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COSIMPLICIAL SPACES AND COCYCLES

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Abstract Standard results from non-abelian cohomology theory specialize to a theory of torsors and stacks for cosimplicial groupoids. The space of global sections of the stack completion of a cosimplicial groupoid G is weakly equivalent to the Bousfield–Kan total complex of BG for all cosimplicial groupoids G. The k-invariants for the Postnikov tower of a cosimplicial space X are naturally elements of stack cohomology for the stack associated to the fundamental groupoid $\pi(X)$ of X. Cocycle-theoretic ideas and techniques are used throughout the paper.

Keywords: cosimplicial spaces; non-abelian cohomology; cocycles

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Introduction

This paper is an exposition of the basic homotopy theory of cosimplicial spaces, from a point of view that is informed by sheaf-theoretic homotopy theory.

This discussion interpolates ideas associated with the injective model structure for cosimplicial spaces with classical results of Bousfield and Kan. The injective model structure for cosimplicial spaces is a special case of the injective model structure for all small diagrams of simplicial sets $I \to s\mathbf{Set}$ which are defined on a fixed index category I, and this in turn is a special case of the injective model structure for simplicial sheaves (and presheaves) on a small Grothendieck site.

We effectively lose nothing by working within the injective structure for cosimplicial spaces, as it has the same weak equivalences as the Bousfield–Kan model structure. At the same time, interesting phenomena arise from the injective structure which correspond to well-studied features of the homotopy theory of simplicial sheaves.

In particular, the injective model structure creates an attractive theory of torsors and stacks for cosimplicial groupoids which is displayed in the second section of this paper. As in local homotopy theory, the category of cosimplicial groupoids has a model structure

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which is induced from the injective structure on cosimplicial spaces, for which the fibrant object associated to a cosimplicial groupoid H is its stack completion $\mathrm{St}(H)$. The stack completion may be described in global sections by torsors, suitably defined, and the link between torsors and stacks is achieved by using cocycles. The use of cocycle theory is a recurring theme of this paper.

There is a minor surprise: while the cosimplicial space BG associated to a cosimplicial groupoid G might not be Bousfield–Kan fibrant, any weak equivalence $G \to H$ induces a weak equivalence of the associated Bousfield–Kan total complexes. This is a consequence of Theorem 12 (or Corollary 13) below, which says that the total complex $\mathrm{Tot}(BG)$ and the classifying space $B(G - \mathbf{tors})$ of the groupoid of G-torsors have the same homotopy type.

The set of isomorphism classes of G-torsors, or equivalently the set of path components of the groupoid $G - \mathbf{tors}$, coincides with the set of morphisms $[*, BG]$ in the homotopy category of cosimplicial spaces, just as in sheaf theory.

The rest of the paper (especially §4) is taken up with an analysis of Postnikov towers and k-invariants.

The Postnikov tower of a cosimplicial space X is used to construct an analogue of the cohomological descent spectral sequence for the homotopy inverse limit of X, as one would expect [6], modulo the catch that X has to have a non-trivial cocycle for this approach to say anything at all. In fact, one can prove easily (Lemma 6) that X has a non-empty homotopy inverse limit if and only if there is a cocycle

$$* \xleftarrow{\simeq} U \to X.$$

There are injective fibrant cosimplicial spaces which do not have cocycles; see Example 5.

To analyse the Postnikov tower of a cosimplicial space X away from the cocycles of X, a different method is required.

The k-invariant of the standard fibration $P_nY \to P_{n-1}Y$ for a simply connected Kan complex Y can be described as the composite

$$P_{n-1}Y \to P_{n-1}Y/P_nY \to P_{n+1}(P_{n-1}Y/P_nY) =: Z_nY,$$

for $n \geqslant 2$, and Z_nY has a functorial base point. It follows (Lemma 23) that, for any diagram of simply connected Kan complexes X, there is a weak equivalence of diagrams

$$Z_nX \simeq K(H_n(Z_nX), n+1).$$

The resulting fibre homotopy sequence

$$P_nX \to P_{n-1}X \to Z_nX$$

specializes to a fibre sequence of diagrams which is indexed by the stack completion of the fundamental groupoid $\pi(X)$. The homotopy Cartesian square which is given by Theorem 26 is the result of applying a homotopy colimit functor to that sequence. From this point of view, the k-invariants of X are elements of stack cohomology groups that are associated to the fundamental groupoid $\pi(X)$.

There is nothing special about cosimplicial spaces in the k-invariant construction; that same construction applies to all diagram-theoretic homotopy types.

1. The injective model structure

Suppose that Δ is the category of finite ordinal numbers $\mathbf{n} = \{0, 1, \ldots, n\}$, $n \geqslant 0$, and all order-preserving functions between them.

Write $s\mathbf{Set}^{\Delta}$ for the category of cosimplicial spaces, meaning functors

$$X : \Delta \to s\mathbf{Set}$$

taking values in simplicial sets, and their natural transformations. It is standard practice to write $X^n = X(\mathbf{n})$ for $\mathbf{n} \in \Delta$.

The injective model structure on the category of cosimplicial spaces has weak equivalences and cofibrations defined *sectionwise*[1]: a map $X \to Y$ is a *weak equivalence* (respectively, *cofibration*) if and only if all maps $X^n \to Y^n$ are weak equivalences (respectively, cofibrations) of simplicial sets. The *injective fibrations* are defined by a right lifting property with respect to trivial cofibrations.

The weak equivalences coincide with the weak equivalences of the Bousfield–Kan model structure on cosimplicial spaces [1]. A *Bousfield–Kan cofibration* is a sectionwise cofibration which induces an isomorphism in maximal augmentations. Explicitly, the maximal augmentation X^{-1} of a cosimplicial set is the simplicial set which is defined by the equalizer diagram

$$X^{-1} \rightarrowtail X^0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} X^1$$

Thus, a Bousfield–Kan cofibration is a cofibration $A \to B$ as defined above, such that the map $A^{-1} \to B^{-1}$ is an isomorphism.

It follows that every injective fibration is a Bousfield–Kan fibration.

We also have the following.

Lemma 1. *There is a natural isomorphism*

$$\varprojlim_{n} X^n \xrightarrow{\cong} X^{-1}$$

for cosimplicial sets (hence for cosimplicial spaces) X.

The proof of Lemma 1 is elementary.

The simplicial set $\mathrm{Tot}(Y)$ is usually defined [1] for a Bousfield–Kan fibrant cosimplicial space Y by

$$\mathrm{Tot}(Y) = \mathbf{hom}(\Delta, Y),$$

where Δ is the cosimplicial space $\mathbf{n} \mapsto \Delta^n$ and $\mathbf{hom}(\Delta, Y)$ is the usual diagram-theoretic function complex.

In general, for cosimplicial spaces X and Y, the function complex $\mathbf{hom}(X, Y)$ is the simplicial set whose n-simplices are the cosimplicial space maps

$$X \times \Delta^n \to Y.$$

[1] The term "sectionwise" is commonly used in algebraic geometry. Homotopy theorists more often use "objectwise" or "pointwise" to describe the same concept.

This function complex construction defines a closed simplicial model structure for both the injective model structure and the Bousfield–Kan model structure on cosimplicial spaces.

The notation $*$ is used for the terminal object in cosimplicial spaces: it is the constant diagram on the one-point simplicial set. The cosimplicial space Δ is cofibrant for the Bousfield–Kan model structure, and it is a "fat point" in the sense that the canonical map $\Delta \to *$ is a weak equivalence.

It is now standard to say (see [7], for example) that the homotopy inverse limit $\underleftarrow{\mathrm{holim}}_{\mathbf{n}}\ X^n$ of a cosimplicial space X is defined by taking an injective fibrant model $j : X \to Z$ (a weak equivalence with Z injective fibrant), and then setting

$$\underleftarrow{\mathrm{holim}}_{\mathbf{n}}\ X^n = \underleftarrow{\lim}_{\mathbf{n}} Z^n = \mathbf{hom}(*, Z).$$

In this sense, the homotopy inverse limit is a derived inverse limit.

The injective model structure on cosimplicial spaces is cofibrantly generated, so one can make a natural choice of injective fibrant model. The homotopy inverse limit construction just described is therefore functorial in cosimplicial spaces X.

Lemma 2. *There is a natural weak equivalence*

$$\mathrm{Tot}(Y) \simeq \underleftarrow{\mathrm{holim}}_{\mathbf{n}}\ Y^n$$

for Bousfield–Kan fibrant objects Y.

