REGULARITY AT SPACE-LIKE AND NULL INFINITY

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Abstract   We extend Penrose's peeling model for the asymptotic behaviour of solutions to the scalar wave equation at null infinity on asymptotically flat backgrounds, which is well understood for flat space-time, to Schwarzschild and the asymptotically simple space-times of Corvino–Schoen/Chrusciel–Delay. We combine conformal techniques and vector field methods: a naive adaptation of the 'Morawetz vector field' to a conformal rescaling of the Schwarzschild metric yields a complete scattering theory on Corvino–Schoen/Chrusciel–Delay space-times. A good classification of solutions that peel arises from the use of a null vector field that is transverse to null infinity to raise the regularity in the estimates. We obtain a new characterization of solutions admitting a peeling at a given order that is valid for both Schwarzschild and Minkowski space-times. On flat space-time, this allows larger classes of solutions than the characterizations used since Penrose's work. Our results establish the validity of the peeling model at all orders for the scalar wave equation on the Schwarzschild metric and on the corresponding Corvino–Schoen/Chrusciel–Delay space-times.

Keywords: conformal compactification; Schwarzschild space-time; wave equation; peeling;
             vector field methods; weighted estimates

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Contents

1. Introduction

Penrose's null infinity, $\mathscr{I}$, of a Lorentzian space-time is a powerful tool for studying the asymptotics of massless fields in both flat and curved space-times. The asymptotic series of the physical field in the physical space-time translates, in an unphysical conformally rescaled space-time, into the Taylor series of the field off the finite hypersurface $\mathscr{I}$. This can also be used as a basis for reformulating the scattering theory of massless fields into a Goursat problem based on $\mathscr{I}$ (see [**1, 8, 9, 15, 17**]). However, the use of $\mathscr{I}$ in curved space-times has for a long time been controversial since firstly, it has not been clear that there is a good generic class of space-times with smooth null infinities and secondly, even if so, it is still not known whether interesting massless fields do in fact fall off as proposed in the formalism so as to be amenable to analysis. The former problem has now been resolved in various ways: by Christodoulou and Klainerman [**2**], Corvino [**5**], Chrusciel and Delay [**3, 4**], Corvino and Schoen [**6**], Friedrich [**10**] (which contains a survey of Friedrich's contributions) and Klainerman and Nicolò [**12–14**]. As far as the second issue is concerned, schemes in which fields admit all kinds of singularities at null infinity have been proposed and shown to be consistent, at least in the neighbourhood of $\mathscr{I}$. Such results have often been understood as an indication that the peeling model (introduced in Penrose's 1965 founding paper [**18**]) on non-trivial asymptotically flat space-times may apply to more restrictive sets of data than in the flat case. To this day, even in the relatively simple case of the Schwarzschild metric, there is no precise understanding of the type of initial data that will guarantee the regularity of solutions on null infinity.

Minkowski space can be conformally embedded into the Einstein cylinder, $\mathbb{R} \times S^3$ and compactified by adding $\mathscr{I}^-$ (past null infinity), $\mathscr{I}^+$ (future null infinity), which are respectively the past and future light cones of a point at space-like infinity (denoted i^0) in the conformal compactification and which refocus to the vertices i^- and i^+ (past and future timelike infinities). The points $i^\pm$ and i^0 are smooth points of the conformal structure. In this picture, it is easy to see that data for massless fields on a Cauchy surface through i^0 will propagate smoothly up the Einstein cylinder indefinitely if the data is smooth over the Cauchy surface including i^0 in the conformally rescaled space-time. This

shows that there is a good class of massless fields that are smooth at $\mathscr{I}$ in Minkowski space, consisting of fields whose conformal rescaling admits a smooth extension across i^0. Solutions that are regular on Minkowski space but with arbitrarily bad behaviour at $\mathscr{I}$ can be understood in this picture as arising from data that is singular at i^0 after conformal rescaling.

A non-trivial asymptotically simple curved space-time has almost the same global structure with smooth conformal boundary $\mathscr{I}^{\pm}$ and smooth vertices $i^{\pm}$ but there is necessarily a singularity at i^0. Black hole space-times such as Schwarzschild or Kerr have a more complicated conformal geometry, with singular $i^{\pm}$, but in the neighbourhood of i^0, they are similar to asymptotically simple space-times. The singularity at i^0 is associated with the ADM mass and so cannot be removed in a physical context except in the case of flat space-time. Thus the above conformal argument for the existence of a large class of solutions that are smooth at $\mathscr{I}$ needs to be modified: using naive methods, one can only guarantee that solutions are smooth on $\mathscr{I}$ if they are compactly supported away from i^0. The general understanding in such situations is that if solutions only fall off at some finite rate near i^0 on spacelike slices, the rate of fall-off at spacelike infinity will determine the regularity of the solution at $\mathscr{I}$. However a precise quantitative description of this relation has not previously been available and very different opinions have been expressed as to what it should be.

This paper provides this description completely for scalar fields on Schwarzschild space-time. This immediately extends to the corresponding Corvino–Schoen/Chrusciel–Delay space-times (asymptotically simple space-times that are diffeomorphic to Schwarzschild outside a compact set). The essential tool is energy estimates, also referred to as vector field methods, with the additional feature that we apply these techniques on a conformally rescaled space-time and not on the physical space-time. We choose carefully a vector field to contract with the stress–energy tensor: the associated energy must be positive on spacelike slices and on $\mathscr{I}$, and the asymptotic behaviour of the vector field near $\mathscr{I}$ and i^0 has to be chosen so as to keep a uniform control on the estimates in these regions. The 'Morawetz vector field', introduced by Morawetz in the early 1960s (see [16]), is a well-known tool for obtaining pointwise decay estimates in flat space-time. It can be easily adapted to the Schwarzschild case and has indeed already been used in this setting to obtain pointwise decay rates for spherically symmetric equations by Dafermos and Rodnianski [7]. The Schwarzschild version of the Morawetz vector field allows us to obtain basic energy estimates between $\mathscr{I}$ and some initial data surface. Higher-order estimates are then deduced by commuting into the equation a null vector field transverse to $\mathscr{I}$, and not the Morawetz vector field itself, contrary to what can be done for flat space-time. This provides a rigorous definition of the peeling as well as the precise function spaces of initial data giving rise to solutions that peel at any given order. We show that these function spaces are optimal for our definition. Comparing them to the usual peeling picture on Minkowski space-time, it turns out that our classes of data giving rise to a given regularity of the solution on $\mathscr{I}$ are larger than what was previously known for flat space-time. Our results therefore validate the peeling model for the wave equation on the Schwarzschild metric at all orders: they also provide a new definition of the peeling

that is more precise than the definitions used sofar. A remarkable feature of this new definition is that, for two different values of the mass m of the black hole, the spaces of initial data for a given regularity are canonically isomorphic. Moreover, the equivalence in the norms is uniform on any compact interval in m, typically $m \in [0, M]$, $M > 0$, which includes Minkowski space.

The paper is organized as follows. Section 2 describes the peeling on Minkowski space as it is usually understood, then gives an alternative description in terms of vector field methods. Section 3 contains the basic ingredients that will be used in the rest of the paper: a foliation of a neighbourhood of i^0 in Schwarzschild's space-time is given, a careful choice of identifying vector field is made for this foliation (this will be crucial for higher-order estimates), the Morawetz vector field is introduced and explicit formulae are obtained for energy densities and error terms. The fundamental energy estimates are derived in §4 and it is remarked that they entail a conformal scattering theory for the wave equation on Corvino–Schoen/Chrusciel–Delay space-times. Higher-order estimates are then obtained in §5, giving a complete classification of the spaces of data that give rise to solutions that peel at any order; these spaces are expressed in Definition 5.5, which concludes the section. The results are interpreted in §6. The appendix contains the proofs of the theorems.

Important remarks

(1) All our results are focused on a neighbourhood of spacelike infinity i^0, where the difficulty is localized. Our definition of the peeling is therefore concerned with the regularity on null infinity near i^0. Global regularity on $\mathscr{I}$ can then easily be recovered by assuming in addition that the data are in local Sobolev spaces on a Cauchy hypersurface.

(2) We work with the wave equation in this paper since some pathological schemes have been put forward in this case. We would expect the situation to be simpler for higher helicity massless fields. This will be the subject of a subsequent study.

(3) Throughout the paper, the energy estimates are performed for solutions associated with smooth compactly supported data; a completion of the space of smooth compactly supported functions in the norms given by these estimates then give the function spaces that characterize data giving rise to solutions that peel at a given order.

(4) For more details on the conformal compactifications of the Schwarzschild and Minkowski space-times that we use, see [19].

Notation

(1) We shall use the notation $\lesssim$ to signify that the left-hand side is bounded by a positive constant times the right-hand side, the constant being independent of the parameters and functions appearing in the inequality.

(2) Given $\mathcal{M}$ a smooth manifold and $\mathcal{O}$ an open set of $\mathcal{M}$, we denote by $\mathcal{C}_0^\infty(\mathcal{O})$ the space of smooth functions with compact support on $\mathcal{O}$, and by $\mathcal{D}'(\mathcal{O})$ its topological dual, the space of distributions on $\mathcal{O}$.

2. Peeling on flat space-time

We denote by $\mathbb{M}$ Minkowski space and by η the Minkowski metric. The peeling properties of solutions to field equations on Minkowski space can be easily understood as a consequence of the embedding of $\mathbb{M}$ into the Einstein cylinder. We first recall the conformal compactification of Minkowski space that realizes this embedding, then we proceed with the usual description of the peeling in this framework. Finally, we propose an alternative description based on vector field methods.

2.1. Conformal compactification of Minkowski space

We choose the advanced and retarded coordinates

$$u = t - r, \qquad v = t + r,$$

then we put

$$p = \arctan u, \qquad q = \arctan v,$$
$$\tau = p + q = \arctan(t - r) + \arctan(t + r),$$
$$\zeta = q - p = \arctan(t + r) - \arctan(t - r).$$

Choosing the conformal factor

$$\Omega = \frac{2}{\sqrt{1 + u^2}\sqrt{1 + v^2}}, \tag{2.1}$$

we obtain the rescaled metric

$$\mathfrak{e} = \Omega^2 \eta = \mathrm{d}\tau^2 - \mathrm{d}\zeta^2 - \frac{(v - u)^2}{(1 + u^2)(1 + v^2)}\, \mathrm{d}\omega^2 = \mathrm{d}\tau^2 - \mathrm{d}\zeta^2 - \sin^2 \zeta\, \mathrm{d}\omega^2 \tag{2.2}$$

and Minkowski space is now described by the domain

$$\mathbb{M} = \{|\tau| + \zeta \leqslant \pi,\ \zeta \geqslant 0,\ \omega \in S^2\}.$$

The metric $\mathfrak{e}$ is the Einstein metric $\mathrm{d}\tau^2 - \sigma_{S^3}^2$, where $\sigma_{S^3}^2$ is the Euclidean metric on the 3-sphere, and it extends analytically to the whole Einstein cylinder $\mathfrak{E} = \mathbb{R}_\tau \times S^3_{\zeta,\theta,\varphi}$. The full conformal boundary of Minkowski space can be defined in this framework. Several parts can be distinguished.

- Future and past null infinities:

$$\mathscr{I}^+ = \{(\tau, \zeta, \omega);\ \tau + \zeta = \pi,\ \zeta \in\,]0, \pi[,\ \omega \in S^2\},$$
$$\mathscr{I}^- = \{(\tau, \zeta, \omega);\ \zeta - \tau = \pi,\ \zeta \in\,]0, \pi[,\ \omega \in S^2\}.$$

  They are smooth null hypersurfaces for $\mathfrak{e}$.

- Future and past timelike infinities:

$$i^\pm = \{(\tau = \pm\pi,\ \zeta = 0,\ \omega);\ \omega \in S^2\}.$$

  They are smooth points for $\mathfrak{e}$.

- Spacelike infinity:

$$i^0 = \{(\tau = 0, \ \zeta = \pi, \ \omega); \ \omega \in S^2\}.$$

It is also a smooth point for $\mathfrak{e}$.

