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REVISITING THE OPTIMAL STATIONARY PUBLIC INVESTMENT POLICY IN ENDOGENOUS GROWTH ECONOMIES

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One part of the literature on endogenous growth concerns models where public infrastructure affects the private production process. An unsolved puzzle in this literature concerns observed public investment-to-output ratios for developed economies, which tend to fall short of theoretical model-based optimal ratios. We reexamine the optimal choice of public investment in a more general framework. This setting allows for long-lasting capital stocks, a lower depreciation rate for public capital than for private capital, an elasticity of intertemporal substitution that differs from unity, and the need to finance a nontrivial share of public services in output. Given other fundamentals in the economy, we show that the optimal public investment-to-output ratio is smaller for low-growth economies, for economies populated by consumers with low preferences for substituting consumption intertemporally, and when public capital is durable. For a calibrated economy, we show that a combination of these factors solves the public investment puzzle.

Keywords: Public Investment, Stationary Policy, Balanced Growth Equilibrium

1. INTRODUCTION

We characterize optimal public investment in a calibrated, general equilibrium endogenous growth model. Optimality is understood in this exercise in a policy-constrained Ramsey sense. Current public investment sizes around 3 to 4%, as seen in most OECD countries, tend to fall short of theoretically based optimal ratios in the previous literature. This apparent puzzle motivates this paper. We reexamine the optimal choice of public investment in a more general framework and find that the observed public investment ratios are about right. The main differences between our setup and previous literature are (i) a more realistic calibration of the depreciation rate for public and private capital,[1] (ii) the requirement to finance

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a positive share of public services in output every period,[2] and (iii) a movement away from logarithmic preferences.[3]

Early empirical work by Aschauer (1989) and Munnell (1990) identified the significant impact that public infrastructure has on economic growth. One strand of the literature on endogenous growth relating to models in which public investment affects the private production process has been motivated in part by this empirical finding. Barro (1990) represents an important breakthrough in characterizing the influence of public infrastructure on growth and welfare in an endogenous growth setting. Subsequent works by Futagami et al. (1993), Glomm and Ravikumar (1994), Cassou and Lansing (1998, 1999), Turnovsky (1997), Aschauer (2000), and Marrero and Novales (2005, 2007) are variations of Barro's research. In these studies, public revenues are raised from proportional income taxes and the government decides on a constant ratio of public investment to output.

The public sector has gradually lost influence in the productive activity of most developed economies since 1960. Although total public outlays have represented meaningful shares of GDP during the last four decades, public investment-to-output ratios have generally declined over the same period.[4] Nevertheless, this ratio has remained relatively stable since the 1990s. By 2000, public investment represented 3.7% of total real GDP in the OECD and 3.1% in the U.S. These figures fall well below the optimal public investment-to-output ratios predicted by most recent studies under a standard calibration. In fact, the significant difference between observed and theoretically based public investment ratios has been blamed for the productivity slowdown of the 1970s and 1980s.[5] The goal of the paper is not to discuss the productivity slowdown, on which there is already an extensive literature, but to revisit the optimal choice of public investment in a more general and plausible framework than those mentioned above. This adds to research by Cassou and Lansing (1998, 1999) and Marrero (2005), who use a stylized one-sector growth model carefully calibrated for the postwar U.S. and other OECD economies to find that reduced public investment over the 1970s and 1980s accounted for only a small proportion of the productivity slowdown.

Our economy allows for a gradual transition between different steady states, which Futagami et al. (1993) found to be an important factor in determining optimal public investment policy in a Barro-type framework. Futagami et al. (1993) did not derive an expression for the optimal stationary public investment policy. In this paper, we obtain an *implicit* expression for the optimal stationary public investment-to-output ratio in a more general endogenous growth framework, and we contribute to filling this gap.[6]

A careful examination of the implicit policy expression provides an important insight into the size and determinants of the optimal stationary public investment-to-output ratio. We show that the optimal ratio is lower than the growth-maximizing ratio. Whereas the public capital elasticity and the subjective discount rate positively affect optimal public investment policy, as is well known,[7] the factors mentioned above have a negative effect on the optimal ratio, as we show. The endogenous growth rate is an indirect channel through which the fundamentals of

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the economy may additionally affect optimal policy. When new features (i)–(iii) are simultaneously incorporated into an otherwise standard, carefully calibrated economy, the optimal public investment ratio obtained under our benchmark parameterization approximates the values observed in actual economies.

The rest of the paper is organized as follows. In Section 2, we show the public investment puzzle within the context of existing studies. In Section 3, we describe the economic model. In Section 4, we define the balanced growth competitive equilibrium and show its existence. Section 5 shows the optimal stationary public investment and tax policy. In Section 6, we carry out a numerical illustration. Section 7 concludes.

2. THE PUBLIC INVESTMENT PUZZLE

In this section, we summarize the optimal stationary public investment policy predicted by existing studies. For standard model-based calibrations, we identify a significant difference between these theoretical optimal ratios and the empirically observed ratios in developed economies at the end of the 90s. We then point out a puzzle in this literature.

The pioneering work of Barro (1990) treats the flow of public infrastructure as an input into private production. The development of public infrastructure induces higher future returns to private investment, but also distorts private incentives to consume and save through higher taxes. Optimal policy equalizes the post-tax return of private capital and the return on public infrastructure. Barro obtained the well-known result that the optimal share of output devoted to public investment equals the elasticity of public capital in the production function, θ. Aschauer (2000) obtains the same result in a similar framework. Moreover, this optimal policy corresponds to the public investment ratio that maximizes growth.

In a similar dynamic framework, Glomm and Ravikumar (1994) assume that the stock of public infrastructure is input into private production. Since public capital is productive in subsequent periods, the optimal public investment-to-output ratio is $\beta\theta$, where β, which is between zero and unity, is the subjective discount rate of the representative household.[8] In addition, if the government is limited to financing a constant share of output devoted to public services, $\psi \in (0, 1)$, in each period, the optimal public investment-to-output ratio is $\beta\theta(1 - \psi)$, as in Marrero and Novales (2005). The negative effect of a positive ψ on the net return to public capital makes the optimal ratio lower than those described above.

Public investment-to-output ratios for developed countries have generally declined since 1960, although they have been reasonably stable during the last decade. By 2000, this ratio was about 3.7% for OECD countries and about 3.1% for the U.S. economy.

For a standard calibration, Table 1 shows the optimal public investment-to-output ratios implied by the above works. We assume that $\beta = 0.97$ and $\psi = 0.18$ [as in Cassou and Lansing (1998)]. Given the debate about the calibration of the public capital elasticity, we consider a range of values for this parameter. For

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TABLE 1. Optimal stationary public investment-to-output ratio in selected earlier literature

Public capital elasticity	Barro (1990), Aschauer (2000)	Glomm and Ravikumar (1994)	Marrero and Novales (2005)	Cassou and Lansing (1998, 1999)
0.05	0.050	0.049	0.040	0.038
0.10	0.100	0.097	0.080	0.076
0.15	0.150	0.146	0.119	0.115
0.20	0.200	0.194	0.159	0.153
0.25	0.250	0.243	0.199	0.191
0.30	0.300	0.291	0.239	0.229

Note: We use standard parameter values: a discount factor of 097, a capital depreciation rate of 0.1, and a share of output devoted to public consumption of 0.18. The public investment-to-output ratio is 0.037 for the OECD and 0.031 for the United States by 2000. Since there is not a general concent about the level of the public capital elasticity in the production function, we consider arrange of this parameter (first column).

small values of θ, that is, below 0.05, the model-based optimal ratios are close to current public investment ratios for the OECD and the U.S.. Although some empirical papers[9] obtain estimates of θ that are close to zero, a recent consensus suggests that this elasticity is between 0.1 and 0.2.[10] Hence, either current public investment policies are suboptimal or existing models omit relevant factors and hence offer misleading policy prescriptions.

Related studies assume that public and private capital fully depreciate in one period, and the models reduce to the special case of an AK economy. Thus, they lack transitional dynamics. Moreover, to obtain analytical solutions, they assume a logarithmic utility function (i.e., that the elasticity of intertemporal substitution is unity).