Proof. Every injective fibrant cosimplicial space is Bousfield–Kan fibrant.

The cosimplicial space Δ is cofibrant for the Bousfield–Kan structure, so any weak equivalence $Y \to Y'$ of Bousfield–Kan fibrant objects induces a weak equivalence

$$\mathbf{hom}(\Delta, Y) \to \mathbf{hom}(\Delta, Y').$$

Thus, if $j : Y \to Z$ is an injective fibrant model for a Bousfield–Kan fibrant object Y, then the map

$$\mathbf{hom}(\Delta, Y) \xrightarrow{j_*} \mathbf{hom}(\Delta, Z)$$

is a weak equivalence. At the same time, the map $\Delta \to *$ is a weak equivalence of cofibrant objects for the injective model structure on cosimplicial spaces, so the induced map

$$\mathbf{hom}(*, Z) \to \mathbf{hom}(\Delta, Z)$$

is a weak equivalence since Z is injective fibrant. □

I shall now write

$$\mathrm{Tot}(X) = \underleftarrow{\mathrm{holim}}_{\mathbf{n}}\ X^n$$

for all cosimplicial spaces X.

There are natural identifications

$$\pi_0\,\mathrm{Tot}(X) = [*, X] \tag{1}$$

and

$$\pi_n(\mathrm{Tot}(X), x) \cong [S^n, X]_* \tag{2}$$

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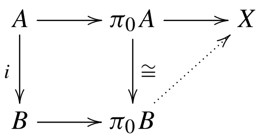
with morphisms in the homotopy category (respectively, pointed homotopy category) of cosimplicial spaces, where x is a cosimplicial space map $x : * \to X$, or a *global* base point for X. The pointed simplicial set $S^n = (S^1)^{\wedge n}$ is identified with a constant cosimplicial space in formula (2).

Here is another elementary statement.

Lemma 3. *Suppose that the cosimplicial space X is a cosimplicial set in the sense that the simplicial set X^n is discrete on a set of vertices for all n. Then X is injective fibrant.*

Proof. If the map $i : A \to B$ is a trivial cofibration of cosimplicial spaces, then the induced map $\pi_0 A \to \pi_0 B$ is an isomorphism of cosimplicial sets, and any map $A \to X$ factors uniquely through a map $\pi_0 A \to X$. Thus, all lifting problems



can be solved. □

Remark 4. Lemma 3 is a special case of a basic sheaf-theoretic fact. If F is a sheaf of sets on a small Grothendieck site $\mathcal{C}$, then the simplicial sheaf $K(F, 0)$ is fibrant for the injective model structure for simplicial presheaves on $\mathcal{C}$ which is defined by the topology; this statement appears, for example, as Lemma 6.10 of [6], with the same proof.

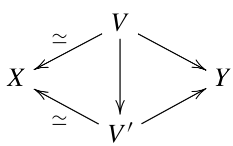
Suppose that I is a small category. In the I-diagram category $s\mathbf{Set}^I$ in simplicial sets, every presheaf is a sheaf, and so every I-diagram of simplicial sets which is simplicially constant is injective fibrant.

Example 5. There are cosimplicial spaces X for which $\mathrm{Tot}(X) = \emptyset$. The cosimplicial space Δ has empty inverse limit, and it follows that the cosimplicial space $\mathrm{sk}_0\,\Delta$ (vertices of Δ^n for all n) has empty inverse limit. This object is an injective fibrant cosimplicial space, by Lemma 3.

A *cocycle* (g, f) from X to Y is a diagram in cosimplicial spaces

$$X \xleftarrow[\simeq]{g} V \xrightarrow{f} Y,$$

such that g is a weak equivalence. A morphism of cocycles is a commutative diagram



These are the objects and morphisms of the cocycle category $h(X, Y)$. It is a basic result [9] for injective model structures on diagram categories that the assignment which sends

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a cocycle (g, f) to the morphism fg^{-1} in the homotopy category defines a bijection

$$\pi_0 Bh(X, Y) \xrightarrow{\cong} [X, Y]$$

between path components of the cocycle category $h(X, Y)$ and the set of morphisms $[X, Y]$ of the homotopy category.

We also have the following.

Lemma 6. *Suppose that X is a cosimplicial space. Then the space* $\mathrm{Tot}(X)$ *is non-empty if and only if there is a cocycle*

$$* \xleftarrow{\simeq} U \to X.$$

Proof. The space $\mathrm{Tot}(X)$ is non-empty if and only if an injective fibrant model $j : X \to Z$ for X has a vertex $* \to Z$. We show that the object Z has a global vertex $* \to Z$ if and only if the cocycle category $h(*, Z)$ is non-empty. The map j is a weak equivalence, so the cocycle category $h(*, Z)$ is non-empty if and only if $h(*, X)$ is non-empty, since these two categories have isomorphic sets of path components.

To see that the injective fibrant object Z has a map $* \to Z$ if the cocycle category $h(*, Z)$ is non-empty, let

$$* \xleftarrow{\simeq} U \xrightarrow{f} Z$$

be a cocycle, and observe that there is a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & Z \\ {\scriptstyle j}\downarrow & \nearrow{\scriptstyle f'} & \\ U' & & \end{array}$$

where j is a trivial cofibration and U' is injective fibrant. The map $U' \to *$ is a trivial injective fibration, and therefore has a section $* \to U'$, and there is a map

$$* \to U' \xrightarrow{f'} Z. \qquad \square$$

Lemma 7. *Suppose that the map $p : X \to Y$ is an injective fibration between injective fibrant cosimplicial spaces. Suppose that* $\mathrm{Tot}(Y) \neq \emptyset$, *and let* $x \in \mathrm{Tot}(Y)$ *be a vertex. Let F be the fibre over x, so the diagram*

$$\begin{array}{ccc} F & \xrightarrow{i} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ * & \xrightarrow[x]{} & Y \end{array}$$

is a pullback in cosimplicial spaces. Then $\mathrm{Tot}(F) \neq \emptyset$ *if and only if there is a cocycle* $* \xleftarrow{\simeq} U \xrightarrow{f} X$ *such that the composite cocycle*

$$* \xleftarrow{\simeq} U \xrightarrow{pf} Y$$

is in the path component of the cocycle $x : * \to Y$.

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Proof. If $\mathrm{Tot}(F) \neq \emptyset$ then there is a cocycle

$$* \xleftarrow{\simeq} U \xrightarrow{g} F,$$

and then the diagram

$$\begin{array}{ccc} U & \xrightarrow{ig} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ * & \xrightarrow[x]{} & Y \end{array}$$

commutes.

Conversely, suppose that we are given a map of cocycles

$$\begin{array}{ccc} U_1 & & \\ & \overset{f_1}{\searrow} & \\ \downarrow & & Y \\ & \underset{f_2}{\nearrow} & \\ U_2 & & \end{array}$$

and write $F_i = U_i \times_Y X$ for $i = 1, 2$. If either of the maps f_1 or f_2 lifts to X, then one of the objects F_i has a non-trivial cocycle. It follows that both of the objects F_i have non-trivial cocycles since the map $F_1 \to F_2$ is a weak equivalence. Thus, if the cocycle $x : * \to Y$ is in the path component of a cocycle $U \to Y$ that lifts to X, then the fibre F has a non-trivial cocycle. □

We finish this section by recalling some basic notation and concepts from [1], which will be needed later.

