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Rooted trees, strong cofinality and ample generics

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Abstract

We characterize those countable rooted trees with non-trivial components whose full automorphism group has uncountable strong cofinality, and those whose full automorphism group contains an open subgroup with ample generics.

1. *Introduction*

In this paper, we study full automorphism groups of countable rooted trees, equipped with the topology of pointwise convergence. We are mainly interested in the notions of strong cofinality and ample generics.

Recall that a group G has uncountable strong cofinality if whenever G is a union of a countable chain of symmetric subsets $A_0 \subseteq A_1 \subseteq \cdots$, then $A_m^k = G$ for some $k, m \in \mathbb{N}$. This property was introduced by Bergman in [**2**], and studied by authors such as Droste and Holland [**5**] or Kechris and Rosendal [**10**]. It can also be viewed as a type of fixed point property, linking it to geometric group theory: it is well known that G has uncountable strong cofinality if and only if every isometric action of G on a metric space has bounded orbits. In particular, uncountable strong cofinality implies Serre's property (FA).

A separable and completely metrizable (that is, Polish) topological group G has ample generics if the diagonal action of G on G^n by conjugation has a comeagre orbit for every $n \in \mathbb{N}$. This notion was first studied by Hodges, Hodkinson, Lascar and Shelach in [**9**], and later by Kechris and Rosendal in [**10**]. It is a very strong property: a group G with ample generics, or even containing an open subgroup with ample generics, has the small index property ([**10**, theorem 1·6]), every homomorphism from G into a separable group is continuous ([**10**, theorem 1·10]), every isometric action of G on a separable metric space is continuous, and there is only one Polish group topology on G ([**10**, corollary 6·25].)

These two seemingly unrelated concepts have in fact something in common. For example, if G has ample generics, and G is a union of a countable chain of *non-open subgroups* $G_0 \leqslant G_1 \leqslant \cdots$, then $G_m = G$ for some $m \in \mathbb{N}$ (see [**10**, theorem 1·7].) However, in general none of them is implied by the other.

Our main results show that in the context of automorphism groups of countable rooted trees, both of these properties are strictly related to the behavior of the algebraic closures of finite sets, and that in a sense having ample generics is a strong form of having uncountable strong cofinality.

For a tree T, $X \subseteq T$, and $G = \mathrm{Aut}(T)$, by $\mathrm{ACL}_T(X)$ we denote the algebraic closure of X in T, that is, the set of all elements of T contained in a finite orbit under the action of the pointwise stabilizer $G_{\langle X \rangle}$ of X. Summing up the main results of the paper, we have:

THEOREM 1. *Let T be a countable rooted tree with non-trivial components, $G =$* $\mathrm{Aut}(T)$.

(1) *G has uncountable strong cofinality iff* $\mathrm{ACL}_T(\emptyset)$ *is finite*;
(2) *G has an open subgroup with ample generics iff* $\mathrm{ACL}_T(X)$ *is finite for every finite $X \subseteq T$.*

It turns out that this sheds light on the relationship between certain known results on rigidity of groups of automorphisms of trees, which we will discuss in the last section of the paper. In fact, this investigation was inspired by them.

2. *Notation and basic facts*

Trees. A *tree* is a connected graph with no cycles. A *rooted tree* T with root r is a tree with a distinguished vertex $r \in T$. For every rooted tree T and every $t \in T$, the unique path $(t_0, \dots, t_n)$ from $t_0 = t$ to $t_n = r$ determines the *successor* $s(t) = t_1$ of t (where we put $s(r) = r$). Similarly, for $t \in T$, a *predecessor* of t is any element $t' \in T$ such that $s(t') = t$. A *leaf* in $T' \subseteq T$ is an element with no predecessors in T'. By a *subtree* of T, we mean a subset $T' \subseteq T$ that is closed under the successor function $s(t)$.

Thus, a rooted tree can also be viewed as a structure (T, s), or as an ordered set $(T, <)$, with the ordering defined by

$$t < t' \iff s^n(t) = t' \text{ for some natural } n > 0,$$

and the largest element r.

For a rooted tree T and $t \in T$, T_t denotes a rooted tree with root t defined by

$$T_t = \{t' \in T : t' \leqslant t\}.$$

Every full automorphism group $G = \mathrm{Aut}(T)$ of a countable (rooted) tree T, where T is regarded as a discrete space, is assumed to be equipped with the pointwise convergence topology. This topology is easily seen to be separable and completely metrizable, that is, *Polish*. For $X \subseteq T$, the symbol $G_{\langle X \rangle}$ stands for the pointwise stabilizer of X in G.

In this paper (except for in the final section), all trees are assumed to be rooted and countable.

Wreath products and rooted trees. Following [**8**], we define the unrestricted generalized wreath product of groups of permutations.

Let Δ be an ordered set, and let (G_δ, N_δ), $\delta \in \Delta$, be transitive permutation groups. Groups G_δ are called *components*.