Futagami et al. (1993) point out the importance of accounting for transitional dynamics in optimal policy design. Turnovsky (2004) emphasizes this issue in a nonscale economy that exhibits exogenous growth. In an economy with transitional dynamics, there is a trade-off between consumption during the transition and long-run growth, which causes the optimal public investment-to-output ratio to be lower than the growth-maximizing ratio. Indeed, assuming a log-linear accumulation rule for public capital and a logarithmic utility function, Cassou and Lansing (1998) find the optimal stationary public investment-to-output ratio to be $\beta\theta\delta_g/[1-\beta(1-\delta_g)]$, where δ_g is the depreciation rate of public capital, which is between zero and unity. If public capital fully depreciates in one period, the economy lacks transitional dynamics and the optimal policy is as stated by Glomm and Ravikumar (1994). However, the lower the depreciation rate, the slower are the transition dynamics and the more important is the welfare trade-off referred to above. Effectively, for $\delta_g < 1$, the optimal ratio is always lower than $\beta\theta$. Nevertheless, for standard parameter values ($\delta_g = 0.1$), the optimal ratio remains well above 4% when the public capital elasticity exceeds 0.05 (see Table 1).

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In subsequent sections, we revisit the optimal choice of public investment under a more general and plausible framework. Finally, we reconsider the public investment puzzle based on a calibrated economy within our new setting.

3. THE ECONOMY

We consider a general one-sector economy incorporating a large but fixed number of identical infinitely lived households. Each household is the owner of a unique firm that produces the single, nonstorable consumption good in the economy, y. There is a benevolent government that solves an optimal fiscal policy problem in a constrained Ramsey sense. The model is similar to those of Ai and Cassou (1995), Cassou and Lansing (1998, 1999), Marrero (2005) and Glomm and Ravikumar (1999), which incorporate durable capital and productive public expenditure. We assume that public capital might depreciate at a slower rate than private capital (Ai and Cassou, 1995), that the government is limited to financing a constant share of output devoted to public consumption in each period,[11] and that the elasticity of intertemporal substitution may differ from unity (as suggested by Prescott and others). Population growth rate is assumed to be zero, population size is normalized to one, labor is supplied inelastically, and all variables are defined in per capita terms.[12] We restrict attention to Cobb–Douglas technology and CES utility, because these functional forms are needed for existence of a balanced steady-state equilibrium.

3.1. Firms

Each firm produces y according to a Cobb–Douglas technology,

$$f\left(k_t, l_t \cdot z_t, g_t\right) = A_0 k_t^{\alpha} \left(l_t z_t\right)^{\phi} g_t^{\theta}, \qquad \alpha, \theta, \phi \in (0, 1)\,, A_0 > 0, \tag{1}$$

where k is private capital stock, g is public capital, l is labor, z is an index of knowledge available to each firm that augments the productive capacity of labor, and $l \cdot z$ is effective labor; A_0 is a technological scale factor, and α, θ, and ϕ are the elasticities of output with respect to k, g, and $l \cdot z$, respectively. The function $f(\cdot)$ is increasing, strictly concave, and twice continuously differentiable, and all factors are essential in the production process and satisfies Inada conditions. We assume that $f(\cdot)$ is homogeneous of degree one, which implies $\alpha + \theta + \phi = 1$.[13] This condition is required for the existence of a balanced growth equilibrium, as commented on in Section 4.

The average capital stock across firms is taken as a proxy for z (Romer, 1986). Although firms decide on private factors, public capital and the knowledge index are outside of their control and are taken as exogenous. Because firms are identical, the average capital stock is equal to k, and per capita output is produced

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according to

$$y_t = F(k_t, l_t, g_t) = f(k_t, l_t \cdot k_t, g_t) = A_0 k_t^{\alpha+\phi} g_t^{\theta}. \quad \textbf{(2)}$$

We note that $F(k_t, l_t, g_t) = f(k_t, l_t \cdot k_t, g_t)$ and $\partial f(\cdot)/\partial g = \partial F(\cdot)/\partial g = \theta y_t/g_t$ for a particular allocation. However, because each firm neglects its own contribution on the aggregate capital stock, $\partial F(\cdot)/\partial k$ and $\partial f(\cdot)/\partial k$ are not equal. More precisely, $\partial f(\cdot)/\partial k = \alpha y_t/k_t$ and $\partial F(\cdot)/\partial k = (\alpha + \phi) y_t/k_t$.

The firm's problem [P1]. Because investment decisions are made by households, the firm's problem is static. Firms demand k and l, whereas g and z are taken as exogenous variables. Each firm pays the competitively determined wage w on the labor it hires and the rate r on the capital it rents. Taking g and z as given optimally leads to the usual marginal productivity conditions and the resultant firm profits, Ω, every period:

$$r_t = \partial f(\cdot)/\partial k = \alpha \frac{y_t}{k_t}, \quad \textbf{(3)}$$

$$w_t = \partial f(\cdot)/\partial (l \cdot z) = \phi \frac{y_t}{l_t}, \quad \textbf{(4)}$$

$$\Omega_t = (1 - \alpha - \phi) y_t = \theta y_t. \quad \textbf{(5)}$$

Because the production function is homogeneous of degree one, from (3)–(5), we have that $y_t = w_t l_t + k_t r_t + \Omega_t$, for all periods t.

3.2. Households

There exist a large number of identical infinitely lived households, which allocate their resources between consumption, c, and investment in physical capital, i. Households are the owners of physical capital and firms, and they receive firms' profits as exogenous income.[14] The single commodity good is valued by the household according to a CES utility,

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$
$$u(c_t) = \frac{c_t^{1-1/\sigma}}{1 - 1/\sigma}, \quad \sigma > 0, \quad \beta \in (0, 1), \quad \textbf{(6)}$$

where β is the discount factor and σ is the elasticity of intertemporal substitution. The function $u(\cdot)$ is increasing, strictly concave, and twice continuously differentiable and satisfies Inada conditions. Each household is endowed with one unit of time.

Private capital accumulates over time according to

$$k_{t+1} = (1 - \delta) k_t + i_t, \quad t = 0, 1, \ldots, \quad \textbf{(7)}$$

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where δ is a linear depreciation rate of private capital, between zero and one. Because $\delta < 1$, this specification allows for long-lived private capital, which is a relevant and realistic element in the model.[15]

$$c_t + k_{t+1} \leq l_t w_t(1-\tau_t) + k_t[1-\delta + r_t(1-\tau_t)] + \Omega_t(1-\tau_t), \qquad \textbf{(8)}$$

every period, where τ_t is a tax rate applied to total household income, which is determined outside of its control.

The household's problem [P2]. A representative household maximizes (6) subject to (8), $c_t \geq 0$, and $k_{t+1} \geq 0$ and the transversality condition

$$\lim_{t\to\infty} \beta^t k_{t+1} \partial u(c_t)/\partial c_t = 0, \qquad \textbf{(9)}$$

which places a limit on the accumulation of private capital. The standard optimal consumption–saving decision is given by

$$\left(\frac{c_{t+1}}{c_t}\right)^{1/\sigma} = \beta[1-\delta + r_{t+1}(1-\tau_{t+1})], \quad t = 0, 1, \ldots. \qquad \textbf{(10)}$$

Because $u(c_t)$ is strictly increasing for all $c_t \geq 0$, (8) holds with equality at equilibrium. Finally, because the household does not value leisure, the unit of labor is supplied inelastically every period.

3.3. Government

The government claims a constant proportion, ψ, of output to fund public consumption, c_g, every period,

$$c_{gt} = \psi y_t, \qquad \textbf{(11)}$$

which is taken as given. This assumption ensures that public expenditure continues to represent a significant and realistic share of economic output as the economy grows.[16] Public consumption does not contribute to either production or consumer welfare.

Infrastructure evolves according to

$$g_{t+1} = i_{gt} + (1-\delta_g)g_t, \quad t = 0, 1, \ldots, \qquad \textbf{(12)}$$

where i_g is public investment and $\delta_g \in [0, 1]$ is the linear depreciation rate of public capital, which might be lower than that of private capital.[17]

We assume that issuing debt is not allowed and a proportional tax on aggregate private income is the only way to finance total public expenses,

$$c_{gt} + i_{gt} = \tau_t y_t, \quad t = 0, 1, \ldots. \qquad \textbf{(13)}$$

Combining (11) with (13), the public investment-to-output ratio would be given by $\tau_t - \psi$, which is denoted by x_t henceforth. A *feasible policy* is a trio of nonnegative

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sequences, $\pi = \{\tau_t, g_{t+1}, c_{gt}\}_{t=0}^{\infty}$, with $\tau_t \in [0,1]$, $c_{gt} \geq 0$, and $g_{t+1} > 0$ and satisfying (12) and (13) for any period t.