The hypersurface $\{t = 0\}$ in Minkowski space is described by the 3-sphere $\{\tau = 0\}$ minus the point i^0 on the Einstein cylinder.

The scalar curvature of $\mathfrak{e}$ can be calculated easily using the conformal factor Ω:

$$\tfrac{1}{6}\mathrm{Scal}_{\mathfrak{e}} = \Omega^{-3}\Box_\eta \Omega = 1. \tag{2.3}$$

In this framework, the vector field ∂_τ is Killing since the metric $\mathfrak{e}$ does not depend on τ.

2.2. Conformal rescaling of the wave equation

The conformal invariance of the wave equation entails the equivalence of the two properties:

(i) $\tilde{\psi} \in \mathcal{D}'(\mathbb{M})$ satisfies

$$\Box_\eta \tilde{\psi} = 0, \quad \Box_\eta = \partial_t^2 - \frac{1}{r^2}\partial_r r^2 \partial_r - \frac{1}{r^2}\Delta_{S^2}. \tag{2.4}$$

(ii) $\psi := \Omega^{-1}\tilde{\psi}$ satisfies

$$\Box_{\mathfrak{e}}\psi + \psi = 0, \quad \Box_{\mathfrak{e}} = \partial_\tau^2 - \Delta_{S^3}. \tag{2.5}$$

2.3. The usual description of peeling

The observation of the peeling in Minkowski space is usually derived from the property that the Cauchy problem on the Einstein cylinder is well-posed in the space of $\mathcal{C}^\infty$ functions. This is a completely standard result, we just quote it here.

Proposition 2.1. *For any initial data $\psi_0, \psi_1 \in \mathcal{C}^\infty(S^3)$, there exists a unique solution $\psi \in \mathcal{C}^\infty(\mathfrak{E})$ of (2.5) such that $\psi(0) = \psi_0$ and $\partial_\tau \psi(0) = \psi_1$.*

This provides a natural definition of solutions that peel at all orders.

Definition 2.2. A solution $\tilde{\psi}$ of (2.4) is said to peel at all orders if $\psi = \Omega^{-1}\tilde{\psi}$ extends as a $\mathcal{C}^\infty$ function on the whole Einstein cylinder. The latter property is satisfied by solutions ψ of (2.5) arising from initial data $\psi|_{\tau=0} \in \mathcal{C}^\infty(S^3)$ and $\partial_\tau \psi|_{\tau=0} \in \mathcal{C}^\infty(S^3)$. Going back to Minkowski space and to the physical field $\tilde{\psi}$, this gives us a corresponding class of data for (2.4), giving rise to solutions that peel at all orders.

2.4. Description by means of vector field methods

Although it is not commonly used, we can give a description of the peeling in Minkowski space in terms of Sobolev spaces. This has the advantage of allowing a precise description

at all orders of regularity, which is trickier in $\mathcal{C}^k$ spaces that are less well controlled by the evolution. To do so, we write Equation (2.5) in its Hamiltonian form,

$$\frac{\partial}{\partial \tau} \begin{pmatrix} \psi \\ \partial_\tau \psi \end{pmatrix} = \mathrm{i} H \begin{pmatrix} \psi \\ \partial_\tau \psi \end{pmatrix}, \qquad H = -\mathrm{i} \begin{pmatrix} 0 & 1 \\ \Delta_{S^3} - 1 & 0 \end{pmatrix},$$

and work on the Hilbert space

$$\mathcal{H} = H^1(S^3) \times L^2(S^3)$$

with the usual inner product

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_{S^3} (\nabla_{S^3} f_1 \cdot \nabla_{S^3} \bar{g}_1 + f_1 \bar{g}_1 + f_2 \bar{g}_2) \, \mathrm{d}\mu_{S^3},$$

where ∇_{S^3} is the Levi-Civita connection and μ_{S^3} the measure induced by the Euclidean metric on S^3. We have the following proposition.

Proposition 2.3. *The operator H with its natural domain $D(H) = H^2(S^3) \times H^1(S^3)$ is self-adjoint on $\mathcal{H}$ and its successive domains are*

$$D(H^k) = H^{k+1}(S^3) \times H^k(S^3).$$

Let $k \in \mathbb{N}$. For any initial data $\psi_0 \in H^{k+1}(S^3)$, $\psi_1 \in H^k(S^3)$, there exists a unique solution

$$\psi \in \bigcap_{l=0}^{k+1} \mathcal{C}^l(\mathbb{R}_\tau; H^{k+1-l}(S^3))$$

of (2.5) such that $\psi(0) = \psi_0$ and $\partial_\tau \psi(0) = \psi_1$. In particular, $\psi \in H^{k+1}_{\mathrm{loc}}(\mathfrak{E})$. Moreover, for any $0 \leqslant l \leqslant k$, $\|\psi(\tau)\|^2_{H^{l+1}(S^3)} + \|\partial_\tau \psi(\tau)\|^2_{H^l(S^3)}$ is constant throughout time.

This gives us a definition of solutions that peel at a given order $k \in \mathbb{N}$ in terms of Sobolev spaces.

Definition 2.4. A solution $\tilde{\psi}$ of (2.4) is said to peel at order $k \in \mathbb{N}$ if $\psi = \Omega^{-1}\tilde{\psi}$ extends as a function that is in H^{k+1}_{loc} on the whole Einstein cylinder. The latter property is satisfied by solutions ψ of (2.5) arising from initial data $\psi|_{\tau=0} \in H^{k+1}(S^3)$ and $\partial_\tau \psi|_{\tau=0} \in H^k(S^3)$. Going back to Minkowski space and to the physical field $\tilde{\psi}$, this gives us a corresponding class of data for (2.4), giving rise to solutions that peel at order k.

What is essentially unsatisfactory in Definition 2.4 is that it merely provides an inclusion: we know a class of data that gives rise to peeling at order k, but we do not know whether it is the largest possible class. An alternative approach consists in using vector field methods (energy estimates). Such techniques allow to prove easily the last property in Proposition 2.3 but are much more flexible than a purely spectral result: we can just as naturally obtain estimates between the initial data surface and $\mathscr{I}^+$. This will provide us with a third description of the peeling on flat space-time. It will be more precise than

the first two in that the optimal set of suitable data for a peeling at order k will be completely characterized.

We consider the stress–energy tensor for Equation (2.5),

$$T_{ab} = T_{(ab)} = \partial_a\psi\partial_b\psi - \tfrac{1}{2}\mathfrak{e}_{ab}\mathfrak{e}^{cd}\partial_c\psi\partial_d\psi + \tfrac{1}{2}\psi^2\mathfrak{e}_{ab}, \tag{2.6}$$

and contract it with the Killing vector field ∂_τ. This yields the conservation law

$$\nabla^a(K^b T_{ab}) = 0. \tag{2.7}$$

The energy 3-form $K^a T_{ab}\,\mathrm{d}^3 x^b = K^a T_a^b \partial_b \lrcorner \mathrm{dvol}^4$ has the expression

$$K^a T_{ab}\,\mathrm{d}^3 x^b = \psi_\tau \nabla\psi\lrcorner\mathrm{dvol}^4 + \tfrac{1}{2}(-\psi_\tau^2 + |\nabla_{S^3}\psi|^2 + \psi^2)\partial_\tau\lrcorner\mathrm{dvol}^4. \tag{2.8}$$

Integrating (2.8) on an oriented hypersurface S defines the energy flux across this surface, denoted $\mathcal{E}_S(\psi)$. For instance, denoting $X_\tau = \{\tau\} \times S^3$ the level hypersurfaces of the function τ

$$\mathcal{E}_{X_\tau}(\psi) = \frac{1}{2}\int_{X_\tau}(\psi_\tau^2 + |\nabla_{S^3}\psi|^2 + \psi^2)\,\mathrm{d}\mu_{S^3},$$

and parametrizing $\mathscr{I}^+$ as $\tau = \pi - \zeta$,

$$\mathcal{E}_{\mathscr{I}^+}(\psi) = \frac{1}{\sqrt{2}}\int_{\mathscr{I}^+}(-2\psi_\tau\psi_\zeta + \psi_\tau^2 + |\nabla_{S^3}\psi|^2 + \psi^2)\,\mathrm{d}\mu_{S^3}$$

$$= \frac{1}{\sqrt{2}}\int_{\mathscr{I}^+}\left(|\psi_\tau - \psi_\zeta|^2 + \frac{1}{\sin^2\zeta}|\nabla_{S^2}\psi|^2 + \psi^2\right)\mathrm{d}\mu_{S^3}.$$

This is a natural H^1 norm of ψ on $\mathscr{I}^+$, involving only the tangential derivatives of ψ along $\mathscr{I}^+$.

Now consider a smooth solution ψ of (2.5). The conservation law (2.7) tells us that (2.8) is closed, hence, integrating it on the closed hypersurface made of the union of X_0 and $\mathscr{I}^+$, we obtain

$$\mathcal{E}_{\mathscr{I}^+}(\psi) = \mathcal{E}_{X_0}(\psi)$$

and since ∂_τ is a Killing vector, for any $k \in \mathbb{N}$, $\partial_\tau^k\psi$ satisfies equation (2.5), whence

$$\mathcal{E}_{\mathscr{I}^+}(\partial_\tau^k\psi) = \mathcal{E}_{X_0}(\partial_\tau^k\psi).$$

Using Equation (2.5), for $k = 2p$, $p \in \mathbb{N}$, we have

$$\mathcal{E}_{X_0}(\partial_\tau^k\psi) = \|\partial_\tau^{2p}\psi\|_{H^1(X_0)}^2 + \|\partial_\tau^{2p+1}\psi\|_{L^2(X_0)}^2$$

$$= \|(1 - \Delta_{S^3})^p\psi\|_{H^1(X_0)}^2 + \|(1 - \Delta_{S^3})^p\partial_\tau\psi\|_{L^2(X_0)}^2$$

$$\simeq \|\psi\|_{H^{2p+1}(X_0)}^2 + \|\partial_\tau\psi\|_{H^{2p}(X_0)}^2, \tag{2.9}$$

and for $k = 2p + 1$, $p \in \mathbb{N}$,

$$\mathcal{E}_{X_0}(\partial_\tau^k\psi) = \|\partial_\tau^{2p+1}\psi\|_{H^1(X_0)}^2 + \|\partial_\tau^{2p+2}\psi\|_{L^2(X_0)}^2$$

$$= \|(1 - \Delta_{S^3})^p\partial_\tau\psi\|_{H^1(X_0)}^2 + \|(1 - \Delta_{S^3})^{p+1}\psi\|_{L^2(X_0)}^2$$

$$\simeq \|\psi\|_{H^{2p+2}(X_0)}^2 + \|\partial_\tau\psi\|_{H^{2p+1}(X_0)}^2. \tag{2.10}$$

Hence, we have, for each $k \in \mathbb{N}$,

$$\|\psi\|^2_{H^{k+1}(X_0)} + \|\partial_\tau \psi\|^2_{H^k(X_0)} \simeq \mathcal{E}_{X_0}(\partial_\tau^k \psi) = \mathcal{E}_{\mathscr{I}^+}(\partial_\tau^k \psi) \simeq \|\partial_\tau^k \psi\|^2_{H^1(\mathscr{I}^+)},$$

and using the fact that the H^k norm controls all the lower Sobolev norms, this gives us the apparently stronger equivalence

$$\|\psi\|^2_{H^{k+1}(X_0)} + \|\partial_\tau \psi\|^2_{H^k(X_0)} \simeq \sum_{p=0}^k \|\partial_\tau^p \psi\|^2_{H^1(\mathscr{I}^+)}. \tag{2.11}$$

Remark 2.5. This equivalence should not in principle be understood as providing a solution to a Goursat problem on $\mathscr{I}^+$. Indeed, in Lars Hörmander's paper on the Goursat problem for the wave equation [**11**], it is made very clear that such an equivalence only provides us with a trace operator on $\mathscr{I}^+$ that is a partial isometry, it is then necessary to prove the surjectivity of this operator in order to solve the Goursat problem. However, we know from the same paper that the Goursat problem for Equation (2.5) with data $\psi|_{\mathscr{I}^+} \in H^1(\mathscr{I}^+)$ is well posed and gives rise to solutions $\psi \in \mathcal{C}^0(\mathbb{R}_\tau; H^1(S^3)) \cap \mathcal{C}^1(\mathbb{R}_\tau; L^2(S^3))$. Hence equivalence (2.11) indeed provides us with a regularity result for the Goursat problem: data on $\mathscr{I}^+$ for which the norm on the right-hand side is finite give rise to solutions that are in $\mathcal{C}^l(\mathbb{R}_\tau; H^{k+1-l}(S^3))$ for all $0 \leqslant l \leqslant k+1$.

