and hyperbolic discounting are special cases of generalized hyperbolic discounting: (3) converges to exponential discounting with rate β as $\alpha \to 0$ and (3) simplifies to hyperbolic discounting for $\alpha = \beta$. When decisions about a nuclear waste facility must be made, Atherton and French [1] claimed that hyperbolic discounting is more reasonable and justifiable than exponential discounting. Other environmental decision problems to which hyperbolic discounting might better be applied concern global climate change, loss of biodiversity, thinning of stratospheric ozone, groundwater pollution, minerals depletion, and many others (Weitzman [24]).

In financial mathematics, a discount function can be expressed in terms of a time-dependent discount rate as follows. Let $B(t)$ be the value at time t of an investment of one currency unit at time 0 [i.e., $B(0) = 1$]. The value is accrued continuously at the risk-free rate function r:

$$dB(t) = r(t)B(t)\,dt, \qquad B(0) = 1.$$

It follows that

$$B(t) = \exp\left(\int_0^t r(s)\,ds\right).$$

The discount factor $D(t)$, $t \geq 0$, is defined as the amount of money that we have to deposit in the bank at time 0 such that the value at time t is equal to 1. It follows that

$$D(t) = 1/B(t) = \exp\left(-\int_0^t r(s)\,ds\right).$$

In economic applications, $B(t)$ is called the future value of money and $D(t)$ is called the present value. The rate functions for the exponential, hyperbolic, and generalized hyperbolic discounting are as follows. For exponential discounting, the rate is constant, $r(t) \equiv r$. For generalized hyperbolic discounting, the rate function is dependent on time and is given by $r(t) = \beta/(1 + \alpha t)$, with hyperbolic discounting $r(t) = \beta/(1 + \beta t)$ as a special case. To assure socioeconomically sustainable civil engineering infrastructures, Rackwitz, Lentz, and Faber [15] proposed using the discount rate function $r(t) = \rho e^{-at} + \delta$, where ρ, a, $\delta > 0$.

In this article, we focus on discount functions that are decreasing in time. For example, we study the generalized hyperbolic discount function and its special cases: exponential and hyperbolic discounting. We thus study discount functions $D(t)$ for which $D(0) = 1$. An example of a discount function that is not unity at time 0 was proposed by Phelps and Pollak [12]. This function is known as the quasi-hyperbolic discount function and it is for rate $r > 0$ defined as $D(t) = \alpha < 1$

for $t = 0$ and $D(t) = \alpha e^{-rt}$ for $t > 0$. Fortunately, the results in this article can be easily extended to quasi-hyperbolic discounting as well.

3. MODEL AND NOTATION

Let T, T_j, C and C_j, $j \geq 1$, be random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the random variables T and T_j are positive and such that the sequence $\{(T, C), (T_j, C_j), j \geq 1\}$ is an independent and identically distributed (i.i.d.) sequence of random vectors with cumulative distribution function H:

$$H(x, y) = \mathbb{P}(T \leq x, C \leq y), \qquad x, y \in \mathbb{R}^+.$$

It follows that the sequence $\{T_j, j \geq 1\}$ is also i.i.d., with cumulative distribution function $F(x) = H(x, +\infty)$ and $F(0) = 0$. Let $N = \{N(t) : t \geq 0\}$ be the renewal process associated with the sequence of partial sums $(S_j)_{j \geq 1}$:

$$N(t) = \max\{j \,|\, S_j \leq t\} = \sum_{k=1}^{\infty} 1_{\{S_k \leq t\}}, \qquad t \geq 0,$$

where $S_j = T_1 + \cdots + T_j, j \geq 1$, and where 1_A denotes the indicator function of the set A. The random variables S_j, $j \geq 1$, can be interpreted as the times at which maintenance actions take place and C_j is the cost of the maintenance action at time S_j.

The expected number of renewals $M(t)$ in the time interval $[0, t]$ can be written as

$$M(t) = \mathbb{E}[N(t)] = \sum_{k=1}^{\infty} F_k(t) = \sum_{k=0}^{\infty} F * f^{k*}(t), \qquad t \geq 0,$$

where F_k is the cumulative distribution function of S_k, $*$ denotes the convolution product, and f^{k*} is the k-fold convolution of f with itself. We will assume that the distribution function F is absolutely continuous with probability density function f, such that the renewal process N has a renewal density m; that is,

$$M(t) = \int_0^t m(u)\, du.$$

For more information about renewal processes, see Tijms [20, Chaps. 2 and 8]. For our purposes, it is more convenient to use a slightly different definition for the renewal measure as the measure associated with the increasing function

$$U(t) = \sum_{k=0}^{\infty} F_k(t), \qquad t \geq 0.$$

Here, F_0 denotes the distribution function of $S_0 \equiv 0$. So $U(t) = M(t) + 1$ and

$$dU(x) = \delta(dx) + m(x)\, dx,$$

where δ is the Dirac measure in zero. In particular, for any nonnegative Borel function g,

$$\int_0^t g(x)\,dU(x) = g(0) + \int_0^t g(x)m(x)\,dx = \sum_{k=0}^{\infty} \mathbb{E}[g(S_k)].$$

The present value of cost C at time t is given by $D(t)C$, where $D(t)$ is a discount factor. For example, if we discount with constant rate $r > 0$, the discount factor is given by $D(t) = e^{-rt}$.

The total discounted cost over the bounded time horizon $[0, t]$ is then given by

$$K(t,D) = \sum_{j=1}^{\infty} D(S_j)C_j 1_{\{S_j \le t\}}. \tag{4}$$

In the case that the interoccurrence times T_j and the cost C_j are independent, the process $\{K(t, D), t \ge 0\}$ is known as a compound renewal process; see Morey [11]. In the special case with discount rate $r \equiv 0$, the process $\{K(t, D), t \ge 0\}$ is also known as a renewal-reward process; see Tijms [20, Chap. 2]. For applications in maintenance engineering, the special case $C_j = c(T_j)$ is important, where $c : \mathbb{R} \mapsto \mathbb{R}^+$ is a given (nonrandom) Borel function; see van Noortwijk [21].