Write $\Delta_{\leqslant n}$ for the full subcategory of Δ on the ordinal numbers $\mathbf{k}$ with $k \leqslant n$. Write $\mathrm{Tr}_n X$ for the composite functor

$$\Delta_{\leqslant n} \subset \Delta \xrightarrow{X} s\mathbf{Set},$$

and let L_n be the left adjoint of the truncation functor $X \mapsto \mathrm{Tr}_n X$. The n-skeleton $\mathrm{sk}_n\, Y$ of a simplicial set Y can be defined by

$$\mathrm{sk}_n\, Y = \varinjlim_{\Delta^k \to Y, k \leqslant n} \Delta^k.$$

It follows that there is an isomorphism of cosimplicial spaces

$$\mathrm{sk}_n\, \Delta \cong L_n \mathrm{Tr}_n \Delta.$$

This relationship between skeleta and truncations is used to show that a map $f : \mathrm{sk}_{n-1}\, \Delta \to X$ can be extended to a map $f' : \mathrm{sk}_n\, \Delta \to X$ if and only if there is a map (simplex) $f' : \Delta^n \to X^n$ such that the diagram

$$\begin{array}{ccccc} \mathrm{sk}_{n-1}\, \Delta^n & \xrightarrow{i} & \Delta^n & \xrightarrow{s} & M^{n-1}\Delta \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f_*} \\ & & X^n & \xrightarrow[s]{} & M^{n-1}X \end{array}$$

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commutes. Here, the *matching space* $M^{n-1}X$ is defined by the assignment

$$M^{n-1}X = \varprojlim_{\mathbf{n}\to\mathbf{k},\, k<n} X^k \cong \varprojlim_{\mathbf{n}\overset{s}{\twoheadrightarrow}\mathbf{k},\, k<n} X^k,$$

which inverse limit can also be defined by the equalizer

$$M^{n-1}X \longrightarrow \prod_{0\leqslant i\leqslant n-1} X^{n-1} \rightrightarrows \prod_{0\leqslant i\leqslant j\leqslant n-1} X^{n-2}$$

which arises from the cosimplicial identities $s^j s^j = s^{j+1}s^i$, $i \leqslant j$. The canonical map $s : X^n \to M^{n-1}X$ is induced by the map

$$(s^i) : X^n \to \prod_{0\leqslant i\leqslant n-1} X^{n-1}$$

that is defined by the codegeneracies s^i.

2. Torsors

Suppose that H is a cosimplicial groupoid, with source and target maps $s, t : \mathrm{Mor}(H) \to \mathrm{Ob}(H)$ and identity $e : \mathrm{Ob}(H) \to \mathrm{Mor}(H)$.

An H-*diagram* X in sets can be defined in multiple equivalent ways.

(1) The internal definition: an H-diagram X is a cosimplicial set map $\pi : X \to Ob(H)$, together with an H-action

$$\begin{array}{ccc} \mathrm{Mor}(H) \times_{s,\pi} X & \xrightarrow{m} & X \\ \downarrow & & \downarrow \pi \\ \mathrm{Mor}(H) & \xrightarrow[t]{} & \mathrm{Ob}(H) \end{array}$$

which respects composition laws and identities of H.

(2) The H-diagram X consists of functors $X^k : H^k \to \mathbf{Set}$ and natural transformations $h_\theta : X^m \to X^n\theta$ for each $\theta : \mathbf{m} \to \mathbf{n}$, such that the usual compatibility conditions are satisfied.

The compatibility conditions amount to the following: the transformation h_1 associated to an identity morphism is the identity, and if one is given composable ordinal number maps

$$\mathbf{m} \xrightarrow{\theta} \mathbf{n} \xrightarrow{\gamma} \mathbf{k}$$

then the diagram of natural transformations

$$\begin{array}{ccc} X^m & \xrightarrow{h_\theta} & X^n\theta \\ {\scriptstyle h_{\gamma\theta}}\downarrow & & \downarrow {\scriptstyle h_\gamma\theta} \\ X^k(\gamma\theta) & \xrightarrow[=]{} & (X^k\gamma)\theta \end{array}$$

commutes.

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(3) Write $E_\Delta H$ for the Grothendieck construction of the cosimplicial diagram of groupoids H. The category $E_\Delta H$ has as objects all pairs $(\mathbf{n}, x)$ such that $\mathbf{n}$ is an ordinal number and x is an object of the groupoid H^n. A morphism $(\gamma, f) : (\mathbf{n}, x) \to (\mathbf{m}, y)$ of $E_\Delta H$ consists of an ordinal number morphism $\gamma : \mathbf{n} \to \mathbf{m}$ and a morphism $f : \gamma(x) \to y$ of the groupoid H^m. An H-diagram X in sets is a set-valued functor $X : E_\Delta H \to \mathbf{Set}$.

In the internal functor description (1), the elements of $\mathrm{Mor}(H)^n \times_{s,t} X^n$ over an object i of H^n are pairs (α, x) such that $\alpha : i \to j$ is a morphism of H^n and x is a member of the fibre $X^n(i)$ over i of the map $X^n \to \mathrm{Ob}(H^n)$. Then $m(\alpha, x) = \alpha_*(x) \in X(j)$ defines the corresponding functor $X^n : H^n \to \mathbf{Set}$ in description (2). The transformations $h_\theta : X^m(i) \to X^n(\theta(i))$ in description (2) correspond to the functions $\theta : X^m \to X^n$ in the commutative diagrams

$$\begin{CD} X^m @>{\theta}>> X^n \\ @V{\pi}VV @VV{\pi}V \\ \mathrm{Ob}(H^m) @>>{\theta}> \mathrm{Ob}(H^n) \end{CD} \tag{3}$$

by restriction to fibres.

Diagram (3) is the simplicial degree-0 part of the commutative diagram

$$\begin{CD} \underrightarrow{\mathrm{holim}}_{H^m}\, X^m @>{\theta}>> \underrightarrow{\mathrm{holim}}_{H^n}\, X^n \\ @VVV @VVV \\ BH^m @>>{\theta}> BH^n \end{CD}$$

of simplicial set maps which arises from description (2). The respective homotopy colimits define a cosimplicial space $\underrightarrow{\mathrm{holim}}_H X$ and a canonical cosimplicial space map $\underrightarrow{\mathrm{holim}}_H X \to BH$.

The homotopy colimit $\underrightarrow{\mathrm{holim}}_{H^n} X^n$ is the standard Bousfield–Kan homotopy colimit. It is the nerve $B(E_{H^n} X^n)$ of the translation category $E_{H^n} X^n$ for the functor $X^n : H^n \to \mathbf{Set}$. The objects of this category are pairs (i, x) with $i \in \mathrm{Ob}(H^n)$ and $x \in X^n(i)$, and its morphisms $(i, x) \to (j, y)$ consist of pairs (α, f) such that $\alpha : i \to j$ is a morphism of H^n and $f : X^n(\alpha)(x) \to y$ is a function.

The corresponding internally defined functor $X \to \mathrm{Ob}(H)$ is the part of the simplicial cosimplicial set map $\underrightarrow{\mathrm{holim}}_H X \to BH$ in simplicial degree 0, the identities are defined by the degeneracy

$$s_0 : X \to (\underrightarrow{\mathrm{holim}}_H X)_1 = \mathrm{Mor}(H) \times_{s,\pi} X,$$

and the multiplication map

$$m : \mathrm{Mor}(H) \times_{s,\pi} X \to X$$

is the face map d_0. The requirement that the multiplication map for the internal functor respects laws of composition amounts to the simplicial identity $d_0 d_1 = d_0 d_0$, and multiplication respects identities by the relation $d_1 s_0 = d_1 s_1 = 1$.

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An H-diagram X in sets is said to be an *H-torsor* if the cosimplicial space $\underrightarrow{\text{holim}}_H\ X$ is weakly equivalent to the terminal object $*$. A *morphism of H-torsors* $f : X \to Y$ is a natural transformation

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \text{Ob}(H) & \end{array}$$

in the usual sense.

The diagram of cosimplicial spaces

$$\begin{array}{ccc} X & \longrightarrow & \underrightarrow{\text{holim}}_H\ X \\ \downarrow & & \downarrow \\ \text{Ob}(H) & \longrightarrow & BH \end{array}$$

is sectionwise homotopy Cartesian for each H-diagram X, by the technology around Quillen's Theorem B; see [2, IV.5.7]. It follows that, if $f : X \to Y$ is a map of H-torsors, then the map $f : X \to Y$ of cosimplicial sets is a weak equivalence of cosimplicial spaces, and is therefore an isomorphism. The category $H - \mathbf{tors}$ of H-torsors and natural transformations between them therefore forms a groupoid.

The functor

$$\underrightarrow{\text{holim}}_H : H - \mathbf{tors} \to h(*, BH) \tag{4}$$

takes an H-torsor X to its *canonical cocycle*

$$* \xleftarrow{\simeq} \underrightarrow{\text{holim}}_H\ X \to BH$$

in cosimplicial spaces. The canonical cocycle functor has a left adjoint

$$\text{pb} : h(*, BH) \to H - \mathbf{tors} \tag{5}$$

which is defined in sections by taking path components of pullbacks along the canonical maps $B(H^n/x) \to BH^n$ (see [8], [10, Lem 9.16]), and we have the following.