This is however stronger than the information we are interested in. Equivalence (2.11) simply says that for smooth solutions, the control of the transverse regularity on $\mathscr{I}^+$ described by $\mathcal{E}_{\mathscr{I}^+}(\partial_\tau^p \psi)$, $0 \leqslant p \leqslant k$, is equivalent to that of the H^{k+1} norm of the restriction of ψ to X_0 and the H^k norm of the restriction of $\partial_\tau \psi$ to X_0. By a standard density argument, this shows that if we wish to guarantee, by means of some control on the initial data, that the restriction to $\mathscr{I}^+$ of $\partial_\tau^p \psi$, $0 \leqslant p \leqslant k$, is in $H^1(\mathscr{I}^+)$, the optimal condition to impose is that $\psi|_{\tau=0} \in H^{k+1}(X_0)$ and $\partial_\tau \psi|_{\tau=0} \in H^k(X_0)$.

This provides us with our third definition of peeling at order k and a characterization by a function space of initial data.

Definition 2.6. A solution $\tilde\psi$ of (2.4) is said to peel at order $k \in \mathbb{N}$ if the traces on $\mathscr{I}^+$ of $\partial_\tau^p \psi$, $0 \leqslant p \leqslant k$, are in $H^1(\mathscr{I}^+)$. The optimal function space of initial data giving rise to such property is defined by $\psi|_{\tau=0} \in H^{k+1}(S^3)$ and $\partial_\tau \psi|_{\tau=0} \in H^k(S^3)$.

Going back to Minkowski space and to the physical field $\tilde\psi$, this gives us the exact function space of data for (2.4), giving rise to solutions that peel at order k.

Remark 2.7. The description given in Definition 2.4 corresponds to the slightly weaker approach, via the equality

$$\mathcal{E}_{X_\tau}(\partial_\tau^k \psi) = \mathcal{E}_{X_0}(\partial_\tau^k \psi) \quad \forall \tau \in \mathbb{R},$$

which entails

$$\|\psi\|_{H^{k+1}(\Omega^+)} \lesssim \|\psi\|^2_{H^{k+1}(X_0)} + \|\partial_\tau \psi\|^2_{H^k(X_0)}, \tag{2.12}$$

where Ω^+ is the 4-volume in the future of X_0 and the past of $\mathscr{I}^+$. It is slightly weaker in the way we understand the transverse regularity at $\mathscr{I}^+$ (implicitly in terms of trace

theorems for Sobolev spaces), hence the fact that we have merely inequalities instead of equivalences. But the spaces of initial data for which regularity at a given order is guaranteed near $\mathscr{I}^+$ are the same.

3. Basic formulae

The Schwarzschild metric is

$$g = \left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 - r^2 \,\mathrm{d}\omega^2,$$

where $\mathrm{d}\omega^2$ is the Euclidean metric on the unit sphere S^2. The associated d'Alembertian is

$$\Box_g = \left(1 - \frac{2m}{r}\right)^{-1} \frac{\partial^2}{\partial t^2} - \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \left(1 - \frac{2m}{r}\right) \frac{\partial}{\partial r} - \frac{1}{r^2} \Delta_{S^2},$$

where Δ_{S^2} is the Laplacian on S^2 endowed with the Euclidean metric. Putting

$$R = 1/r, \quad u = t - r_*, \quad \text{with } r_* = r + 2m \log(r - 2m),$$

the metric can be transformed and conformally rescaled to give

$$\hat{g} = R^2 g = R^2 (1 - 2mR) \,\mathrm{d}u^2 - 2 \,\mathrm{d}u \,\mathrm{d}R - \mathrm{d}\omega^2,$$

and in this form of the metric, $\mathscr{I}^+$ is given by $R = 0$. We denote by G the function that with r associates r_*:

$$G : \,]2m + \infty[\longrightarrow \mathbb{R}, \qquad G(r) = r + 2m \log(r - 2m); \tag{3.1}$$

it is an analytic diffeomorphism between $]2m, +\infty[$ and $\mathbb{R}$. The scalar curvature of $\hat{g}$ is given by

$$\tfrac{1}{6}\mathrm{Scal}_{\hat{g}} = R^{-3}\Box_g R = 2mR.$$

The inverse metric is

$$\hat{g}^{-1} = -2\partial_u \partial_R - R^2(1 - 2mR)\partial_R^2 - \eth\bar{\eth},$$

which gives the d'Alembertian

$$\Box_{\hat{g}} = -2\partial_u \partial_R - \partial_R R^2 (1 - 2mR)\partial_R - \Delta_{S^2}.$$

The equation that we study is the conformally invariant wave equation

$$(\Box_{\hat{g}} + \tfrac{1}{6}\mathrm{Scal}_{\hat{g}})\phi = (\Box_{\hat{g}} + 2mR)\phi = 0. \tag{3.2}$$

It is such that, denoting by $\mathcal{B}_I$ (for block I) the exterior of the black hole in Schwarzschild's space-time, the two following properties are equivalent:

(1) $\tilde{\phi} \in \mathcal{D}'(\mathcal{B}_I)$ satisfies $\Box_g \tilde{\phi} = 0$;

(2) $\phi := R^{-1}\tilde{\phi} = r\tilde{\phi}$ satisfies (3.2) on $\mathcal{B}_I$.

3.1. Stress–energy tensor and conservation laws

We use the stress–energy tensor for the free wave equation $\Box_{\hat{g}}\phi = 0$:

$$T_{ab} = T_{(ab)} = \partial_a\phi\partial_b\phi - \tfrac{1}{2}\hat{g}_{ab}\hat{g}^{cd}\partial_c\phi\partial_d\phi \qquad (3.3)$$

(although we could equivalently use the one for the Klein–Gordon equation $\Box_{\hat{g}}\phi + \phi = 0$, obtained by adding to T_{ab} the term $\tfrac{1}{2}\phi^2\hat{g}_{ab}$). It satisfies

$$\nabla^a T_{ab} = \Box_{\hat{g}}\phi\partial_b\phi = -2mR\phi\partial_b\phi.$$

This gives rise to approximately conserved quantities for our equation by considering the integral of the energy 3-form

$$K^a T_{ab}\,\mathrm{d}^3 x^b = K^a T^b_a \partial_b \lrcorner \mathrm{dvol}^4, \qquad (3.4)$$

where K^a is a Killing or approximate Killing vector. We can calculate the error in the conservation law as follows:

$$\nabla^a(K^b T_{ab}) = \Box_{\hat{g}}\phi K^b\partial_b\phi + \nabla^a K^b T_{ab} = -2mR\phi K^b\partial_b\phi + (\nabla_a K_b)T^{ab}. \qquad (3.5)$$

If K^a is Killing, the Killing form, also referred to as deformation tensor, $\nabla_{(a}K_{b)}$, vanishes and so does $(\nabla_a K_b)T^{ab}$, but this still does not give us an exact conservation law because of the term involving the scalar curvature.

We shall use this to perform energy estimates in a neighbourhood of i^0: for a choice of foliation, we estimate the error term by the energy on each slice and then invoke Gronwall's lemma. We also, however, need to be clear about the choice of vector field used to identify the different slices.

3.2. Foliation and identifying vector field

We shall work on a small neighbourhood of i^0

$$\Omega^+_{u_0} := \{u < u_0,\ t > 0\}, \quad \text{with } u_0 \ll -1.$$

Our essential foliation of $\Omega^+_{u_0}$ is by the spacelike hypersurfaces

$$\mathcal{H}_s = \{u = -sr_*\}. \qquad (3.6)$$

For $s = 1$, $\mathcal{H}_1$ is merely the $\{t = 0\}$ surface. For $s = 0$, the part of $\mathcal{H}_0$ within $\Omega^+_{u_0}$ is to be understood as the limit of the hypersurface $\mathcal{H}_s$ as $s \to 0$; it coincides with the part of $\mathscr{I}^+$ within $\Omega^+_{u_0}$. Indeed, for u and ω fixed, as s tends to zero, r_* is constrained to tend towards $+\infty$: we thus go to infinity along an outgoing principal null geodesic, we approach the point $(u, R = 0, \omega) \in \mathscr{I}^+$.

We record the relations

$$\frac{\mathrm{d}r_*}{\mathrm{d}R} = \frac{-1}{R^2(1-2mR)}, \quad \mathrm{d}R|_{\mathcal{H}_s} = \frac{R^2(1-2mR)}{s}\,\mathrm{d}u|_{\mathcal{H}_s} = \frac{r_* R^2(1-2mR)}{|u|}\,\mathrm{d}u|_{\mathcal{H}_s} \quad \text{near } i^0.$$

The foliation $\mathcal{H}_s$ is not smooth in the sense that r_*^{-1} is not a smooth function of R at $R = 0$; although the first derivative is bounded and tends to 1, the second is logarithmically divergent as $R \to 0$.

In this case, a natural identifying vector field ν needs to satisfy $\nu(s) = 1$; we choose

$$\nu = r_*^2 R^2 (1 - 2mR)|u|^{-1} \partial_R. \tag{3.7}$$

The 4-volume measure dvol^4, against which the error terms will be integrated, is thus decomposed into the product of $\mathrm{d}s$ and $\nu \lrcorner \mathrm{dvol}^4|_{\mathcal{H}_s} = r_*^2 R^2 (1 - 2mR)|u|^{-1} \, \mathrm{d}u \, \mathrm{d}^2\omega|_{\mathcal{H}_s}$, the former being the measure along the integral lines of ν^a and the latter our 3-volume measure on each $\mathcal{H}_s$.

Such a choice of identifying vector field ν^a, which is really associated with the choice of parameter s for the foliation, will lead to error terms that cannot be controlled by the energy density* and therefore to the impossibility of performing a priori estimates. All we need to do in order to solve this problem is the following change of parameter:

$$\tau := -2(\sqrt{s} - 1), \quad \text{so that} \quad \frac{\mathrm{d}\tau}{\mathrm{d}s} = -\frac{1}{\sqrt{s}}. \tag{3.8}$$

The change of sign and the -1 term are there purely for aesthetic reasons, the important part is $2\sqrt{s}$. This new parameter varies from 0 to 2 as s varies from 1 to 0. We denote

$$\Sigma_{\tau(s)} = \mathcal{H}_s. \tag{3.9}$$

The natural new identifying vector field is

$$V = -\sqrt{s}\nu = -\sqrt{\frac{|u|}{r_*}} r_*^2 R^2 (1 - 2mR)|u|^{-1}\partial_R = -(r_* R)^{3/2}(1 - 2mR)\sqrt{\frac{R}{|u|}}\partial_R. \tag{3.10}$$

We shall use both notations $\{\mathcal{H}_s\}_s$ and $\{\Sigma_\tau\}_\tau$, i.e. both parameters s and τ related by (3.8), for our foliation by spacelike hypersurfaces. The parameter s has a straightforward definition, it is useful for calculations on a single hypersurface and for the sake of simplicity we systematically use it for expressing the main results. The parameter τ on the other hand is adapted to a priori estimates and will be used in the proof of energy estimates.

We also need to consider in $\Omega_{u_0}^+$ the $u = \text{constant}$ surface:

$$\mathcal{S}_u = \{(u, R, \omega); \ 0 < R < 1/G^{-1}(-u), \ \omega \in S^2\},$$

that forms, for $u = u_0$, the part of the null boundary of $\Omega_{u_0}^+$ in the finite Schwarzschild space-time (G is the function (3.1) defining r_* in terms of r). The natural vector field connecting one surface of constant u to the next is

$$W := \partial_u + \frac{R^2(1 - 2mR)}{s}\partial_R = \partial_u + \frac{(r_* R)^2(1 - 2mR)}{|u|r_*}\partial_R, \quad \text{where } s = -\frac{u}{r_*}. \tag{3.11}$$

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* To be more precise, this does not occur for the fundamental estimates, because the scalar curvature $2mR$ gives us some extra fall-off at $\mathscr{I}$. For higher-order estimates, commuting ∂_R into the equation will give error terms without any fall-off. So the problem will occur as soon as we try to gain one extra degree of regularity from the fundamental estimates.