4. FIRST AND SECOND MOMENTS OF DISCOUNTED COST

Let D be a given discount function; that is, D is a continuous, nonnegative, nonincreasing function with $D(0) = 1$. The next theorem gives a formula for the mean value of the total discounted cost $K(t, D)$ over the finite time interval $[0, t]$.

Theorem 4.1: *For any discount function D,*

$$\mathbb{E}[K(t,D)] = \int_0^t \mathbb{E}[D(x+T)C1_{\{x+T \le t\}}]\,dU(x).$$

Proof: The expected value of the term with $j = 1$ in the right-hand side of (4) can be written as

$$\begin{aligned}\mathbb{E}[D(S_1)C_1 1_{\{S_1 \le t\}}] &= \mathbb{E}\left[D(T)C1_{\{T \le t\}}\right] \\ &= \int_0^t \mathbb{E}\left[D(x+T)C1_{\{x+T \le t\}}\right] dF_0(x),\end{aligned}$$

and, for $j > 1$,

$$\begin{aligned}\mathbb{E}[D(S_j)C_j 1_{\{S_j \le t\}}] &= \mathbb{E}\left[D(S_{j-1}+T_j)C_j 1_{\{S_{j-1}+T_j \le t\}}\right] \\ &= \int_0^t \mathbb{E}\left[D(x+T)C1_{\{x+T \le t\}}\right] dF_{j-1}(x),\end{aligned}$$

since (T_j, C_j) and S_{j-1} are independent and (T, C) has the same distribution as (T_j, C_j). The Theorem follows now by summation over j. ■

In the special case that T and C are independent, we get the simpler equation

$$\mathbb{E}[K(t,D)] = \mathbb{E}[C]\int_0^t D(x)m(x)\,dx. \tag{5}$$

Note that the integral in (5) can be interpreted as the expected total discounted cost up to time t for the case with constant unit cost, $C_j \equiv 1$,

$$\mathbb{E}\left[\sum_{j=1}^{\infty} D(S_j)1_{\{S_j \le t\}}\right] = \int_0^t D(x)m(x)\,dx.$$

We continue with the behavior of $\mathbb{E}[K(t,D)]$ as $t \to \infty$.

THEOREM 4.2: *Let* $\mathbb{E}[C] < \infty$. *Then for any discount function* D *with*

$$\int_0^{\infty} D(x)m(x)\,dx < \infty,$$

the expected discounted cost over an unbounded horizon is finite:

$$\lim_{t\to\infty} \mathbb{E}[K(t,D)] = \int_0^{\infty} \mathbb{E}[D(x+T)C]\,dU(x). \tag{6}$$

PROOF: Since

$$D(x+T)C1_{\{x+T\le t\}}m(x) \le D(x)Cm(x),$$

we get the result by dominated convergence. ■

Now we consider the case that

$$\int_0^{\infty} D(x)m(x)\,dx = +\infty.$$

Equation (5), for the case that T and C are independent, shows that we get a nontrivial limit if we normalize by dividing through the factor $\int_0^t D(x)m(x)\,dx$. The next theorem shows that under some technical condition on the density of the interoccurrence times of the renewal process, the same normalization can be used if T and C are dependent. We need convergence of the renewal density $m(x)$ as $x \to \infty$. The next lemma contains a sufficient condition, which covers all cases of practical interest.

LEMMA 4.3: *Let f be the density of the interoccurrence times. If* $\lim_{t\to\infty} f(x) = 0$ *and if* $f \in L^{1+\delta}$*, that is,*

$$\int_0^\infty |f(x)|^{1+\delta}\, dx < \infty$$

for some $\delta > 0$*, then*

$$\lim_{x\to\infty} m(x) = 1/\mathbb{E}[T]. \tag{7}$$

For a proof, we refer to Smith [18]. The necessary and sufficient conditions for (7) are also known; see Smith [19]. From now, we will always, without explicitly mentioning, assume that the conditions in Lemma 4.3 are satisfied.

THEOREM 4.4: *Let* $\mathbb{E}[C] < \infty$*. Then for any discount function D with*

$$\int_0^\infty D(x)m(x)\, dx = +\infty,$$

the long-term expected cost per renewal is given by

$$\lim_{t\to\infty} \frac{\mathbb{E}[K(t,D)]}{\int_0^t D(x)m(x)\, dx} = \mathbb{E}[C].$$

PROOF: As in the proof of Theorem 4.2,

$$\frac{\int_0^t D(x+T)C1_{\{x+T\le t\}} m(x)\, dx}{\int_0^t D(x)m(x)\, dx} \le C.$$

Since, by l'Hôpital's rule and (7),

$$\begin{aligned}\lim_{t\to\infty} & \frac{\int_0^t D(x+T)C1_{\{x+T\le t\}} m(x)\, dx}{\int_0^t D(x)m(x)\, dx} \\ &= \lim_{t\to\infty} \frac{\int_0^{t-T} D(x+T)Cm(x)\, dx}{\int_0^t D(x)m(x)\, dx} \\ &= C \lim_{t\to\infty} \frac{m(t-T)}{m(t)} = C,\end{aligned}$$

we get, by dominated convergence,

$$\begin{aligned}\lim_{t\to\infty} \frac{\mathbb{E}[K(t,D)]}{\int_0^t D(x)m(x)\, dx} &= \lim_{t\to\infty} \mathbb{E}\left[\frac{\int_0^t D(x+T)C1_{\{x+T\le t\}} m(x)\, dx}{\int_0^t D(x)m(x)\, dx}\right] \\ &= \mathbb{E}[C].\end{aligned}$$