4. COMPETITIVE EQUILIBRIUM AND BALANCED GROWTH PATH

DEFINITION 1. *Given a feasible policy π and initial conditions $k_0, g_0 > 0$, a π-*competitive equilibrium *under a balanced budget is a vector of sequences $\{c_t, c_{gt}, k_{t+1}, l_t, g_{t+1}, i_t, i_{gt}, y_t\}_{t=0}^{\infty}$ together with a price system $\{r_t, w_t\}_{t=0}^{\infty}$ such that (i) $\{k_{t+1}, l_t\}_{t=0}^{\infty}$ solve the profit maximizing problem of the firms* [P1]; (ii) *$\{c_t, k_{t+1}, l_t\}_{t=0}^{\infty}$ solve the household's problem* [P2]; (iii) *the technology constraints (2), (7), (12) hold; and (iv) markets clear every period,*

$$y_t = c_t + c_{gt} + i_t + i_{gt}, \tag{14}$$

$$l_t = 1. \tag{15}$$

Because our goal is to examine the long-run implications of fiscal policy, the rest of the section lays down the properties of a balanced steady-state equilibrium and the conditions for its existence. This analysis is based on King et al. (1988), Jones and Manuelli (1990, 1997), and Caballé and Santos (1993), among others.[18]

DEFINITION 2. A balanced growth path, *BGP* (or steady-state equilibrium) *is a competitive equilibrium allocation such that y_t, c_t, k_t, and g_t grow at constant rates and the output/capital ratio is constant.*

Since the output/capital ratio is constant in steady state and the production function is homogeneous of degree one (i.e., $\alpha + \theta + \phi = 1$), y_t, k_t, and g_t grows at the same constant rate in a balanced equilibrium, which is denoted by γ.[19] Dividing (14) by k_t, we see that c_t grows at the same rate γ in a BGP. Combining (10) and (3), it is easy to show that the following standard conditions are necessary for the existence of a constant private consumption growth rate in a balanced growth equilibrium: (i) the production function shows constant elasticities of productive factors to output, that is, α, θ, and ϕ are constant; (ii) the utility function exhibits a constant elasticity of intertemporal substitution, that is, σ is constant; (iii) fiscal policy is stationary, that is, $\tau_t = \tau$ and $x_t = x$.

Condition (i) implies a constant real interest rate at the BGP [condition (3)]. Condition (ii) is equivalent to saying that preferences exhibit constant marginal rates of substitution between current and future consumption; that is, the left-hand side in (10) is constant. We note that the Cobb–Douglas technology and the CES utility are the only functional forms satisfying these properties.[20] From (10), it is clear that, given (ii) and (iii), condition (i) must be satisfied for the existence of a constant consumption growth rate.

We write BGP conditions in terms of stationary ratios: $\hat{k} = k/g$, $\hat{z} = z/g$, $\hat{y} = y/g$, $\hat{c} = c/g$, and $\hat{c}_g = c_g/g$. Because these ratios are constant at a BGP, we omit the time subscript if the economy is at this equilibrium. We customize

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competitive equilibrium conditions to be along a BGP, and we get the system of nonlinear equations

$$\hat{c} + (\gamma + \delta)\,\hat{k} = (1 - \tau)A_0\hat{k}^{\alpha}\hat{z}^{\phi}, \qquad \textbf{(16)}$$

$$1 + \gamma = \beta^{\sigma}\left[1 - \delta + (1 - \tau)\,\alpha A_0\hat{k}^{\alpha-1}\hat{z}^{\phi}\right]^{\sigma}, \qquad \textbf{(17)}$$

$$\gamma = xA_0\hat{k}^{\alpha}\hat{z}^{\phi} - \delta^{g}, \qquad \textbf{(18)}$$

$$\hat{c}^{g} = \psi A_0\hat{k}^{\alpha}\hat{z}^{\phi}, \qquad \textbf{(19)}$$

where (16) and (17) are the balanced growth versions of the resource constraint of the economy and of the Euler equation of the representative household, respectively, and (18) and (19) follow directly from the public investment and public consumption rules respectively.

The following proposition refers to necessary and sufficient conditions for the existence of an interior balanced equilibrium in our economic setting: [21]

PROPOSITION 3. *Given a CES utility, a Cobb–Douglas technology, and a stationary fiscal policy, the following conditions are necessary and sufficient for the existence of a unique interior balanced path with positive growth rate*:

(C1) $r > \dfrac{1 - \beta(1 - \delta)}{(1 - \tau)\beta}$,

(C2) $1 + \gamma < \beta^{\sigma/(1-\sigma)}$.

From (17), condition C1 guarantees that γ is positive. This condition is equivalent to *condition* [*G*] in Jones and Manuelli (1990). Condition C2 says that γ cannot be so large that it allows households to follow a chain-letter action; that is, the transversality condition (9) must hold along the BGP.[22] This condition guarantees $\sum_{t=0}^{\infty}\beta^{t}u(c_t)$ to be bounded above on the set of feasible allocations in a balanced growth competitive equilibrium, a necessary condition for its existence. In terms of γ, conditions C1 and C2 can be seen as

$$1 < 1 + \gamma < \beta^{\sigma/(1-\sigma)}. \qquad \textbf{(20)}$$

Thus, given a feasible policy π, the combination of (17) and (18) leads to

$$\Phi(\hat{k}, \hat{z}) = \beta^{\sigma}[\alpha(1 - \tau)A_0\hat{k}^{\alpha-1}\hat{z}^{\phi} + 1 - \delta]^{\sigma} - xA_0\hat{k}^{\alpha}\hat{z}^{\phi} + 1 - \delta^{g} = 0, \qquad \textbf{(21)}$$

and positive roots of $\Phi(\hat{k}, \hat{z})$ are potential candidates to be steady-state values of $\hat{k}$. The solution takes the form of $\hat{k} = \kappa(\hat{z})$. But $\hat{z} = \hat{k}$ at equilibrium. Hence, there must exist some fixed point, $\hat{k}^{*} > 0$, of $\kappa(\hat{k})$ solving (21): $\kappa(\hat{k}^{*}) = \hat{k}^{*}$, such that $\Phi(\kappa(\hat{k}^{*}), \hat{k}^{*}) = \Phi(\hat{k}^{*}, \hat{k}^{*}) = 0$.[23] Given $\sigma > 0$, $\theta \in (0, 1)$, $\tau \in [0, 1]$, and, $\hat{k} > 0$, and setting $\hat{z} = \hat{k}$ into (21), it is easy to show that $\Phi(\hat{k}, \hat{k})$ is continuous and strictly decreasing in $\hat{k}$, with $\lim_{\hat{k}\to 0^{+}}\Phi(\hat{k}, \hat{k}) = +\infty$ and $\lim_{\hat{k}\to +\infty}\Phi(\hat{k}, \hat{k}) = -\infty$. Hence, there exists a single $\hat{k}^{*} > 0$ such that $\Phi(\hat{k}^{*}, \hat{k}^{*}) = 0$, which defines the

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steady state of the economy and proves existence and uniqueness of an interior balanced path with positive growth. Given $\hat{k}^*$, the other endogenous variables are easily recovered from (16)–(19).

5. OPTIMAL PUBLIC INVESTMENT POLICY ON THE BGP

There are many competitive equilibria implied by different government policies. The Ramsey problem chooses the π-competitive equilibrium allocation that maximizes consumer welfare. In this section, we follow Chapter 12 in Ljungqvist and Sargent (2000) to define and solve the Ramsey problem. In the Appendix (part 1), we show that the assumptions of Theorem 2 in Le Van and Saglam (2004) are fulfilled for our Ramsey problem; hence Lagrangian multiplier techniques can be used to solve it.[24]

The Ramsey problem is

$$\max_{\{c_t, k_{t+1}, g_{t+1}, \tau_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-1/\sigma}}{1-1/\sigma}, \text{ subject to}$$

$$c_t^{-1/\sigma} = \beta c_{t+1}^{-1/\sigma} \left[1 - \delta + \alpha A_0 k_{t+1}^{-\theta} g_{t+1}^{\theta} (1 - \tau_{t+1})\right],$$

$$g_{t+1} = (\tau_t - \psi) A_0 k_t^{1-\theta} g_t^{\theta} - (1 - \delta_g) g_t,$$

$$(1 - \tau_t) A_0 k_t^{1-\theta} g_t^{\theta} = c_t + k_{t+1} - (1 - \delta) k_t,$$

with $c_t, g_{t+1}, k_{t+1} \geq 0$ and $\tau_t \in [0, 1]$ for all t, and satisfying the transversality condition (9). The first constraint corresponds to the household's Euler equation, the second represents the accumulation of public capital, and the third is the global resource constraint of the economy. The household's budget constraint is not explicitly included, because it is redundant when the resource constraint holds and the government's budget constraint is satisfied.