The vector field W is tangent to each hypersurface $\mathcal{H}_s$ and on $\mathscr{I}^+$ reduces to ∂_u. The corresponding decomposition of the 4-volume measure will be the product of $\mathrm{d}u$ along the integral lines of W and of the 3-volume measure on each $\mathcal{S}_u$:

$$W \lrcorner \mathrm{dvol}^4|_{\mathcal{S}_u} = \left(\mathrm{d}R\,\mathrm{d}^2\omega - \frac{R^2(1-2mR)}{s}\,\mathrm{d}u\,\mathrm{d}^2\omega \right)\bigg|_{\mathcal{S}_u} = \mathrm{d}R\,\mathrm{d}^2\omega|_{\mathcal{S}_u}.$$

3.3. The Morawetz vector field

For $m = 0$, g is the Minkowski metric and the Morawetz vector field (see [**16**])

$$K^a \partial_a = u^2 \partial_u + v^2 \partial_v \quad (u = t - r, \ v = t + r)$$
$$= (r^2 + t^2)\partial_t + 2tr\partial_r$$

is a Killing vector for the rescaled metric $\hat{g}$. Its expression in terms of the variables $u = t - r$ and $R = 1/r$ is

$$K^a \partial_a = u^2 \partial_u - 2(1 + uR)\partial_R. \tag{3.12}$$

It is a classic vector field for pointwise decay estimates in flat space-time and a version of K^a on the Schwarzschild space-time has been used recently for pointwise decay estimates on the Schwarzschild metric by Dafermos and Rodnianski [**7**]. One of its key features is that it is transverse to $\mathscr{I}^+$, it will therefore give us more information on $\mathscr{I}^+$ than other Killing vector fields such as for instance $\partial_u = \partial_t$.

Remark 3.1. Expressing the time translation on the Einstein cylinder ∂_τ in the coordinate basis u, R, ω, we obtain

$$\partial_\tau = \tfrac{1}{2}[\partial_u + u^2\partial_u - 2(1 + uR)\partial_R].$$

We see that

$$2\partial_\tau = \partial_u + K^a \partial_a,$$

or in terms of variables t, r, ω,

$$2\partial_\tau = \partial_t + K^a \partial_a,$$

where K is the Morawetz vector field (3.12). So the two vectors ∂_τ and K^a are very close. We could have chosen to use ∂_τ instead of K^a, the results and their proofs would have been essentially unchanged.

We choose our approximate Killing vector field on the Schwarzschild metric as follows: we keep the expression (3.12) of the Morawetz vector field, but now in terms of the variables $u = t - r_*$ and $R = 1/r$; we put

$$T^a \partial_a := u^2 \partial_u - 2(1 + uR)\partial_R. \tag{3.13}$$

Expressed in the coordinate system (t, r_*), this gives

$$T^a \partial_a = (t - r_*)^2 \partial_t - 2(1 + uR)\frac{-r^3}{r - 2m}(\partial_t + \partial_{r_*})$$

$$= (t^2 + r_*^2)\partial_t - 2tr_*\partial_t + 2\left(1 - \frac{2m}{r}\right)^{-1}(tr + r^2 - rr_*)(\partial_t + \partial_{r_*}).$$

This is no longer a Killing vector field for g, its Killing form is

$$\nabla_{(a}T_{b)} = 4mR^2(3 + uR)\,\mathrm{d}u^2, \tag{3.14}$$

which has a good fall-off at infinity.

It is important to note that the vector field T^a is uniformly timelike in a neighbourhood of i^0 and can therefore be used for obtaining energy estimates with positive definite energies on spacelike hypersurfaces.

Remark 3.2. The vector field T^a is everywhere timelike near i^0, indeed we have

$$\hat{g}_{ab}T^aT^b = u^2(4(1 + uR) + u^2R^2(1 - 2mR)).$$

This vanishes for the two values of uR:

$$(uR)_\pm = -2\frac{1 \mp \sqrt{2mR}}{1 - 2mR},$$

that for small R are arbitrarily close to -2, and $\hat{g}_{ab}T^aT^b$ is positive for uR outside $[(uR)_-, (uR)_+]$. In a small enough neighbourhood of i^0, we shall have

$$-r_* < u \ll -1, \quad \text{hence} \quad -\frac{r_*}{r} < Ru \ll -R,$$

and Ru therefore lives in an interval of the form $[-1-\varepsilon, 0]$, where $\varepsilon > 0$ is as small as we wish it to be, since at infinity $r_* \simeq r$. Consequently, in a small enough neighbourhood of i^0, the vector T^a is uniformly timelike.

3.4. Energy density and error terms for $K^a = T^a$

The energy density 3-form $E(\phi)$ associated with T^a is given by

$$E(\phi) := T^aT_{ab}\,\mathrm{d}^3x^b = T^aT_a^b\partial_b\lrcorner\mathrm{dvol}^4$$
$$= [u^2\phi_u^2 + R^2(1 - 2mR)(u^2\phi_u\phi_R - (1 + uR)\phi_R^2) + (1 + uR)|\nabla_{S^2}\phi|^2]\,\mathrm{d}u \wedge \mathrm{d}^2\omega$$
$$+ \tfrac{1}{2}[((2 + uR)^2 - 2mu^2R^3)\phi_R^2 + u^2|\nabla_{S^2}\phi|^2]\,\mathrm{d}R \wedge \mathrm{d}^2\omega. \tag{3.15}$$

For a hypersurface $\mathcal{S}$, we define

$$\mathcal{E}_{\mathcal{S}}(\phi) := \int_{\mathcal{S}} E(\phi).$$

For instance,

$$\mathcal{E}_{\mathscr{I}^+}(\phi) = \int_{\mathscr{I}^+} [u^2\phi_u^2 + |\nabla_{S^2}\phi|^2]\,\mathrm{d}u \wedge \mathrm{d}^2\omega,$$

$$\mathcal{E}_{\mathcal{S}_u}(\phi) = \int_{\mathcal{S}_u} \tfrac{1}{2}[((2+uR)^2 - 2mu^2R^3)\phi_R^2 + u^2|\nabla_{S^2}\phi|^2]\,\mathrm{d}R \wedge \mathrm{d}^2\omega. \qquad (3.16)$$

The energy on the surface $\mathcal{H}_s$ is given by

$$\mathcal{E}_{\mathcal{H}_s}(\phi) = \int_{\mathcal{H}_s} \Bigg[u^2\phi_u^2 + R^2(1-2mR)u^2\phi_u\phi_R$$
$$+ R^2(1-2mR)\left(\frac{(2+uR)^2}{2s} - \frac{mu^2R^3}{s} - (1+uR)\right)\phi_R^2$$
$$+ \left(\frac{u^2R^2(1-2mR)}{2s} + 1 + uR\right)|\nabla_{S^2}\phi|^2 \Bigg]\,\mathrm{d}u \wedge \mathrm{d}^2\omega. \qquad (3.17)$$

Using the expression (3.14) of the Killing form of T^a, the error term (3.5) takes the form

$$T_{ab}\nabla^a T^b - 2mR\phi T^b\partial_b\phi = 4mR^2(3+uR)\phi_R^2 - 2mR\phi(u^2\partial_u\phi - 2(1+uR)\partial_R\phi).$$

Working on the foliation $\{\Sigma_\tau\}_\tau$, this yields the error terms:

$$(T_{ab}\nabla^a T^b)V\lrcorner\mathrm{dvol}^4 = 4mR^2(3+uR)\phi_R^2(r_*R)^{3/2}(1-2mR)\sqrt{\frac{R}{|u|}}\,\mathrm{d}u \wedge \mathrm{d}^2\omega, \quad (3.18)$$

$$(-2mR\phi T^b\partial_b\phi)V\lrcorner\mathrm{dvol}^4 = -2mR\phi(u^2\partial_u\phi - 2(1+uR)\partial_R\phi)$$
$$\times (r_*R)^{3/2}(1-2mR)\sqrt{\frac{R}{|u|}}\,\mathrm{d}u \wedge \mathrm{d}^2\omega. \quad (3.19)$$

4. Fundamental estimates

In this part, we obtain the basic energy estimates for Equation (3.2). We work on the neighbourhood of i^0,

$$\Omega_{u_0}^+ := \{(u, R, \omega);\ u < u_0,\ 0 < s < 1,\ \omega \in S^2\} \quad (s = -u/r^*)$$
$$= \{(u, R, \omega);\ u < u_0,\ t > 0,\ \omega \in S^2\},$$

for $u_0 \ll -1$ such that T^a is uniformly timelike in $\Omega_{u_0}^+$. In order to prove the energy estimates, we use the approximate conservation law (3.5); however, we must first show that the energy on Σ_τ (3.17) uniformly dominates the error term (3.18). On $\Omega_{u_0}^+$ with $|u_0|$ large enough, this error term is uniformly equivalent to $R^2\sqrt{R/|u|}\phi_R^2$, which is in turn uniformly controlled by $R|u|^{-1}\phi_R^2$. The following lemma shows that this is controlled by the energy density restricted to the hypersurface $\mathcal{H}_s$.

Lemma 4.1. *On $\Omega_{u_0}^+$, for $u_0 < 0$, $|u_0|$ large enough, the energy density on $\mathcal{H}_s$ associated with T^a, i.e.*

$$T^a T_{ab}\, \mathrm{d}^3 x^b|_{\mathcal{H}_s \cap \Omega_{u_0}^+},$$

is uniformly equivalent to

$$\left\{ \left(u^2 \phi_u^2 + \frac{R}{|u|} \phi_R^2 + |\nabla_{S^2}\phi|^2 \right) \mathrm{d}u \wedge \mathrm{d}^2\omega \right\}\Big|_{\mathcal{H}_s \cap \Omega_{u_0}^+}. \tag{4.1}$$

For the error term (3.19) involving the scalar curvature, the factor

$$(T^b \partial_b \phi) V \lrcorner \mathrm{dvol}^4 = (u^2 \partial_u \phi - 2(1 + uR)\partial_R \phi)(r_* R)^{3/2}(1 - 2mR)\sqrt{\frac{R}{|u|}}\, \mathrm{d}u\, \mathrm{d}^2\omega$$

is naturally controlled by the energy, thanks to the choice of parameter τ, which gives us the factor $\sqrt{R/|u|}$. Indeed its two terms are respectively equivalent to $u^2\sqrt{R/|u|}\phi_u$, which is uniformly controlled by $|u\phi_u|$, and $\sqrt{R/|u|}\phi_R$. So we just need to control the factor $2mR\phi$. The following Poincaré-type estimate and its corollary allow us to do so. They in fact prove more than is strictly necessary for this step, but this will be useful for higher-order estimates. We first need to introduce some notation.

Definition 4.2. *We denote for $0 \leqslant s \leqslant 1$,*

$$\mathcal{H}_{s,u_0} := \mathcal{H}_s \cap \{u < u_0\},$$

for $s = 0$, we use the alternative notation

$$\mathscr{I}_{u_0}^+ := \mathscr{I}^+ \cap \{u < u_0\}.$$

For $0 \leqslant s_0 \leqslant 1$, in addition to the hypersurface $\mathcal{H}_{s_0,u_0}$, we also consider the part of $\mathcal{S}_{u_0}$ in the past of $\mathcal{H}_{s_0}$:

$$\mathcal{S}_{u_0,s_0} = \mathcal{S}_{u_0} \cap \{s_0 < s < 1\}.$$

Lemma 4.3. *Given $u_0 < 0$, there exists a constant $C > 0$ such that for any $f \in \mathcal{C}_0^\infty(\mathbb{R})$, we have*

$$\int_{-\infty}^{u_0} (f(u))^2\, \mathrm{d}u \leqslant C \int_{-\infty}^{u_0} u^2 (f'(u))^2\, \mathrm{d}u.$$

This has an important consequence.[*]

Corollary 4.4. *For $u_0 < 0$, $|u_0|$ large enough, there exists a constant $C > 0$ such that for any smooth compactly supported initial data at $t = 0$, the associated rescaled solution ϕ satisfies*

$$\forall s \in [0,1], \quad \int_{\mathcal{H}_{s,u_0}} \phi^2\, \mathrm{d}u\, \mathrm{d}^2\omega \leqslant C\mathcal{E}_{\mathcal{H}_{s,u_0}}(\phi).$$

From this, we infer the first fundamental estimate.

