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# RELIABILITY STUDIES OF BIVARIATE BIRNBAUM–SAUNDERS DISTRIBUTION

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In this paper, we study the bivariate Birnbaum–Saunders (BVBS) distribution from a reliability point of view. The monotonicity of the hazard rates of the univariate as well as the conditional distributions is discussed. Clayton's association measure is obtained in terms of the hazard gradient and its value in the case of the BVBS distribution is derived. The probability distributions, in the case of series and parallel systems, are derived and the monotonicity of the failure rate, in the case of series system, is discussed.

## 1. INTRODUCTION

It is well known that Birnbaum–Saunders (BS) distribution, here after called as BS distribution, is a versatile model for analyzing lifetime data. This two-parameter BS distribution was originally proposed by Birbaum and Saunders [4] as a fatigue failure model. Later on Desmond [6] strengthened the physical justification of the use of this model by relaxing some of the assumptions made by Birnbaum and Saunders [4] and later established the relationship between the BS distribution and the inverse Gaussian distribution. In fact, it has been shown that the BS distribution is a special case of the mixture of inverse Gaussian and its length-biased version when the mixing proportion is equal to 0.5; see Gupta and Akman [14,15] in this connection.

In the last 25 years, different aspects of the BS distribution have been studied by various researchers. A comprehensive treatment of the BS distribution can be found in Johnson, Kotz, and Balakrishnan [21]. More recently, Balakrishnan et al. [1] studied some inference problems pertaining to mixture models based on the BS distribution. Some other recent references include Ng, Kundu, and Balakrishnan [27,28] and Kundu, Kannan, and Balakrishnan [22]. The cumulative distribution function (CDF) of the BS distribution is defined through the CDF of a standard normal variable, and it can be obtained as a monotone transformation from a standard normal variable. This helps us to investigate the monotonicity of the density function and the hazard function. Independent of this fact, Gupta and Akman [14] and Kundu et al. [22] showed that the hazard function of the BS distribution is not monotone and is unimodal.

Recently Kundu, Balakrishnan, and Jamalizadeh [23] derived a bivariate BS (BVBS) distribution which is an absolutely continuous distribution whose marginals and conditionals

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have BS distribution. This new family of distributions has five unknown parameters whose inference problems have been studied by these authors.

In this paper, we are interested in studying the class of BVBS distribution from a reliability point of view. More specifically, we study the association between the variables and obtain conditions for which this class of distributions is $TP_2$ (totally positive of order 2) or $RR_2$ (reverse rule of order 2). This enables us to study the dependence properties of the model. We study the hazard components of the hazard gradient in the sense of Johnson and Kotz [20] and their monotonic structure. An association measure $\theta(x, y)$ defined by Oakes [29], is investigated for this class of bivariate distributions. Some of the results presented here are general and would be useful in studying the association in other classes of bivariate distributions.

The organization of this paper is as follows: In Section 1, we present some general results for bivariate distributions. Also we give some definitions and background of reliability functions. The monotonicty of the failure rates of the conditional distributions of the BVBS distribution is discussed in Section 2. In Section 3, we investigate the association measure, presented by Clayton [5], and its relationship with some dependence notions in reliability. The value of the association measure is derived for our model. Section 4 contains the distributions of the series and parallel systems and the investigation of the monotonicity of the failure rate of the series system. Finally, in Section 5, we present some comments and conclusions.

### 1.1. Some Definitions and Background of Reliability Functions

Let $T$ be a non-negative random variable denoting the life length of a component having distribution function $F(t)$ with $F(0) = 0$ and the probability density function (pdf) $f(t)$. Then the failure rate of $T$ is given by $r(t) = f(t)/R(t)$, where $R(t) = 1 - F(t)$ is the survival (reliability) function of $T$. We also assume that $f(t)$ is strictly positive, continuous and twice differentiable on $(0, \infty)$.

Let $h : \mathbb{R} \to \mathbb{R}$ be a real-valued differentiable function. Then $h(t)$ is said to be

(1) increasing if $h'(t) > 0$ for all $t$ and is denoted by I;
(2) decreasing if $h'(t) < 0$ for all $t$ and is denoted by D;
(3) bathtub shaped if $h'(t) < 0$ for $t \in (0, t_0)$, $h'(t_0) = 0$, $h'(t) > 0$ for $t > t_0$ and is denoted by B;
(4) upside down bathtub shaped if $h'(t) > 0$ for $t \in (0, t_0)$, $h'(t_0) = 0$, $h'(t) < 0$ for $t > t_0$ and is denoted by U.

