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Sets of Finite Perimeter and Functions of Bounded Variation

This topic is well documented in many books, see [203, Section 4.5] and [15, 189, 227, 297, 299, 397, 448]. The structure theory of sets of finite perimeter is due to De Giorgi in the 1950s based on the earlier work of Caccioppoli, see, in particular, [156]. English translations of many of De Giorgi's papers can be found in [157]. Maggi [299] follows rather closely De Giorgi's original ideas.

12.1 Sets of Finite Perimeter

What is the perimeter of an arbitrary Lebesgue measurable set E in $\mathbb{R}^n$? Even for open sets the right notion clearly is not the $\mathcal{H}^{n-1}$ measure of the topological boundary ∂E. For example, if E is a countable union of balls $B_i \subset B(0,1)$ with $\sum_i \mathcal{H}^{n-1}(\partial B_i) < \infty$, the topological boundary could be almost anything, in particular, $B(0,1)$ if the centres are dense, but a more reasonable notion of perimeter would seem to be $\sum_i \mathcal{H}^{n-1}(\partial B_i)$. The Gauss–Green theorem gives a hint about how to define a good general notion of perimeter. If E has a smooth boundary, then for any compactly supported C^1 vectorfield ϕ, $\phi \in C_c^1(\mathbb{R}^n)$,

$$\int_E \operatorname{div} \phi = \int_{\partial E} \phi \cdot n_E \, d\mathcal{H}^{n-1}, \tag{12.1}$$

where n_E is the outer unit normal of E. Among ϕ with $|\phi| \leq 1$, the right-hand side is maximized when $\phi = n_E$ on ∂E and yields $\mathcal{H}^{n-1}(\partial E)$. The left-hand side is defined for all measurable sets E, so we can define

Definition 12.1 The *perimeter* of a Lebesgue measurable set $E \subset \mathbb{R}^n$ is

$$P(E) = \sup\left\{\int_E \operatorname{div} \phi \colon \phi \in C_c^1(\mathbb{R}^n), |\phi| \leq 1\right\}.$$

If $P(E) < \infty$, we say that E is a set of finite perimeter.

That E is a set of finite perimeter means that the characteristic function χ_E is of bounded variation:

Definition 12.2 A Lebesgue integrable function u on $\mathbb{R}^n$ is of *bounded variation*, $u \in BV(\mathbb{R}^n)$, if

$$\sup\left\{\int u \operatorname{div}\phi \colon \phi \in C_c^1(\mathbb{R}^n), |\phi| \le 1\right\} < \infty.$$

Then $u \in BV(\mathbb{R}^n)$ if and only if its distributional partial derivatives are finite Radon measures. That is, Du is a vector-valued Radon measure. We shall discuss them a bit more in the next section.

It is easy to see that the perimeter is lower semicontinuous: if $\chi_{E_i} \to \chi_E$ in $L^1(\mathbb{R}^n)$, then $P(E) \le \liminf_{i\to\infty} P(E_i)$. From this we see that if, as at the beginning of this chapter, E is a countable union of balls $B_i \subset B(0,1)$ with $\sum_i \mathcal{H}^{n-1}(\partial B_i) < \infty$, then E has finite perimeter. Moreover, so does $B(0,1) \setminus E$.

By abstract arguments based on the Riesz representation theorem, we have

Theorem 12.3 *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter. There are $\mu_E \in \mathcal{M}(\mathbb{R}^n)$ and a Borel function $\nu_E \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $|\nu_E(x)| = 1$ for μ_E almost all $x \in \mathbb{R}^n$ and*

$$\int_E \operatorname{div}\phi = \int \phi \cdot \nu_E \, d\mu_E \text{ for } \phi \in C_c^1(\mathbb{R}^n). \tag{12.2}$$

Recalling (12.1), it is natural to call μ_E the generalized perimeter measure of E and ν_E its generalized outer normal.