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[*] Not as immediate as one may think. Indeed, ϕ_u is the partial derivative of ϕ with respect to the variable u, it differs from $(\mathrm{d}/\mathrm{d}u)(\phi|_{\mathcal{H}_s})$ (see Equation (A.1)).

Theorem 4.5. *For $u_0 < 0$, $|u_0|$ large enough, there exists a constant $C > 0$ such that, for any smooth compactly supported initial data at $t = 0$, the associated rescaled solution ϕ satisfies for all $0 \leqslant s_0 \leqslant 1$,*

$$\mathcal{E}_{\mathcal{H}_{s_0,u_0}}(\phi) \leqslant C\mathcal{E}_{\mathcal{H}_{1,u_0}}(\phi),$$

and in particular, for $s_0 = 0$,

$$\mathcal{E}_{\mathscr{I}^+_{u_0}}(\phi) \leqslant C\mathcal{E}_{\mathcal{H}_{1,u_0}}(\phi).$$

We also prove a converse inequality in order to ensure the optimality of our estimates.

Theorem 4.6. *For $u_0 < 0$, $|u_0|$ large enough, there exists a constant $C > 0$ such that, for any smooth compactly supported initial data at $t = 0$, the associated rescaled solution ϕ satisfies for all $0 \leqslant s_0 \leqslant 1$,*

$$\mathcal{E}_{\mathcal{H}_{s_0,u_0}}(\phi) \leqslant C(\mathcal{E}_{\mathscr{I}^+_{u_0}}(\phi) + \mathcal{E}_{\mathcal{S}_{u_0,s_0}}(\phi)),$$

and in particular, for $s_0 = 1$,

$$\mathcal{E}_{\mathcal{H}_{1,u_0}}(\phi) \leqslant C(\mathcal{E}_{\mathscr{I}^+_{u_0}}(\phi) + \mathcal{E}_{\mathcal{S}_{u_0}}(\phi)).$$

Using the foliation $\{\mathcal{S}_u\}_u$ and the results of Theorem 4.5, we can prove another fundamental estimate that gives control over the 4-volume local L^2 norm of ϕ_R in $\Omega^+_{u_0}$, it is analogous to the point of view on peeling developed in Remark 2.7 for Minkowski space.

Theorem 4.7. *Let ϕ be a smooth solution of (3.2) associated with smooth compactly supported data at $t = 0$, we have for all $u \leqslant u_0$,*

$$\int_{\mathcal{S}_u} \phi_R^2 \, \mathrm{d}R \, \mathrm{d}^2\omega \leqslant C\mathcal{E}_{\mathcal{H}_{1,u_0}}(\phi),$$

where the constant $C > 0$ is independent of ϕ and u. In particular, for any compact subset K of $\bar{\Omega}^+_{u_0}$, where $\bar{\Omega}^+_{u_0} = \Omega^+_{u_0} \cup \mathscr{I}^+_{u_0} \cup \mathcal{H}_{1,u_0} \cup \mathcal{S}_{u_0}$, there exists a constant $C_K > 0$ independent of ϕ such that

$$\int_K \phi_R^2 \, \mathrm{d}u \, \mathrm{d}R \, \mathrm{d}^2\omega \leqslant C_K\mathcal{E}_{\mathcal{H}_{1,u_0}}(\phi).$$

Remark 4.8. The estimates of Theorems 4.5 and 4.6 allow us to solve the Goursat problem on $\mathscr{I}$ for Equation (3.2) and therefore entail a complete conformal scattering theory as defined in [**15**].

5. Higher-order estimates and peeling

In order to obtain estimates on the higher-order derivatives of ϕ, we commute differential operators into Equation (3.2). Because of the symmetries of Schwarzschild's space-time, we have for any $k \in \mathbb{N}$,

$$(\Box + 2mR)\partial_u^k \phi = 0, \tag{5.1}$$

$$(\Box + 2mR)\nabla_{S^2}^k \phi = 0. \tag{5.2}$$

Equations (5.1) and (5.2) give us on $\partial_u^k \phi$ and $\nabla_{S^2}^k \phi$ respectively, exactly the same energy estimates that we have obtained on ϕ.

We now need some control over the derivatives of ϕ with respect to R. Mimicking the proof of the peeling in flat space given in § 2 would lead to the use of the Morawetz vector field for this purpose. However, we quickly realize that this leads to serious problems in the Schwarzschild case. Indeed, we have

$$[T, 2mR] = -4m(1 + uR),$$
$$[T, [T, 2mR]] = 4mu(2 + uR),$$
$$[T, [T, [T, 2mR]]] = 0.$$

Although the third commutator is zero, the second blows up near i^0, which leads to error terms that cannot be controlled as soon as we reach two orders of differentiation (other terms with similar behaviour will come from repeated commutations of T with the d'Alembertian).

Instead, we commute the operator ∂_R into the equation to obtain

$$(\Box + 2mR)\phi_R = 2(1 - 3m)R\partial_R\phi_R - 2(1 - 6mR)\phi_R - 2m\phi. \tag{5.3}$$

We obtain the following approximate conservation law for $\psi := \phi_R$:

$$\nabla^a(T^b T_{ab}(\psi)) = \Box\psi T^b \partial_b \psi + \nabla^a T^b T_{ab}(\psi)$$
$$= (2(1 - 3m)R\partial_R\psi + 2(1 - 5mR)\psi - 2m\phi)T^b \partial_b \psi + 4mR^2(3 + uR)\psi^2. \tag{5.4}$$

Using Corollary 4.4 for ϕ_R as well as ϕ and following the proof of Theorems 4.5 and 4.6, we obtain the following proposition.

Proposition 5.1. *There exists a constant $C > 0$ such that, for any solution ϕ of (3.2) associated with smooth compactly supported initial data, we have for all $0 \leqslant s_0 \leqslant 1$,*

$$\mathcal{E}_{\mathcal{H}_{s_0,u_0}}(\phi_R) \leqslant C(\mathcal{E}_{\mathcal{H}_{1,u_0}}(\phi) + \mathcal{E}_{\mathcal{H}_{1,u_0}}(\phi_R)),$$
$$\mathcal{E}_{\mathcal{H}_{s_0,u_0}}(\phi_R) \leqslant C(\mathcal{E}_{\mathscr{I}_{u_0}^+}(\phi) + \mathcal{E}_{\mathcal{S}_{u_0,s_0}}(\phi) + \mathcal{E}_{\mathscr{I}_{u_0}^+}(\phi_R) + \mathcal{E}_{\mathcal{S}_{u_0,s_0}}(\phi_R)).$$

Further commutations of ∂_R into Equation (3.2) will always lead to controllable terms, simply because the coefficients of the error terms are polynomials in R. We therefore have the following general result.

Theorem 5.2. *For each $k \in \mathbb{N}$, there exists a constant $C_k > 0$ such that, for any solution ϕ of (3.2) associated with smooth compactly supported initial data, we have for all $0 \leqslant s_0 \leqslant 1$,*

$$\mathcal{E}_{\mathcal{H}_{s_0,u_0}}(\partial_R^k \phi) \leqslant C_k \sum_{p=0}^{k} \mathcal{E}_{\mathcal{H}_{1,u_0}}(\partial_R^p \phi),$$

$$\mathcal{E}_{\mathcal{H}_{s_0,u_0}}(\partial_R^k \phi) \leqslant C_k \sum_{p=0}^{k} (\mathcal{E}_{\mathscr{I}_{u_0}^+}(\partial_R^p \phi) + \mathcal{E}_{\mathcal{S}_{u_0,s_0}}(\partial_R^p \phi)),$$

and in particular

$$\mathcal{E}_{\mathscr{I}_{u_0}^+}(\partial_R^k \phi) \leqslant C_k \sum_{p=0}^{k} \mathcal{E}_{\mathcal{H}_{1,u_0}}(\partial_R^p \phi),$$

$$\mathcal{E}_{\mathcal{H}_{1,u_0}}(\partial_R^k \phi) \leqslant C_k \sum_{p=0}^{k} (\mathcal{E}_{\mathscr{I}_{u_0}^+}(\partial_R^p \phi) + \mathcal{E}_{\mathcal{S}_{u_0}}(\partial_R^p \phi)).$$

An similarly to Theorem 4.7, we can prove the following result.

Theorem 5.3. *Let ϕ be a smooth solution of (3.2) associated with smooth compactly supported data at $t = 0$. For all $k \in \mathbb{N}$ and $u \leqslant u_0$, there exists a constant $C_k > 0$ independent of ϕ and u such that*

$$\int_{\mathcal{S}_u} |\partial_R^k \phi|^2 \, \mathrm{d}R \, \mathrm{d}^2\omega \leqslant C_k \sum_{p=0}^{k} \mathcal{E}_{\mathcal{H}_{1,u_0}}(\partial_R^p \phi).$$

In particular, for any compact subset K of $\bar{\Omega}_{u_0}^+$, where $\bar{\Omega}_{u_0}^+ = \Omega_{u_0}^+ \cup \mathscr{I}_{u_0}^+ \cup \mathcal{H}_{1,u_0} \cup \mathcal{S}_{u_0}$, there exists a constant $C_{K,k} > 0$ such that

$$\int_K |\partial_R^k \phi|^2 \, \mathrm{d}u \, \mathrm{d}R \, \mathrm{d}^2\omega \leqslant C_{K,k} \sum_{p=0}^{k} \mathcal{E}_{\mathcal{H}_{1,u_0}}(\partial_R^p \phi).$$

We can now define the function spaces of initial data that guarantee peeling at order k for solutions of (3.2). First we need to be clear about what we mean by peeling at a certain order.

Definition 5.4. We say that a solution ϕ of (3.2) peels at order $k \in \mathbb{N}$ if for all polynomial P in ∂_R and ∇_{S^2} of order lower than or equal to k, we have

$$\mathcal{E}_{\mathscr{I}_{u_0}^+}(P\phi) < +\infty.$$

This means than for all $p \in \{0, 1, \dots, k\}$ we have for all $q \in \{0, 1, \dots, p\}$,

$$\mathcal{E}_{\mathscr{I}_{u_0}^+}(\partial_R^q \nabla_{S^2}^{p-q} \phi) < +\infty.$$

By Theorem 5.2, the condition on initial data that guarantees peeling at order k is therefore that

$$\forall p \in \{0, 1, \dots, k\}, \ \forall q \in \{0, 1, \dots, p\}, \quad \mathcal{E}_{\mathcal{H}_{1,u_0}}(\partial_R^q \nabla_{S^2}^{p-q} \phi) < +\infty.$$

This can be expressed as a condition purely on the initial data that no longer involves the full solution ϕ. First note that Equation (3.2) in terms of variables t, r_*, ω becomes

$$\left(\partial_t^2 - \partial_{r_*}^2 + F\left(\frac{2m}{r^3} - \frac{1}{r^2}\Delta_{S^2}\right)\right)\phi = 0, \quad \text{where } F(r) := 1 - \frac{2m}{r}.$$

Whence,

$$
\partial_R \begin{pmatrix} \phi \\ \partial_t \phi \end{pmatrix} = -\frac{r^3}{r-2m}(\partial_t + \partial_{r_*}) \begin{pmatrix} \phi \\ \partial_t \phi \end{pmatrix}
$$

$$
= -\frac{r^3}{r-2m} \begin{pmatrix} \partial_{r_*} & 1 \\ \partial_{r_*}^2 - \frac{2mF}{r^3} + \frac{F}{r^2}\Delta_{S^2} & \partial_{r_*} \end{pmatrix} \begin{pmatrix} \phi \\ \partial_t \phi \end{pmatrix}
$$

$$
=: L \begin{pmatrix} \phi \\ \partial_t \phi \end{pmatrix}.
$$

The operator L purely involves spacelike derivatives. We can now express the spaces of initial data that entail peeling at a given order.