■

Theorem 4.4 determines the long-term expected cost per renewal. For the purpose of reserving budget for performing future maintenance actions, it is important to determine how much money these actions cost per unit time while taking the discounting into account. In finance, this cost is known as the equivalent average cost per unit time (see, e.g., Wagner [22, Chap. 11] and Brealey and Myers [2, Chap. 6]). The expected equivalent average cost (EEAC) per unit time computed over a bounded time horizon of length t is defined as

$$\text{EEAC} = \frac{\mathbb{E}[K(t,D)]}{\int_0^t D(x)\,dx}. \tag{8}$$

Washburn [23] denoted the EEAC with the equivalent rate of spending. For a bounded horizon and an unbounded horizon with $\int_0^\infty D(x)\ dx < \infty$, the equivalent average cost per unit time can also be interpreted as a stream of fixed identical costs per unit time sufficient to recover all the necessary discounted costs. In this situation, the present values of the expected equivalent average cost per unit time summed over a bounded time horizon is equal to the total expected discounted costs over the whole time horizon. Under the same assumptions as Theorem 4.4, the long-term expected equivalent average cost per unit time can be written as follows.

Corollary 4.5: *If* $\int_0^\infty D(x)\,dx = +\infty$, *then the long-term expected equivalent average cost per unit time is*

$$\lim_{t\to\infty} \frac{\mathbb{E}[K(t,D)]}{\int_0^t D(x)\,dx} = \frac{\mathbb{E}[C]}{\mathbb{E}[T]}.$$

Proof: It is sufficient to remark that

$$\lim_{t\to\infty} \frac{\int_0^t D(x)m(x)\,dx}{\int_0^t D(x)\,dx} = \lim_{t\to\infty} m(t) = \frac{1}{\mathbb{E}[T]},$$

so

$$\int_0^\infty D(x)\,dx = +\infty \quad \text{if and only if} \quad \int_0^\infty D(x)m(x)\,dx = +\infty. \qquad \blacksquare$$

We continue with the second moment of the discounted cost over a bounded time horizon of length t.

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Theorem 4.6: *Let (T′, C′) and (T,C) be i.i.d. For any discount function D,*

$$\mathbb{E}[K^2(t,D)] = \int_0^t \mathbb{E}\left[D^2(x+T)C^2 1_{\{x+T\le t\}}\right] dU(x) + 2\int_0^t\int_0^t \mathbb{E}\left[D(x+T)CD(x+T+y+T')\right.$$
$$\left.\times C' 1_{\{x+T+y+T'\le t\}}\right] dU(x)\, dU(y).$$

Proof: We can write

$$K^2(t,D) = \sum_{i=1}^\infty \sum_{j=1}^\infty D(S_i)C_i 1_{\{S_i\le t\}} D(S_j)C_j 1_{\{S_j\le t\}}$$
$$= \sum_{i=1}^\infty D^2(S_i)C_i^2 1_{\{S_i\le t\}} + 2\sum_{i=1}^\infty \sum_{j=i+1}^\infty D(S_i)C_i D(S_j)C_j 1_{\{S_j\le t\}}.$$

We now calculate the terms

$$\mathbb{E}\left[D(S_i)C_i D(S_j)C_j 1_{\{S_j\le t\}}\right], \qquad i\le j.$$

For the case $i=j$,

$$\mathbb{E}\left[D^2(S_i)C_i^2 1_{\{S_i\le t\}}\right] = \mathbb{E}\left[D^2(S_{i-1}+T_i)C_i^2 1_{\{S_{i-1}+T_i\le t\}}\right]$$
$$= \int_0^t \mathbb{E}\left[D^2(x+T)C^2 1_{\{x+T\le t\}}\right] dF_{i-1}(x),$$

and it follows that

$$\sum_{i=1}^\infty \mathbb{E}\left[D^2(S_i)C_i^2 1_{\{S_i\le t\}}\right] = \int_0^t \mathbb{E}\left[D^2(x+T)C^2 1_{\{x+T\le t\}}\right] dU(x).$$

For the case $i<j$, define

$$S(m,n) = \begin{cases} \sum_{j=m}^n T_j & \text{if } m\le n \\ S_0 & \text{if } m = n+1. \end{cases}$$

Then

$$\mathbb{E}[D(S_i)C_i D(S_j)C_j 1_{\{S_j\le t\}}]$$
$$= \mathbb{E}\left[D(S_{i-1}+T_i)C_i D(S_{i-1}+T_i+S(i+1,j-1)+T_j)\right.$$
$$\left.\times C_j 1_{\{S_{i-1}+T_i+S(i+1,\,j-1)+T_j\le t\}}\right]$$
$$= \int_0^t\int_0^t \mathbb{E}[D(x+T)CD(x+T+y+T')C' 1_{\{x+T+y+T'\le t\}}]$$
$$\times dF_{i-1}(x)\, dF_{j-i-1}(y).$$

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It follows that

$$\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}\mathbb{E}[D(S_i)C_iD(S_j)C_j1_{\{S_j\le t\}}]$$
$$=\int_0^t\int_0^t\mathbb{E}[D(x+T)CD(x+T+y+T')\times C'1_{\{x+T+y+T'\le t\}}]\,dU(x)\,dU(y).$$

The theorem follows now. ∎

In the special case that T and C are independent, we obtain the equation

$$\mathbb{E}[K^2(t,D)]=\mathbb{E}[C^2]\int_0^t D^2(x)m(x)\,dx$$
$$+2(\mathbb{E}[C])^2\int\int_{x+y\le t}D(x)D(x+y)m(x)m(y)\,dx\,dy.$$

Now, we consider $\lim_{t\to\infty}\mathbb{E}[K^2(t,D)]$ in Theorem 4.6.