For every $\delta \in \Delta$, fix $n_\delta \in N_\delta$. Let $S \subseteq \prod_{\delta \in \Delta} N_\delta$ be the set of all elements satisfying *the maximum condition*, that is, $x \in S$ if and only if there are no infinite strictly increasing sequences in the support $\mathrm{supp}\,(x)$ of x defined by

$$\mathrm{supp}\,(x) = \{\delta \in \Delta : x(\delta) \neq n_\delta\}.$$

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For $x \in S$, $\delta, \gamma \in \Delta$, $i \in N_\delta$, define $x_i^\delta \in S$ by

$$x_i^\delta(\gamma) = \begin{cases} x(\gamma) & \text{if } \delta \neq \gamma, \\ i & \text{if } \delta = \gamma. \end{cases}$$

Now, $g \in \mathrm{Sym}(S)$ is an element of $Wr_{\delta \in \Delta} G_\delta$ if for every $x, y \in S$, and $\delta \in \Delta$:

(i) $x(\gamma) = y(\gamma)$ for all $\gamma > \delta \Rightarrow g(x)(\gamma) = g(y)(\gamma)$ for all $\gamma > \delta$;
(ii) there is $\sigma \in G_\delta$ such that $g(x_i^\delta)(\delta) = \sigma(i)$ for all $i \in N_\delta$.

It is not hard to see that since all G_δ, $\delta \in \Delta$, are transitive, this definition does not depend (up to permutational isomorphism) on the choice of n_δ.

For $\Delta = \{\delta_0, \delta_1\}$ with $\delta_0 < \delta_1$, we write

$$Wr_{\delta \in \Delta} G_\delta = G_{\delta_0}\, Wr\, G_{\delta_1}.$$

Then the *base group* G^{base} of $G = G_{\delta_0}\, Wr\, G_{\delta_1}$ is a normal group of all the permutations $g \in G$ such $g(n_0, n_1) = (n_0', n_1)$ for every $(n_0, n_1) \in N_{\delta_0} \times N_{\delta_1}$, that is, the second coordinate of (n_0, n_1) stays fixed. It is easy to see that $G^{base} = G_0^{N_{\delta_1}}$, and that every element $g \in G$ is of the form $g = g_0 g_1$, where $g_0 \in G^{base}$, $g_1 \in G_1$. In other words, $G = G^{base} G_1$.

Now, let T be a rooted tree, $G = \mathrm{Aut}(T)$, and let $\overline{T}$ be a fundamental domain of T, that is, a lift of the projection $T \to T/G$. The tree $\overline{T}$ contains exactly one element of each orbit of the action of G on T. It is well known (see for example [**12**, p. 25]) that such $\overline{T}$ exists. It is also easy to see that any two fundamental domains are isomorphic as rooted trees. Therefore, we will always assume that T comes equipped with a fixed fundamental domain $\overline{T}$.

For $t \in T$, let $\mathcal{N}_t$ denote the orbit of t under the action of $G_{\langle s(t) \rangle}$ on T. It is easy to see that the group $G_{\langle s(t) \rangle}$, when restricted to $\mathcal{N}_t$, is simply the full symmetric group $\mathrm{Sym}(\mathcal{N}_t)$, so put $G_t = \mathrm{Sym}(\mathcal{N}_t)$, $t \in \overline{T}$. Associate with $\overline{T}$ the ordering inherited from T, and define $n_t = t$ for every $t \in \overline{T}$. Observe that every element in $\prod_{t \in \overline{T}} \mathcal{N}_t$ satisfies the maximum condition because there are no infinite strictly increasing sequences in $\overline{T}$. It is a straightforward exercise to show that $G \cong Wr_{t \in \overline{T}} G_t$.

As a matter of fact, if $\mathcal{N}_t = \{t\}$ for some $t \in \overline{T}$, then G_t is trivial and such t does not contribute to the automorphism group of T. Therefore, in this paper we will restrict our attention to trees T such that $|\mathcal{N}_t| > 1$ for every $t \in \overline{T}$, $t \neq r$, that is *trees with non-trivial components*. This will allow for cleaner statements of the results but one could also modify our arguments so that they work for all countable rooted trees.

Ample generics. Recall that a countable structure M is *ultrahomogeneous* if every isomorphism ϕ between finite substructures of M can be extended to an automorphism of M. It is *locally finite* if all finitely generated substructures are finite.

For a countable locally finite structure M in a fixed countable signature, $\mathrm{Age}(M)$ is the family of all finite substructures of M. Also, for $n \in \mathbb{N}$, the family $\mathcal{K}_p^n$ consists of all the objects of the form

$$\langle A, \phi_0 : B_0 \longrightarrow C_0, \ldots, \phi_n : B_n \longrightarrow C_n \rangle,$$

where $A, B_i, C_i \in \mathrm{Age}(M)$, $B_i, C_i \subseteq A$, and ϕ_i are isomorphisms, $i \leqslant n$.

There is a natural notion of embedding associated with every $\mathcal{K}_p^n$. For

$$\mathcal{S} = \langle A, \phi_0 : B_0 \longrightarrow C_0, \ldots, \phi_n : B_n \longrightarrow C_n \rangle,$$

$$\mathcal{T} = \langle D, \psi_0 : E_0 \longrightarrow F_0, \ldots, \psi_n : D_n \longrightarrow F_n \rangle,$$

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$f : A \to D$ embeds $\mathcal{S}$ into $\mathcal{T}$ if it is an embedding of A into D as structures, and

$$f \circ \phi_i \subseteq \psi_i \circ f$$

for $i \leqslant n$.