For some definitions given above, see Gupta and Warren [17] and Barlow and Proschan [2]. Also see Barlow, Marshall, and Proschan [3] for some properties of probability distributions with the monotone hazard rate.

In order to determine the monotonicity of the failure rates, we proceed as follows: Define

$$\eta(t) = -f'(t)/f(t). \tag{1.1}$$

This function contains useful information about $r(t)$ and is simpler because it does not involve $R(t)$. In particular, the shape of $\eta(t)$ (I, D, B, etc.) often determines the shape of the failure rate. The relation between $r(t)$ and $\eta(t)$ is given by

$$\frac{d}{dt}\ln r(t) = r(t) - \eta(t). \tag{1.2}$$

The above equation also suggests that the turning point of $r(t)$ is a solution of the equation $\eta(t) = r(t)$. Also it can be verified that $\lim_{t\to\infty} r(t) = \lim_{t\to\infty} \eta(t)$. For more relations between $\eta(t)$ and $r(t)$, see Marshall and Olkin [25].

It can be easily seen that $\eta'(t) > 0$ if and only if $f(t)$ is logconcave, and thus, the distribution is increasing failure rate (IFR). In fact $r'(t) > 0$ is equivalent to the logconcavity of $R(t)$, where $R(t)$ is the reliability function. So logconcavity of $f(t)$ is a stronger condition than the logconcavity of $R(t)$.

In order to determine the monotonicity of $r(t)$, we present a modification, due to Marshall and Olkin [25], of Glaser [9] which helps us to determine the shape of the failure rates of the four types described above.

Theorem 1.1: *Let $f$ be a density strictly positive and differentiable on $(0,\infty)$ such that $\lim_{x\to\infty} f(x) = 0$. Then*

(a) *If $\eta(t) \in I$, then $r(t) \in I$ (IFR).*
(b) *If $\eta(t) \in D$, then $r(t) \in D$ (DFR).*
(c) *If $\eta(t) \in B$, then $r(t) \in B$ (bathtub-shaped failure rate).*
(d) *If $\eta(t) \in U$, then $r(t) \in U$ (upside bathtub-shaped failure rate).*

Proof: See Marshall and Olkin ([25], page 134).

## 2. BVBS DISTRIBUTION

The CDF of a two-parameter BS random variable $T$, for $\alpha > 0$, $\beta > 0$ can be written as

$$F_T(t:\alpha,\beta) = \Phi\left[\frac{1}{\alpha}\left\{\left(\frac{t}{\beta}\right)^{1/2} - \left(\frac{\beta}{t}\right)^{1/2}\right\}\right], \quad t > 0, \tag{2.1}$$

where $\Phi(.)$ is the CDF of a standard normal variable.

Using the same idea as (2.1), Kundu et al. [23] introduced the BVBS, hereafter called BVBS, model as follows:

Let us define the following functions:

$$A(t_1) = \frac{1}{\alpha_1}\left\{\left(\frac{t_1}{\beta_1}\right)^{1/2} - \left(\frac{\beta_1}{t_1}\right)^{1/2}\right\}$$

and

$$B(t_2) = \frac{1}{\alpha_2}\left\{\left(\frac{t_2}{\beta_2}\right)^{1/2} - \left(\frac{\beta_2}{t_2}\right)^{1/2}\right\}$$

The bivariate random vector $(T_1, T_2)$ is said to have a BVBS distribution with parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \rho$, if the CDF of $(T_1, T_2)$ can be expressed as

$$P(T_1 \le t_1, T_2 \le t_2) = \Phi_2[A(t_1), B(t_2); \rho], \tag{2.2}$$

for $t_1 > 0, t_2 > 0$, and zero otherwise. Here $\alpha_1 > 0, \alpha_2 > 0, \beta_1 > 0, \beta_2 > 0, -1 < \rho < 1$ and $\Phi_2(u, v, \rho)$ is the CDF of a standard normal vector $(Z_1, Z_2)$ with correlation coefficient $\rho$.