A fairly easy but important result is the compactness theorem:

Theorem 12.4 *If $E_j \subset B(0,1)$, $j =, 1, 2, \dots$ are Lebesgue measurable with $\sup_j P(E_j) < \infty$, then there is a subsequence (E_{j_i}) and a set E with finite perimeter such that $\chi_{E_{j_i}} \to \chi_E$ in $L^1(\mathbb{R}^n)$ and $\mu_{E_{j_i}} \to \mu_E$ weakly.*

Definition 12.5 The *reduced boundary* $\partial^* E$ of E is the set of points $x \in \operatorname{spt}\mu_E$ such that $|\nu_E(x)| = 1$ and

$$\lim_{r\to 0} \frac{1}{\mu_E(B(x,r))} \int_{B(x,r)} \nu_E \, d\mu_E \tag{12.3}$$

exists and has norm 1.

It follows from the general theory of differentiation of measures that

$$\mu_E(\mathbb{R}^n \setminus \partial^* E) = 0. \tag{12.4}$$

Formula (12.2) is a very general, but abstract, form of the Gauss–Green theorem. In order to make it more concrete, we should understand better what μ_E, ν_E and $\partial^* E$ really are. This is included in De Giorgi's structure theorem:

Theorem 12.6 *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter. Then $\partial^* E$ is $(n-1)$-rectifiable, $\mu_E = \mathcal{H}^{n-1} \llcorner \partial^* E$, for $\mathcal{H}^{n-1}$ almost all $x \in \partial^* E$ the approximate tangent $(n-1)$-plane of $\partial^* E$ is $\{y\colon (y-x)\cdot \nu_E(x) = 0\}$, and*

$$\int_E \operatorname{div} \phi = \int_{\partial^* E} \phi \cdot \nu_E \, d\mathcal{H}^{n-1} \text{ for } \phi \in C_c^1(\mathbb{R}^n).$$

I say a few words about the main steps of the proof, which themselves are of independent interest. First, there are the isoperimetric inequality

$$\mathcal{L}^n(E)^{(n-1)/n} \lesssim P(E) = \mu_E(\mathbb{R}^n) \tag{12.5}$$

and the local isoperimetric inequality for every ball $B \subset \mathbb{R}^n$,

$$\min\left\{\mathcal{L}^n(B\cap E)^{(n-1)/n}, \mathcal{L}^n(B \setminus E))^{(n-1)/n}\right\} \lesssim \mu_E(B). \tag{12.6}$$

These follow from, and are in fact equivalent to, Sobolev and Poincaré inequalities for BV functions, which in turn follow from the classical inequalities and the fact that the smooth functions are dense in $BV(\mathbb{R}^n)$.

The isoperimetric inequalities lead to density estimates for the Lebesgue measure and μ_E. The key for deriving these is the identity

$$\int_{E\cap B(x,r)} \operatorname{div} \phi = \int_{B(x,r)} \phi \cdot \nu_E \, d\mu_E + \int_{E\cap \partial B(x,r)} \phi \cdot \nu \, d\mathcal{H}^{n-1}$$

for $\phi \in C_c^1(\mathbb{R}^n)$, where ν is the outward unit normal of $B(x,r)$. This follows applying (12.2) to a C^1 approximation of the characteristic function of $B(x,r)$. Then we have

Lemma 12.7 *There are positive constants c and C depending only on n such that if $x \in \partial^* E$, then for all sufficiently small $r > 0$,*

$$\mathcal{L}^n(E \cap B(x,r)) \ge cr^n,\ \mathcal{L}^n(B(x,r)\setminus E) \ge cr^n,$$

$$cr^{n-1} \le \mu_E(B(x,r)) \le Cr^{n-1}.$$

These ingredients can be used to prove the blow-up theorem:

Theorem 12.8 *Let $x \in \partial^* E$ and $H(x) = \{y\colon y \cdot \nu_E(x) \le 0\}$. Then $\chi_{r^{-1}(E-x)} \to \chi_{H(x)}$ as $r \to 0$ locally in $L^1(\mathbb{R}^n)$.*