Theorem 8. *There are induced isomorphisms*

$$\pi_0(H - \mathbf{tors}) \cong \pi_0 h(*, BH) \cong [*, BH].$$

Again, $[*, BH]$ denotes the set of morphisms in the homotopy category of cosimplicial spaces, and is also isomorphic to the set $\pi_0\,\text{Tot}(BH)$.

Theorem 8 is a special case of a principle which identifies non-abelian cohomology with isomorphism classes of torsors. The torsors that are considered here are torsors for groupoids; this concept is a generalization of torsors for groups, or classical principal homogeneous spaces.

Remark 9. Every torsor X for a cosimplicial groupoid H consists of functors $X^n : H^n \to \mathbf{Set}$, which are themselves torsors in the sense that the simplicial set maps $\underrightarrow{\text{holim}}_{H^n} X^n \to *$

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are weak equivalences. The simplicial set $\underrightarrow{\text{holim}}_{H^n} X^n$ is the nerve of the translation groupoid $E_{H^n} X^n$ which has objects consisting of pairs (x, v) with $x \in X^n(v)$. The set of objects of $E_{H^n} X^n$ is non-empty, and so there is a natural transformation

$$H^n(\ ,v) \xrightarrow{x} X^n$$

of functors defined on the groupoid H^n. This natural transformation is a map of H^n-torsors, and is therefore an isomorphism. This transformation is a *trivialization* of the torsor X^n in the geometric sense.

The collection of H-torsors therefore consists of functors $X \to \mathrm{Ob}(H)$ such that X has cardinality bounded above by $|\mathrm{Mor}(H)|$. We can thus assume that the groupoid $H - \mathbf{tors}$ is a small groupoid, and so the nerve $B(H - \mathbf{tors})$ is a simplicial set.

There is a way [10, Proposition 6.7] to replace the cocycle category $h(*, BH)$ by a small category up to "weak equivalence", but we shall not need this here.

Example 10. There is a cosimplicial groupoid H for which the associated cosimplicial space BH is not Bousfield–Kan fibrant.

To see this, observe first of all that, if $f : K \to H$ is a morphism between contractible groupoids, then the induced map $f : BK \to BH$ is a fibration if and only if f is surjective on objects.

If H is a cosimplicial contractible groupoid, then all of the groupoids $M^n H$ are contractible: if $(x_0, \ldots, x_n)$ and $(y_0, \ldots, y_n)$ are objects of $M^n H$, then there is a unique morphism $f_i : x_i \to y_i$ in H^{n-1}, and these morphisms f_i are consistent with the cosimplicial identities because they specialize to unique morphisms of H^{n-2} under the codegeneracy maps. It follows that the maps

$$s : BH^n \to M^{n-1}BH = B(M^{n-1}H)$$

are fibrations if and only if the functors $s : H^n \to M^{n-1}H$ are surjective on objects.

There are cosimplicial sets X for which the functions $s : X^n \to M^{n-1}X$ are not surjective in general. In the cosimplicial category Δ, the category $M^1\Delta \cong \mathbf{1} \times \mathbf{1}$ has four objects, so the functor

$$s : \mathbf{2} \to M^1\Delta$$

cannot be surjective on objects.

Suppose that X is a cosimplicial set, and let $C(X)$ be the degreewise contractible groupoid on X. Then the groupoid $C(X)^n$ has the set X^n as objects, and has exactly one morphism between any two elements of X^n. The groupoid $M^n C(X)$ is the contractible groupoid on the set $M^n X$ for all n, and so there are cosimplicial sets X for which the functors $s : C(X)^n \to M^{n-1}C(X)$ are not all surjective on objects.

The cosimplicial space BH for a cosimplicial groupoid H is not Bousfield–Kan fibrant in general, but we can form the function complex $\mathbf{hom}(\Delta, BH)$. This object is the nerve of a groupoid H^Δ whose objects are all cosimplicial functors $\Delta \to H$ and whose morphisms are the natural transformations of these functors.

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Lemma 11. *Suppose that U is a cosimplicial groupoid such that the map $BU \to *$ is a weak equivalence of cosimplicial spaces. Then the function complex* $\mathbf{hom}(\Delta, BU)$ *is a contractible space.*

Proof. One shows that there is an isomorphism of groupoids

$$U^{\Delta} \cong U^0,$$

while U^0 is a contractible groupoid by assumption.

The groupoid U^0 is non-empty. Pick $a \in U^0$, and let it define a functor $\mathbf{0} \xrightarrow{a} U^0$. The image of the vertex $i \in \mathbf{n}$ in U^n is determined by the composite

$$\mathbf{0} \xrightarrow{a} U^0 \xrightarrow{i_*} U^n,$$

where $i : \mathbf{0} \to \mathbf{n}$ is the ordinal number morphism which picks out the vertex i. A functor $a : \mathbf{n} \to U^n$ is defined by sending the morphism $i \leqslant j$ to the unique morphism $i_*(a) \to j_*(a)$ of the groupoid U^n. The functors $a : \mathbf{n} \to U^n$, $n \geqslant 0$, define a map $\Delta \to U$ of cosimplicial categories. Conversely, a morphism $\Delta \to U$ is completely determined by the object $\mathbf{0} \to U^0$ in cosimplicial degree 0.

The groupoids $(U^n)^{\mathbf{1}}$ of morphisms in U^n are contractible, so a morphism of the groupoid U^{Δ} is completely determined by the part in cosimplicial degree 0. □

Recall that a map $G \to G'$ of groupoids is a weak equivalence if and only if the induced map $BG \to BG'$ of classifying spaces is a weak equivalence.

There is a model structure on the category $\mathbf{Gpd}^{\Delta}$ of the cosimplicial groupoids for which the weak equivalences (respectively, fibrations) are those maps $f : G \to H$ for which the induced maps $BG \to BH$ are weak equivalences (respectively, injective fibrations) of cosimplicial spaces. This is a special case of general results about sheaves and/or presheaves of groupoids, for which the usual references are [3, 11].

This model structure has an associated definition of cocycles and cocycle categories $h(G, H)$ in cosimplicial groupoids: a cocycle in this category is a diagram

$$G \underset{\simeq}{\overset{g}{\leftarrow}} K \to H$$

in cosimplicial groupoids for which the map g is a weak equivalence.

A cocycle

$$* \overset{\simeq}{\leftarrow} V \to BH$$

in cosimplicial spaces defines a cocycle

$$* \overset{\simeq}{\leftarrow} \pi(V) \to H$$

in cosimplicial groupoids, by an adjointness argument, where $\pi(V)$ is the result of applying the fundamental groupoid functor in all cosimplicial degrees. The fundamental groupoid functor therefore defines a functor of cocycle categories

$$\pi : h(*, BG) \to h(*, G).$$

This functor has a right adjoint

$$B : h(*, G) \to h(*, BG)$$

which sends a cocycle

$$* \xleftarrow{\simeq} H \to G$$

in cosimplicial groupoids to the cocycle

$$* \xleftarrow{\simeq} BH \to BG$$

in cosimplicial spaces. It follows that there are isomorphisms

$$\pi_0 h(*, G) \cong \pi_0 h(*, BG) \cong [*, BG].$$

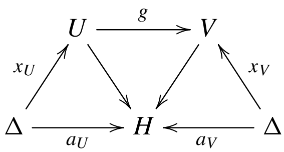
Make a fixed choice of morphism $x_U : \Delta \to U$ for all cosimplicial groupoids U such that $U \to *$ is a weak equivalence, as in the proof of Lemma 11.

Suppose that we are given a cocycle

$$* \xleftarrow{\simeq} U \xrightarrow{f} H$$

in cosimplicial groupoids. We have our fixed choice of morphism $x_U : \Delta \to U$ of cosimplicial categories. Write a_U for the composite $f \cdot x_U$.

Suppose that $g : U \to V$ is a morphism of cocycles, and consider the diagram



Then V^Δ is a contractible groupoid by Lemma 11, and so there is a unique natural transformation $g \cdot x_U \to x_V$, which induces a morphism $g_* : a_U \to a_V$ in H^Δ.

The assignment $g \mapsto g_*$ is functorial. We have therefore defined a functor

$$s : h(*, H) \to H^\Delta.$$

The functor

$$\underrightarrow{\text{holim}}_H : H - \mathbf{tors} \to h(*, BH)$$

factors through (and is defined by) a functor

$$E_H : H - \mathbf{tors} \to h(*, H),$$

where $(E_H X)^n$ for a torsor X is the translation groupoid which is associated to the functor $X^n : H^n \to \mathbf{Set}$.