The corresponding pdf of $T_1$ and $T_2$ is given by

$$f_{T_1,T_2}(t_1,t_2) = \phi_2[A(t_1), B(t_2); \rho]A'(t_1)B'(t_2),$$

where $\phi_2(u, v, \rho)$ denote the joint pdf of $Z_1$ and $Z_2$ given by

$$\phi_2(u, v; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(u^2 - 2\rho uv + v^2)\right\}.$$

The following theorem provides the marginal and conditional distributions of the BVBS distribution

THEOREM 2.1 (Kundu et al. [23]): *If* $(T_1, T_2)$ ˜$BVBS$ $(\alpha_1, \alpha_2, \beta_1, \beta_2, \rho)$, *then*

(a) $T_i$˜$BS(\alpha_i, \beta_i), i = 1, 2$

(b) *The conditional pdf of* $T_1$ *given* $T_2 = t_2$ *is given by*

$$f_{T_1|T_2=t_2}(t_1|t_2) = \frac{A'(t_1)}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(A(t_1) - \rho B(t_2))^2\right\}.$$

(c) *The conditional CDF of* $T_1$ *given* $T_2 = t_2$ *is given by*

$$P(T_1 \leq t_1|T_2 = t_2) = \Phi\left\{\frac{(A(t_1) - \rho B(t_2))}{\sqrt{1-\rho^2}}\right\}.$$

### 2.1. Conditional Failure Rate of $T_1$ Given $T_2 = t_2$

The conditional survival function of $T_1$ given $T_2 = t_2$ can be written as

$$R(t_1|t_2) = 1 - \Phi\left\{\frac{A(t_1) - \rho B(t_2)}{\sqrt{1-\rho^2}}\right\},$$

*2.1.1. The conditional failure rate of* $T_1$ *given* $T_2 = t_2$. The conditional failure rate of $T_1$ given $T_2 = t_2$ is given by

$$\begin{aligned} r_{T_1|T_2}(t_1|t_2) &= -\frac{d}{dt_1} \ln\left\{1 - \Phi\left\{\frac{A(t_1) - \rho B(t_2)}{\sqrt{1-\rho^2}}\right\}\right\} \\ &= h_N\left[\frac{A(t_1) - \rho B(t_2)}{\sqrt{1-\rho^2}}\right] \frac{A'(t_1)}{\sqrt{1-\rho^2}}, \end{aligned} \tag{2.3}$$

where $h_N(.)$ is the failure rate of a standard normal variable.

In order to investigate the monotonicity of the above failure rate, we have

$$\frac{d}{dt_1} r_{T_1|T_2}(t_1|t_2) = h'_N\left[\frac{A(t_1) - \rho B(t_2)}{\sqrt{1-\rho^2}}\right] \frac{(A'(t_1))^2}{1-\rho^2} + h_N\left[\frac{A(t_1) - \rho B(t_2)}{\sqrt{1-\rho^2}}\right] \frac{A''(t_1)}{\sqrt{1-\rho^2}}.$$

Now the failure rate of $T_1$ is given by

$$r_{T_1}(t_1) = h_N(A(t_1)A'(t_1)).$$

Knowing the fact that $r_{T_1}(t_1)$ is of the type $U$, $r_{T_1|T_2}(t_1|t_2)$, as a function of $t_1$ is of the type $U$, see Gupta and Akman [14]. The turning point of the failure rate of the conditional distribution is given by the solution of the equation

$$h'_N\left[\frac{A(t_1)-\rho B(t_2)}{\sqrt{1-\rho^2}}\right]\frac{(A'(t_1))^2}{1-\rho^2}+h_N\left[\frac{A(t_1)-\rho B(t_2)}{\sqrt{1-\rho^2}}\right]\frac{A''(t_1)}{\sqrt{1-\rho^2}}=0. \qquad \textbf{(2.4)}$$

For investigating the monotonicity of the above conditional failure rate as a function of $t_2$, we proceed as follows:

In order to study the dependence between the two variables $X$ and $Y$, we define a local dependence function

$$\gamma_f(x,y)=\frac{\partial^2}{\partial x\partial y}\ln f(x,y),$$

where $f(x,y)$ is the joint pdf of $(X,Y)$, see Holland and Wang [19] for definition and properties of the local dependence function. One important property of the local dependence function is the $TP_2$ $(RR_2)$ property defined as:

Definition 2.2: *A function $f(x,y)$ is said to be $TP_2$ (totally positive of order 2) or $RR_2$ (reverse rule of order 2) if*

$$f(x_1,y_1)f(x_2,y_2)\geq(\leq)f(x_1,y_2)f(x_2,y_1), x_1<x_2, y_1<y_2.$$

The following theorem ties the local dependence function and the $TP_2$ $(RR_2)$ property.

Theorem 2.3: *The density of $(X,Y)$ is $TP_2$ (totally positive of order 2) or $RR_2$ (reverse rule of order 2) according as the local dependence function $\gamma_f(x,y)>(<)0$.*

Proof: See Theorem 7.1 of Holland and Wang [19].

We now show that the BVBS has the $\text{TP}_2$ $(RR_2)$ property according to $\rho>0$ $(\rho<0)$.

Theorem 2.4: *The BVBS distribution has the $TP_2(RR_2)$ property according to $\rho>0$ $(\rho<0)$.*

Proof: The pdf of the BVBS distribution can be written as

$$f_{T_1,T_2}(t_1,t_2)$$
$$=\frac{1}{2\pi\sqrt{1-\rho^2}}A'(t_1)B'(t_2)\exp\left\{-\frac{1}{2(1-\rho^2)}(A^2(t_1)-2\rho A(t_1)B(t_2)+B^2(t_2))\right\}.$$

This gives

$$\frac{\partial^2}{\partial t_1\partial t_2}\ln f_{T_1,T_2}(t_1,t_2)=\frac{\rho}{1-\rho^2}A'(t_1)B'(t_2). \qquad \textbf{(2.5)}$$

It can now be verified that $A'(t_1)>0$ and $B'(t_2)>0$. The result is proved using Theorem 2.3.

We now state the following result due to Shaked [30].

Lemma 2.5: *If $f(x,y)$ is $TP_2(RR_2, resp.)$, the conditional failure rate of $X$ given $Y=y$ is decreasing (increasing, resp.) in $y$.*

Using the above result, we conclude that

Theorem 2.6: *For the BVBS, the failure rate of the conditional distribution of $T_1$ given $T_2=t_2$ is decreasing (increasing) in $t_2$ according as $\rho>(<)0$.*

### 2.2. Conditional Failure Rate of $T_1$ Given $T_2 > t_2$

As stated before, the conditional CDF of $(T_1, T_2)$ is given by

$$P(T_1 \le t_1, T_2 \le t_2) = \Phi_2[A(t_1), B(t_2); \rho].$$

The failure rate of $T_1$ given $T_2 > t_2$ is given by

$$\begin{aligned} h_1(t_1, t_2) &= -\frac{\partial}{\partial t_1} \ln P(T_1 > t_1 | T_2 > t_2) \\ &= \frac{A'(t_1)\phi(A(t_1))[1 - \Phi(\frac{B(t_2) - \rho A(t_1)}{\sqrt{1-\rho^2}})]}{P(T_1 > t_1, T_2 > t_2)}. \end{aligned} \tag{2.6}$$

The corresponding pdf of $T_1$ given $T_2 > t_2$ is given by

$$f_{T_1|T_2>t_2}(t_1|t_2) = \phi(A(t_1))A'(t_1)\frac{\left[1 - \Phi(\frac{B(t_2) - \rho A(t_1)}{\sqrt{1-\rho^2}}\right]}{P(T_2 > t_2)}. \tag{2.7}$$

For the standard normal distribution, see Gupta and Gupta [10].