Definition 5.5. Given $\phi_0, \phi_1 \in \mathcal{C}_0^\infty([-u_0, +\infty[_{r_*} \times S_\omega^2)$, we define the following squared norm of order k:

$$
\left\| \begin{pmatrix} \phi \\ \partial_t \phi \end{pmatrix} \right\|_k^2 := \sum_{p=0}^{k} \sum_{q=0}^{p} \mathcal{E}_{\mathcal{H}_{1,u_0}} \left(L^q \nabla_{S^2}^{p-q} \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \right), \tag{5.5}
$$

where we have denoted by

$$
\mathcal{E}_{\mathcal{H}_{1,u_0}} \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}
$$

the energy $\mathcal{E}_{\mathcal{H}_{1,u_0}}(\phi)$ where ϕ is replaced by ϕ_0 and $\partial_t \phi = \phi_u$ is replaced by ϕ_1.

The space of initial data (on $[-u_0, +\infty[_{r_*} \times S_\omega^2)$ for which the associated solution peels at order k is the completion of $\mathcal{C}_0^\infty([-u_0, +\infty[_{r_*} \times S_\omega^2) \times \mathcal{C}_0^\infty([-u_0, +\infty[_{r_*} \times S_\omega^2)$ in the norm (5.5). The fact that we have estimates both ways at all orders in Theorem 5.2 guarantees that this setting is optimal in our framework.

Remark 5.6. It is important to note that all estimates in this paper are locally uniform in m. In particular, u_0 and all the constants in all the theorems can be fixed independently of m for m in a fixed compact domain $[0, M]$.

Remark 5.7. Similarly, for two different values of the mass m, the function spaces of initial data for a given regularity are canonically isomorphic. The equivalence in the norms is uniform for m in a fixed compact domain $[0, M]$. This is because the quantity $r_* - r$ never appears in these norms. Hence, we do not see any logarithmic divergence between the norms for two different values of m.

6. Interpretation of the results

The characterization of the peeling given in Definition 5.5 proves the existence of large classes of data that guarantee peeling at any given order on the Schwarzschild space-time. It is interesting to see whether these classes are natural extensions to the Schwarzschild case of the classes of data for Minkowski space-time given in Definition 2.6.

For $m = 0$, all our estimates proved for Schwarzschild's space-time are still valid and give the corresponding estimates for Minkowski space-time. Accordingly, Definition 5.5 for $m = 0$ gives a fourth definition of the peeling in flat space-time that is as follows.

Definition 6.1. Given $\phi_0, \phi_1 \in \mathcal{C}_0^\infty([-u_0, +\infty[_r \times S_\omega^2)$, we define the following squared norm of order k:

$$\left\| \begin{pmatrix} \phi \\ \partial_t \phi \end{pmatrix} \right\|_k^2 := \sum_{p=0}^k \sum_{q=0}^p \mathcal{E}_{\mathcal{H}_{1,u_0}} \left(L^q \nabla_{S^2}^{p-q} \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \right), \tag{6.1}$$

where

$$L = -r^2 \begin{pmatrix} \partial_r & 1 \\ \partial_r^2 + \frac{1}{r^2}\Delta_{S^2} & \partial_r \end{pmatrix} \begin{pmatrix} \phi \\ \partial_t \phi \end{pmatrix}.$$

The space of initial data (on $[-u_0, +\infty[_r \times S_\omega^2)$ for which the associated solution peels at order k is the completion of $\mathcal{C}_0^\infty([-u_0, +\infty[_r \times S_\omega^2) \times \mathcal{C}_0^\infty([-u_0, +\infty[_r \times S_\omega^2)$ in the norm (6.1).

Of course Definitions 2.6 and 6.1 are of a different nature since the former considers spaces of data on the whole $\{t = 0\}$ slice whereas the latter gives only data in a neighbourhood of i^0. However, we can compare the spaces of data on the domain $[-u_0, +\infty[_r \times S_\omega^2$. At the level of basic energies, we show that the two definitions are equivalent. First, we introduce some notations so that we can clearly identify the physical solution and the rescaled fields obtained using Ω and R. We shall denote the physical field, i.e. the solution of (2.4) on $\mathbb{M}$, by $\tilde{\phi}$. The field rescaled using the conformal factor Ω defined in (2.1) will be denoted by $\psi := \Omega^{-1}\tilde{\phi}$ and when rescaling using $R = 1/r$, we shall use the notation $\phi := R^{-1}\tilde{\phi} = r\tilde{\phi}$.

Proposition 6.2. *In terms of $\tilde{\phi}$, the energy at $\tau = 0$ on the Einstein cylinder reads*

$$\mathcal{E}_{X_0}(\psi) = \int_{\{t=0\}} \left(\tilde{\phi}_t^2 + \left(\tilde{\phi}_r + \frac{2r}{1+r^2}\tilde{\phi} \right)^2 + \left| \frac{\nabla_{S^2}\tilde{\phi}}{r} \right|^2 + \frac{4\tilde{\phi}^2}{(1+r^2)^2} \right) \frac{1+r^2}{4} r^2 \, \mathrm{d}r \, \mathrm{d}^2\omega \tag{6.2}$$

$$\simeq \int_{\Sigma_0} \left((1+r^2)(\partial_t\tilde{\phi})^2 + (1+r^2)(\partial_r\tilde{\phi})^2 + (1+r^2)\left| \frac{\nabla_{S^2}\tilde{\phi}}{r} \right|^2 + \tilde{\phi}^2 \right) r^2 \, \mathrm{d}r \, \mathrm{d}^2\omega; \tag{6.3}$$

while the energy at $t = 0$ for ϕ satisfies

$$\mathcal{E}_{\mathcal{H}_{1,u_0}}(\phi) \simeq \int_{\{t=0, \, r > -u_0\}} \left\{ r^2\tilde{\phi}_t^2 + r^2\tilde{\phi}_r^2 + r^2\left| \frac{\nabla_{S^2}\tilde{\phi}}{r} \right|^2 + \tilde{\phi}^2 \right\} r^2 \, \mathrm{d}r \, \mathrm{d}^2\omega. \tag{6.4}$$

It is for higher orders that the definitions differ. On the Einstein cylinder, we raise regularity with ∂_τ, which, on the hypersurface $\{t = 0\}$, coincides with $\frac{1}{2}(1 + r^2)\partial_t$. As we can see from identities (2.9) and (2.10), the application of ∂_τ amounts to that of

the elliptic operator $(1 - \Delta_{S^3})^{1/2}$, which can be understood, thanks to the ellipticity, as providing a control over $r^2 \partial_r$ and angular derivatives on S^2 independently. In the partial compactification obtained when rescaling with R, the vector we use for raising regularity in the estimates is $\partial_R = -r^2(\partial_t + \partial_r)$; it is a null vector field whose action cannot be understood as that of an elliptic operator. Although it also acts as a combination of $r^2 \partial_r$ and angular derivatives, it gives a weaker control on initial data.

So, our characterization of the peeling on Schwarzschild's space-time is a natural generalization of a definition on Minkowski space that is not equivalent to the usual description of the peeling: it provides larger classes of data giving rise to solutions that peel at a certain order. This is because we pinpoint the relevant null derivative to control near i^0 instead of controlling all derivatives. Our paper thus not only proves the validity of the peeling model for the wave equation on the Schwarzschild metric at all orders, it also provides a description of the peeling on flat space-time that is more general than the one used since Penrose's paper in 1963.

Appendix A. Proofs of the main results

The following lemma contains trivial but essential observations.

Lemma A.1. *Let $\varepsilon > 0$, then for $u_0 < 0$, $|u_0|$ large enough, in the domain $\Omega_{u_0}^+$, we have*

$$r < r_* < r(1 + \varepsilon), \qquad 1 < Rr_* < 1 + \varepsilon, \qquad 0 < R|u| < 1 + \varepsilon, \qquad 1 - \varepsilon < 1 - 2mR < 1,$$

and of course

$$0 < s = \frac{|u|}{r_*} < 1.$$

A.1. Proof of Lemma 4.1

The restriction of the 3-form $T^a T_{ab} \, \mathrm{d}^3 x^b$ to the hypersurface $\mathcal{H}_s$ is given by

$$
\begin{aligned}
T^a &T_{ab} \, \mathrm{d}^3 x^b |_{\mathcal{H}_s} \\
&= \left\{ u^2 \phi_u^2 + R^2(1 - 2mR) \left(\frac{(2 + uR)^2}{2s} - 1 - uR - \frac{mu^2 R^3}{s} \right) \phi_R^2 \right. \\
&\qquad \left. + R^2(1 - 2mR)u^2 \phi_u \phi_R + \left(\frac{R^2 u^2(1 - 2MR)}{2s} + 1 + uR \right) |\nabla_{S^2} \phi|^2 \right\} \mathrm{d}u \wedge \mathrm{d}^2 \omega,
\end{aligned}
$$

and using the definition of $\mathcal{H}_s = \{u = -sr_*\}$, we get

$$
\begin{aligned}
T^a T_{ab}\, \mathrm{d}^3 x^b|_{\mathcal{H}_s} \\
= \Big\{ & u^2 \phi_u^2 + R^2(1-2mR)\Big(\frac{(2+uR)^2}{2} \frac{r_*}{|u|} - 1 - uR - m|u|R^3 r_* \Big) \phi_R^2 \\
& + R^2(1-2mR)u^2 \phi_u \phi_R + \Big(\frac{R^2(1-2MR)}{2}|u|r_* + 1 + uR \Big)|\nabla_{S^2}\phi|^2 \Big\}\, \mathrm{d}u \wedge \mathrm{d}^2\omega.
\end{aligned}
$$

Let $\varepsilon > 0$ small, we choose $u_0 \ll -1$ such as in Lemma A.1. We first deal with the angular term:

$$
\begin{aligned}
\tfrac{1}{2} R^2(1-2MR)|u|r_* + 1 + uR &= 1 + |u|R(-1 + \tfrac{1}{2}Rr_*(1-2mR)) \\
&= 1 + |u|R(-1 + \tfrac{1}{2}(1 + 2mR\log(r-2m))(1-2mR)) \\
&= 1 + |u|R(-\tfrac{1}{2} + mR(-1 + (1-2mR)\log(r-2m)))
\end{aligned}
$$

and taking $|u_0|$ large enough, we get

$$
1 \geqslant \tfrac{1}{2} R^2(1-2MR)|u|r_* + 1 + uR \geqslant \tfrac{1}{2}(1-\varepsilon).
$$

Now we estimate above and below the term in ϕ_R^2:

$$
\begin{aligned}
R^2(1-2mR)&\Big(\frac{(2+uR)^2}{2} \frac{r_*}{|u|} - 1 - uR - m|u|R^3 r_* \Big) \\
&= \frac{R}{|u|}(1-2mR)\Big(\frac{(2+uR)^2}{2} Rr_* - |u|R + (uR)^2 - mR(uR)^2 Rr_* \Big) \\
&= \frac{R}{|u|}(1-2mR)\Big(\Big(2 + 2uR + \frac{(uR)^2}{2} \Big) Rr_* + uR + (uR)^2 - mR(uR)^2 Rr_* \Big) \\
&= \frac{R}{|u|}(1-2mR)\Big(2 + 2uR + \frac{(uR)^2}{2} + uR + (uR)^2 \Big) \\
&\quad + \frac{R}{|u|}(1-2mR)\Big(\Big(2 + 2uR + \frac{(uR)^2}{2} \Big) 2mR\log(r-2m) - mR(uR)^2 Rr_* \Big).
\end{aligned}
$$