To get this, one first uses compactness to show that some subsequence converges to the characteristic function of a set F with locally finite perimeter for which ν_F is constant μ_F almost everywhere. Moreover, the distributional derivatives of χ_F vanish in directions orthogonal to ν_F, whereas the derivative in the direction of ν_F is non-zero. From this it is not trivial but not very difficult either to show that F is a half-space. Moreover, it follows that $\nu_E(x)$ is the approximate normal of E at $x \in \partial^* E$ in the following sense:

$$\lim_{r\to 0} r^{-n} \mathcal{L}^n (\{y \in E \cap B(x,r) \colon (y-x)\cdot \nu_E(x) > 0\}) = 0, \tag{12.7}$$

$$\lim_{r\to 0} r^{-n} \mathcal{L}^n (\{y \in B(x,r) \setminus E \colon (y-x)\cdot \nu_E(x) < 0\}) = 0. \tag{12.8}$$

From the blow-up theorem, one can proceed to show that $\{y \colon y \cdot \nu_E(x) = 0\}$ is the approximate tangent plane of $\partial^* E$ at x, from which the rectifiability follows.

The *essential boundary* of E,

$$\partial_e E = \{x \colon \Theta^{*n}(E,x) > 0 \text{ and } \Theta^{*n}(\mathbb{R}^n \setminus E, x) > 0\},$$

gives a different view of the finite perimeter sets and the reduced boundary. We have, see Theorems 4.5.6 and 4.5.11 in [203] and also [299, Theorem 16.2],

Theorem 12.9 *If $E \subset \mathbb{R}^n$ is a set of finite perimeter, then $\partial^* E \subset \partial_e E$ and $\mathcal{H}^{n-1}(\partial_e E \setminus \partial^* E) = 0$.*

Theorem 12.10 *A measurable set $E \subset \mathbb{R}^n$ has finite perimeter if and only if $\mathcal{H}^{n-1}(\partial_e E) < \infty$.*

Lahti [284] and Eriksson-Bique [188] gave different proofs and metric space versions for the last theorem.

12.2 Plateau-Type Problems

Sets of finite perimeter give a convenient setting to define and study codimension one generalized minimal surfaces. Other settings will be discussed later. The classical Plateau problem asks us to find and describe the surface with minimal area among surfaces with a given boundary. Many variants of this, often of very general type, have been studied, and some of them will be discussed later. Usually there are several non-trivial subproblems: what is surface, what is area, what is boundary?

In the case of finite perimeter sets, boundary is not defined; instead, one considers sets that agree with a given set outside a fixed set:

Definition 12.11 Let $A \subset \mathbb{R}^n$ and let $E_0, E \subset \mathbb{R}^n$ be sets of finite perimeter. We say that E is *perimeter minimizing* in A with boundary data E_0 if $E \setminus A = E_0 \setminus A$ and $P(E) \leq P(F)$ for all sets F of finite perimeter such that $F \setminus A = E_0 \setminus A$.

The existence of perimeter-minimizing sets follows using the usual direct method of calculus of variations: choose a minimizing sequence; $P(E_i) \to \inf\{P(F)\colon F \setminus A = E_0 \setminus A\}$ and use compactness to select a converging subsequence, then the limit is a minimizer by lower semicontinuity:

Theorem 12.12 *Let $A \subset \mathbb{R}^n$ be bounded and let $E_0 \subset \mathbb{R}^n$ be a set of finite perimeter. Then there exists a perimeter-minimizing set in A with boundary data E_0.*

The real problem then is the regularity of the minimizers. We shall say something about this in Chapter 15.

Sets of finite perimeter can be used to model many other geometric variational problems too, see [299].

12.3 Functions of Bounded Variation

Above we already defined these. The book [15] contains a lot of detailed information about them. Here I only discuss some properties related to rectifiability.