In order to investigate the monotonicity of this conditional failure rate as a function of $t_1$, we present the following two results:

LEMMA 2.7: *Let $X$ be a continuous random variable with density function $f(x)$ and the corresponding eta function $\eta(x)$. Let $f^*(x) = w(x)f(x)$ be the weighted density with weight function $w(x)$ and the corresponding eta function $\eta^*(x)$.*

A. *Suppose $\eta(x)$ is increasing and $w(x)$ is logconcave. Then $f^*(x)$ is logconcave.*
B. *Suppose (i) $\eta(x)$ is of the type $U$ (ii) $w(x)$ is logconcave (iii) $\partial^3/\partial x^3[\ln(w(x)] > 0$. Then $\eta^*(x)$ is of the type $U$*

PROOF: See Gupta and Arnold [16]

LEMMA 2.8: *Suppose $X$ is a standard normal variable with failure rate $h_N(t)$. Then $h''_N(t) > 0$.*

PROOF: The truncated variance is given by

$$\mathrm{Var}(X|X > t) = 1 - h'_N(t),$$

see McGill [26]. This gives

$$\sigma_F^2(t) = \mathrm{Var}(X - t|X > t) = 1 - h'_N(t).$$

Since $X$ has the IFR, it has decreasing mean residual life function and decreasing variance residual life function, see Gupta [13]. This means that $\sigma_F^{2\prime}(t) < 0$ or $h''_N(t) > 0$.

Using the above two Lemmas, we present the following result.

THEOREM 2.9: *For the BVBS distribution, the failure rate of $T_1$ given $T_2 > t_2$ is of the type $U$, assuming $\rho > 0$.*

PROOF: The conditional *pdf* of $T_1$ given $T_2 > t_2$ is given by

$$f_{T_1|T_2>t_2}(t_1|t_2) = w(t_1)\phi(A(t_1))A'(t_1),$$

where the weight function $w(t_1)$ is given by

$$w(t_1) = \frac{\left[1 - \Phi\left(\frac{B(t_2)-\rho A(t_1)}{\sqrt{1-\rho^2}}\right)\right]}{P(T_2 > t_2)}.$$

It can be verified that $A'(t_{1)} > 0$, $A''(t_1) < 0$ and $A'''(t_1) > 0$. Also

$$\begin{aligned}\frac{d}{dt_1}\ln w(t_1) &= \frac{\rho}{\sqrt{1-\rho^2}} h_N\left(\frac{B(t_2)-\rho A(t_1)}{\sqrt{1-\rho^2}}\right)A'(t_1),\\ \frac{d^2}{dt_1^2}\ln w(t_1) &= \frac{\rho}{\sqrt{1-\rho^2}}\left[h'_N\left(\frac{B(t_2)-\rho A(t_1)}{\sqrt{1-\rho^2}}\right)\left(\frac{-\rho A'^2(t_1)}{\sqrt{1-\rho^2}}\right)\right]\\ &\quad + h_N\left(\frac{B(t_2)-\rho A(t_1)}{\sqrt{1-\rho^2}}\right)A''(t_1)\\ &< 0\end{aligned}$$

and

$$\begin{aligned}\frac{d^3}{dt_1^3}\ln w(t_1) = \frac{-\rho^2}{1-\rho^2}&\left[2A'(t_1)A''(t_1)h'_N\left(\frac{B(t_2)-\rho A(t_1)}{\sqrt{1-\rho^2}}\right)\right.\\ &- \frac{\rho}{\sqrt{1-\rho^2}}A'^3(t_1)\left[h'''_N\left(\frac{B(t_2)-\rho A(t_1)}{\sqrt{1-\rho^2}}\right)\right]\\ &+ \frac{\rho}{\sqrt{1-\rho^2}}\left[A'''(t_1)h_N\left(\frac{B(t_2)-\rho A(t_1)}{\sqrt{1-\rho^2}}\right)\right]\\ &\left.-\frac{\rho}{\sqrt{1-\rho^2}}A'(t_1)A''(t_1)h_N\left(\frac{B(t_2)-\rho A(t_1)}{\sqrt{1-\rho^2}}\right)\right]\\ &> 0.\end{aligned}$$

Now, for the univariate BS model, $\eta(x)$ is of the type $U$, see Gupta and Akman [14]. Also all the conditions of Lemma 2.7 are satisfied, we can conclude that $\eta^*(t_1|T_2 > t_2)$ is of the type $U$. Thus, the failure rate of $T_1$ given $T_2 > t_2$ is of the type $U$.

In order to study the monotonicity of this failure rate as a function of $t_2$, we state the following result due to Shaked [30].