Theorem 12. *The composite functor*

$$H - \mathbf{tors} \xrightarrow{E_H} h(*, H) \xrightarrow{s} H^\Delta \tag{6}$$

is a weak equivalence of groupoids.

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Proof. Suppose that X is an H-torsor. Following the choices made above, write $x = x_{E_H X}$ and $a = a_{E_H X}$. Consider the functor $x : \mathbf{n} \to E_{H^n} X^n$ in cosimplicial degree n. Then, for each $i \in \mathbf{n}$, $x(i) = (a(i), x_i)$ with $x_i \in X(a(i))$, and there is an induced isomorphism of H^n-torsors

$$H^n(\ ,a(i)) \xrightarrow[\cong]{x_i} X^n.$$

If $i \leqslant j$, then the diagram

$$\begin{array}{ccc} H^n(\ ,a(i)) & & \\ & \searrow^{x_i} & \\ \downarrow{\scriptstyle \alpha_*} & & X^n \\ & \nearrow_{x_j} & \\ H^n(\ ,a(j)) & & \end{array}$$

commutes, where $\alpha : a(i) \to a(j)$ is defined by the functor a.

Suppose that $g : X \to Y$ is a morphism of H-torsors. Then all diagrams

$$\begin{array}{ccc} H^n(\ ,a(i)) & \xrightarrow[\cong]{x_i} & X^n \\ {\scriptstyle s(g_*)}\downarrow & & \downarrow{\scriptstyle g} \\ H^n(\ ,b(i)) & \xrightarrow[y_i]{\cong} & Y^n \end{array}$$

commute, where $y = x_{E_H Y}$ and $b = a_{E_H Y}$. It follows that the morphism of torsors g is completely determined by the morphism $s(g_*)$ of H^Δ. The composite functor (6) is therefore fully faithful.

If $\tau : \Delta \to H$ is an object of H^Δ, then τ is a cocycle, and there is a cocycle morphism

$$\begin{array}{ccc} \Delta & \xrightarrow{\eta} & E_H\,\mathrm{pb}(\tau) \\ & {\scriptstyle \tau}\searrow & \downarrow \\ & & H \end{array}$$

for an H-torsor $\mathrm{pb}(\tau)$ which arises from the adjunction of (4) and (5). If $x = x_{E_H\,\mathrm{pb}(\tau)} : \Delta \to E_H\,\mathrm{pb}(\tau)$ with image $a : \Delta \to H$, then there is a unique morphism $\eta \to x$ in the trivial groupoid $(E_H\,\mathrm{pb}(\tau))^\Delta$, whose image is a morphism $\tau \to a$ of H^Δ. $\square$

Corollary 13. *The composite functor (6) induces a weak equivalence*

$$B(H - \mathbf{tors}) \xrightarrow{\simeq} B(H^\Delta) = \mathbf{hom}(\Delta, BH).$$

The functor

$$H \mapsto B(H - \mathbf{tors})$$

takes weak equivalences of cosimplicial groupoids to weak equivalences of spaces, so we also have the following.

Corollary 14. *Any weak equivalence $f : G \to H$ of cosimplicial groupoids induces a weak equivalence*

$$f_* : \mathbf{hom}(\Delta, BG) \to \mathbf{hom}(\Delta, BH).$$

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We say that a cosimplicial groupoid G is a *stack* if the cosimplicial space BG is injective fibrant. The injective model structure for cosimplicial groupoids is cofibrantly generated, so there is a functorial fibrant model $j : G \to St(G)$ for a cosimplicial groupoid G, such that the map j is a trivial cofibration and $St(G)$ is injective fibrant. This fibrant model $St(G)$ is a functorial stack completion for G. More generally, a weak equivalence $G \to H$ of cosimplicial groupoids such that H is a stack is called a *stack completion* of G.

If G is a cosimplicial groupoid and $j : G \to H$ is a stack completion of G, then Corollary 14 implies that the induced map

$$\mathbf{hom}(\Delta, BG) \xrightarrow{j_*} \mathbf{hom}(\Delta, BH)$$

is a weak equivalence. At the same time, the weak equivalence $\Delta \to *$ induces a weak equivalence

$$\mathbf{hom}(*, BH) \to \mathbf{hom}(\Delta, BH),$$

since BH is injective fibrant. Thus, in sheaf-theoretic language, the groupoid G^Δ is equivalent to the groupoid of global sections of the stack completion of the cosimplicial groupoid G.

3. Abelian cohomology

The results of this section appear in [1] for the most part, and are essentially well known. They are included here for the sake of completeness, and the overall description is from a sheaf-theoretic point of view.

Lemma 15. *Suppose that A is a cosimplicial abelian group. Then there is an abelian group homomorphism $j : M^{n-1}A \to A^n$ such that the composite $s \cdot j$ is the identity on $M^{n-1}A$. The homomorphism j is natural in cosimplicial abelian groups A.*

Proof. Suppose that $(0, \dots, 0, a_i, \dots, a_{n-1})$ is an element of $M^{n-1}A$. Then $s^j a_i = 0$ for $j < i$, and we have

$$\begin{aligned} sd^{i+1}a_i &= (s^0 d^{i+1}a_i, \dots, s^{i-1}d^{i+1}a_i, a_i, a_i, \dots) \\ &= (d^i s^0 a_i, \dots, d^i s^{i-1} a_i, a_i, a_i, \dots) \\ &= (0, \dots, 0, a_i, a_i, \dots). \end{aligned}$$

It follows that

$$(0, \dots, 0, a_i, \dots, a_n) - sd^{i+1}a_i = (0, \dots, 0, a_{i+1} - a_i, \dots),$$

and we construct $j(a_0, \dots a_n)$ inductively by setting

$$j(0, \dots, 0, a_i, \dots, a_n) = d^{i+1}a_i + j(0, \dots, 0, a_{i+1} - a_i, \dots). \qquad \square$$

Corollary 16. *Suppose that $f : A \to B$ is a map of cosimplicial objects in simplicial abelian groups such that each morphism $f : A^n \to B^n$ is a fibration of simplicial abelian groups. Then f is a Bousfield–Kan fibration.*

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Proof. We use the Dold–Kan correspondence [2, III.2.3], [2, III.2.11] to suppose that $p : A \to B$ is a morphism of cosimplicial objects in chain complexes such that each chain map $f : A^n \to B^n$ is surjective in non-zero degrees.

Suppose that $m > 0$, and consider the map

$$A^{n+1}_m \to B^{n+1}_m \times_{M^n B_m} M^n A_m.$$

We want to show that this homomorphism is surjective.

For this, it suffices to show that the indicated map p_* in the comparison of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \longrightarrow & A^{n+1}_m & \xrightarrow{s} & M^n A_m & \longrightarrow & 0 \\ & & {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & K_2 & \longrightarrow & B^{n+1}_m & \xrightarrow[s]{} & M^n B_m & \longrightarrow & 0 \end{array}$$

is surjective. This map p_* is a direct summand of the surjective map $p : A^{n+1}_m \to B^{n+1}_m$ by Lemma 15, and is therefore surjective. □

Corollary 17. *Suppose that A is a cosimplicial object in simplicial abelian groups. Then there are isomorphisms*

$$[*, A] \cong \pi_0 \mathbf{hom}(\Delta, A) \cong \pi_{ch}(N\mathbb{Z}\Delta, NA),$$

where N is the normalized chains functor and $\pi_{ch}(,)$ is the group of chain homotopy classes of maps in cosimplicial chain complexes.

Proof. The cosimplicial object A in simplicial abelian groups is Bousfield–Kan fibrant by Corollary 16, and so the group of homotopy classes of maps $\Delta \to A$ coincides with the group of morphisms $[\Delta, A] \cong [*, A]$ in the homotopy category of cosimplicial spaces.