Theorem 4.7: *Let* $\mathbb{E}[C^2]<\infty$. *For any discount function D with*

$$\int_0^\infty D^2(x)m(x)\,dx<\infty,$$

we have

$$\lim_{t\to\infty}\mathbb{E}[K^2(t,D)]=\int_0^\infty\mathbb{E}[D^2(x+T)C^2]\,dU(x)$$
$$+2\int_0^\infty\int_0^\infty\mathbb{E}\left[D(x+T)CD(x+T+y+T')C'\right]\,dU(x)\,dU(y).$$

5. EXPONENTIAL DISCOUNTING

For exponential discounting with constant rate $r>0$ as in (1),

$$\int_0^\infty D(x)\,dx=\frac{1}{r},$$

and we will denote the discounted cost over $[0,t]$ by $K(t,r)$. Since

$$\int_0^\infty e^{-rx}\,dU(x)=\sum_{k=0}^{\infty}\int_0^\infty e^{-rx}\,dF^{k*}(x)=\sum_{k=0}^{\infty}\left(\mathbb{E}\left[e^{-rT}\right]\right)^k,$$

it follows from (6) that

$$\lim_{t\to\infty}\mathbb{E}[K(t,r)]=\frac{\mathbb{E}[e^{-rT}C]}{1-\mathbb{E}[e^{-rT}]}$$

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and, hence, the limit value of the EEAC [see (8) for the definition] is

$$\lim_{t\to\infty} \frac{\mathbb{E}[K(t,r)]}{\int_0^t D(x)\,dx} = \lim_{t\to\infty} \frac{\mathbb{E}[K(t,r)]}{(1-e^{-rt})/r} = \frac{r\mathbb{E}[e^{-rT}C]}{1-\mathbb{E}[e^{-rT}]}.$$

If $E[C^2] < \infty$, it follows from Theorem 4.7 that

$$\lim_{t\to\infty} \mathbb{E}[K^2(t,r)] = \frac{\mathbb{E}[C^2 e^{-2rT}](1-\mathbb{E}[e^{-rT}]) + 2\mathbb{E}[Ce^{-rT}]\mathbb{E}[Ce^{-2rT}]}{(1-\mathbb{E}[e^{-rT}])(1-\mathbb{E}[e^{-2rT}])}.$$

Consider the case that the interoccurrence times are exponentially distributed with parameter $m > 0$. The renewal measure is then equal to $dU(x) = \delta(dx) + m dx$. By Theorem 4.1, for the expected discounted cost over the finite time interval $[0, t]$, we obtain

$$\begin{aligned}\mathbb{E}[K(t,r)] &= \mathbb{E}\left[e^{-rT}C1_{\{T\le t\}}\right] + \int_0^t \mathbb{E}\left[e^{-r(x+T)}C1_{\{x+T\le t\}}\right] m\,dx \\ &= \frac{m+r}{r}\mathbb{E}\left[e^{-rT}C1_{\{T\le t\}}\right] - \frac{m}{r}e^{-rt}\mathbb{E}[C1_{\{T\le t\}}].\end{aligned}$$

The difference between the asymptotic discounted cost and the expected discounted cost over the finite time interval $[0, t]$ is given by

$$\frac{m+r}{r}\mathbb{E}\left[e^{-rT}C1_{\{T>t\}}\right] + \frac{m}{r}e^{-rt}\mathbb{E}[C1_{\{T\le t\}}].$$

It follows that for exponential interoccurrence times, the asymptotic expression is a very accurate upper bound for the total expected cost over a bounded time interval $[0, t]$, the error being less than $(m+r)e^{-rt}\mathbb{E}[C]/r$. For exponential discounting, $D(s+t) = D(s)D(t)$, $s, t \ge 0$. Therefore, it is possible to derive simple expressions for the Laplace transform of the first and second moments of the total cost $K(t, r)$ over the bounded time horizon $[0, t]$. Using numerical methods developed by den Iseger [4], we can get numerical approximations for the mean and variance of the total cost over a finite time interval. The formula for the Laplace transform of the expected value $\mathbb{E}[K(t, r)]$ of the total discounted cost over $[0, t]$ in the next proposition holds also for $r = 0$, being the case of no discounting.

PROPOSITION 5.1: *For any* $r \ge 0$,

$$\int_0^\infty \mathbb{E}[K(t,r)]e^{-st}\,dt = \frac{\mathbb{E}[Ce^{-(s+r)T}]}{s(1-\mathbb{E}[e^{-(s+r)T}])}. \quad \textbf{(9)}$$

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Proof: It follows from Theorem 4.1 that

$$\begin{aligned}\int_0^\infty \mathbb{E}[K(t,r)]e^{-st}\,dt &= \mathbb{E}\left[\int_0^\infty \int_0^t e^{-r(x+T)} C 1_{\{x+T\le t\}}\,dU(x)e^{-st}\,dt\right]\\ &= \mathbb{E}\left[\frac{1}{s}e^{-(s+r)T} C\int_0^\infty e^{-(s+r)x}\,dU(x)\right]\\ &= \frac{\mathbb{E}[Ce^{-(s+r)T}]}{s(1-\mathbb{E}[e^{-(s+r)T}])}.\end{aligned}$$

■

In the same way, using Theorem 4.6, we find the Laplace transform of the second moment:

Proposition 5.2: *For any $r \ge 0$,*

$$\int_0^\infty \mathbb{E}[K^2(t,r)]e^{-st}\,dt = \frac{\mathbb{E}[C^2 e^{-(s+2r)T}](1-\mathbb{E}[e^{-(s+r)T}]) + 2\mathbb{E}[Ce^{-(s+r)T}]\mathbb{E}[Ce^{-(s+2r)T}]}{s(1-\mathbb{E}[e^{-(s+r)T}])(1-\mathbb{E}[e^{-(s+2r)T}])}. \quad \textbf{(10)}$$

It is interesting to note that the equivalent average cost per unit time and the average cost per unit time are related as follows (see van Noortwijk [21]). As r tends to zero from above, the EEAC approaches the expected average cost per unit time; that is,

$$\lim_{r\downarrow 0} \frac{\mathbb{E}[K(t,r)]}{\int_0^t e^{-rx}\,dx} = \frac{\mathbb{E}[K(t,0)]}{t}.$$

6. HYPERBOLIC DISCOUNTING

Consider hyperbolic discounting, where the discount factor is given by

$$D_H(t) = \frac{1}{1+\beta t},$$

with $\beta > 0$ a constant (Herrnstein [7] and Mazur [10]). In this case,

$$\int_0^t D_H(x)\,dx = \frac{1}{\beta}\ln(1+\beta t) \to \infty \quad \text{as } t \to \infty.$$

It follows that

$$\lim_{t\to\infty} \frac{\mathbb{E}[K(t,D_H)]}{\int_0^t D_H(x)\,dx} = \lim_{t\to\infty} \frac{\beta\mathbb{E}[K(t,D_H)]}{\ln(1+\beta t)} = \frac{\mathbb{E}[C]}{\mathbb{E}[T]}.$$

In the special case that the interoccurrence times are exponentially distributed with parameter $m > 0$, we get a constant renewal density m, and if we also assume that the cost are independent of the interoccurrence times, we get from (5) that

$$\mathbb{E}[K(t, D_H)] = \mathbb{E}[C] \int_0^t \frac{1}{1 + \beta x} m\, dx = \mathbb{E}[C] \frac{m}{\beta} \ln(1 + \beta t).$$

According to Loewenstein and Prelec [9], hyperbolic discounting can be generalized as follows. The generalized hyperbolic discount factor is given by

$$D_{GH}(t) = (1 + \alpha t)^{-\beta/\alpha}, \qquad \alpha, \beta > 0.$$

If $\alpha = \beta$, we get the hyperbolic discount function D_H. The limit, as $\alpha \to 0$, is exponential discounting. The generalized hyperbolic discount factor has a kind of Bayesian interpretation as the expected value of an exponential discount factor with uncertain rate r. If the uncertainty in r is modeled by a gamma distributed random variable R with mean β and variance $\alpha\beta$, we get

$$\mathbb{E}\left[e^{-Rt}\right] = \int_0^\infty e^{-rt} \frac{(r/\alpha)^{\beta/\alpha - 1}}{\alpha \Gamma(\beta/\alpha)} e^{-r/\alpha}\, dr = (1 + \alpha t)^{-\beta/\alpha};$$

see Weitzman [25]. In this case, the asymptotic behavior of the expected value of the discounted sum depends on the parameter values α and β. Note that

$$\int_0^\infty D_{GH}(x)\, dx = \begin{cases} 1/(\beta - \alpha) & \text{if } \alpha < \beta \\ +\infty & \text{if } \alpha \geq \beta. \end{cases}$$

So, for $\alpha > \beta$, it follows from Corollary 4.5 that

$$\lim_{t \to \infty} \frac{\mathbb{E}[K(t, D_{GH})]}{\int_0^t D_{GH}(x)\, dx} = \lim_{t \to \infty} \frac{(\alpha - \beta)\mathbb{E}[K(t, D_{GH})]}{(1 + \alpha t)^{-(\beta/\alpha)+1} - 1} = \frac{\mathbb{E}[C]}{\mathbb{E}[T]}.$$

If $\alpha = \beta$, we have hyperbolic discounting. Finally, consider the case $\alpha < \beta$, where we have to apply (6). Assume that T has an exponential distribution with parameter m. The renewal density $m(x) \equiv m$ is constant in this case, and we get

$$\begin{aligned} \lim_{t \to \infty} \mathbb{E}[K(t, D_{GH})] &= \mathbb{E}\left[C(1 + \alpha T)^{-\beta/\alpha}\right] + \mathbb{E}\left[\frac{mC}{\beta - \alpha}(1 + \alpha T)^{-(\beta/\alpha)+1}\right] \\ &= \mathbb{E}\left[C \frac{1}{\beta - \alpha} \frac{\beta - \alpha + m(1 + \alpha T)}{(1 + \alpha T)^{\beta/\alpha}}\right]. \end{aligned}$$

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If the distribution of T is exponential with parameter m, the conclusions for the three cases $\alpha < \beta$, $\alpha = \beta$, and $\alpha > \beta$ respectively are

$$\lim_{t\to\infty}\frac{\mathbb{E}[K(t,D_{GH})]}{\int_0^t D_{GH}(x)\,dx} = \begin{cases} \lim_{t\to\infty}\dfrac{(\alpha-\beta)\mathbb{E}[K(t,D_{GH})]}{(1+\alpha t)^{-(\beta/\alpha)+1}-1} = \mathbb{E}\left[C\dfrac{\beta-\alpha+m(1+\alpha T)}{(1+\alpha T)^{\beta/\alpha}}\right], \\ \lim_{t\to\infty}\dfrac{\beta\mathbb{E}[K(t,D_{GH})]}{\ln(1+\beta t)} = \dfrac{\mathbb{E}[C]}{\mathbb{E}[T]}, \\ \lim_{t\to\infty}\dfrac{(\alpha-\beta)\mathbb{E}[K(t,D_{GH})]}{(1+\alpha t)^{-(\beta/\alpha)+1}-1} = \dfrac{\mathbb{E}[C]}{\mathbb{E}[T]}. \end{cases}$$

For $\alpha < \beta$ and $\alpha > \beta$, we have the same normalization, but in the first case, the normalization factor tends to $1/(\beta - \alpha)$ as $t \to \infty$, and in the second case, it tends to $+\infty$.