LEMMA 2.10: *If $f(x, y)$ is $TP_2(RR_2)$, the conditional failure rate of $X$ given $Y > y$ is decreasing (increasing) in $y$.*

Using the above result, we conclude that

THEOREM 2.11: *For the BVBS, the failure rate of the conditional distribution of $T_1$ given $T_2 > t_2$ is decreasing (increasing) in $t_2$ according as $\rho > (<)0$.*

## 3. ASSOCIATION MEASURE

In the context of bivariate survival models induced by frailties, Oakes [29] studied the following association measure:

$$\theta(t_1,t_2)) = \frac{SS_{12}}{S_1S_2},$$

where $S = S(t_1,t_2)$ is the survival function, $S_{12} = \partial^2 S(t_1,t_2)/\partial t_1\partial t_2, S_1 = (\partial/\partial t_1)S(t_1,t_2)$ and $S_2 = (\partial/\partial t_2)S(t_1,t_2)$; see also Clayton [5].

Clayton [5] presented the above association measure, deriving from the Cox model, in a study of the association between the lifespans of fathers and their sons.

It can be easily seen that

$$\theta(t_1,t_2) = \frac{r(t_1|T_2 = t_2)}{h_1(t_1,t_2)}.$$

The numerator is the hazard rate for sons at time $t_1$ given that their fathers died at $t_2$. The denominator is the hazard rate for sons at time $t_1$ given that their fathers live past $t_2$. Also

$$r(t_1|T_2 = t_2) = -S_{12}/S_2 \text{ and } h_1(t_1,t_2) = -S_1/S.$$

It can now be verified that

$$\begin{aligned}\frac{\partial}{\partial t_2}h_1(t_1,t_2) &= \frac{S_2}{S}\left[-h_1(t_1,t_2) + r(t_1|T_2 = t_2\right] \\ &= \frac{S_2}{S}h_1(t_1,t_2)(\theta - 1),\end{aligned}$$

suppressing the argument of $\theta$.

Since $S_2\langle 0,\theta\rangle(<)1$ is equivalent to $\frac{\partial}{\partial t_2}h_1(t_1,t_2) < (>)0$.

Thus, in the case of BVBS, $\theta > (<)1$ according as $\rho > (<)0$, see Theorem 2.11

Note that, in general, $TP_2$ property (positive dependence) implies $\theta > 1$. Likewise $RR_2$ property implies $\theta < 1$.

We now express $\theta(t_1,t_2)$ in terms of the hazard components derived earlier

Using the definitions given above, it can be verified that

$$\begin{aligned}\frac{\partial^2}{\partial t_1\partial t_2}\ln S(t_1,t_2) &= \frac{S_1S_2}{S^2}(\theta - 1) \\ &= h_1(t_1,t_2)h_2(t_1,t_2)(\theta - 1),\end{aligned}$$

where

$$h_2(t_1,t_2) = -\frac{\partial}{\partial t_2}\ln S(t_1,t_2).$$

This gives

$$\theta(t_1,t_2) = 1 - \frac{\frac{\partial}{\partial t_2}h_1(t_1,t_2)}{h_1(t_1,t_2)h_2(t_1,t_2)}.$$

By symmetry, we also have

$$\theta(t_1,t_2) = 1 - \frac{\frac{\partial}{\partial t_1}h_2(t_1,t_2)}{h_1(t_1,t_2)h_2(t_1,t_2)}. \tag{3.1}$$

*Remark 3.1*: It can be proved that $T_1$ and $T_2$ are independent if and only if $\theta(t_1,t_2) = 1$.

Using the values of $h_1(t_1,t_2)$ and $h_2(t_1,t_2)$ derived earlier, $\theta(t_1,t_2)$ can be obtained for the $BVBS$ distribution; see Eq. (2.6).