By Lemma A.1, the term in the penultimate line above dominates the term in the last line. It can be written as

$$
\frac{1}{2}\frac{R}{|u|}(1-2mR)(3(uR)^2 + 6uR + 4).
$$

Recall that $-1 - \varepsilon < Ru < 0$ in $\Omega_{u_0}^+$. Noting that the function $f(x) = 3x^2 + 6x + 4$ is everywhere positive on $\mathbb{R}$, takes its minimum at $x = -1$, $f(-1) = 1$ and $f(x) \in [1,4]$ for $x \in [-2,0]$, we infer that for $|u_0|$ large enough, $u_0 < 0$, we have in $\Omega_{u_0}^+$ say

$$
\frac{1}{3}\frac{R}{|u|} \leqslant R^2(1-2mR)\Big(\frac{(2+uR)^2}{2}\frac{r_*}{|u|} - 1 - uR - m|u|R^3 r_* \Big) \leqslant 5\frac{R}{|u|}.
$$

Then we treat the term in $\phi_u \phi_R$ in the following way:

$$|R^2(1-2mR)u^2\phi_u\phi_R| \leqslant (R|u|)^{3/2}|u\phi_u|\left|\sqrt{\frac{R}{|u|}}\phi_R\right|$$

$$\leqslant (1+\varepsilon)^{3/2}|u\phi_u|\left|\sqrt{\frac{R}{|u|}}\phi_R\right|$$

$$\leqslant (1+\varepsilon)^{3/2}\frac{1}{2}\left(\lambda^2 u^2\phi_u^2 + \frac{1}{\lambda^2}\frac{R}{|u|}\phi_R^2\right), \quad \lambda \in \mathbb{R}^*.$$

This guarantees that the energy density on $\mathcal{H}_s$ is estimated above by (4.1) uniformly in $\Omega_{u_0}^+$. Now in order to establish the estimate below, we need to choose λ such that

$$\frac{\lambda^2}{2} < 1 \quad \text{and} \quad \frac{1}{2\lambda^2} < \frac{1}{3},$$

i.e. $\sqrt{3/2} < \lambda < \sqrt{2}$. Taking for example $\lambda = \frac{1}{2}(\sqrt{3/2} + \sqrt{2})$, this concludes the proof of the lemma, provided ε is small enough. $\qquad\square$

A.2. Proof of Lemma 4.3

This is a simple integration by parts:

$$\int_{-\infty}^{u_0} f^2 \, \mathrm{d}u = [(u-u_0)f^2]_{-\infty}^{u_0} - \int_{-\infty}^{u_0} (u-u_0)2ff' \, \mathrm{d}u.$$

The boundary term vanishes (recall that f is assumed to be compactly supported, which gets rid of the boundary term at $-\infty$) and using $u_0 < 0$, we get

$$\int_{-\infty}^{u_0} f^2 \, \mathrm{d}u \leqslant \int_{-\infty}^{u_0} |u-u_0|2|f|\,|f'| \, \mathrm{d}u$$

$$\leqslant \int_{-\infty}^{u_0} 2|f|\,|uf'| \, \mathrm{d}u$$

$$\leqslant \frac{1}{2}\int_{-\infty}^{u_0} f^2 \, \mathrm{d}u + 2\int_{-\infty}^{u_0} u^2(f')^2 \, \mathrm{d}u,$$

which gives the result. $\qquad\square$

A.3. Proof of Corollary 4.4

We simply need to show that

$$\int_{\mathcal{H}_{s,u_0}} \left|u\frac{\mathrm{d}}{\mathrm{d}u}(\phi|_{\mathcal{H}_s})\right|^2 \, \mathrm{d}u\, \mathrm{d}^2\omega$$

is controlled uniformly by $\mathcal{E}_{\mathcal{H}_{s,u_0}}(\phi)$. For a given $s \in [0,1]$, we have

$$\phi|_{\mathcal{H}_s}(u,\omega) = \phi\left(u, R = \frac{1}{G^{-1}(-u/s)}, \omega\right),$$

where G is the function $r \mapsto r_* = G(r)$ (see (3.1)). Hence,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}(\phi|_{\mathcal{H}_s}) &= \left(\phi_u - \frac{1}{s} \frac{-1}{G^{-1}(-u/s)} (G^{-1})'(-u/s) \phi_R \right)\Big|_{\mathcal{H}_s} \\
&= \left(\phi_u - \frac{R}{u} \frac{r_* R}{1 - 2mR} \phi_R \right)\Big|_{\mathcal{H}_s}.
\end{aligned}
\tag{A.1}
$$

By Lemma A.1, it follows that

$$
\int_{\mathcal{H}_{s,u_0}} \left| u \frac{\mathrm{d}}{\mathrm{d}u}(\phi|_{\mathcal{H}_s}) \right|^2 \mathrm{d}u\, \mathrm{d}^2\omega \lesssim \int_{\mathcal{H}_{s,u_0}} \left(u^2 \phi_u^2 + \frac{R}{|u|} \phi_R^2 \right) \mathrm{d}u\, \mathrm{d}^2\omega \lesssim \mathcal{E}_{\mathcal{H}_{s,u_0}}(\phi).
$$

This concludes the proof. $\qquad\qquad\square$

A.4. Proof of Theorem 4.5

The surfaces $\mathcal{H}_{s,u_0}$, $s_0 \leqslant s \leqslant 1$, parametrized by τ, will be denoted Σ_{τ,u_0}, $0 \leqslant \tau \leqslant \tau(s_0)$. We also consider $\mathcal{S}_{u_0}^{s_0}$ which is the part of $\mathcal{S}_{u_0}$ situated in the past of $\mathcal{H}_{s_0}$, i.e. for which[*] we have $s_0 < s < 1$. We denote by Err the total error term on each hypersurface $\mathcal{H}_s$:

$$
\begin{aligned}
\mathrm{Err} := 4mR^2(3+uR)\phi_R^2(r_*R)^{3/2}(1-2mR)\sqrt{\frac{R}{|u|}} \\
- 2mR\phi(u^2\partial_u\phi - 2(1+uR)\partial_R\phi)(r_*R)^{3/2}(1-2mR)\sqrt{\frac{R}{|u|}}.
\end{aligned}
$$

This is estimated uniformly on $\Omega_{u_0}^+$ by

$$
|\mathrm{Err}| \lesssim \frac{R}{|u|}\phi_R^2 + |\phi|\,|u\phi_u| + |\phi|\sqrt{\frac{R}{|u|}}|\phi_R| \lesssim u^2\phi_u^2 + \frac{R}{|u|}\phi_R^2 + \phi^2.
$$

We use the approximate conservation law (3.5) on the domain $\Omega_{u_0}^+ \cap \{s_0 < s < 1\}$ to obtain

$$
\begin{aligned}
\mathcal{E}_{\mathcal{H}_{s_0,u_0}}(\phi) + \mathcal{E}_{\mathcal{S}_{u_0}^{s_0}}(\phi) - \mathcal{E}_{\mathcal{H}_{1,u_0}}(\phi) &= \int_0^{\tau(s_0)} \int_{\Sigma_{\tau,u_0}} \mathrm{Err}\,\mathrm{d}\tau\,\mathrm{d}u\,\mathrm{d}^2\omega \\
&\lesssim \int_0^{\tau(s_0)} \mathcal{E}_{\Sigma_{\tau,u_0}}(\phi)\,\mathrm{d}\tau + \int_0^{\tau(s_0)} \left(\int_{\Sigma_{\tau,u_0}} \phi^2\,\mathrm{d}u\,\mathrm{d}^2\omega \right)\mathrm{d}\tau.
\end{aligned}
$$

---

[*] This hypersurface is not the part of $\mathcal{S}_{u_0}$ introduced in Definition 4.2 and considered in Theorem 4.6; instead, up to a negligible set, it is the complement of $\mathcal{S}_{u_0,s_0}$ in $\mathcal{S}_{u_0}$.

Furthermore, since $(2 + uR)^2 - 2mu^2R^3 \geqslant 0$ in $\Omega_{u_0}^+$, we have by (3.16) that $\mathcal{E}_{\mathcal{S}_{u_0}^{s_0}}(\phi) \geqslant 0$. Thus, using Lemma 4.3, we get

$$
\begin{aligned}
\mathcal{E}_{\Sigma_{\tau(s_0)}, u_0}(\phi) &\leqslant \mathcal{E}_{\Sigma_{\tau(s_0)}, u_0}(\phi) + \mathcal{E}_{\mathcal{S}_{u_0, s_0}}(\phi) \\
&\lesssim \mathcal{E}_{\Sigma_{0, u_0}}(\phi) + \int_0^{\tau(s_0)} \mathcal{E}_{\Sigma_{\tau(s)}, u_0}(\phi)\,\mathrm{d}\tau + \int_0^{\tau(s_0)} \left(\int_{\Sigma_{\tau, u_0}} \phi^2\,\mathrm{d}u\,\mathrm{d}^2\omega \right) \mathrm{d}\tau \\
&\lesssim \mathcal{E}_{\Sigma_{0, u_0}}(\phi) + \int_0^{\tau(s_0)} \mathcal{E}_{\Sigma_{\tau(s)}, u_0}(\phi)\,\mathrm{d}\tau.
\end{aligned}
$$

The result follows via a standard Gronwall estimate. □

A.5. Proofs of Proposition 5.1 and Theorems 5.2 and 5.3

They are straightforward extensions by induction of the proofs of Theorems 4.5–4.7. □

A.6. Proof of Theorem 4.6

It is very similar to that of Theorem 4.5. On the domain $\Omega_{u_0}^+ \cap \{0 < s < s_0\}$, we use the approximate conservation law (3.5)

$$
\mathcal{E}_{\mathcal{I}_{u_0}^+}(\phi) + \mathcal{E}_{\mathcal{S}_{u_0, s_0}}(\phi) - \mathcal{E}_{\Sigma_{\tau(s_0)}, u_0}(\phi) = \int_{\tau(s_0)}^2 \int_{\Sigma_{\tau(s)}, u_0} \mathrm{Err}\,\mathrm{d}u\,\mathrm{d}^2\omega\,\mathrm{d}\tau.
$$

We obtain

$$
\mathcal{E}_{\Sigma_{\tau(s_0)}, u_0}(\phi) \lesssim \mathcal{E}_{\mathcal{I}_{u_0}^+}(\phi) + \mathcal{E}_{\mathcal{S}_{u_0, s_0}}(\phi) + \int_{\tau(s_0)}^2 \mathcal{E}_{\Sigma_{\tau(s)}, u_0}(\phi)\,\mathrm{d}\tau.
$$

The result follows via a standard Gronwall estimate. □

A.7. Proof of Theorem 4.7

Given $u_1 < u_0$, we use the approximate conservation law (3.5) on the domain

$$
\Omega_{u_1}^+ = \{0 \leqslant s \leqslant 1\} \cap \{u < u_1\}.
$$

We obtain

$$
\begin{aligned}
\mathcal{E}_{\mathcal{I}_{u_1}^+}(\phi) &+ \mathcal{E}_{\mathcal{S}_{u_1}}(\phi) - \mathcal{E}_{\mathcal{H}_{1, u_1}}(\phi) \\
&= \int_{\Omega_{u_1}^+} (4mR^2(3 + uR)\phi_R^2 - 2mR\phi(u^2\phi_u - 2(1 + uR)\partial_R\phi))\,\mathrm{d}u\,\mathrm{d}R\,\mathrm{d}^2\omega.
\end{aligned}
$$

Foliating $\Omega_{u_1}^+$ by $\{\Sigma_\tau\}_{\tau, u_0}$, we gain in the integral a factor of $(r_*R)^{3/2}(1 - 2mR)\sqrt{R/|u|}$ and so all the terms are controlled by the energy density on Σ_{τ, u_1} (see proof of Theorem 4.5 for details), which is itself controlled uniformly by the energy density of $\Sigma_{0, u_1} = \mathcal{H}_{1, u_1}$ (applying the result of Theorem 4.5 with u_1 instead of u_0). Whence,

$$
\mathcal{E}_{\mathcal{I}_{u_1}^+}(\phi) + \mathcal{E}_{\mathcal{S}_{u_1}}(\phi) - \mathcal{E}_{\mathcal{H}_{1, u_1}}(\phi) \lesssim \mathcal{E}_{\mathcal{H}_{1, u_1}}(\phi).
$$