7. NO DISCOUNTING

Consider the case without discounting where $D(t) \equiv 1$. For this discount function, we use the notation $K(t, 0)$ for the total cost up to time t. No discounting is sometimes also referred to as zero discounting (i.e., exponential discounting with rate $r = 0$). Theorem 4.1 implies that the expected total cost over $[0, t]$ can be represented as

$$\mathbb{E}[K(t,0)] = \mathbb{E}[CU(t-T)]. \tag{11}$$

Different from $r > 0$, we have for $r = 0$ that

$$\int_0^\infty D(x)\,dx = +\infty.$$

It follows from Corollary 4.5 that the expected average cost per unit time is

$$\lim_{t\to\infty}\frac{\mathbb{E}[K(t,0)]}{\int_0^t D(x)\,dx} = \lim_{t\to\infty}\frac{\mathbb{E}[K(t,0)]}{t} = \frac{\mathbb{E}[C]}{\mathbb{E}[T]}.$$

This follows also directly from (11) and an application of the Elementary Renewal Theorem. This result is also known as the Renewal-Reward Theorem; see Tijms [20, Chap. 2].

The second moment can be expressed as

$$\mathbb{E}[K^2(t,0)] = \mathbb{E}[C^2U(t-T)] + 2\mathbb{E}[U*U(t-T-T')CC'],$$

where $U * U$ denotes the convolution of the renewal measure U with itself. Consider the renewal measure $dU(x) = \delta(dx) + m\,dx$; then

$$\mathbb{E}[K(t,0)] = \mathbb{E}[C1_{\{T\le t\}}] + m\mathbb{E}[C(t-T)1_{\{T\le t\}}].$$

We continue with a derivation of the asymptotic expansion of the first and the second moments of the total cost $K(t, 0)$ as $t \to \infty$. These expansions are well known for renewal processes; see Tijms [20, Chap. 8]. We need a (straightforward) generalization of these results to delayed renewal processes. Let $\tilde{N}$ be a delayed

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renewal process associated with the sequence of independent nonnegative random variables $\{\tilde{T}_n : n \geq 1\}$, where $\tilde{T}_1$ has cumulative distribution function G and $\tilde{T}_k$, $k \geq 2$, have identical cumulative distribution functions F. We assume that the distribution functions F and G are absolutely continuous and we will denote the corresponding probability densities by f and g, respectively. Denote the kth moment of the distributions F and G with μ_k and ν_k, respectively; $k = 1, 2$. Let $\{\tilde{S}_k : k \geq 1\}$ be the sequence of partial sums of the sequence $\{\tilde{T}_n\}$. It follows that

$$\mathbb{E}\left[\tilde{N}(t)\right] = \sum_{k=1}^{\infty} \mathbb{E}\left[1_{\{\tilde{S}_k \leq t\}}\right] = \sum_{k=0}^{\infty} G * f^{k*}(t).$$

In the same way as for renewal processes, we can apply the key renewal theorem to get

$$\lim_{t\to\infty}\left\{\sum_{k=0}^{\infty} G * f^{k*}(t) - \frac{t}{\mu_1}\right\} = \frac{\mu_2}{2\mu_1^2} - \frac{\nu_1}{\mu_1} \tag{12}$$

and

$$\lim_{t\to\infty}\int_0^t\left\{\sum_{k=0}^{\infty} G * f^{k*}(x) - \left[\frac{x}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - \frac{\nu_1}{\mu_1}\right]\right\} dx$$
$$= \frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3 + 3\nu_1\mu_2}{6\mu_1^2} + \frac{\nu_2}{2\mu_1}.$$

Let us now return to the first moment of the total cost $K(t, 0)$ and define $M_1(t) = \mathbb{E}[K(t, 0)]$.

Proposition 7.1:

$$\lim_{t\to\infty}\left\{M_1(t) - \frac{\mu_C}{\mu_T}t\right\} = \frac{\mu_C\mu_{T^2}}{2\mu_T^2} - \frac{\mu_{CT}}{\mu_T}.$$

Proof: Since the sequence $\{(T, C), (T_j, C_j), j \geq 1\}$ is an i.i.d. sequence of random vectors, we get, for $k > 1$,

$$\begin{aligned}\mathbb{E}[C_k 1_{\{S_k \leq t\}}] &= \mathbb{E}[C_k 1_{\{T_k \leq t - S_{k-1}\}}] \\ &= \int_0^t \mathbb{E}[C_k 1_{\{T_k \leq t-u\}}]\, f^{(k-1)*}(u)\, du \\ &= G * f^{(k-1)*}(t),\end{aligned}$$

where

$$G(t) = \mathbb{E}[C 1_{\{T \leq t\}}].$$

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It follows that the expectation $M(t)$ of the total cost $K(t, 0)$ is given by

$$M_1(t) = \mathbb{E}[K(t,0)] = \sum_{k=1}^{\infty} \mathbb{E}[C_k 1_{\{S_k \le t\}}] = \sum_{k=0}^{\infty} G * f^{k*}(t). \tag{13}$$

Note that $(1/\mu_C)G$ is a cumulative distribution function with first moment given by

$$\nu_1 = \int_0^\infty \left(1 - \left(\frac{1}{\mu_C}\right) G(t)\right) dt = \frac{\mu_{CT}}{\mu_C}.$$

So (12) implies the result. ■

In the special case $C \equiv 1$, Proposition 7.1 can be simplified to

$$\lim_{t\to\infty} \left\{ \mathbb{E}(K(t,\ 0)) - \frac{t}{\mu_T} \right\} = \frac{\mu_{T^2}}{2\mu_T^2} - 1. \tag{14}$$

This result was proved by Smith [17]. For T exponentially distributed, the right-hand side of (14) equals zero and the asymptotic expansion is exact.