We say that $\mathcal{K}_p^n$ satisfies the *weak amalgamation property* (WAP) if for every $\mathcal{S} \in \mathcal{K}_p^n$ there exists $\mathcal{T} \in \mathcal{K}_p^n$, and an embedding $e : \mathcal{S} \to \mathcal{T}$ such that for every $\mathcal{F}, \mathcal{G} \in \mathcal{K}_p^n$, and embeddings $i : \mathcal{T} \to \mathcal{F}$, $j : \mathcal{T} \to \mathcal{G}$, there exists $\mathcal{E} \in \mathcal{K}_p^n$ and embeddings $k : \mathcal{F} \to \mathcal{E}$, $l : \mathcal{G} \to \mathcal{E}$ such that $\mathcal{E}$ amalgamates $\mathcal{F}$ and $\mathcal{G}$ over $\mathcal{T}$, that is,

$$k \circ i \circ e = l \circ j \circ e.$$

The family $\mathcal{K}_p^n$ satisfies the *joint embedding property* (JEP) if any two $\mathcal{S}, \mathcal{T} \in \mathcal{K}_p^n$ can be embedded in some $\mathcal{E} \in \mathcal{K}_p^n$.

A Polish group G has *ample generics* if each diagonal action of G on G^n, $n \in \mathbb{N}$, by conjugation:

$$g.(g_0, \ldots, g_n) = (gg_0g^{-1}, \ldots, gg_ng^{-1}),$$

$g, g_0, \ldots, g_n \in G$, has a comeagre orbit. We have ([**10**, theorem 6·2]):

THEOREM 2. *Let M be a countable, locally finite, ultrahomogeneous structure, $\mathcal{K} = \mathrm{Age}(M)$, and $G = \mathrm{Aut}(M)$. Then G has ample generics if and only if $\mathcal{K}_p^n$ satisfies WAP and JEP for every $n \in \mathbb{N}$.*

More information on this topic can be found in [**10**].

Obviously, not all rooted trees are ultrahomogeneous. Therefore, we add to T a family of unary predicates $\mathcal{O}_t$, $t \in T$, such that the structure $(T, s, \{\mathcal{O}_t\}_{t\in T})$ is locally finite, ultrahomogeneous, and has the same automorphisms as T:

$$t' \in \mathcal{O}_t \longleftrightarrow t' = g(t) \text{ for some } g \in \mathrm{Aut}(T),$$

for $t, t' \in T$.

PROPOSITION 3. *Let T be a countable rooted tree. The structure $(T, s, \{\mathcal{O}_t\}_{t\in T})$ is locally finite, ultrahomogeneous, and*

$$\mathrm{Aut}(T) = \mathrm{Aut}(T, s, \{\mathcal{O}_t\}_{t\in T}).$$

Proof. First of all, local finiteness follows form the fact that each set $\{s^n(t) : n \in \mathbb{N}\}$, $t \in T$, is finite. We show that if $f : A \to B$ is an isomorphism between finite subtrees of T, and $t \in T \setminus A$ is a predecessor of some $a \in A$, then there exists $t' \in T$ such that $f \cup \{(t, t')\}$ is an isomorphism. By the standard back-and-forth argument this implies ultrahomogeneity of $(T, s, \{\mathcal{O}_t\}_{t\in T})$.

Let $f : A \to B$, $t, a \in T$ be as above, and fix $g \in \mathrm{Aut}(T)$ with $g(a) = f(a)$. Then t witnesses that $\mathcal{N}_t \setminus A \neq \emptyset$, so there exists $t' \in g[\mathcal{N}_t] \setminus f[A]$. Clearly, t' is as required.

Obviously, the predicates $\mathcal{O}_t$ do not affect automorphisms of T.

Using the same argument, one can prove the following.

COROLLARY 4. *Suppose that $T' \subseteq T$ is a finite substructure of $(T, s, \{\mathcal{O}_t\}_{t\in T})$ such that for every $t, t' \in T'$ we have that*

$$t' \in \mathcal{O}_t \Longrightarrow |\mathcal{N}_t \cap T'| = |\mathcal{N}_{t'} \cap T'|.$$

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Then for any finite substructures $A, B \subseteq T'$ and every isomorphic $f : A \to B$, there exists an automorphism $g : T' \to T'$ that extends f.

LEMMA 5. *Let T'' be a finite subtree of a countable rooted tree T. Then there exists a finite subtree T' such that $T'' \subseteq T'$ and T' satisfies the assumption of Corollary* 4.

Proof. This is a simple induction on the height of T''. For $t \in T$, let $h(t)$ be the size of the unique path joining t and r, and let $H = \max\{h(t) : t \in T''\}$. Suppose that T'' is finite and such that the assumption of Corollary 4 is satisfied for every $t \in T''$ with $h(t) \leqslant n$. Then, clearly, by adding finitely many elements to T'', we can find a subtree T' such that the assumption of Corollary 4 is satisfied for elements $t \in T'$ with $h(t) \leqslant n + 1$, and $h(t) \leqslant H$ for every $t \in T'$.