### 3.1. Effect of the association measure

The deviation of the ratio of these hazards from 1 characterizes the measure of mutual dependence of respective lifespans. The stronger the dependence between $T_1$ and $T_2$ is, the higher is the value of $|\theta(t_1,t_2)-1|$. If $\theta(t_1,t_2)$ decreases to one when $t_1$ and $t_2$ tend to $+\infty$, then the dependence of the pair $(T_1,T_2)$ is at least $DTP(0,1)$ or $DTP(1,0)$. This means that the conditional hazard of $T_2$ given $T_1=t_1$ and the conditional hazard of $T_1$ given $T_2=t_2$ decrease in $t_1$ (respectively in $t_2$); see Shaked [30] for the definitions of $DTP(0,1)$ and $DTP(1,0)$. Clayton [5] proposed this measure by assuming that association arises because the two members of a pair share some common influence and not because one event influences the other. Thus $\theta(t_1,t_2)$ explains an association between two non-negative survival times with continuous joint distribution by their common dependence on an unobserved random variable. This unobserved random variable is commonly known as frailty or environmental effect, see Oakes [29] and Manatunga and Oakes [24] for more details on frailty models. Clayton [5] describes the estimation of the parameter $\theta(x,y)$ from longitudinal studies. He also provides a numerical example in case–control studies. The estimation of $\theta(t_1,t_2)$ in a discrete form of the model is considered by Oakes [29] even in the presence of censoring in either or both components and it is shown that these estimates can be used to test the independence of the two variables. He uses the data of Hanley and Parnes [18] to illustrate this methodology.

## 4. SERIES AND PARALLEL SYSTEMS OF TWO COMPONENTS

In this section, we shall obtain the density functions of $U_1=\min(T_1,T_2)$ and $U_2=\max(T_1,T_2)$. Also we study the monotonicity of the failure rate of $U_1$.

We know that for any bivariate vector $(T_1,T_2)$, the density functions of $U_1$ and $U_2$ are given by

$$f_{U_1}(t)=f_{T_1}(t)P(T_2>t|T_1=t)+f_{T_2}(t)P(T_1>t|T_2=t) \tag{4.1}$$

and

$$f_{U_2}(t)=f_{T_1}(t)P(T_2<t|T_1=t)+f_{T_2}(t)P(T_1<t|T_2=t), \tag{4.2}$$

see Gupta and Gupta [11].

For the BVBS, using (4.1), it can be verified that

$$f_{U_1}(t)=f_{T_2}(t)\left[1-\Phi\left(\frac{A(t)-\rho B(t)}{\sqrt{1-\rho^2}}\right)\right]+f_{T_1}(t)\left[1-\Phi\left(\frac{B(t)-\rho A(t)}{\sqrt{1-\rho^2}}\right)\right]$$

and

$$f_{U_2}(t)=f_{T_2}(t)\Phi\left(\frac{A(t)-\rho B(t)}{\sqrt{1-\rho^2}}\right)+f_{T_1}(t)\Phi\left(\frac{B(t)-\rho A(t)}{\sqrt{1-\rho^2}}\right).$$

For the standard bivariate normal distribution, it reduces to

$$f_{U_1}(t)=2\phi(t)\left[1-\Phi\left(t\sqrt{\frac{1-\rho}{1+\rho}}\right)\right] \tag{4.3}$$

and

$$f_{U_2}(t)=2\phi(t)\Phi\left(t\sqrt{\frac{1-\rho}{1+\rho}}\right).$$

### 4.1. Monotonicity of the failure rate

In most practical applications, the failure rate is quite complicated and so the straight derivative method is very complex. In such cases, we work with the density function and use Glaser's [9] approach described earlier. In our case, even the expression for $\eta_{U_1}(t)$ is quite involved to yield an analytic solution of the problem. So we proceed as follows:

Let us denote by $h(t)$ the failure rate of $U_1$ with survival function $S_{U_1}(t)$. Then $hh(th)(t)$

$$h(t) = -\frac{d}{dt} \ln S(t) = h_1(t,t) + h_2(t,t), \tag{4.4}$$

where $S(t_{1,}t_2)$ is the survival function of $(T_{1,}T_2)$.

Note that $h_i(t,t), i = 1,2$ is proportional to the failure rate $h_i^*(t,t)$ of the conditional distribution of $U_1$ given $T_1 < T_2(T_2 < T_1)$. In the context of competing risks, $h_i(t,t)$ describes the (instantaneous) rate of dying from cause $i$ when both the causes are acting simultaneously, see Gupta [12] and Elandt-Johnson and Johnson [7].