Let $M_2(t) = \mathbb{E}[K^2(t,\ 0)]$ be the second moment of the total cost.

Proposition 7.2:

$$\lim_{t\to\infty} \left\{ M_2(t) - \left[\left(\frac{\mu_C}{\mu_T}\right)^2 t^2 + \left(\frac{2\mu_C^2 \mu_{T^2}}{\mu_T^3} + \frac{\mu_{C^2}\mu_T - 4\mu_C \mu_{CT}}{\mu_T^2} \right) t \right] \right\}$$
$$= \frac{3\mu_C^2 \mu_{T^2}^2}{2\mu_T^4} - \frac{2\mu_C^2 \mu_{T^3} + 12\mu_C \mu_{CT}\mu_{T^2}}{3\mu_T^3}$$
$$+ \frac{4\mu_C \mu_{CT^2} + 4\mu_{CT}^2 + \mu_{C^2}\mu_{T^2}}{2\mu_T^2} - \frac{\mu_{C^2 T}}{\mu_T}.$$

Proof: Since the sequence $\{(T,\ C), (T_j,\ C_j), j \ge 1\}$ is an i.i.d. sequence of random vectors, we get

$$M_2(t) = \mathbb{E}\left[\left(C_1 1_{\{S_1 \le t\}} + \sum_{k=2}^{\infty} C_k 1_{\{S_k \le t\}} \right)^2 \right]$$
$$= H(t) + 2\mathbb{E}\left[C_1 \sum_{k=2}^{\infty} C_k 1_{\{S_k \le t\}} \right] + \mathbb{E}\left[\left(\sum_{k=2}^{\infty} C_k 1_{\{S_k \le t\}} \right)^2 \right], \tag{15}$$

where

$$H(t) = \mathbb{E}[C^2 1_{\{T \le t\}}].$$

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The third term on the right-hand side of (15) can be rewritten as

$$\mathbb{E}\left[\left(\sum_{k=2}^{\infty} C_k 1_{\{S_k \le t\}}\right)^2\right] = \mathbb{E}\left[\left(\sum_{k=2}^{\infty} C_k 1_{\{S(2,k) \le t - T_1\}}\right)^2\right]$$
$$= \int_0^t M_2(t-x) f(x)\, dx.$$

Defining

$$N(t) = \mathbb{E}[C_1 C_2 1_{\{S_2 \le t\}}],$$

we can rewrite the second term on the right-hand side of (15) as

$$\mathbb{E}\left[C_1 \sum_{k=2}^{\infty} C_k 1_{\{S_k \le t\}}\right] = \sum_{k=2}^{\infty} \mathbb{E}[C_1 C_k 1_{\{S_k \le t\}}]$$
$$= N(t) + \sum_{k=3}^{\infty} \mathbb{E}[C_1 C_k 1_{\{T_1 + S(2,k-1) + T_k \le t\}}]$$
$$= N(t) + \sum_{k=3}^{\infty} \int_0^t N(t-x) f^{(k-2)*}(x)\, dx$$
$$= \sum_{k=0}^{\infty} N * f^{k*}(t).$$

Note that $(1/\mu_C^2)N$ is a cumulative distribution function with first moment

$$\nu_1 = \int_0^{\infty} \left(1 - \left(\frac{1}{\mu_C^2}\right) N(t)\right) dt$$
$$= \frac{1}{\mu_C^2} \int_0^{\infty} \mathbb{E}[C_1 C_2 1_{\{S_2 > t\}}]\, dt$$
$$= \frac{2\mu_{CT}}{\mu_C}$$

and second moment

$$\nu_2 = 2\int_0^{\infty} t \left(1 - \left(\frac{1}{\mu_C^2}\right) N(t)\right) dt$$
$$= \frac{2}{\mu_C^2} \mathbb{E}\left[C_1 C_2 \int_0^{\infty} t 1_{\{S_2 > t\}}\, dt\right]$$
$$= \frac{2\mu_C \mu_{CT^2} + 2\mu_{CT}^2}{\mu_C^2}.$$

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It follows that

$$M_2(t) = H(t) + 2\sum_{k=0}^{\infty} N * f^{k*}(t) + \int_0^t M_2(t-x)f(x)\,dx. \tag{16}$$

Using the notation $\mathcal{L}_\phi$ for the Laplace transform of a real function ϕ, that is,

$$\mathcal{L}_\phi(s) = \int_0^\infty e^{-st}\phi(t)\,dt, \qquad s \geq 0,$$

we get, as $s \to 0$,

$$\mathcal{L}_{M_2}(s) = \frac{\mathbb{E}[C^2 e^{-sT}]\big(1-\mathcal{L}_f(s)\big) + 2\big(\mathbb{E}[Ce^{-sT}]\big)^2}{s\big(1-\mathcal{L}_f(s)\big)^2}$$
$$= \frac{2\gamma}{s^3} + \frac{\eta}{s^2} + \cdots,$$

where

$$\gamma = \left(\frac{\mu_C}{\mu_T}\right)^2, \qquad \eta = \frac{2\mu_C^2\mu_{T^2}}{\mu_T^3} - \frac{4\mu_C\mu_{CT}}{\mu_T^2} + \frac{\mu_{C^2}}{\mu_T}.$$