Chain homotopy is defined by a natural path object for chain complexes, which therefore defines a path object for cosimplicial objects in simplicial abelian groups through the Dold–Kan correspondence, again by Corollary 16. It follows that the morphisms $\Delta \to A$ are homotopic if and only if the corresponding morphisms $N\mathbb{Z}\Delta \to NA$ are chain homotopic. □

Now suppose that A is a cosimplicial abelian group. For $k \leqslant n-1$, let $M^{n-1}_k A$ be the set of tuples $(a_0, \dots, a_k)$ with $a_i \in A^{n-1}$ and $s^i a_j = s^{j-1} a_i$ for $i < j$. There is a natural map $s : A^n \to M^{n-1}_k A$ which is defined by

$$s(a) = (s^0 a, s^1 a, \dots, s^k a).$$

This morphism s has a natural splitting and is therefore surjective, as in the proof of Lemma 15.

Write $cN_k A^n$ for the kernel of the map $s : A^n \to M^{n-1}_k A$. Then $cN_k A^n$ is the intersection of the kernels of the $s^i : A^n \to A^{n-1}$, for $0 \leqslant i \leqslant k$.

The coboundary

$$\delta = \sum_{i=0}^{n} (-1)^i d^i : A^n \to A^{n+1}$$

induces a morphism $cN_kA^n \to cN_kA^{n+1}$. We therefore have a natural cochain inclusion $cN_kA \subset A$. Write cNA for the intersection of the complexes cN_kA in A.

Lemma 18. *Suppose that A is a cosimplicial abelian group. Then the cochain complex map $cN_kA \subset A$ is a cohomology isomorphism for all k. The inclusion $cNA \subset A$ is also a cohomology isomorphism.*

Proof. There are short exact sequences of cochain complexes

$$0 \to cN_{k+1}A^n \to cN_kA^n \xrightarrow{s_*} C^n \to 0,$$

where $C^n = 0$ if $k \geqslant n-1$ and the map s_* is the map

$$s^{k+1} : cN_kA^n \to cN_kA^{n-1}$$

if $k < n-1$. In the latter case, the map s^{k+1} has a section given by d^{k+2}.

Suppose that $n > k+1$, and form the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & cN_{k+1}A^n & \longrightarrow & cN_kA^n & \xrightarrow{s^{k+1}} & cN_kA^{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta_*} & & \\
0 & \longrightarrow & cN_{k+1}A^{n+1} & \longrightarrow & cN_kA^{n+1} & \xrightarrow[s^{k+1}]{} & cN_kA^n & \longrightarrow & 0
\end{array}$$

Then

$$s^{k+1}\left(\sum_{j=0}^{n+1}(-1)^j d^j\right)(x) = \left(\sum_{j=k+3}^{n+1}(-1)^j d^{j-1}s^{k+1}\right)(x)$$

for $x \in cN_kA^n$, so

$$\delta_* = \sum_{j=k+2}^{n}(-1)^{j+1}d^j$$

for $n > k+1$. The cochain complex C^* has a contracting homotopy defined by the maps s_{k+1} in degrees where it is non-zero.

It follows that all inclusions $cN_{k+1}A \subset cN_kA$ are cohomology isomorphisms.

A similar argument shows that the inclusion $cN_0A \subset A$ is a cohomology isomorphism. The quotient complex for this inclusion is isomorphic to the cochain complex

$$0 \to A^0 \xrightarrow{d^1} A^1 \xrightarrow{d^1-d^2} A^2 \xrightarrow{d^1-d^2+d^3} A^3 \to \cdots,$$

and the quotient map is the map s^0 in positive degrees. The contracting homotopy is the map s^1. □

Suppose again that A is a cosimplicial abelian group, and form the cosimplicial space $K(A,n)$, as a cosimplicial object in simplicial abelian groups.

Lemma 19. *Suppose that A is a cosimplicial abelian group. Then there is a natural isomorphism*

$$\pi_0\mathbf{hom}(\Delta, K(A,n)) \cong H^n(A),$$

where $H^n(A)$ is the nth cohomology group of the cochain complex associated to A.

Proof. A cosimplicial chain map $f : N\mathbb{Z}\Delta \to A[-n]$ is uniquely specified by the chain complex morphisms

$$f^k : N\mathbb{Z}\Delta^k \to A^k[-n]$$

for $k \geqslant n$, which morphisms respect the cosimplicial identities. These cochain complex morphisms are completely determined by the morphism f^n, and in particular the image $f(\iota_n) \in A^n$ of the classifying simplex. The requirement that the diagram

$$\begin{array}{ccc}
N\mathbb{Z}\Delta^{n+1} & \xrightarrow{f^{n+1}} & A^{n+1}[-n] \\
{\scriptstyle d^i}\uparrow & & \uparrow{\scriptstyle d^i} \\
N\mathbb{Z}\Delta^{n} & \xrightarrow{f^{n}} & A^{n}[-n] \\
{\scriptstyle s^j}\downarrow & & \downarrow{\scriptstyle s^j} \\
N\mathbb{Z}\Delta^{n-1} & \xrightarrow[f^{n-1}=0]{} & A^{n-1}[-n]
\end{array}$$

commutes forces $f(\iota_n) \in cNA^n$ and $\sum_{i=0}^{n+1}(-1)^i d^i f(\iota_n) = 0$. Conversely, a cycle $z \in cNA^n$ completely determines the map f.

Similarly, a cosimplicial chain homotopy s between chain maps $f, g : N\mathbb{Z}\Delta^n \to A[n]$ is defined by the element $s(\iota_{n-1}) \in cNA^{n-1}$ such that

$$\delta s(\iota_{n-1}) = f(\iota_n) - g(\iota_n)$$

in cNA^n.

It follows that there is an isomorphism

$$\pi_{ch}(N\mathbb{Z}\Delta, A[-n]) \cong H^n(cNA)$$

which is natural in cosimplicial abelian groups A. There are natural isomorphisms

$$[*, K(A, n)] \cong \pi_{ch}(N\mathbb{Z}\Delta, A[-n])$$

and

$$H^n(cNA) \cong H^n(A)$$

by Corollary 17 and Lemma 18, respectively. □

Corollary 20. *Suppose that A is a cosimplicial abelian group. Then there are natural isomorphisms*

$$\pi_k \mathbf{hom}(\Delta, K(A, n)) \cong \begin{cases} H^{n-k}(A) & \text{if } 0 \leqslant k \leqslant n \\ 0 & \text{if } k > n. \end{cases}$$

Suppose that $i : A \to J^*$ is an injective resolution of A in the category of cosimplicial abelian groups, thought of as a morphism of unbounded chain complexes with A concentrated in chain degree 0. Then there is a induced weak equivalence of cosimplicial chain complexes

$$i : A[-n] \to \mathrm{Tr}(J^*[-n]),$$

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where Tr is the good truncation functor in degree 0 (which preserves homology isomorphisms). Write $K(J,n)$ for the cosimplicial object in simplicial abelian groups which is given by applying the Dold–Kan correspondence to $\mathrm{Tr}(J^*[-n])$. Then there is an induced weak equivalence

$$i : K(A,n) \to K(J,n).$$

The weak equivalences i and $\Delta \to *$ induce isomorphisms

$$\pi_{ch}(N\mathbb{Z}\Delta, A[-n]) \cong \pi_{ch}(N\mathbb{Z}\Delta, \mathrm{Tr}(J^*[-n])) \cong \pi_{ch}(N\mathbb{Z}*, \mathrm{Tr}(J^*[-n])).$$

The first of these isomorphisms is a consequence of Corollary 17.

For the latter, a cosimplicial space X defines a bicomplex $\hom(X_n, J^p)$ with associated spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^q(H_pX, A) \Rightarrow \pi_{ch}(\mathbb{Z}X, J^*[-p-q]) = \pi_{ch}(\mathbb{Z}X, \mathrm{Tr}(J^*[-p-q])). \tag{7}$$

It follows that any weak equivalence $X \to Y$ of cosimplicial spaces induces an isomorphism

$$\pi_{ch}(N\mathbb{Z}Y, \mathrm{Tr}(J^*[-n])) \xrightarrow{\cong} \pi_{ch}(N\mathbb{Z}X, \mathrm{Tr}(J^*[-n]))$$

for all n.

Remark 21. The ideas of the last few paragraphs, leading to the “universal coefficients” spectral sequence (7) and the displayed application, are again essentially sheaf theoretic. The spectral sequence (7) is a special case of a spectral sequence which relates homology sheaves to cohomology groups for simplicial sheaves and presheaves. Many of these ideas originated in [4], and the theory is discussed in some detail in [10].