Groups. A group G has *uncountable strong cofinality* if for any $A_0 \subseteq A_1 \subseteq \cdots$ such that each A_m is symmetric and $G = \bigcup_m A_m$, we have $A_m^k = G$ for some k, m. If $G = \bigcup_m G_m$ for some strictly increasing infinite sequence of subgroups $G_0 < G_1 < \cdots$, then we say that G has *countable cofinality*. Finally, a topological group G has the *small index property* if any subgroup of index less than $2^{\aleph_0}$ is open in G.

3. *Uncountable strong cofinality*

The first two lemmas are straightforward, so we omit their proofs.

LEMMA 6. *Let T be a countable rooted tree,* $\mathrm{Aut}(T) = Wr_{t\in\overline{T}}\, G_t$, *and $\overline{S} \subseteq \overline{T}$ be a subtree of $\overline{T}$. Then $\overline{S}$ corresponds to an invariant subtree $S \subseteq T$,* $\mathrm{Aut}(S) = Wr_{t\in\overline{S}}\, G_t$, *and $g \mapsto g{\restriction_{\overline{S}}}$, $g \in Wr_{t\in\overline{T}}\, G_t$, defines a continuous and surjective homomorphism $\phi : Wr_{t\in\overline{T}}\, G_t \to Wr_{t\in\overline{S}}\, G_t$.*

LEMMA 7. *Let $T = \{r\} \bigcup \{t_n\}$ be a tree such that r is the successor of each t_n, and let G_t, $t \in T$, be permutation groups. Then $Wr_{t\in T}\, G_t$ is isomorphic to $(\prod_n G_{t_n}) Wr\, G_r$.*

LEMMA 8. *Let T be a finite rooted tree, and let G_t, $t \in T$, be permutation groups. If every G_t, $t \in T$, has uncountable strong cofinality (in particular, if G_t is finite), then $Wr_{t\in T}\, G_t$ has uncountable strong cofinality.*

Proof. This is an easy induction on the size of T. Let $t_0 \in T$ be such that all predecessors of t_0, say $t_1, \ldots, t_n$, are leaves in T. Let $T' = T \setminus \{t_1, \ldots, t_n\}$, and let $G'_{t_0} = (\prod_{i=1}^n G_{t_i}) Wr\, G_{t_0}$, $G'_t = G_t$ if $t \in T'$, $t \neq t_0$. Clearly, G'_{t_0} has uncountable strong cofinality, and $|T'| < |T|$. By induction hypothesis, $Wr_{t\in T'}\, G'_t$ has uncountable strong cofinality, and Lemma 7 implies that $Wr_{t\in T'}\, G'_t$ is isomorphic to $Wr_{t\in T} G_t$

The next lemma contains folklore facts.

LEMMA 9. *The group $(\mathbb{Z}_2)^{\mathbb{N}}$ has countable cofinality, and does not have the small index property. Therefore, every Polish group G that maps homomorphically, continuously and surjectively onto $(\mathbb{Z}_2)^{\mathbb{N}}$ has countable cofinality, and does not have the small index property.*

Proof. Select a Hamel basis B for $(\mathbb{Z}_2)^{\mathbb{N}}$ regarded as a linear space over the field $\mathbb{Z}_2$, and build a countable strictly increasing sequence $B_0 \subsetneq B_1 \subsetneq \cdots$ such that $B = \bigcup_n B_n$. The linear spaces H_n generated by B_n are groups witnessing countable cofinality of $(\mathbb{Z}_2)^{\mathbb{N}}$.

Now, let K be the complement of a non-principal ultrafilter on $\mathbb{N}$. Then K is a subgroup of $(\mathbb{Z}_2)^{\mathbb{N}}$ of index 2 that is dense in $(\mathbb{Z}_2)^{\mathbb{N}}$, and so it cannot be open. Thus, K witnesses that $(\mathbb{Z})^{\mathbb{N}}$ does not have the small index property.

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If G is Polish, and $\phi : G \to (\mathbb{Z}_2)^{\mathbb{N}}$ is homomorphic, continuous and surjective, then ϕ is open (see [7, theorem 2·3·3]), so $\phi^{-1}[K]$ is not open in G, and has index 2 in G. Also, groups $\phi^{-1}[H_n]$ form a strictly increasing sequence, whose union is G.

LEMMA 10. *Let S_0 be a countable rooted tree defined by one of the following*:

(i) *S_0 is an infinite branch, or*

(ii) *$S_0 = \{t_0, \ldots, t_n\} \bigcup \{s_0, s_1, \ldots\}$, where $r = t_0, t_1, \ldots, t_n$ is the unique path joining the root r and the only non-trivially branching element t_n, and t_n is the successor of each $s_0, s_1, \ldots$.*

Suppose that T is a tree with non-trivial components, $\mathrm{Aut}(T) = Wr_{t\in\overline{T}}\, G_t$, *and $\overline{T}$ contains a subtree S_0 as above such that each G_t, $t \in S_0$, is finite. Then* $\mathrm{Aut}(T)$ *has countable cofinality, and it does not have the small index property.*

Proof. Suppose first that $\overline{T} = S_0$. We consider the case that S_0 is an infinite branch. Let $S_0 = \{s_0, s_1, \ldots\}$ be the increasing enumeration of S_0, and let $H_n = Wr_{s\in\{s_0,\ldots,s_n\}}\, G_s$. Each H_n is a finite, non-trivial permutation group of a finite set, so each of its elements g can be homomorphically assigned its sign, $\mathrm{sgn}(g)$. In this manner, we get a continuous and surjective homomorphism $\mathrm{Aut}(T) \to (H_n)^{\mathbb{N}} \to (\mathbb{Z}_2)^{\mathbb{N}}$, so $Wr_{s\in S_0}\, G_s$ has countable cofinality and does not have the small index property by Lemma 9.