Optimal (interior) conditions for this problem are shown in the Appendix (part 2) [condition (A.15)–(A.21)].[25] We then focus on the optimal stationary policy in a balanced growth competitive equilibrium [condition (A.22)–(A.28) in the Appendix (part 2)]. For the Cobb–Douglas technology, combining (A.22) and (A.25), this policy must satisfy the following condition:

$$\beta \left[1 + (1 - \psi)\theta A_0 \hat{k}^{1-\theta} - \delta_g\right] = (1 + \gamma)^{1/\sigma}. \tag{22}$$

On the left-hand side of the equation is the marginal rate of transformation between current output and available public capital—net of depreciation and discounted by the factor β—which must be set equal to the rate at which the household wishes to substitute current and future consumption (the right-hand side of the equation).[26] A high level of the right-hand side—that is, a low level of σ—means that the household places little value on future consumption relative to current consumption. Thus, given other economic fundamentals and the proportion ψ of total resources used by the government to fund public services, optimal public

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investment policy should hardly crowd out current consumption, in which case the public investment-to-output ratio should be low. The productivity of public capital is high in this case and the left-hand side of (22) equals the right-hand side. Similar arguments apply to economies with high levels of σ.

In addition, from the equation for public capital accumulation (A.27), we have

$$A_0\hat{k}^{1-\theta} = \frac{\delta_g + \gamma}{x}. \tag{23}$$

Substituting (23) into (22) reveals that the optimal stationary public policy, x^+ and τ^+, satisfies

$$x^+ = \beta\theta(1-\psi)\frac{\gamma+\delta_g}{(1+\gamma)^{1/\sigma} - \beta(1-\delta_g)}, \tag{24}$$

$$\tau^+ = x^+ + \psi. \tag{25}$$

Nevertheless an explicit expression for the optimal stationary policy cannot be obtained, because γ depends on fundamentals and policy parameters in a nontrivial manner.[27] The unique environment in which an explicit expression for the optimal public investment ratio can be determined is whenever $u(c) = \log c$ (i.e., $\sigma = 1$) and $\delta_g = 1$, where the optimal public investment-to-output ratio is the standard $\beta\theta\,(1-\psi)$ (Marrero and Novales, 2005).

Although, in general, we cannot obtain an explicit expression for x^+, several important results arise from a careful examination of (24). As a point of reference, we take the standard optimal ratio, $\beta\theta\,(1-\psi)$. From (24), it is easy to show that $x^+ < (>)\beta\theta\,(1-\psi)$ whenever the following condition holds:

$$1+\gamma < (>)(1+\gamma)^{1/\sigma} + (1-\delta_g)(1-\beta). \tag{26}$$

Because $(1+\gamma)^{1/\sigma} > 1$ and $\beta, \delta_g < 1$, it is worth noting that the right-hand side term in (20) is greater than the right-hand side expression in (26) for any parameter values. Thus, in theory, x^+ might be above or below the threshold $\beta\theta\,(1-\psi)$. The following proposition states this result in accordance with the relationship among key parameters σ and δ_g.

PROPOSITION 4. *Under a CES utility, a Cobb–Douglas technology, a stationary fiscal policy, and conditions* C1 *and* C2: (*i*) *if* $\sigma < 1$ *the optimal public investment-to-output ratio,* x^+, *is lower than* $\beta\theta(1-\psi)$ *for any level of* δ_g; (*ii*) *if* $\sigma = 1$ *and* $\delta_g < 1$, x^+ *is also below* $\beta\theta(1-\psi)$; (*iii*) *if* $\sigma = 1$ *and* $\delta_g = 1$, x^+ *is equal to* $\beta\theta(1-\psi)$; (*iv*) *if* $\sigma > 1$ *and* $\delta_g = 1$, x^+ *is higher than* $\beta\theta(1-\psi)$; (*v*) *finally, if* $\sigma > 1$ *and* $\delta_g < 1$, *condition* (26) *must be checked to determine whether* x^+ *is greater than or less than* $\beta\theta(1-\psi)$.

Although the optimal stationary public investment-to-output ratio may exceed $\beta\theta\,(1-\psi)$, it cannot do so by much. Indeed, the next proposition shows that the

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ratio cannot exceed $\theta\,(1-\psi)$, which is the ratio that maximizes the steady-state growth rate.[28]

PROPOSITION 5. *Under a CES utility, a Cobb–Douglas technology, a stationary fiscal policy, and conditions* C1 *and* C2, *the optimal public investment-to-output ratio along a BGP equilibrium is less than* $\theta(1-\psi)$.

Proof. Rewrite condition (24) as $x^{+} = \theta(1-\psi)\frac{\bar{\gamma}+\delta_g}{1/\beta\bar{u}_c-(1-\delta_g)}$. From conditions C1 and C2, $\frac{\bar{\gamma}+\delta_g}{1/\beta\bar{u}_c-(1-\delta_g)}$ is less than unity; hence x^{+} is lesser than $\theta(1-g)$. ■

In an economy in which consumers place little value on current consumption in terms of future consumption, a benevolent government sets a high level of x^{+}, generally above $\beta\theta(1-\psi)$ and close to the level that maximizes growth, $\theta(1-\psi)$. Consequently, the resulting difference between economic growth and the marginal rate of substitution between future and current consumption would be large. However, the optimal public investment-to-output ratio does not exceed the ratio that maximizes growth, because under this policy there would be less consumption and growth on the BGP. On the other hand, the optimal public investment-to-output ratio would be well below $\beta\theta(1-\psi)$ and the resulting difference between γ and $(1+\gamma)^{1/\sigma}$—that is the marginal rate of substitution between next-period and current consumption—would be small. In short, the optimal public investment-to-output ratio would be smaller for low-growth economies, economies populated by consumers with low preferences for substituting consumption intertemporally, and economics with long-lasting public infrastructures.

From (24), it is clear that the following fundamentals of the economy directly affect optimal public investment policy: β, θ, ψ, δ_g, and σ. Although the effects on x^{+} of β, θ, δ_g, and σ are positive, that of ψ on x^{+} is negative. The intuition behind these relationships is straightforward. A lower discount factor, β, and a smaller marginal rate of substitution between present and future consumption means that households have a higher preference for current consumption relative to future consumption. On the other hand, a lower depreciation rate implies slower transition dynamics. Because $\beta < 1$, the future (i.e., the long run) is less important than the short run for aggregate welfare under these circumstances. Last, θ is positively related to the rate of return on public capital, whereas a higher ψ implies that a higher proportion of output must be financed by distortionary taxes, which reduces the return to private investment.

In addition, changes in these fundamentals might have indirect effects on the optimal policy through their effects on the endogenous growth rate. Indeed, the private capital depreciation rate can only affect the optimal policy through this channel. The total effect of all the other factors on the optimal policy is the sum of the direct and indirect effects. For a calibrated economy, we evaluate these effects in the next section. We show that the strength of the indirect effects depends largely on the depreciation rate of public capital and on household preferences between present and future consumption. For instance, if the elasticity of intertemporal

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substitution is unity and public capital fully depreciates in one period, the optimal public investment ratio is $\beta\theta\,(1-\psi)$, and there is no indirect effect.

6. A NUMERICAL ILLUSTRATION

In order to illustrate the relationship between the optimal stationary policy and the fundamentals of the economy, we examine numerical solutions for a calibrated economy that resembles steady-state characteristics of the U.S. postwar economy. Given parameter values, the solution of (21) gives the steady-state value of $\hat{k}$. We then obtain γ and x^{+} from (23) and (24), respectively.

6.1. Values of the Parameters

The parametrization of the baseline economy is standard. The time unit is the natural year. Based on King et al. (1988) and others, we choose $\phi = 0.58$, the labor elasticity in the Cobb–Douglas technology. We use $\sigma = 0.5$, as considered by Prescott and others. For depreciation rates, we take those levels estimated by Ai and Cassou (1995): $\delta = 0.094$ and $\delta_g = 0.038$. The value $\psi = 0.18$ implies that the government spending on nonproductive goods and services is 18% of output, which is consistent with its postwar average. The elasticity of public capital is the most controversial parameter to assign. Consequently, we use a range of values for this parameter.[29] For the benchmark economy, however, θ is chosen together with A_0 and β to match a steady-state per capita growth rate of 2.9% and to reproduce a public-to-private capital ratio of 0.55 with $x = 0.052$ and an after-tax interest rate of 6.9%,[30] which are consistent with their levels in the 60s for the U.S. economy. Finally, recall that θ, ϕ, and α are not independent, because $\alpha + \theta + \phi = 1$.

For our model economy, the baseline calibration sets the private investment-to-output ratio to be a little under 18%, the share of output that is devoted to private consumption to be about 60%, and total public receipts as a percentage of output to be a little under 25%. These ratios are common for the postwar U.S. and other developed economies.

We report the values of the parameters for the baseline economy in Table 2. Notice that $\theta = 0.093$, similar to that estimated by Shioji (2001), and the resultant value of α is 0.327, similar to the capital share used in the literature.