We have, finally, an isomorphism

$$\pi_{ch}(N\mathbb{Z}*, \mathrm{Tr}(J^*[-n])) \cong H^{-n}(\varprojlim\ J^*) = \mathbf{R}\varprojlim{}^n(A),$$

where $\mathbf{R}\varprojlim{}^n(A)$ is the nth derived functor of the inverse limit functor on cosimplicial abelian groups.

Lemma 19 therefore implies the following.

Lemma 22. *There is an isomorphism*

$$H^n(A) \cong \mathbf{R}\varprojlim{}^n(A)$$

which is natural in cosimplicial abelian groups A.

A cosimplicial abelian group A is the analogue of a sheaf of abelian groups in the present context, and it is a consequence of Lemma 22 that the groups $H^n(A)$ are the corresponding sheaf cohomology groups. Sheaf cohomology groups are defined to be the higher derived functors of global sections, while “global sections” is a different name for the inverse limit functor.

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4. Postnikov towers

Suppose that

$$X_0 \xleftarrow{p} X_1 \xleftarrow{p} X_2 \leftarrow \cdots$$

is a tower of sectionwise fibrations of sectionwise fibrant cosimplicial spaces, and let $X = \varprojlim_n X_n$. Take an injective fibrant model

$$GX_0 \xleftarrow{q} GX_1 \xleftarrow{q} GX_2 \leftarrow \cdots$$

of the original tower by first taking an injective fibrant model $j : X_0 \to GX_0$ (j a trivial cofibration, GX_0 injective fibrant), and then inductively form diagrams

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{j} & GX_{n+1} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X_n & \xrightarrow[j]{} & GX_n \end{array}$$

such that all maps j are trivial cofibrations and all q are injective fibrations. Then the diagram of simplicial set maps

$$\begin{array}{ccc} X^m_{n+1} & \xrightarrow{j} & GX^m_{n+1} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X^m_n & \xrightarrow[j]{} & GX^m_n \end{array}$$

in each cosimplicial degree consists of trivial cofibrations j and Kan fibrations p and q. The maps j form a weak equivalence of injective fibrant towers in simplicial sets, so the maps

$$X^m = \varprojlim_n X^m_n \xrightarrow{j_*} \varprojlim_n GX^m_n$$

are weak equivalences for all m. It follows that the map

$$X = \varprojlim_n X_n \xrightarrow{j_*} \varprojlim_n GX_n$$

is a weak equivalence of cosimplicial spaces. The object $\varprojlim_n GX_n$ is injective fibrant, so j_* is an injective fibrant model of X.

Suppose that X is a cosimplicial Kan complex, and form the Postnikov tower

$$P_1X \xleftarrow{p} P_2X \xleftarrow{p} P_3X \xleftarrow{p} \cdots$$

by making a sectionwise construction. Then the canonical map $X \to \varprojlim_n P_nX$ is a weak equivalence, and the composite

$$X \xrightarrow{\simeq} \varprojlim_n P_nX \xrightarrow{j_*} \varprojlim_n GP_nX$$

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is an injective fibrant model of X. Write

$$GX = \varprojlim_n GP_nX.$$

Suppose, more generally, that I is a small category. It is well known (see also Remark 4) that the category of I-diagrams $X : I \to s\mathbf{Set}$, with natural transformations, has an injective model structure for which the weak equivalences and cofibrations are defined sectionwise, while the injective fibrations are defined by a right lifting property. The injective model structure on cosimplicial spaces is a special case of this object. Again, the terminal object for the I-diagram category is denoted by $*$, and we have cocycles

$$* \xleftarrow{\simeq} U \to X$$

and a cocycle category $h(*, X)$ for an I-diagram X. It is again a consequence of general results [9] that there is a bijection

$$\pi_0 h(*, X) \cong [*, X]$$

relating path components in the cocycle category and morphisms in the homotopy category for the injective model structure on $s\mathbf{Set}^I$.

We also have the following.

Lemma 23. *Suppose that the I-diagram F is a diagram of Eilenberg–Mac Lane spaces in the sense that there are weak equivalences $F(i) \simeq K(B(i), n)$ for all objects i of I, and for some fixed $n \geqslant 2$. Suppose that F has a cocycle*

$$* \xleftarrow{\simeq} U \to F$$

in I-diagrams. Then F is weakly equivalent to the I-diagram $K(H_nF, n)$.

Proof. Take a factorization

$$\begin{array}{ccc} U & \xrightarrow{i} & V \\ & \searrow & \downarrow{\scriptstyle p} \\ & & F \end{array}$$

where i is a cofibration and p is a trivial injective fibration. There are maps

$$F \xleftarrow{p} V \xrightarrow{h} \mathbb{Z}V \to \mathbb{Z}V/\mathbb{Z}U \to P_n(\mathbb{Z}V/\mathbb{Z}U) \xleftarrow{\simeq} K(H_n, n),$$

where h is the Hurewicz map and P_n is the nth Postnikov section functor in simplicial abelian groups. The composite

$$V \xrightarrow{h} \mathbb{Z}V \to \mathbb{Z}V/\mathbb{Z}U \to P_n(\mathbb{Z}V/\mathbb{Z}U)$$

is a sectionwise weak equivalence by the Hurewicz theorem. The I-diagram H_n in abelian groups can be identified with the integral homology H_nF of the diagram F. □

Corollary 24. *Suppose that F is a diagram of cosimplicial Eilenberg–Mac Lane spaces in the sense of Lemma 23. Then F is weakly equivalent in the I-diagram category to $K(A, n)$ for some cosimplicial abelian group A if and only if F has a cocycle.*

Proof. The object $K(A, n)$ has a global base point $* \to K(A, n)$ (and hence a cocycle) which is defined by the element 0. Thus, if F is weakly equivalent to $K(A, n)$, then F has a cocycle. The converse assertion is proved in Lemma 23. □

Suppose that the cosimplicial space X has a cocycle $U \to X$, or equivalently that there is a global point $* \to GX$ for GX. Let $x : * \to GX$ be a choice of base point, and write $x : * \to GP_nX$ for its images in the objects GP_nX. Define the cosimplicial space F_n by the pullback diagram

$$\begin{array}{ccc} F_n & \longrightarrow & GP_nX \\ \downarrow & & \downarrow{\scriptstyle q} \\ * & \xrightarrow[x]{} & GP_{n-1}X \end{array}$$

for $n \geqslant 2$. Then F_n is injective fibrant, and there is a pullback

$$\begin{array}{ccc} \mathbf{hom}(*, F_n) & \longrightarrow & \mathbf{hom}(*, GP_nX) \\ \downarrow & & \downarrow{\scriptstyle q_*} \\ * & \xrightarrow[x]{} & \mathbf{hom}(*, GP_{n-1}X) \end{array}$$

The space $\mathbf{hom}(*, F_n)$ is non-empty, and it follows from Lemma 23 that there is a weak equivalence

$$F_n \xrightarrow{\simeq} K(\pi_n(GX, x), n).$$

We know how to compute the homotopy groups of the space $\mathbf{hom}(*, F_n)$ on account of Corollary 20 in the presence of a cocycle for X. The spectral sequence for the tower of fibrations

$$\mathbf{hom}(*, GP_1X) \xleftarrow{q_*} \mathbf{hom}(*, GP_2X) \xleftarrow{q_*} \mathbf{hom}(*, GP_3X) \leftarrow \cdots$$

is a special case of the descent spectral sequence for a simplicial presheaf, with E_1-terms given by sheaf cohomology groups [5]. Thomason's reindexing trick [12, 5.54] converts these E_1-terms to E_2-terms of the form appearing in the Bousfield–Kan spectral sequence for the tower of fibrations $\{\mathrm{Tot}_s(GX)\}$; see also [6].

Remark 25. Suppose that Y is a simply connected Kan complex, with Postnikov tower $\{P_nY\}$. Form the natural cofibre sequence

$$P_nY \to P_{n-1}Y \to P_{n-1}Y/P_nY.$$

By this, we mean that we take a functorial replacement of the fibration $p : P_nY \to P_{n-1}Y$ by a cofibration $P_nY \to A_{n-1}Y$ and then we write $P_{n-1}Y/P_nY$ for the natural fibrant replacement of the quotient $A_{n-1}Y/P_nY$. Then let

$$P_{n-1}Y/P_nY \to P_{n+1}(P_nY/P_{n-1}Y)$$

be the usual fibration to the $(n+1)$th Postnikov section of the homotopy cofibre of p. Then the composite

$$P_{n-1}Y \to P_{n-1}Y/P_nY \to P_{n+1}(P_{n-1}Y/P_nY)$$

is the k-invariant k_q of the fibration q, and there is a natural homotopy fibre sequence

$$P_nY \xrightarrow{q} P_{n-1}Y \xrightarrow{k_q} P_{n+1}(P_{n-1}Y/P_nY) \tag{8}$$

which identifies P_nY with the homotopy fibre of k_q over the base point of the diagram $P_{n+1}(P_{n-1}Y/P_nY)$ which is defined by the image of P_nY.