Let us consider the other case now. Let $H = G_{t_n} Wr \cdots Wr\, G_{t_0}$. By Lemma 7, the group G can be written as $G = (\prod_n G_{s_n}) Wr\, H$, where H is a group of permutations of a finite set of size $N+1$, and G_{s_n} are non-trivial symmetric groups. Hence, every element $g \in G^{base}$ is of the form

$$((g_0^0, g_1^0, \ldots), (g_0^1, g_1^1, \ldots), \ldots, (g_0^N, g_1^N, \ldots)),$$

where $g_j^i \in G_{s_i}$. Therefore, we can define $\phi : G \to (\mathbb{Z}_2)^{\mathbb{N}}$ by

$$\phi(hg) = (\mathrm{sgn}(g_n^0 \cdots g_n^N))_{n\in\mathbb{N}}$$

for $h \in H$, $g \in G^{base}$. The mapping ϕ is a homomorphism. To see this, note that for every $h \in H$, $g \in G^{base}$, $gh = h\bar{g}$, where $\bar{g}$ is a coordinate permutation of g of the form

$$((g_0^{i_0}, g_1^{i_0}, \ldots), (g_0^{i_1}, g_1^{i_1}, \ldots), \ldots, (g_0^{i_N}, g_1^{i_N}, \ldots)),$$

so

$$\phi(h_0g_0h_1g_1) = \phi(h_0h_1\bar{g}_0g_1) = \phi(\bar{g}_0g_1) = \phi(\bar{g}_0)\phi(g_1) = \phi(g_0)\phi(g_1) = \phi(h_0g_0)\phi(h_1g_1)$$

for every $h_0, h_1 \in H$, $g_0, g_1 \in G^{base}$.

As all G_s, $s \in S_0$, are symmetric groups, it is also surjective, so, as before, G can be homomorphically, continuously, and surjectively mapped onto $(\mathbb{Z}_2)^{\mathbb{N}}$.

If S_0 is a subtree of $\overline{T}$, then by Lemma 6, $Wr_{t\in\overline{T}}\, G_t$ maps homomorphically, continuously and surjectively onto $Wr_{t\in S_0}\, G_t$. An application of Lemma 9 finishes the proof.

Now we prove the main technical lemma.

LEMMA 11. *Suppose that $G = \prod_{n=0}^{N} (G_n)^{\mathbb{N}}$, where G_n are any groups, $N \in \{0, 1, \ldots, \mathbb{N}\}$, and that $G = \bigcup_m A_m$ for some $A_0 \subseteq A_1 \subseteq \cdots$. Suppose also that the following condition is satisfied:*

there exists l such that if $g = (g_0, g_1, \ldots) \in A_m^k$ for some k, m, and $\bar{g} = (\bar{g}_0, \bar{g}_1, \ldots) \in G$ is such that each $\bar{g}_n \in G_n^{\mathbb{N}}$ is a coordinate permutation of g_n, then $\bar{g} \in A_m^{k+l}$.

Then $G = A_m^k$ for some k, m.

Proof. Without loss of generality we can assume that groups G_n are pairwise disjoint, and that $e_G \in A_0$. We start with a claim.

Claim. There exists k such that for every countable $C \subseteq \bigcup_n G_n$ (that is, $C \in [\bigcup_n G_n]^\omega$) there exists $g^C \in G$ such that for every $g \in G$ with $\mathrm{range}(g) \subseteq C$, and every m, if $g^C \in A_m$, then $g \in A_m^k$.

In the proof of the claim, by saying that $g_{\restriction I}$ is a coordinate permutation of h, where $g, h \in G$, $I \subseteq \mathbb{N}^N$, we mean, abusing terminology slightly, that there exists a bijection $f : I \to \mathbb{N}^N$ such that $(g_{f(i)}) = h$.

For C as above, take $g^C = (g_0^C, g_1^C, \ldots)$ to be some fixed element such that each g_n^C contains infinitely many copies of every element from $(C \cap G_n) \cup (C \cap G_n)^{-1} \cup \{e_{G_n}\}$.

Suppose first that $g = (g_0, g_1, \ldots) \in G$ is such that each g_n is the identity on infinitely many coordinates.

We can partition $\mathbb{N}^N$ into 3 infinite subsets I, I', I'' such that $g^C_{\restriction I}$ is a coordinate permutation of g, $g^C_{\restriction I'}$ is a coordinate permutation of g^C, and $g^C_{\restriction I''}$ is the identity. Then, by the definition of g^C, the element $g^C_{\restriction I \cup I'}$ is a coordinate permutation of $(g^C)^{-1}$, and $g^C_{\restriction I' \cup I''}$ is a coordinate permutation of g^C. Fix a permutation σ of $\mathbb{N}^N$ such that $\sigma[I''] = I$, $\sigma[I \cup I'] = I' \cup I''$, and $(g^C_\sigma)_{\restriction I' \cup I''} = (g^C)^{-1}_{\restriction I' \cup I''}$, where g^C_σ is a coordinate permutation of g^C induced by σ. Then, by our assumption, $g^C_\sigma \in A_m^{1+l'}$ for some l', $g^C g^C_\sigma$ is a coordinate permutation of g, and $g^C g^C_\sigma \in A_m^{2+l'}$, so, by our assumption again, $g \in A_m^{2+l'+l''}$ for some l''. Here, l', l'' are independent of the choice of C and g.