TABLE 2. The benchmark calibration

Parameters									
β	σ	ψ	x	A_0	α	ϕ	θ	δ	δ_g
0.965	0.500	0.180	0.052	0.779	0.327	0.580	0.093	0.094	0.038
Main steady-state and model-based ratios for the baseline economy									
γ	c/y	i_k/y	i_g/y	c_g/y	g/k	k/y			
0.029	0.603	0.167	0.052	0.18	0.55	1.36			

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6.2. Simulation Results

We first show the quantitative properties of the baseline economy. Then we conduct a sensitivity analysis with respect to the following important parameters: the elasticity of public capital, θ; the elasticity of intertemporal substitution, σ; the share of output devoted to public services, ψ; and the public and private capital depreciation rates, δ_g and δ. The sensitivity analysis facilitates understanding of the determinants of the optimal stationary policy and of the level of the optimal public investment-to-output ratio. In general, the numerical exercise helps solve the public investment puzzle described in Section 2. For each parameterization, we solve the optimal stationary public investment policy numerically, as stated above. We limit the sensitivity analysis to parameterizations satisfying condition (20).[31]

Table 3 reports the optimal stationary public investment-to-output ratio under alternative parameterizations. The table is divided into blocks, one for each

TABLE 3. Optimal stationary public investment-to-output ratio. Sensitivity analysis

	Elasticity of public capital							
	0.097(2)	0.02	0.05	0.10	0.15	0.20	0.25	0.30
x^+ (%)(1)	3.94	0.84	2.12	4.24	6.36	8.48	10.56	12.60
	3.94	0.84	2.12	4.24	6.36	8.48	10.62	12.74
	Elasticity of intertemporal substitution							
	0.500(2)	0.10	0.25	0.75	1.00	1.25	1.50	1.625(3)
x^+ (%)(1)	3.94	2.56	3.08	4.80	5.62	6.44	7.24	7.62
	3.94	1.26	2.66	4.68	5.16	5.48	5.72	5.82
	Public Consumption-to-output ratio							
	0.180(2)	0.00	0.05	0.10	0.15	0.25	0.30	0.35
x^+ (%)(1)	3.94	4.70	4.50	4.28	4.08	3.66	3.44	3.02
	3.94	4.82	4.58	4.34	4.10	3.62	3.36	2.88
	Public capital depreciation rate							
	0.038(2)	0.00	0.10	0.15	0.25	0.50	0.75	1.00
x^+ (%)(1)	3.94	2.50	5.22	5.80	6.42	7.02	7.22	7.34
	3.94	2.40	5.14	5.66	6.24	6.82	7.06	7.20
	Private capital depreciation rate							
	0.094(2)	0.000	0.025	0.051	0.100	0.125	0.150	0.165(4)
x^+ (%)(1)	3.94	3.78	3.82	3.86	3.96	4.03	4.13	4.56
	3.94	3.94	3.94	3.94	3.94	3.94	3.94	3.94

Notes: (1) The optimal ratio keeping the growth rate unchanged. Its difference with respect to the value in the above column measures the indirect effect caused by the parameter change throughout the endogenous growth rate. (2) Benchmark calibration and optimal policy under the baseline economy. (3) The policy satisfying the optimal interior condition breaks the NPG condition for $\sigma > 1.625$. (4) The policy satisfying the optimal interior condition breaks that $\gamma > 0$ for $\delta > 0.165$.

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parameter. The first row of each block shows the optimal policy for the associated parameterization. The second row reports the public investment ratio that is consistent with a constant growth rate. They may differ from each other because of the indirect effect commented above.

The first column in the table reports the optimal stationary policy for the baseline economy. Thus, public investment must be 3.94% of real output. It is worth noting that the optimal ratio is between 3% and 4% for minor changes in all important parameters. For the baseline economy, the optimal ratio is much less than the standard $\beta\theta(1-\psi) = 7.4\%$. The associated growth rate is 2.8% and the marginal rate of substitution between future and current consumption is 1.057. All these results are consistent with Proposition 3.

Other important macro ratios under the optimal policy are $c/y = 61.1\%$, $i_k/y = 17\%$, and $g/k = 43\%$. With respect to the baseline economy, the public sector turns out to be less important for private production, and the public-to-private capital ratio falls from 55% to 43%. On the other hand, private consumption and private investment represent a greater share of output. These features are consistent with the macroeconomic trends in developed economies during recent decades.

For the baseline economy, it is worth noting that the optimal public investment-to-output ratio is very similar to the average ratio in OECD countries and slightly above the ratio for the United States, based on data for 2000 (recall from Section 2). Hence, economic elements may explain the public investment puzzle described in Section 2. Moreover, our results indicate that all elements combine to generate an optimal public investment-to-output ratio of between 3% and 4% under a realistic calibration.

The effect of β and θ on the optimal policy is clearly positive from (24), as in existing studies. Thus, we focus on the new effects of δ_g, σ, ψ, and δ. The optimal ratio rises with the public capital depreciation rate. Moreover, x^+ approaches $\beta\theta(1-\psi)$ and $\theta(1-\psi)$ if public capital fully depreciates in one period. In principle, a sufficiently low δ_g is needed to achieve a level of x^+ below 4% for a reasonable calibration of the economy. However, this condition is not sufficient. In addition, the elasticity of intertemporal substitution must be less than unity. Indeed, using the benchmark level of δ_g, the optimal ratio is 5.62 and just below $\theta(1-\psi)$ for $\sigma = 1$ and $\sigma = 1.625$, respectively. In addition, ψ must be higher than about 0.15 for the optimal ratio to be less than 4%. For instance, if $\psi = 0$, and all other parameters are unchanged, the optimal ratio is about 5%.

If, in Table 3, we compare the first and second column of each block, we can discuss the importance of the indirect effect relative to the direct effect each parameter has on the optimal public investment ratio. In doing so, we conclude that the overall effects of δ_g and ψ on the optimal policy are mainly direct effects, similarly to the effect of public capital elasticity. On the other hand, the indirect effect predominates in the overall effect of a change in σ, especially when σ differs greatly from unity. The effect of δ is completely indirect.

In short, in our economic model, a public consumption-to-output ratio of above about 0.15, an elasticity of intertemporal substitution of less than unity, a public

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capital elasticity of below about 0.15, and a depreciation rate of capital of less than 0.125 are necessary for the optimal public investment-to-output ratio to be less than 4%. Parameter values in these ranges are commonly used in studies of economic growth in developed economies.

7. FINAL REMARKS

An unsolved puzzle in the growth literature concerned observed public investment-to-output ratios of about 3% or 4% for developed economies, which tended to fall short of theoretical model-based optimal ratios. We have reexamined the optimal choice of public investment in a more general and plausible framework than those considered by earlier papers in this literature. We have combined the following elements in a standard dynamic setting that incorporates public capital: (i) public and private capital are durable, (ii) public capital depreciates at a lower rate than private capital, (iii) the elasticity of intertemporal substitution is less than unity, and (iv) a significant proportion of output is devoted to public consumption. We have used a calibrated economy to estimate the optimal public investment-to-output ratio and have shown that features (i)–(iv) must be simultaneously assumed to produce optimal public investment-to-output ratios of less than 4%.

We have derived a general condition characterizing optimal public investment policy in this framework. The condition involves the rate of endogenous growth. Even though we cannot obtain an explicit expression for the optimal public investment ratio, careful examination of the implicit condition reveals important findings. The optimal public investment-to-output ratio is below the growth-maximizing ratio. The optimal public investment ratio is lower for low-growth economies, as well as for economies populated by consumers with a low preference for substituting consumption intertemporally. In general, given the fundamentals of the economy, a developed country with an initially high growth rate tends to stabilize its growth rate. Inflation and interest rates tend to fall and the financial sector becomes more competitive and efficient. Low interest rates and the development of flexible financial and credit markets tend to reduce the marginal rate of intertemporal substitution of consumption. Our findings suggest that this trend should be accompanied by an optimal strategy that reduces the share of output devoted to public investment. This pattern is consistent with recent trends in most developed economies.

The public capital elasticity and the discount factor in the utility function are two important determinants of the optimal policy, as earlier papers have already shown. The negative effect of the share of output devoted to public consumption is worth noting. Finally, the elasticity of intertemporal substitution and the public capital depreciation rate have positive and important effects on the optimal public investment-to-output ratio. In addition to influencing the optimal public investment ratio directly, these parameters may also affect optimal policy indirectly, through their effect on the endogenous rate of growth. This indirect channel is particularly important in the case of the elasticity of intertemporal substitution and of the private capital depreciation rate, whereas, for other parameters, the direct effect on optimal policy dominates.