A function $u \in L^1(\mathbb{R}^n)$ has an *approximate limit* a at x if

$$\lim_{r\to 0} r^{-n} \int_{B(x,r)} |u(y) - a|\, dy.$$

The set S_u where u does not have any approximate limit is called the approximate discontinuity set of u. At jump points, the nature of the discontinuity is more specific: x is called an *approximate jump point* of u if there are $a, b \in \mathbb{R}$, $a \neq b$, and $\nu \in S^{n-1}$ such that

$$\lim_{r\to 0} r^{-n}\int_{\{y\in B(x,r)\colon (y-x)\cdot\nu>0\}} |u(y)-a|\, dy = \lim_{r\to 0} r^{-n}\int_{\{y\in B(x,r)\colon (y-x)\cdot\nu<0\}} |u(y)-b|\, dy = 0.$$

The set of approximate jump points is denoted by J_u. By [203] and [434], we have, see [15, Theorem 3.78],

Theorem 12.13 *If $u \in BV(\mathbb{R}^n)$, then S_u is $(n-1)$-rectifiable and $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$.*

For the characteristic functions of sets of finite perimeter this follows from Theorems 12.6 and 12.9 and from (12.7), (12.8). There is a coarea formula for BV-functions which implies that for almost all $t \in \mathbb{R}$ the sets $\{x\colon u(x) > t\}$ have finite perimeter. Letting D be a countable dense set of such t, one can show that up to $\mathcal{H}^{n-1}$ measure zero S_u is contained in $\cup_{t\in D}\partial^*\{x\colon u(x) > t\}$, from which the rectifiability of S_u follows.

The book [15] contains much more about the structure of the derivative measure Du. First $|Du| \ll \mathcal{H}^{n-1}$, where $|Du|$ is the total variation measure of Du. For $u = \chi_E$ this follows from Theorem 12.6 and after that for general u from the coarea formula. Then, see the proof of [15, Proposition 3.92],

Theorem 12.14 *If $u \in BV(\mathbb{R}^n)$, then $Du \llcorner \{x\colon \Theta^{*n-1}(|Du|, x) > 0\}$ is $(n-1)$-rectifiable.*

A vector-valued function $u\colon \mathbb{R}^n \to \mathbb{R}^k$ is in $BV(\mathbb{R}^n, \mathbb{R}^k)$ if its coordinate functions are of bounded variation. Then the derivative Du is a $k \times n$ matrix valued measure. Let $D_s u$ denote its singular part in the Lebesgue decomposition. The following Alberti rank one theorem [2] has applications in many areas:

Theorem 12.15 *If $u \in BV(\mathbb{R}^n, \mathbb{R}^k)$, then the Radon–Nikodym derivative $D(D_s u, |D_s u|)(x)$ has rank 1 for $|D_s u|$ almost all $x \in \mathbb{R}^n$.*

Massaccesi and Vittone [315] have given a very simple and elegant proof using sets of finite perimeter, and De Philippis and Rindler derived it as a special case of their more general result in [175], see Section 15.5. But since Alberti's original proof gives a lot of information about the structure, also related to rectifiability, of singular measures on $\mathbb{R}^n$, we shall briefly discuss it. As already indicated in Section 4.8, this is closely connected to the work of Alberti, Csörnyei and Preiss, [3,4]. They describe in [3] how to prove Theorem 12.15 by their tangent field results.

Partially relying on the ideas of [315], Don, Massacessi and Vittone [180] proved the rank one theorem in the Heisenberg group, and a larger class of Carnot groups, and Antonelli, Brena and Pasqualetto [21] proved it in RCD spaces.