Formal inversion of this suggests that as $t \to \infty$,

$$M_2(t) = \gamma t^2 + \eta t + \cdots.$$

Define

$$Z_0(t) = M_2(t) - (\gamma t^2 + \eta t).$$

Using (16), it follows that Z_0 satisfies the following renewal equation:

$$Z_0(t) = a(t) + \int_0^t Z_0(t-x)f(x)\,dx,$$

where, by partial integration,

$$a(t) = -(\gamma t^2 + \eta t) + H(t) + 2\sum_{k=0}^{\infty} N * f^{k*}(t)$$
$$+ \gamma\int_0^t (t-x)^2 f(x)\,dx + \eta\int_0^t (t-x)f(x)\,dx$$
$$= H(t) - \mu_{C^2} + 2\left\{\sum_{k=0}^{\infty} N * f^{k*}(t) - \left[\frac{\mu_C^2}{\mu_T}t + \frac{\mu_{T^2}\mu_C^2 - 2\mu_C\mu_T\mu_{CT}}{2\mu_T^2}\right]\right\}$$
$$+ 2\gamma\int_t^\infty (t-x)(1-F(x))\,dx + \eta\int_t^\infty (1-F(x))\,dx.$$

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The function a is directly Riemann integrable, and since

$$\int_0^\infty (H(t) - \mu_{C^2})\,dt = -\mathbb{E}\left[\int_0^\infty C^2 1_{\{T>t\}}\,dt\right] = -\mu_{C^2T},$$

and, by (13),

$$\begin{aligned}2\int_0^\infty &\left\{\sum_{k=0}^\infty N * f^{k*}(t) - \left[\frac{\mu_C^2}{\mu_T}t + \frac{\mu_{T^2}\mu_C^2 - 2\mu_C\mu_T\mu_{CT}}{2\mu_T^2}\right]\right\}dt\\ &= \frac{\mu_C^2\mu_{T^2}^2}{2\mu_T^3} - \frac{\mu_C^2\mu_{T^3} + 6\mu_C\mu_{CT}\mu_{T^2}}{3\mu_T^2} + \frac{2\mu_C\mu_{CT^2} + 2\mu_{CT}^2}{\mu_T},\\ \int_0^\infty \int_t^\infty (t-x)(1-F(x))\,dx\,dt &= -\frac{1}{2}\int_0^\infty x^2(1-F(x))\,dx\\ &= -\frac{1}{6}\mu_{T^3},\end{aligned}$$

and

$$\int_0^\infty \int_t^\infty (1-F(x))\,dx\,dt = \frac{1}{2}\mu_{T^2},$$

we get

$$\begin{aligned}\int_0^\infty a(t)\,dt = {}& \frac{3\mu_C^2\mu_{T^2}^2}{2\mu_T^3} - \frac{2\mu_C^2\mu_{T^3} + 12\mu_C\mu_{CT}\mu_{T^2}}{3\mu_T^2}\\ &+ \frac{4\mu_C\mu_{CT^2} + 4\mu_{CT}^2 + \mu_{C^2}\mu_{T^2}}{2\mu_T} - \mu_{C^2T}.\end{aligned}$$

The proposition follows from an application of the key renewal theorem. ■

COROLLARY 7.3: *If in Proposition 7.1 the convergence is sufficiently fast, that is,*

$$M_1(t) - \frac{\mu_C}{\mu_T}t = \frac{\mu_C\mu_{T^2}}{2\mu_T^2} - \frac{\mu_{CT}}{\mu_T} + o\left(\frac{1}{t}\right)$$

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as $t \to \infty$, we get from Propositions 7.1 and 7.2 that

$$\lim_{t\to\infty}\left\{\mathrm{Var}(K(t,0)) - \frac{\mu_C^2\mu_{T^2} - 2\mu_C\mu_T\mu_{CT} + \mu_{C^2}\mu_T^2}{\mu_T^3}t\right\}$$
$$= \frac{5\mu_C^2\mu_{T^2}^2}{4\mu_T^4} - \frac{2\mu_C^2\mu_{T^3} + 9\mu_C\mu_{CT}\mu_{T^2}}{3\mu_T^3}$$
$$+ \frac{4\mu_C\mu_{CT^2} + 2\mu_{CT}^2 + \mu_{C^2}\mu_{T^2}}{2\mu_T^2} - \frac{\mu_{C^2T}}{\mu_T}.$$

In the special case $C \equiv 1$, Corollary 7.3 can be simplified to

$$\lim_{t\to\infty}\left\{\mathrm{Var}(K(t,0)) - \frac{\mu_{T^2} - \mu_T^2}{\mu_T^3}t\right\} = \frac{5\mu_{T^2}^2}{4\mu_T^4} - \frac{2\mu_{T^3}}{3\mu_T^3} - \frac{\mu_{T^2}}{2\mu_T^2}. \quad \textbf{(17)}$$

This result was proved by Smith [18]. For T exponentially distributed, the right-hand side of (17) equals zero and the asymptotic expansion is exact.

8. CONCLUSIONS

Using renewal theory, mathematical expressions for the first and second moments of the discounted costs over a bounded or unbounded time horizon are derived for general forms of discounting. As special cases, the following types of discounting are considered: exponential discounting, (generalized) hyperbolic discounting, and no discounting. For exponential discounting, analytic expressions are derived for the first and second moments of the discounted costs over an unbounded time horizon. For generalized hyperbolic discounting, a striking new result arises for the equivalent average costs per unit time. When the integral of the generalized hyperbolic discount function over an unbounded time horizon is infinite, the expected equivalent average costs per unit time tends to the expected average costs per unit time as the time horizon approaches infinity. For no discounting, asymptotic expansions are derived for the first and second moments of the expected costs over a bounded time horizon for possibly dependent renewal costs and renewal interoccurrence times.

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