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NOTES

1. Ai and Cassou (1995); Cassou and Lansing (1998).
2. Cassou and Lansing (1998); Marrero and Novales (2005, 2007).
3. Mehra and Prescott (1985); King et al. (1988).
4. See Cassou and Lansing (1999) and Marrero (2005) for a description of public investment downsizing in the OECD and the United States, respectively. See also Kamps (2005).
5. Aschauer (1989) and Munnell (1990), among others.
6. The condition we deduce is similar to that derived by Turnovsky (2004) in a continuous-time nonscale growing economy with public and private capital. Nevertheless, fiscal policy does not affect growth in Turnovsky's framework.
7. See, e.g., Barro (1990) and Glomm and Ravikumar (1994).
8. Glomm and Ravikumar (1994) consider the case in which private capital has a harmful effect on public capital under congestion (see also Barro and Sala-i-Martín, 1992). Thus, the optimal income tax rate and the optimal public investment-to-output ratio would be higher than $\beta\theta$ as a result.
9. See, e.g., Holtz-Eakin (1994), Hulten and Schwab (1991), and Tatom (1991).
10. See, e.g., Ai and Cassou (1995), Cassou and Lansing (1998, 1999), and Shioji (2001).
11. Cassou and Lansing (1998); Marrero and Novales (2005, 2007).
12. Specifying the public capital stock in per capita terms ensures that there are no scale effects associated with the number of firms (Barro and Sala-i-Martín, 1992; Cassou and Lansing, 1998).
13. For a similar technology, Aschauer (1989), Munnell (1990), and Ai and Cassou (1995) found support that $\alpha + \theta + \phi = 1$ for the post-war U.S. economy.
14. For simplicity, we assume that the number of firms is identical to the number of households. This assumption implies that all households receive equal amounts of total profits (Cassou and Lansing, 1998).
15. Many papers have assumed that public and private capital fully depreciates in one period [Barro (1990); Glomm and Ravikumar (1994); Turnovsky (2000); Marrero and Novales (2005), among many others]. We show that this assumption is in general very important for an optimal stationary public investment policy. Assuming $\delta < 1$, Cassou and Lansing (1998) use a log-linear capital accumulation rule, $k_{t+1} = A_1 k_t^{1-\delta} i_t^{\delta}$, which is less standard but allows them to obtain, together with additional restrictive assumptions on utility and production functions, a closed-form solution for the dynamics of the model. Instead, we consider a more standard linear rule for public and private capital accumulation [Ai and Cassou (1995); Glomm and Ravikumar (1999); Marrero 2005].
16. Cassou and Lansing (1998); Turnovsky (2004); Marrero and Novales (2005, 2007), among others.
17. Auerbach and Hines (1987) estimated a depreciation rate in the United States of 0.137 for equipment and 0.033 for structures. Since private capital includes a larger share of equipment than public capital, the estimated depreciation rate for private capital is expected to be larger. Ai and Cassou (1995) found support for this in the form of an estimated δ_g of just over half that of δ.
18. See also Solow (1988), Barro (1990), Lucas (1988), and Rebelo (1991).
19. With $\alpha+\theta+\phi = 1$, we are in the AK economy. If $\alpha+\theta+\phi < 1$, we would be in a neoclassical growth environment, where $\lim_{k\to\infty} \partial f(\cdot)/\partial k = 0$ and no endogenous growth is possible. Finally, if $\alpha + \theta + \phi > 1$, the growth rate of consumption would be explosive. See Jones and Manuelli (1997) for more details about this point.
20. "... whether the resulting consumption series is "trend-stationary" (i.e., constant growth rates) or "difference stationary" (i.e., the difference $c_{t+1} - c_t$ is stationary) is determined by the properties of the utility function" [Jones and Manuelli (1997, p. 83)]. For instance, under an exponential utility function, that is, $u(c) = -e^{-\lambda c}$, $\lambda > 0$, there is no guarantee that the economy could even display positive asymptotic growth. The consumption growth rate converges to zero as $c \to \infty$, and it is just the difference $c_{t+1} - c_t$ that converges to a constant value. Another function exhibiting nonconstant elasticity of substitution is $u(c_t - c^*) = \frac{(c_t - c^*)^{1-1/\sigma} - 1}{1-1/\sigma}$, for $\sigma \neq 1$, and $\ln(c_t - c^*)$, for $\sigma = 1$, where c^* would denote a subsistent level of consumption [Chatterjee (1994); Álvarez-Pelaez and Díaz (2005), among others]. If c^* is constant, it is easy to show that there exists sustained and positive

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growth if $\beta[1-\delta+(1-\tau)r_{t+1}] > 1$ for all t. However, the rate of growth is constant just in the limit, when $k, c \to \infty$. Thus, although a quasi-BGP might exist, a necessary condition for the existence of a BGP is not satisfied.

21. The author gratefully acknowledges the comments of an anonymous referee at this point.

22. All aggregate variables grow at the rate γ along the BGP. Hence, $C_t = C_0(1+\gamma)^t$ and $k_{t+1} = k_0(1+\gamma)^{t+1}$. For the CES utility function and BGP, condition (9) becomes $\frac{k_0}{c_0^{1/\sigma}}(1+\gamma)$ $\lim_{t\to\infty}(\beta(1+\gamma)^{1-1/\sigma})^t = 0$. Hence $1+\gamma < \beta^{\sigma/(1-\sigma)}$ for this condition to be satisfied.

23. Le Van et al. (2002) shows detailed and rigorous proofs of existence of optimal solutions and competitive equilibrium in a discrete-time version of the Romer (1986) model. Gourdel et al. (2004) shows existence and uniqueness in a discrete-time Lucas (1988) model. They use the idea that fixed points are equilibria. Durán and Le Van (2003) prove existence of equilibrium in a simple one-sector growth model. Glomm and Ravikumar (1994, 1999) show existence and uniqueness of a π-competitive equilibrium in a dynamic setting similar to that in this paper, with public and private capital in the production function but without spillover externalities. In this sense, our analysis is somewhat limited because we have focused on the existence of BGP equilibrium, in which we need to assume a Cobb–Douglas technology and a CES utility. The elaboration of detailed proofs for the existence and uniqueness of generic competitive equilibrium and optimal solutions in our dynamic setting is a hard task; it goes beyond the scope of this paper and is left for a future research.

24. The author gratefully acknowledges the comments of an anonymous referee at this point.

25. This Ramsey problem has the noteworthy feature that its solution could be time-inconsistent. Nevertheless, since we will restrict fiscal policy to be stationary, we can ignore the time-inconsistency problem. See chapter 12 in Ljungqvist and Sargent (2000) for more details about this point.

26. For the CES function and BGP, $u_c(c_t)/u_c(c_{t+1}) = (1+\gamma)^{1/\sigma}$.

27. Futagami et al. (1993) pointed out that the optimal stationary public investment policy might differ substantially in an economy with transition dynamics, but they did not specify the condition for this. Condition (24) contributes in this respect.

28. From (17) and (18), and using $\hat{z} = \hat{k}$, it is easy to show that γ is maximized by setting this level of x, which was also found in Marrero and Novales (2005).

29. For instance, Aschauer (1989) and Munnell (1990) estimate very high values of θ, equal to 0.39 and 0.34 respectively. Lynde and Richmond (1992) and Ai and Cassou (1995) account for nonstationarity in the data their and estimate are smaller but still significant: the former estimates $\theta = 0.2$ using time series techniques, whereas the latter estimates φ between 0.15 and 0.2, using a GMM estimation process. In a more recent paper, Shioji (2001) uses dynamic panel techniques and estimates the elasticity of output with respect to infrastructure to be somewhere around 0.1 and 0.15. On the other hand, papers by Holtz-Eakin (1994), Hulten and Schwab (1991), and Tatom (1991), among others, put that estimate very close to zero. Sturm et al. (1997) offer a selective review of these empirical studies.

30. See Cooley and Prescott (1995). When privately issued real bonds are introduced into the consumer budget constraint, the optimally condition for bonds leads to $1 + r^* = \exp(\gamma - \ln\beta)$. We calibrate β by setting $r^* = 0.069$ and $\gamma = 0029$ in this expression.

31. For example, given other parameters unchanged from the benchmark parametrization, $\theta < 0.01$ does not satisfy $\gamma > 0$, $\sigma > 1.625$ does not satisfy the transversality condition, $\delta > 0.165$ does not satisfy $\gamma > 0$, etc. Notice that if the benchmark level of δ would be initially higher that 0.098, the calibrated level of A_0 would be also higher than 0.779 (to ensure a growth rate of 2.9%) and the value of δ breaking $\gamma > 0$ would be above 0.165.