To prove Theorem 12.15, Alberti studied tangential properties of general measures. For $\mu \in \mathcal{M}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let $E(\mu, x)$ be the set of vectors $v \in \mathbb{R}^n$ such that for some real valued $u \in BV(\mathbb{R}^n)$,

$$\lim_{r\to 0} \frac{|Du - v\mu|(B(x,r))}{\mu(B(x,r))} = 0.$$

Then $E(\mu, x)$ is a linear subspace of $\mathbb{R}^n$. One can show that if $\mu = \mathcal{H}^{n-1} \llcorner E$, where E is $\mathcal{H}^{n-1}$ measurable with $\mathcal{H}^{n-1}(E) < \infty$, then for μ almost all $x \in \mathbb{R}^n$, $E(\mu, x) = \operatorname{apTan}(E, x)^{\perp}$ if E is $(n-1)$-rectifiable, and $E(\mu, x) = \{0\}$ if E is purely $(n-1)$-unrectifiable. For $n = 2$, $E(\mu, x)$ is closely related to the decomposition bundle $V(\mu, x)$ of Alberti and Marchese [5], which we discussed in Section 4.8.

For $v \in E(\mu, x), v \neq 0$, $v\mu$ is kind of tangential to a derivative at x of a BV-function. The key to the proof of Theorem 12.15 is that for any singular measure there is at most one direction for which this can happen:

Theorem 12.16 *If $\mu \in \mathcal{M}(\mathbb{R}^n)$ is singular, then the dimension of $E(\mu, x)$ is either 0 or 1 for μ almost all $x \in \mathbb{R}^n$.*

The proof of this theorem requires most of the effort. Alberti proved it first for $n = 2$ with a calculus-type argument. Then he used disintegration of μ with two-dimensional slices.

If $\mu \in \mathcal{M}(\mathbb{R}^n)$ is any singular Borel measure, we can decompose by Theorem 12.16,

$$\mu = \mu_1 + \mu_0 = \mu \llcorner B_1 + \mu \llcorner B_0, \text{ where } B_i = \{x\colon \dim E(\mu, x) = i\}, i = 0, 1. \tag{12.9}$$

Alberti showed that $\dim E(\mu, x) > 0$ for μ almost all $x \in \mathbb{R}^n$ if and only if $\mu = |Du| \llcorner B$ for some $u \in BV(\mathbb{R}^n)$ and some Borel set $B \subset \mathbb{R}^n$. So $\mu_1 = |Du| \llcorner B_1$ for some $u \in BV(\mathbb{R}^n)$ and μ_0 is orthogonal Du for every $u \in BV(\mathbb{R}^n)$. From this, it follows that for any singular measure μ and $u \in BV(\mathbb{R}^n, \mathbb{R}^k)$, the rank of the Radon–Nikodym derivative $D(Du, \mu)(x)$ is 0 or 1 for μ almost all $x \in \mathbb{R}^n$. Theorem 12.15 follows, applying this to $\mu = |D_s u|$.

In the decomposition (12.9) μ_1 has an Alberti representation (recall Definition 4.25 and (7.6)) $\mu_1 = \int \mathcal{H}^{n-1} \llcorner E_t \, dt$ where each E_t is $(n-1)$-rectifiable.

A subclass SBV of BV, special functions of bounded variation, consists of functions $u \in BV(\mathbb{R}^n)$ for which Du is a sum of an absolutely continuous and a rectifiable measure, the latter being concentrated on J_u. That is, $u \in SBV$ if the so-called Cantor part of Du vanishes. SBV and its many applications are extensively discussed in [15].

In [13], Ambrosio developed a theory of metric space-valued functions of bounded variation. Let X be a locally compact metric space. A Borel function $u\colon \mathbb{R}^n \to X$ belongs to $BV(\mathbb{R}^n, X)$ if there is $\mu \in \mathcal{M}(\mathbb{R}^n)$ such that for every 1-Lipschitz function $\phi\colon X \to \mathbb{R}$, $\phi \circ u \in BV(\mathbb{R}^n)$ with $|D(\phi \circ u)|(A) \le \mu(A)$ for $A \subset \mathbb{R}^n$. In particular, he proved an analogue of Theorem 12.13 in this setting.