The fibre sequence (8) is functorial in simply connected Kan complexes Y.

Suppose that H is a cosimplicial groupoid. For the discussion of weak equivalences that appears below, an H-diagram in simplicial sets is best viewed as a functor $Y : E_\Delta H \to s\mathbf{Set}$ which is defined on the Grothendieck construction for H.

There are functors $H^n \to E_\Delta H$ which are defined by $x \mapsto (\mathbf{n}, x)$, and the functors Y^n associated to the functor Y are the composites

$$H^n \to E_\Delta H \xrightarrow{Y} \mathbf{Set}.$$

The spaces $\underrightarrow{\text{holim}}_{H^n} Y^n$ define a cosimplicial space $\underrightarrow{\text{holim}}_H Y$ and a canonical map $\underrightarrow{\text{holim}}_H Y \to BH$. In this way we define a functor

$$\underrightarrow{\text{holim}}_H : s\mathbf{Set}^{E_\Delta H} \to s\mathbf{Set}^\Delta/BH$$

from the category of $E_\Delta H$ diagrams in simplicial sets to the category of cosimplicial spaces $Z \to BH$ over BH. Conversely, starting with a cosimplicial space map $X \to BH$, we form the pullbacks

$$\begin{array}{ccc} \text{pb}(X)_x & \longrightarrow & X^n \\ \downarrow & & \downarrow \\ B(H^n/x) & \longrightarrow & BH^n \end{array}$$

for each object $(\mathbf{n}, x)$ of the Grothendieck construction $E_\Delta H$. Then the simplicial sets $\text{pb}(X)_x$ define a functor $\text{pb}(X) : E_\Delta H \to s\mathbf{Set}$. The construction is plainly functorial in objects $X \to BH$, and so we have a functor

$$\text{pb} : s\mathbf{Set}^\Delta/BH \to s\mathbf{Set}^{E_\Delta H}.$$

The pullback functor pb is left adjoint to the homotopy colimit functor $\underrightarrow{\text{holim}}_H$ since H is a cosimplicial groupoid. The homotopy colimit functor preserves weak equivalences, while the pullback functor preserves weak equivalences by a Quillen Theorem B argument. The pullback functor also preserves cofibrations, so the pullback and homotopy colimit functors form a Quillen adjunction. This adjunction is a Quillen equivalence, since the counit map ϵ and unit map η are weak equivalences for all objects in the respective categories. See also [8, Lemma 18].

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The homotopy colimit functor $\underrightarrow{\text{holim}}_H$ preserves homotopy Cartesian diagrams, since it preserves weak equivalences and is the right adjoint part of a Quillen adjunction.

Suppose that X is a cosimplicial Kan complex. There are canonical maps

$$\begin{array}{ccc} P_n X & \xrightarrow{q} & P_{n-1} X \\ & \searrow \quad \swarrow & \\ & P_1 X & \\ & \simeq \downarrow j & \\ & BH & \end{array}$$

for $n \geqslant 2$, where H is a stack completion of the fundamental groupoid

$$\pi(P_1 X) \cong \pi(X)$$

of X. Let $j : \pi(X) \to H$ also denote the map induced by $j : P_1 X \to BH$.

Take $x \in H^m$. There is a morphism $\alpha : x \to j(y)$ for some $y \in \pi(X^m)$, and so there is a weak equivalence

$$\text{pb}(P_n(X))_x \simeq \text{pb}(P_n(X))_y, \tag{9}$$

where the latter space is determined by the pullback diagram

$$\begin{array}{ccc} \text{pb}(P_n X)_y & \longrightarrow & P_n X^m \\ \downarrow & & \downarrow \\ B(\pi(X^m)/y) & \longrightarrow & B\pi(X^m) \end{array}$$

The weak equivalence (9) is a special case of a weak equivalence

$$\text{pb}(Z)_x \simeq \text{pb}(Z)_y$$

which is defined and natural for objects $Z \to B\pi(X)$. It follows that the map

$$q_* : \text{pb}(P_n X)_x \to \text{pb}(P_{n-1} X)_x$$

is weakly equivalent to the map

$$q_* : \text{pb}(P_n X)_y \to \text{pb}(P_{n-1} X)_y. \tag{10}$$

The pullback $\text{pb}(P_n X)_y$ is naturally weakly equivalent to the Postnikov section $P_n\,\text{pb}(X)_y$ of the universal cover $\text{pb}(X)_y$ of X^m, and the induced fibrations (10) are weakly equivalent to the fibrations

$$q : P_n\,\text{pb}(X)_y \to P_{n-1}\,\text{pb}(X)_y.$$

It therefore follows from Remark 25 that there are homotopy fibre sequences of diagrams

$$\text{pb}(P_n X) \xrightarrow{q_*} \text{pb}(P_{n-1} X) \xrightarrow{k_q} P_{n+1}(\text{pb}(P_{n-1} X)/\text{pb}(P_n X)) =: Z_n X$$

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over $E_\Delta H$. The image of $\mathrm{pb}(P_nX)$ in Z_nX defines a global base point, and so Lemma 23 implies that Z_nX is weakly equivalent to $K(H_{n+1}(Z_nX), n+1)$ in the $E_\Delta H$-diagram category.

There is an isomorphism of groups

$$H_{n+1}(Z_nX)(\mathbf{m}, y) \cong \pi_n(X^m, y)$$

for each $y \in X^m$. For more general $x \in H^m$ there is a non-canonical isomorphism

$$H_{n+1}(Z_nX)(\mathbf{m}, x) \cong \pi_n(X^m, y)$$

which is induced by a morphism $x \to j(y)$ of H^m.

Taking homotopy colimits preserves homotopy Cartesian squares, and we have proved the following.

Theorem 26. *Suppose that X is a cosimplicial Kan complex, and suppose that $n \geqslant 2$. Then there is a homotopy Cartesian diagram*

$$\begin{array}{ccc} P_nX & \longrightarrow & B\,\mathrm{St}(\pi(X)) \\ {\scriptstyle q}\downarrow & & \downarrow \\ P_{n-1}X & \xrightarrow[k_{q*}]{} & \underrightarrow{\mathrm{holim}}_H K(H_n(Z_nX), n+1) \end{array} \qquad (11)$$

in cosimplicial spaces, where $Z_nX = P_{n+1}(\mathrm{pb}(P_{n-1}X)/\mathrm{pb}(P_nX))$ as a diagram over the Grothendieck construction $E_\Delta H$ of the stack completion H of the fundamental groupoid $\pi(X)$.

The k-invariant $k_q : \mathrm{pb}(P_{n-1}X) \to K(H_n(Z_nX), n+1)$ represents an element of the stack cohomology group

$$[\mathrm{pb}(P_{n-1}X), K(H_n(Z_nX), n+1)],$$

where stack cohomology is interpreted to mean abelian group cohomology for diagrams over the category $E_\Delta H$; see [8].

The ideas displayed in this section admit substantial generalization. One could, for example, start with an I-diagram X of Kan complexes, and observe that its Postnikov tower is defined over the I-diagram $\pi(X)$ of fundamental groupoids, as well over the stack completion H, which is an injective fibrant model of $\pi(X)$. Then one shows that the comparison $q_* : \mathrm{pb}(P_nX) \to \mathrm{pb}(P_{n-1}X)$ of associated diagrams on the Grothendieck construction E_IH has the formal properties that we saw in the proof of Theorem 26, so the sequence

$$\mathrm{pb}(P_nX) \xrightarrow{q_*} \mathrm{pb}(P_{n-1}X) \xrightarrow{k_q} P_{n+1}(\mathrm{pb}(P_{n-1}X)/\mathrm{pb}(P_nX)) =: Z_nX$$

is a homotopy fibre sequence of diagrams, and Z_nX is a diagram of Eilenberg–Mac Lane spaces having a global base point. It follows that there are homotopy Cartesian diagrams of the form (11) for all such I-diagrams X.

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