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APPENDIX

1. THE RAMSEY PROBLEM AND THE LAGRANGIAN REPRESENTATION

In the first part of the Appendix, we show that Lagrangian multipliers techniques can be used to solve our infinite-dimensional constrained optimization Ramsey problem. We show that conditions ensuring Theorem 2 in Le Van and Saglam (2004)—LVS in what follows—are satisfied. Given $k_0, g_0 > 0$, we define $\mathbf{c} = (c_0, c_1, c_2, \ldots)$, $\mathbf{k} = (k_0, k_1, k_2, \ldots)$, $\mathbf{g} = (g_0, g_1, g_2, \ldots)$, and $\boldsymbol{\tau} = (\tau_0, \tau_1, \tau_2, \ldots)$ as feasible sequences of c, k, g, and τ, respectively, in the sense that is defined in LVS. Following LVS, the Ramsey problem can be expressed as

$$\min H(X), \text{ s.t}$$
$$\Psi(X) \leq 0,$$
$$X \in l^{\infty} \text{x} l^{\infty} \text{x} l^{\infty} \text{x}[0,1]^{\infty},$$

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where $X = (\mathbf{c}, \mathbf{k}, \mathbf{g}, \boldsymbol{\tau})$, $H(X) = -\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-1/\sigma}}{1-1/\sigma}$, $H : l^\infty x l^\infty x l^\infty x[0,1]^\infty \to \mathbb{R} \cup \{+\infty\}$, and $\Psi_t(X) = [\Psi_{1t}(X), \Psi_{2t}(X), \ldots, \Psi_{8t}(X)]$, with

$$\Psi_{1t}(X) = \left(\frac{c_{t+1}}{c_t}\right)^{1/\sigma} - \beta\big[1 - \delta + A_0 \alpha k_{t+1}^{-\theta} g_{t+1}^{\theta}(1 - \tau_{t+1})\big], \tag{A.1}$$

$$\Psi_{2t}(X) = g_{t+1} - (\tau_t - \psi) A_0 k_t^{1-\theta} g_t^{\theta} - (1 - \delta_g) g_t, \tag{A.2}$$

$$\Psi_{3t}(X) = c_t + k_{t+1} - (1 - \delta) k_t - (1 - \tau_t) A_0 k_t^{1-\theta} g_t^{\theta}, \tag{A.3}$$

$$\Psi_{4t}(X) = -c_t, \tag{A.4}$$

$$\Psi_{5t}(X) = -k_t, \tag{A.5}$$

$$\Psi_{6t}(X) = -g_t, \tag{A.6}$$

$$\Psi_7(X) = -\tau_t, \tag{A.7}$$

$$\Psi_8(X) = \tau_t - 1, \tag{A.8}$$

for any time period $t = 0, 1, 2, \ldots$. We define the domains of $H(\cdot)$ and $\Psi(\cdot)$, respectively, as

$$\mathbf{C} = \mathrm{dom}(H) = \left\{X \in l_+^\infty x l^\infty x l^\infty x[0,1]^\infty \text{ such that } H(X) < +\infty\right\},$$
$$\Gamma = \mathrm{dom}(\Psi) = \left\{X \in l^\infty x l_+^\infty x l_+^\infty x[0,1]^\infty \text{ such that } \Psi_t(X) < +\infty, \forall t\right\} = \mathrm{dom}(\Psi_t)\ \forall t,$$

and

$$\mathbf{C} \cap \Gamma = l_+^\infty x l_+^\infty x l_+^\infty x[0,1]^\infty.$$

Given two alternative feasible sequences, X and $\tilde{X}$, and any time period, $T \in \mathbf{N}$, we will make use of the following function:

$$\mathbf{X}_t^T(X, \tilde{X}) = \begin{cases} X_t, & \text{if } t \le T \\ \tilde{X}_t, & \text{if } t > T. \end{cases}$$

We first check that the *Slater condition* holds. Because the Inada conditions are satisfied, $\lim_{k \to 0} F_k(t) = \lim_{k \to 0} f_k(t) = \lim_{g \to 0} F_g(t) = +\infty$. Hence, for any $k_0, g_0 > 0$, there exist some feasible $\tilde{k}$,$\tilde{g}$ such that

$$0 < \tilde{k} + \varepsilon_1 < (1 - \varepsilon_2)\, A_0 k_0^{1-\theta} g_0^{\theta} + (1 - \delta)\, k_0, \tag{A.9}$$

$$0 < \tilde{k} + \varepsilon_1 < (1 - \varepsilon_2)\, A_0 \tilde{k}^{1-\theta} \tilde{g}^{\theta} + (1 - \delta)\, \tilde{k}, \tag{A.10}$$

$$0 < \tilde{g} < (\varepsilon_2 - \psi) A_0 k_0^{1-\theta} g_0^{\theta} + (1 - \delta_g) g_0, \tag{A.11}$$

$$0 < \tilde{g} < (\varepsilon_2 - \psi) A_0 \tilde{k}^{1-\theta} \tilde{g}^{\theta} + (1 - \delta_g) \tilde{g}, \tag{A.12}$$

$$1 < \beta[1 - \delta + A_0 \alpha \tilde{k}^{-\theta} \tilde{g}^{\theta}(1 - \varepsilon_2)], \tag{A.13}$$

for some positive and small enough levels of ε_1 and ε_2. Let $\mathbf{c}^* = (\varepsilon_1, \varepsilon_1, \varepsilon_1, \ldots)$, $\mathbf{k}^* = (k_0, \tilde{k}, \tilde{k}, \ldots)$, $\mathbf{g}^* = (g_0, \tilde{g}, \tilde{g}, \ldots)$, $\boldsymbol{\tau}^* = (\varepsilon_2, \varepsilon_2, \varepsilon_2, \ldots)$ and $X^* = (\mathbf{c}^*, \mathbf{k}^*, \mathbf{g}^*, \boldsymbol{\tau}^*)$. Given (A.9)–(A.13), note that, for any time period $t = 0, 1, 2, \ldots$, $\Psi_{1t}(\mathbf{c}^*, \boldsymbol{\tau}^*) < 0$, $\Psi_{2t}(\mathbf{c}^*, \boldsymbol{\tau}^*) < 0$, $\Psi_{3t}(\mathbf{c}^*, \boldsymbol{\tau}^*) < 0$, $\Psi_{4t}(\mathbf{c}^*, \boldsymbol{\tau}^*) = -\varepsilon_1 < 0$, $\Psi_{5t}(\mathbf{c}^*, \boldsymbol{\tau}^*) = -k_0$ for $t = 0$, and $\Psi_{5t}(\mathbf{c}^*, \boldsymbol{\tau}^*) = -\tilde{k}$ for $t > 0$; $\Psi_{6t}(\mathbf{c}^*, \boldsymbol{\tau}^*) = -g_0 < 0$ for $t = 0$; and $\Psi_{6t}(\mathbf{c}^*, \boldsymbol{\tau}^*) = -\tilde{g} < 0$

for $t > 0$, $\Psi_{7t}(\mathbf{c}^*,\boldsymbol{\tau}^*) = -\varepsilon_2 < 0$, and $\Psi_{8t}(\mathbf{c}^*,\boldsymbol{\tau}^*) = \varepsilon_2 - 1 < 0$. Thus, the Slater condition is verified.

LVS point out another sufficient conditions (Assumptions 1 and 2, p. 397). We now check Assumption 1. For any $\bar{X} \in \mathbf{C}$, $\tilde{X} \in l^\infty \text{x} l^\infty \text{x} l^\infty \text{x}[0,1]^\infty$ such that for any $T \in \mathbf{N}$, $\mathbf{X}^T(\bar{X},\tilde{X}) \in \mathbf{C}$, we have

$$H[\mathbf{X}^T(\bar{X},\tilde{X})] = -\sum_{t=0}^{T} \beta^t \frac{\bar{c}_t^{1-1/\sigma}}{1-1/\sigma} - \sum_{t=T+1}^{\infty} \beta^t \frac{\tilde{c}_t^{1-1/\sigma}}{1-1/\sigma}.$$

As $\tilde{X} \in l^\infty \text{x} l^\infty \text{x} l^\infty \text{x}[0,1]^\infty$, the consumption sequence $(\tilde{c}_0, \tilde{c}_1, \tilde{c}_2, \ldots)$ must be bounded from above. Thus, if γ is the long-run growth rate of the economy, there exists some $a > 0$ such that $\tilde{c}_t \leq a(1+\gamma)^t$, for $t > T$, as $T \to +\infty$. The transversality condition implies that $(1+\gamma)^{1-1/\sigma}\beta < 1$; hence $\frac{a^{1-1/\sigma}}{1-1/\sigma}\sum_{t=T+1}^{\infty}[\beta(1+\gamma)^{1-1/\sigma}]^t \to 0$ as $T \to +\infty$ and therefore $\lim_{T\to\infty} H[\mathbf{X}^T(\bar{X},\tilde{X})] = H(\bar{X})$, and Assumption 1 in LVS is satisfied.