Since any $g \in G$ is a product of two elements as above, $g \in A_m^k$, if $k \geqslant 2(2 + l' + l'')$ and $\mathrm{range}(g) \subseteq C$. This finishes the proof of the claim.

Put

$$B_m = \left\{C \in \left[\bigcup_n G_n\right]^\omega : g^C \in A_m\right\}.$$

Clearly, $\bigcup_m B_m = [\bigcup_n G_n]^\omega$. But this means that there exists m such that $B_m = B$. Otherwise, there is some $C_m \notin B_m$ for every m. Since families B_m are closed under taking subsets, we have that $\bigcup_m C_m \notin B_n$ for every n, and $\bigcup_m B_m \neq B$, which is a contradiction. By the claim, $G = A_m^k$ for some k.

LEMMA 12. *Let $N \in \{0, 1, \ldots, \mathbb{N}\}$, and let G_n, $n \in N$, be permutation groups. Then $G = \prod_{n=0}^N (G_n Wr\, \mathrm{Sym}(\mathbb{N}))$ has uncountable strong cofinality.*

Proof. Every element of $G = \prod_{n=0}^N (G_n Wr\, \mathrm{Sym}(\mathbb{N}))$ is of the form

$$(g_0 s_0, g_1 s_1, \ldots),$$

where $g_n \in G_n^{\mathbb{N}}$, $s_n \in \mathrm{Sym}(\mathbb{N})$, $n \in N$, so we can write it as

$$(g_0, g_1, \ldots)(s_0, s_1, \ldots),$$

where $(g_0, g_1, \ldots) \in \prod_{n=0}^N G_n^{\mathbb{N}}$, $(s_0, s_1, \ldots) \in (\mathrm{Sym}(\mathbb{N}))^N$.

Suppose that $G = \bigcup_m A_m$, where $A_0 \subseteq A_1 \subseteq \cdots$, and put $H = \prod_{n=0}^N G_n^{\mathbb{N}}$, $B_m = H \cap A_m$. By [**4**, lemma 3·5], the group $(\mathrm{Sym}(\mathbb{N}))^N$ has uncountable strong cofinality, that is, there exist k, m such that $(\mathrm{Sym}(\mathbb{N}))^N \subseteq A_m^k$. Obviously, without loss of generality we can assume that $m = 0$. Observe that the natural action of $(\mathrm{Sym}(\mathbb{N}))^N$ on H by conjugation gives rise to all possible permutations of coordinates of elements of H. Therefore, Lemma 11 implies that there exist k, m such that $H \subseteq B_m^k$, and G has uncountable strong cofinality.

THEOREM 13. *Let T be a countable rooted tree with non-trivial components. If* $\mathrm{ACL}_T(\varnothing)$ *is finite, then* $\mathrm{Aut}(T)$ *has uncountable strong cofinality. Otherwise, it has countable cofinality and does not have the small index property.*

Proof. Put $G = \mathrm{Aut}(T) = Wr_{t\in\overline{T}}\, G_t$. Observe that $S = \mathrm{ACL}_T(\varnothing)$ is an invariant subtree of T, so $\overline{S} = \overline{T} \cap S$ is a fundamental domain of S. Since every component G_s for $s \in \overline{S}$ is finite, the tree S is finite if and only if $\overline{S}$ is finite.

Suppose that $\overline{S}$ is finite, and fix $s \in \overline{S}$. Let $\{t_n\}$ be an enumeration of all the predecessors of s in $\overline{T} \setminus \overline{S}$, that is, $G_{t_n} = \mathrm{Sym}(\mathbb{N})$. Then, for every t_n, we have

$$Wr_{t\in\overline{T}_{t_n}}\, G_t = H_n\, Wr\, \mathrm{Sym}(\mathbb{N}),$$

for some permutation group H_n. Let $S_s = \{s\} \cup \overline{T}_{t_0} \cup \overline{T}_{t_1} \cup \cdots$. By Lemma 7,

$$Wr_{t\in S_s}\, G_t = \Big(\prod_n (H_n\, Wr\, \mathrm{Sym}(\mathbb{N}))\Big)\, Wr\, G_s.$$

We add a new element s' to $\overline{S}$, which is a predecessor of s, and put

$$G_{s'} = \prod_n (H_n\, Wr\, \mathrm{Sym}(\mathbb{N})).$$

Then, for $S' = \overline{S} \cup \{s' : s \in \overline{S}\}$, the group $Wr_{s\in S'}\, G_s$ is isomorphic to G. By Lemma 12, each $G_{s'}$ has uncountable strong cofinality. Since each G_s, $s \in \overline{S}$, is finite, and S' is finite, by Lemma 8, $Wr_{s\in S'}\, G_s$, and so G, has uncountable strong cofinality.