Then, the positivity of $\mathcal{E}_{\mathscr{I}_{u_1}^+}(\phi)$ gives us the inequality

$$\mathcal{E}_{\mathcal{S}_{u_1}}(\phi) = \int_{\mathcal{S}_{u_1}} \tfrac{1}{2}[((2+uR)^2 - 2mu^2R^3)\phi_R^2 + u^2|\nabla_{S^2}\phi|^2]\,\mathrm{d}R\,\mathrm{d}^2\omega \lesssim \mathcal{E}_{\mathcal{H}_{1,u_0}}(\phi).$$

Finally, Lemma A.1 entails

$$\int_{\mathcal{S}_{u_1}} \phi_R^2 \,\mathrm{d}R\,\mathrm{d}^2\omega \lesssim \mathcal{E}_{\mathcal{S}_{u_1}}(\phi) \lesssim \mathcal{E}_{\mathcal{H}_{1,u_0}}(\phi),$$

which concludes the proof. $\qquad\square$

A.8. Proof of Proposition 6.2

First we express ψ_τ and ψ_ζ in terms of $\tilde{\phi}_t$ and $\tilde{\phi}_r$. We have

$$\frac{\partial}{\partial\tau} = \frac{1}{2}\left((1+t^2+r^2)\frac{\partial}{\partial t} + 2tr\frac{\partial}{\partial r}\right),$$

$$\frac{\partial}{\partial\zeta} = \frac{1}{2}\left(2tr\frac{\partial}{\partial t} + (1+t^2+r^2)\frac{\partial}{\partial r}\right).$$

Hence

$$\partial_\tau\psi = \frac{1+t^2+r^2}{2}\Omega^{-1}\left(\frac{t+r}{1+(t+r)^2} + \frac{t-r}{1+(t-r)^2} + \partial_t\right)\tilde{\phi}$$
$$+ tr\Omega^{-1}\left(\frac{t+r}{1+(t+r)^2} - \frac{t-r}{1+(t-r)^2} + \partial_r\right)\tilde{\phi},$$

$$\partial_\zeta\psi = tr\Omega^{-1}\left(\frac{t+r}{1+(t+r)^2} + \frac{t-r}{1+(t-r)^2} + \partial_t\right)\tilde{\phi}$$
$$+ \frac{1+t^2+r^2}{2}\Omega^{-1}\left(\frac{t+r}{1+(t+r)^2} - \frac{t-r}{1+(t-r)^2} + \partial_r\right)\tilde{\phi}.$$

In particular at $t=0$ (i.e. at $\tau=0$ as well) we obtain

$$\partial_\tau\psi(0) = \frac{(1+r^2)^2}{4}\partial_t\tilde{\phi},$$

$$\partial_\zeta\psi(0) = \frac{(1+r^2)^2}{4}\left(\frac{2r}{1+r^2} + \partial_r\right)\tilde{\phi}.$$

Besides the Euclidean measure on the 3-sphere $\{\tau=0\}$ becomes, in terms of variables (r,θ,φ),

$$\mathrm{d}\mu_{S^3} = \sin^2\zeta\,\mathrm{d}\zeta\,\mathrm{d}^2\omega = \frac{4r^2}{(1+r^2)^2}\frac{2}{1+r^2}\,\mathrm{d}r\,\mathrm{d}^2\omega,$$

$$
\begin{aligned}
\mathcal{E}_{X_0}(\psi) &= \frac{1}{2} \int_{\Sigma_0} (\psi_\tau^2 + |\nabla_{S^3} \psi|^2 + \psi^2) \, \mathrm{d}\mu_{S^3} \\
&= \frac{1}{2} \int_{\Sigma_0} \left(\psi_\tau^2 + \psi_\zeta^2 + \frac{1}{\sin^2 \zeta} |\nabla_{S^2} \psi|^2 + \psi^2 \right) \mathrm{d}\mu_{S^3} \\
&= \frac{1}{2} \int_{\{t=0\}} \left(\frac{(1+r^2)^4}{16} (\partial_t \tilde{\phi})^2 + \frac{(1+r^2)^4}{16} \left(\partial_r \tilde{\phi} + \frac{2r}{1+r^2} \tilde{\phi} \right)^2 \right. \\
&\quad \left. + \frac{(1+r^2)^2}{4r^2} \frac{(1+r^2)^2}{4} |\nabla_{S^2} \tilde{\phi}|^2 + \frac{(1+r^2)^2}{4} \tilde{\phi}^2 \right) \frac{4r^2}{(1+r^2)^2} \frac{2}{1+r^2} \, \mathrm{d}r \, \mathrm{d}^2\omega \\
&= \frac{1}{2} \int_{\{t=0\}} \left(\frac{1+r^2}{2} (\partial_t \tilde{\phi})^2 + \frac{1+r^2}{2} \left(\partial_r \tilde{\phi} + \frac{2r}{1+r^2} \tilde{\phi} \right)^2 \right. \\
&\quad \left. + \frac{1+r^2}{2} \frac{1}{r^2} |\nabla_{S^2} \tilde{\phi}|^2 + \frac{2}{1+r^2} \tilde{\phi}^2 \right) r^2 \, \mathrm{d}r \, \mathrm{d}^2\omega.
\end{aligned}
$$

This gives (6.2).

Let us now consider only the two following terms:

$$
\begin{aligned}
I &:= \int_{\Sigma_0} \left(\frac{1+r^2}{2} \left(\partial_r \tilde{\phi} + \frac{2r}{1+r^2} \tilde{\phi} \right)^2 + \frac{2}{1+r^2} \tilde{\phi}^2 \right) r^2 \, \mathrm{d}r \, \mathrm{d}^2\omega \\
&= \int_{\Sigma_0} \left(\frac{1+r^2}{2} \tilde{\phi}_r^2 + 2r \tilde{\phi} \tilde{\phi}_r + \frac{2r^2}{1+r^2} \tilde{\phi}^2 + \frac{2}{1+r^2} \tilde{\phi}^2 \right) r^2 \, \mathrm{d}r \, \mathrm{d}^2\omega \\
&= \int_{\Sigma_0} \left(\frac{1+r^2}{2} \tilde{\phi}_r^2 + 2r \tilde{\phi} \tilde{\phi}_r + 2 \tilde{\phi}^2 \right) r^2 \, \mathrm{d}r \, \mathrm{d}^2\omega.
\end{aligned}
$$

By integration by parts, we have[*]

$$
\int_0^{+\infty} 2r^3 \tilde{\phi} \tilde{\phi}_r \, \mathrm{d}r = -3 \int_0^{+\infty} \tilde{\phi}^2 r^2 \, \mathrm{d}r,
$$

whence

$$
I = \int_{\Sigma_0} \left(\frac{1+r^2}{2} \tilde{\phi}_r^2 - \tilde{\phi}^2 \right) r^2 \, \mathrm{d}r \, \mathrm{d}^2\omega.
$$

It remains to show that the first term in the above integral compensates for the second and gives us a control over the integral of $\tilde{\phi}^2$. We proceed similarly as for the proof of Lemma 4.3:

$$
\begin{aligned}
\int_0^{+\infty} \tilde{\phi}^2 r^2 \, \mathrm{d}r &= -\frac{2}{3} \int_0^{+\infty} r^3 \tilde{\phi} \tilde{\phi}_r \, \mathrm{d}r \\
&\leqslant \frac{\lambda}{3} \int_0^{+\infty} \tilde{\phi}^2 r^2 \, \mathrm{d}r + \frac{1}{3\lambda} \int_0^{+\infty} \tilde{\phi}_r^2 r^4 \, \mathrm{d}r, \quad \text{for any } \lambda > 0,
\end{aligned}
$$

and therefore

$$
\lambda(3 - \lambda) \int_0^{+\infty} \tilde{\phi}^2 r^2 \, \mathrm{d}r \leqslant \int_0^{+\infty} \tilde{\phi}_r^2 r^4 \, \mathrm{d}r.
$$

---

[*] Note that $\tilde{\phi}_r$ certainly does not have a limit at $r = 0$ in the generic case, but it remains bounded in the neighbourhood of $r = 0$, which is enough to justify the integration by parts and prove the equality.

The coefficient $\lambda(3-\lambda)$ is maximum for $\lambda = \frac{3}{2}$ where it takes the value $\frac{9}{4}$. Hence, for any $\varepsilon > 0$,

$$I \geqslant \int_{\Sigma_0} (\varepsilon(1+r^2)\tilde{\phi}_r^2 + (\tfrac{1}{2} - \varepsilon)\tfrac{9}{4}\tilde{\phi}^2 - \tilde{\phi}^2) r^2 \, \mathrm{d}r \, \mathrm{d}^2\omega,$$

and taking $0 < \varepsilon < \frac{1}{18}$, say $\varepsilon = \frac{1}{36}$, we obtain

$$I \geqslant \int_{\Sigma_0} \left(\frac{1+r^2}{36} \tilde{\phi}_r^2 + \tfrac{1}{16}\tilde{\phi}^2 \right) r^2 \, \mathrm{d}r \, \mathrm{d}^2\omega,$$

from which we conclude that

$$\int_{\Sigma_0} \left((1+r^2)(\partial_t \tilde{\phi})^2 + (1+r^2)(\partial_r \tilde{\phi})^2 + (1+r^2)\frac{1}{r^2}|\nabla_{S^2}\tilde{\phi}|^2 + \tilde{\phi}^2 \right) r^2 \, \mathrm{d}r \, \mathrm{d}^2\omega \lesssim \mathcal{E}_{X_0}(\psi).$$

The reciprocal inequality is trivial. This proves equivalence (6.3).

To prove equivalence (6.4), we first use Lemma 4.1 and then the equalities $\phi = r\tilde{\phi}$, $\partial_u = \partial_t$ and $\partial_R = -r^2(\partial_t + \partial_r)$. We obtain

$$\mathcal{E}_{\mathcal{H}_{1,u_0}} \simeq \int_{\{r > -u_0\}} \left(r^2\tilde{\phi}_t^2 + r^2\left(\tilde{\phi}_t + \tilde{\phi}_r + \frac{\tilde{\phi}}{r}\right)^2 + r^2\left|\frac{\nabla_{S^2}\tilde{\phi}}{r}\right|^2 + \tilde{\phi}^2 \right) r^2 \, \mathrm{d}r \, \mathrm{d}^2\omega$$

$$= \int_{\{r > -u_0\}} \left(2r^2\tilde{\phi}_t^2 + r^2\tilde{\phi}_r^2 + r^2\left|\frac{\nabla_{S^2}\tilde{\phi}}{r}\right|^2 + 2\tilde{\phi}^2 \right) r^2 \, \mathrm{d}r \, \mathrm{d}^2\omega$$

$$+ \int_{\{r > -u_0\}} (2r^2\tilde{\phi}_t\tilde{\phi}_r + 2r\tilde{\phi}_t\tilde{\phi} + 2r\tilde{\phi}_r\tilde{\phi}) r^2 \, \mathrm{d}r \, \mathrm{d}^2\omega.$$

Now for any $A, B, C \in \mathbb{R}$, we have

$$2AB + 2AC + 2BC \leqslant 2(A^2 + B^2 + C^2)$$

and using the facts that

$$\left(\sqrt{\tfrac{3}{2}}A + \sqrt{\tfrac{2}{3}}B + \sqrt{\tfrac{2}{3}}C \right)^2 \geqslant 0 \quad \text{and} \quad \left(\frac{1}{\sqrt{6}}B + \sqrt{\tfrac{2}{3}}C \right)^2 \geqslant 0$$

we also have

$$2AB + 2AC + 2BC = 2AB + \tfrac{4}{3}BC + 2AC + \tfrac{2}{3}BC$$
$$\geqslant -(\tfrac{3}{2}A^2 + \tfrac{2}{3}B^2 + \tfrac{2}{3}C^2) - (\tfrac{1}{6}B^2 + \tfrac{2}{3}C^2)$$
$$\geqslant -(\tfrac{3}{2}A^2 + \tfrac{5}{6}B^2 + \tfrac{4}{3}C^2).$$

Taking $A = r\tilde{\phi}_t$, $B = r\tilde{\phi}_r$ and $C = \tilde{\phi}$, this proves equivalence (6.4). $\qquad\square$

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