Assumption 2 in LVS states that if $\bar{X} \in \Gamma$, $\tilde{X} \in \Gamma$, and $\mathbf{X}^T(\bar{X},\tilde{X}) \in \Gamma$, for all $T \in \mathbf{N}$: (a) $\Psi_t[\mathbf{X}^T(\bar{X},\tilde{X})] \to \Psi_t(\bar{X})$ as $T \to +\infty$; (b) there exists a constant M such that for all $T \in \mathbf{N}$, $\|\Psi_t[\mathbf{X}^T(\bar{X},\tilde{X})]\| \leq M$; (c) for all $T \in \mathbf{N}$, $\lim_{t\to\infty}|\Psi_t(\mathbf{X}^T(\bar{X},\tilde{X})) - \Psi_t(\tilde{X})| = 0$. Following the same steps as in LVS, conditions (a)–(c) are clearly satisfied.

Finally, it is obvious that, if $\mathbf{X}^+$ is a solution to the Ramsey problem, $\mathbf{X}^T(X^+, X^*)$ belongs to $l_+^\infty \text{x} l_+^\infty \text{x} l_+^\infty \text{x}[0,1]^\infty$, for all $T \in \mathbf{N}$.

2. OPTIMAL CONDITIONS ALONG THE BGP

In part 1 of the Appendix, we show that if $X^+ = (\mathbf{c}^+, \mathbf{k}^+, \mathbf{g}^+, \boldsymbol{\tau}^+)$ is a solution to the Ramsey problem, there exists a Lagrangian representation associated with it. For any period t, Lagrange multipliers associated with restrictions (A.1)–(A.8) are denoted by λ_{1t}, λ_{2t}, λ_{3t}, λ_{4t}, λ_{5t}, λ_{6t}, λ_{7t}, λ_{8t}, respectively. The Inada conditions imply that the solution to the Lagrangian would be interior; hence $\lambda_{4t} = \lambda_{5t} = \lambda_{6t} = \lambda_{7t} = \lambda_{8t} = 0$ for all t. Moreover, restrictions hold with equality because utility and production functions are strictly monotone. To simplify notation, we define the following ratios: $\tilde{\lambda}_{3t} = \lambda_{3t}/c_t^{-1/\sigma}$, $\tilde{\lambda}_{2t} = \lambda_{2t}/c_t^{-1/\sigma}$, $\tilde{\lambda}_{1t} = \lambda_{1t}/k_{t+1}$, and $\tilde{u}_c(t) = c_t^{-1/\sigma}/c_{t+1}^{-1/\sigma}$. We also use $F(t)$ and $f(t)$ to denote $F(k_t, l_t, g_t)$ and $f(k_t, l_t z_t, g_t)$, respectively, and $f_k(t)$, $F_k(t)$, $f_{kk}(t)$, $F_{kk}(t)$ to denote first and second time derivatives of $f(\cdot)$ and $F(\cdot)$, respectively, with respect to k and so on of the indicated object, evaluated at a particular allocation.

The Ramsey problem in its Lagrangian form is[32]

$$\max_{\{c_t,k_{t+1},g_{t+1},\tau\}_0^\infty} \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{l} \frac{c_t^{1-1/\sigma}}{1-1/\sigma} - \lambda_{1t}\left[c_t^{-1/\sigma} - \beta c_{t+1}^{-1/\sigma}(1-\delta+f_k(t+1)(1-\tau_{t+1}))\right] \\ \qquad - \lambda_{2t}[g_{t+1} - (\tau_t - \psi)F(t) - (1-\delta_g)g_t] \\ \qquad - \lambda_{3t}[c_t + k_{t+1} - (1-\delta)k_t - (1-\tau_t)F(t)] \end{array} \right\}. \quad \textbf{(A.14)}$$

Optimal conditions are[33]

$$\tau_t : -\alpha\tilde{\lambda}_{1t-1} + \tilde{\lambda}_{2t} - \tilde{\lambda}_{3t} = 0; \quad \textbf{(A.15)}$$

$$c_t : 1 + \frac{1}{\sigma}\frac{\lambda_{1t}}{c_t} - \tilde{\lambda}_{3t} - \frac{1}{\sigma}\frac{\lambda_{1t-1}}{c_t}[1-\delta+f_k(t)(1-\tau_t)] = 0; \quad \textbf{(A.16)}$$

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$$k_{t+1} : \beta\tilde{\lambda}_{1t}(1-\tau_{t+1})k_{t+1}\frac{\partial f_k(t+1)}{\partial k}\tilde{u}_c^{-1}(t) - \tilde{\lambda}_{3t} + \tag{A.17}$$
$$: + \beta\tilde{\lambda}_{2t+1}(\tau_{t+1}-\psi)F_k(t+1)\tilde{u}_c^{-1}(t) +$$
$$: + \beta\tilde{u}_c^{-1}(t)\tilde{\lambda}_{3t+1}[1-\delta+(1-\tau_{t+1})\tilde{u}_c^{-1}(t)F_k(t+1)] = 0;$$

$$g_{t+1} : \beta\tilde{\lambda}_{1t}\tilde{u}_c^{-1}(t)(1-\tau_{t+1})k_{t+1}\frac{\partial f_k(t+1)}{\partial g} - \tilde{\lambda}_{2t} + \tag{A.18}$$
$$: + \beta\tilde{u}_c^{-1}(t)\tilde{\lambda}_{2t+1}[(\tau_{t+1}-\psi)F_g(t+1)+1-\delta_g] +$$
$$: + \beta\tilde{u}_c^{-1}(t)\tilde{\lambda}_{3t}(1-\tau_{t+1})F_g(t+1) = 0;$$

$$\lambda_{1t} : c_t^{-1/\sigma} - \beta c_{t+1}^{-1/\sigma}(1-\delta+f_k(t+1)(1-\tau_{t+1})) = 0; \tag{A.19}$$

$$\lambda_{2t} : g_{t+1} - (\tau_t-\psi)F(t) - (1-\delta_g)g_t = 0; \tag{A.20}$$

$$\lambda_{3t} : c_t + k_{t+1} - (1-\delta)k_t - (1-\tau_t)F(t) = 0. \tag{A.21}$$

On the BGP, y, c, k, and g grow at the constant rate γ, while $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{\lambda}_3$, and $\tilde{u}_c$ must be constant. We omit subindex t along the BGP. Linear homogeneity of $f(\cdot)$ implies that $(\partial f_k(\cdot)/\partial g)k = \alpha F_g$ and $(\partial f_k(\cdot)/\partial k)k = \alpha F_k - f_k$. On the BGP, conditions (A.5)–(A.25) reduce to

$$\tau : \tilde{\lambda}_2 - \tilde{\lambda}_3 = \alpha\tilde{\lambda}_1; \tag{A.22}$$

$$c : 1 + \frac{1}{\sigma}\tilde{\lambda}_1(1+\gamma)k/c - \tilde{\lambda}_3 - \frac{1}{\sigma}\tilde{\lambda}_1 k/c\,[1-\delta+f_k\,(1-\tau)] = 0; \tag{A.23}$$

$$k : \beta\tilde{\lambda}_1\tilde{u}_c^{-1}(1-\tau)\,(\alpha F_k - f_k) - \tilde{\lambda}_3 + \beta\tilde{\lambda}_2\,(\tau-\psi)\,\tilde{u}_c^{-1}F_k + \tag{A.24}$$
$$: +\beta\tilde{u}_c^{-1}\tilde{\lambda}_3\left[(1-\delta)+(1-\tau)\tilde{u}_c^{-1}F_k\right] = 0;$$

$$g : \beta\tilde{\lambda}_1\tilde{u}_c^{-1}\,(1-\tau)\,\alpha F_g - \tilde{\lambda}_2 + \beta\tilde{u}_c^{-1}\tilde{\lambda}_2\left[(\tau-\psi)\,F_g + 1 - \delta_g\right] + \tag{A.25}$$
$$: +\beta\tilde{u}_c^{-1}\tilde{\lambda}_3(1-\tau)F_g = 0;$$

$$\lambda_1 : 1 - \beta\tilde{u}_c^{-1}\,[1-\delta+f_k\,(1-\tau)] = 0; \tag{A.26}$$

$$\lambda_2 : \gamma + \delta_g - (\tau-\psi)\,F_g/\theta = 0; \tag{A.27}$$

$$\lambda_3 : \frac{c}{k} + \gamma + \delta - (1-\tau)F_k/(1-\theta) = 0. \tag{A.28}$$

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