If $\overline{S}$ is infinite, then, by König's lemma, $\overline{S}$ contains an infinite branch, or some element of $\overline{S}$ has infinitely many predecessors; in any case, $\overline{S}$ and thus $\overline{T}$ contains a subtree S_0 as in the statement of Lemma 10. Therefore, G has countable cofinality and does not have the small index property.

4. *Ample generics*

THEOREM 14. *Let T be a countable rooted tree with non-trivial components. For $G =$* $\mathrm{Aut}(T)$ *the following conditions are equivalent*:

(1) $\mathrm{ACL}_T(X)$ *is finite for every finite* $X \subseteq T$;
(2) *G contains an open subgroup H with ample generics*;
(3) *G has the small index property.*

Proof. We show (1) $\Rightarrow$ (2). Suppose that $\mathrm{ACL}_T(X)$ is finite for every finite $X \subseteq T$, and let $X_0 = \mathrm{ACL}_T(\varnothing)$. Observe that the stabilizer $G_{\langle X_0\rangle}$ of X_0 is open in G, and $G_{\langle X_0\rangle}$ can be regarded as the automorphism group of an ultrahomogeneous, locally finite structure T' obtained from $(T, s, \{\mathcal{O}_t\}_{t\in T})$ by adding names to T for every $x \in X_0$.

We show that for $\mathcal{K} = \mathrm{Age}(T')$, the classes $\mathcal{K}^n_p$, $n \in \mathbb{N}$, satisfy WAP and JEP. By Theorem 2, this will prove that $G_{\langle X_0\rangle}$ has ample generics.

Fix $n \in \mathbb{N}$ and $\mathcal{S} \in \mathcal{K}^n_p$ of the form

$$\mathcal{S} = \langle A, \phi_0 : B_0 \longrightarrow C_0, \ldots, \phi_n : B_n \longrightarrow C_n\rangle.$$

By Lemma 5, there exists a finite subtree A'' such that $A \subseteq A''$ and A'' satisfies the assumptions of Corollary 4. Put $A' = \mathrm{ACL}_T(A'')$ and observe that A' also satisfies the assumptions of Corollary 4. Therefore, we can extend each ϕ_i to an automorphism $\phi'_i : A' \to A'$.

Let $\mathcal{T} \in \mathcal{K}_p^n$ be defined by

$$\mathcal{T} = \left\langle A', \phi_0' : A' \longrightarrow A', \dots, \phi_n' : A' \longrightarrow A' \right\rangle.$$

Then, by our assumption, for every predecessor $t \in T \setminus A'$ of an element $a \in A'$ the set $\mathcal{N}_t$ is infinite. Therefore, if

$$\mathcal{F} = \langle H, \chi_0 : M_0 \longrightarrow N_0, \dots, \chi_n : M_n \longrightarrow N_n \rangle,$$

$$\mathcal{G} = \langle P, \xi_0 : Q_0 \longrightarrow R_0, \dots, \xi_n : Q_n \longrightarrow R_n \rangle,$$

$\mathcal{F}, \mathcal{G} \in \mathcal{K}_p^n$, and $i : \mathcal{T} \to \mathcal{F}$, $j : \mathcal{T} \to \mathcal{G}$ are embeddings, then we can assume without loss of generality that $H \cap P = A'$, and $(\chi_i)_{\restriction A'} = (\xi_i)_{\restriction A'}$, $i \leqslant n$. It is a little tedious but completely straightforward to check that in this case the structure $\mathcal{E} \in \mathcal{K}_n^p$ defined by

$$\mathcal{E} = \left\langle H \cup P, \chi_0 \cup \xi_0, \dots, \chi_n \cup \xi_n \right\rangle$$

along with natural embeddings k, l of $\mathcal{F}, \mathcal{G}$ into $\mathcal{E}$ amalgamates $\mathcal{F}$ and $\mathcal{G}$ over $\mathcal{T}$.

To show JEP, observe that every embedding $f : A \to B$ between finite substructures of T' fixes all elements in X_0, so we can repeat the above argument.

The implication (2) $\Rightarrow$ (3) follows from [**10**, theorem 6·9], which says that the existence of ample generics implies the small index property. Now observe that since $[G_{\langle X_0 \rangle} : G] \leqslant \aleph_0$, the group G also has the small index property.

Finally, we show $\neg$(1) $\Rightarrow$ $\neg$(3). Let $X_1 \subseteq T$ be a finite set such that $\mathrm{ACL}_T(X_1)$ is infinite. It is not hard to see that $G_{\langle X_1 \rangle} \cong Wr_{s \in S}\, G_s$ for some countable rooted tree S and symmetric groups G_s, $s \in S$. Then we proceed as in the proof of Theorem 13.

By [**10**, theorems 6·24 and 6·25], we get

COROLLARY 15. *Let T be a countable rooted tree with non-trivial components, $G =$ Aut(T). If $\mathrm{ACL}_T(X)$ is finite for every finite $X \subseteq T$, then*:

(1) *every homomorphism from G into a separable topological group H is continuous;*

(2) *the standard product topology is the unique Polish topology on G.*

5. *Rigidity of trees*

Recall that a tree is a connected graph with no cycles. In [**1**, theorem 4·4], Bass and Lubotzky proved a rigidity theorem to the extent that a sufficiently rich group of automorphisms of a locally finite tree T completely determines T. That is, if G is a group of automorphisms of locally finite trees T_1, T_2 that satisfies some additional assumptions, we will not dwell into, then T_1 is isomorphic to T_2.