Ambrosio, Coscia and Dal Maso investigated in [14] mappings $u\colon \mathbb{R}^n \to \mathbb{R}^n$ of bounded deformation, $BD(\mathbb{R}^n)$, that is, $Du + (Du)^T$ is a matrix-valued Radon measure. Clearly, BV implies BD. Among other things, they proved the generalization of Theorem 12.14; the proof uses the Besicovitch–Federer projection Theorem 4.17:

Theorem 12.17 *If $u \in BD(\mathbb{R}^n)$, then $Du \llcorner \{x\colon \Theta^{*n-1}(|Du|, x) > 0\}$ is $(n-1)$-rectifiable.*

Various physically motivated PDE problems lead to functions of BV type for which analogous rectifiability results have been proven, see [18, 165–167, 306, 307].

12.4 Perimeter in Heisenberg and Carnot Groups

The initial motivation of Franchi, Serapioni and Serra Cassano in [212] to study rectifiability in Heisenberg groups was to develop De Giorgi's theory of sets of finite perimeter there. Recall the structure of the Heisenberg group $\mathbb{H}^n$ and the notion of $(m, \mathbb{H})$-rectifiable sets from Chapter 8, with $m = 2n + 1$ in the codimension one case.

Denote by $C_c^1(\mathbb{H}^n, H\mathbb{H}^n)$ the space of compactly supported continuous continuously Pansu differentiable functions with values in the horizontal sections of $\mathbb{H}^n$; $\phi(p) \in \tau_p(\mathbb{H})$, where $\mathbb{H} = \{(z,t) \colon t = 0\}$ is the horizontal plane. Now the perimeter is defined in terms of the Heisenberg divergence: $\operatorname{div}_H \phi = \sum_{j=1}^n (X_j\phi_j + Y_j\phi_{n+j})$.

Definition 12.18 The *Heisenberg perimeter* of a Lebesgue measurable set $E \subset \mathbb{H}^n$ is

$$P_H(E) = \sup\left\{\int_E \operatorname{div}_H \phi \colon \phi \in C_c^1(\mathbb{H}^n, H\mathbb{H}^n), |\phi| \leq 1\right\}.$$

If $P_H(E) < \infty$, we say that E is a set of finite Heisenberg perimeter.

If $P_H(E) < \infty$, we again have by the Riesz representation theorem that there are $\mu_E \in \mathcal{M}(\mathbb{H}^n)$ and a Borel function $\nu_E \colon \mathbb{H}^n \to H\mathbb{H}^n$ such that $|\nu_E(p)| = 1$ for μ_E almost all $p \in \mathbb{H}^n$ and

$$\int_E \operatorname{div}_H \phi = \int \phi \cdot \nu_E \, d\mu_E \text{ for } \phi \in C_c^1(\mathbb{H}^n, H\mathbb{H}^n).$$

The reduced boundary $\partial^* E$ with $\mu_E(\mathbb{H}^n \setminus \partial^* E) = 0$ can then be defined as in the Euclidean case.

This much is true in general Carnot groups. Franchi, Serapioni and Serra Cassano proved De Giorgi's structure theorem first in $\mathbb{H}^n$ in [212] and then in all step 2 Carnot groups in [213]. In particular, we have

Theorem 12.19 *Let $E \subset \mathbb{H}^n$ be a set of finite Heisenberg perimeter. Then $\partial^* E$ is $(2n+1, \mathbb{H})$-rectifiable.*

The proof follows the same main lines as in the Euclidean case, but it is technically much harder. Again, the blow-ups at the points of the reduced boundary converge to vertical subgroups. This statement is false in higher-order groups, at least in the Engel group, which is of step 3, and the analogue of De Giorgi's theorem is not known. Ambrosio, Kleiner and Le Donne [19] proved a partial result in general Carnot groups: some sequences of blow-ups converge to vertical subgroups.

See the survey [396] of Serra Cassano for further comments and references.