Then the pdfs $f_i^*(t)$ of the conditional distributions are given by

$$f_i^*(t) = \frac{1}{\pi_i} h_i(t,t) S_{U_1}(t), \quad i = 1,2, \tag{4.5}$$

where $S_{U_1}(u)$ is given by

$$S_{U_1}(u) = \exp\left\{ -\int_0^u \sum_{i=1}^{2} h_i(x,x)dx \right\} \tag{4.6}$$

and $\pi_i$ is a proper constant of proportionality. For more details and applications, see Gaynor et al. [8]. Note that $f_1^*(t)$ helps us to investigate the monotonicity of the hazard rate of the minimum.

Because of the proportionality mentioned before, the monotonicity of $h(t)$ can be established if $f_1^*(t)$ and $f_2^*(t)$ fulfill the criteria mentioned before.

As seen before

$$h_1(t_1,t_2) = \frac{A'(t_1)\phi(t_1)\left[1 - \Phi\left(\frac{B(t_2)-\rho A(t_1)}{\sqrt{1-\rho^2}}\right)\right]}{P(T_1 > t_1, T_2 > t_2)}. \tag{4.7}$$

This gives

$$f_1^*(t) = \frac{1}{\pi_1} A'(t)\phi(A(t)) \left[1 - \Phi\left(\frac{B(t) - \rho A(t)}{\sqrt{1-\rho^2}}\right)\right]. \tag{4.8}$$

This gives

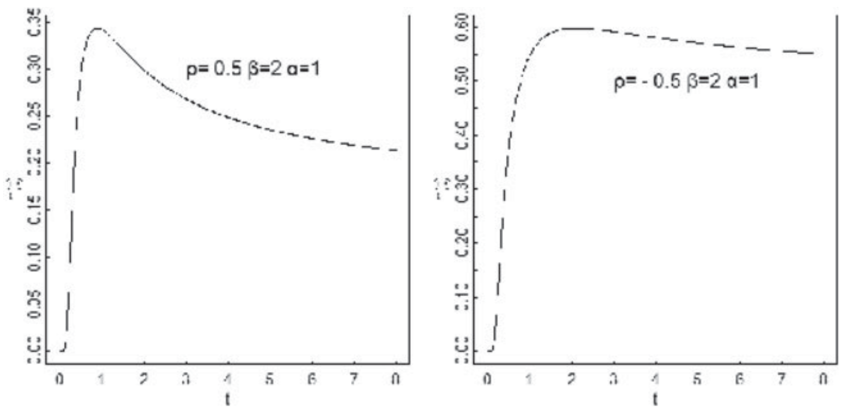
$$\begin{aligned} \eta_1^*(t) &= -\frac{d}{dt} \ln f_1^*(t) \\ &= -\frac{A''(t)}{A'(t)} + A(t)A'(t) + \frac{B'(t) - \rho A'(t)}{\sqrt{1-\rho^2}} h_N\left(\frac{B(t) - \rho A(t)}{\sqrt{1-\rho^2}}\right), \end{aligned} \tag{4.9}$$

where $h_N(.)$ is the failure rate of a standard normal.

Note that, for the case of bivariate normal distribution, it matches with Gupta and Gupta [11].

The general analytical investigation of the monotonicity of the failure rate is not feasible. So we consider the case when $A(t) = B(t)$. In that case $f_1^*(t)$ becomes

$$f_1^*\,(t) = \frac{1}{\pi_1} f_{T_1}(t)[1 - \Phi((A(t)\sqrt{(1-\rho)/(1+\rho)}]$$




The above two graphs indicate that the failure rate of the series system is of the type $U$.

## 5. CONCLUSIONS AND COMMENTS

In this paper, we have presented some properties of the BVBS distribution from a reliability point of view. The dependence properties of the model are studied by examining the local dependence function and the association measure due to Clayton [5]. The determination of the monotonicity of the hazard components is discussed and it is shown that the failure rate of $T_1$ given $T_2 > t_2$ is of the type $U$. In the case of series system, the monotonicity of the failure rate is discussed. In general, the determination of the monotonicity of the hazard is not feasible by analytical means. However, in some particular cases, one can get some information about the monotonicity. We hope that our investigation will be helpful to reliability researchers and theoreticians.

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