As the authors pointed out, the condition of being locally finite is rather restrictive. This limitation was removed, applying two different approaches, by Psaltis ([**11**]) and Forester ([**6**]), however not without some trade-ins. Psaltis managed to get rid of the assumption of local finitiness of T, but had to restrict himself to full automorphism groups. His main result [**11**, theorem 6·9a,b,c] is

THEOREM 16 (Psaltis). *Let T be a tree with countable number of edges incident at each vertex, and $i_G(e) \geqslant 3$ for each edge e. Then* Aut(T) *completely determines T.*

Here $i_G(e)$ denotes $[G_t : G_e]$, where G_e is the stabilizer of edge e in G, and G_t is the stabilizer of vertex t such that $e = (t, s)$ for some $s \in T$.

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On the other hand, Forester's results concern also subgroups of the full automorphism groups, but with more additional assumptions present; in particular, they involve Serre's property (FA). Recall that G has property (FA) if every action of G on a tree without inversions has a fixed point.

THEOREM 17 (Forester). *Let G be a group acting on trees T_1, T_2 without inversions. Let T_1 be a strongly slide-free, and T_2 a proper tree, both cocompact. Suppose that all vertex stabilizers are unsplittable. If either*

(a) *one of the trees has (FA) vertex stabilizers, or*
(b) *one of the trees is locally finite,*

then there is a unique isomorphism of G-trees T_1 and T_2.

We will not define all the technical notions involved in the statement of this theorem. Suffices to say that if $i_G(e) \geqslant 3$ for every edge e in T, then T is strongly slide-free and proper, and the stabilizers of vertices in $\mathrm{Aut}(T)$ are known to be unsplittable. Cocompactness means that there are only finitely many orbits of the action of the stabilizer of t on the set of all children of t, $t \in T$, so it puts extra restrictions on T_1, T_2, compared to Psaltis' theorem. However, one can ask whether the assumption on sharing property (FA) by all stabilizers can be removed if G is the full automorphism group (or, when it is satisfied.) Because uncountable strong cofinality clearly implies property (FA), Theorem 14 shows that this happens only in very special situations, when $i(e) = \aleph_0$ for 'most' edges e in T. It turns out that in this case stabilizers of vertices of T contain an open subgroup with ample generics.

LEMMA 18. *Let T be a countable tree. If $\mathrm{ACL}_T(\{t\})$ is finite for every $t \in T$, then $\mathrm{ACL}_T(X)$ is finite for every non-empty finite subtree $X \subseteq T$.*

Proof. Let $G = \mathrm{Aut}(T)$, and let $X = \{x_0, \ldots, x_n\} \subseteq T$ be a finite, non-empty subtree, that is, a connected subset of T. We fix $r \in X$, and regard T as a rooted tree with root r. For $i \leqslant n$, let S_i be a rooted tree with root x_i defined as S_s in Theorem 13, that is, if $t_0, \ldots, t_m$ is an enumeration of all the predecessors t of x_i such that $t \notin X$, then

$$S_i = \{x_i\} \cup T_{t_0} \cup \cdots \cup T_{t_m}.$$

We show that if there is an infinite orbit under the action of some $G_{\langle x_i \rangle}$ on T, then there is in an infinite orbit of the action of $G_{\langle X \rangle}$ on T. This will finish the proof.

Suppose that for some $i_0 \leqslant n$ there exists an infinite orbit $\mathcal{O}$ under the action of $G_{\langle x_{i_0} \rangle}$ on T. By finiteness of X, there must exist $i_1 \leqslant n$ such that $S_{i_1} \cap \mathcal{O}$ is infinite. Then it is easy to see that there exists an infinite orbit under the action of $\mathrm{Aut}(S_{i_1})$ on S_{i_1}, and this implies that there exists an infinite orbit under the action of $G_{\langle X \rangle}$ on T.

THEOREM 19. *Let T be a countable tree such that fixing any vertex of T results in a rooted tree with non-trivial components, and let $G = \mathrm{Aut}(T)$. The following conditions are equivalent*:

(1) $\mathrm{ACL}_T(X)$ *is finite for every finite non-empty subtree $X \subseteq T$*;
(2) *The stabilizer $G_{\langle t \rangle}$ of every vertex $t \in T$ has property (FA)*;
(3) *the stabilizer $G_{\langle t \rangle}$ of every vertex $t \in T$ contains an open subgroup with ample generics.*

Proof. In view of Theorems 13 and 14, implications (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) are obvious. We show (2) $\Rightarrow$ (1). Since every $G_{\langle t \rangle}$ has property (FA), by [**12**, theorem 15] and remarks following it, no $G_{\langle t\rangle}$ has countable cofinality, that is, by Theorem 13, $\mathrm{ACL}_T(\{t\})$ is finite for every $t \in T$. By Lemma 18, $\mathrm{ACL}_T(X)$ is finite for every finite non-empty $X \subseteq T$.

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