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A FINITELY GENERATED MODULAR ORTHOLATTICE

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By an ortholattice we mean a lattice with 0 and 1 and a complementation operation which is an involutorial antiautomorphism. The free modular ortholattice on two generators has 96 elements—cf. J. Kotas [8]. But

There exists a modular ortholattice with 3 *generators containing an infinite independent sequence of nonzero pairwise perspective* (*and orthogonal*) *elements.*

Due to Kaplansky [6] and Amemiya–Halperin [1] such a lattice cannot be embedded into a countably complete modular ortholattice. Also, this answers a question raised by G. Burns and W. Poguntke: There is a complemented modular lattice of infinite length which (as a lattice) is subdirectly irreducible and finitely generated. One just has to take a subdirectly irreducible ortholattice factor in which at least one (whence all) elements in the sequence stay different from zero. By orthomodularity this will be subdirectly irreducible as a lattice, too, and generated by six elements.

Actually, we construct the above lattice generators a, c, and d such that a is perspective to a' via d and a' is perspective in $[0, a'+c]$ to $a(a'+c)<a$ via c. Even, $0, 1, a, a', c, d$ form a partial lattice J_1^4 as defined in Day, Herrmann, and Wille [3]. Then, defining $a_0 = a$ and, recursively, $a_{n+1} = ((a_n + d)a' + c)a$ the sequence $a_n a'_{n+1}$ $(n \geq 0)$ will have the properties asked for—as is very well known cf. [7].

For lattice theory we refer to [1, 2, 7] for model theory to [9].

Outline of construction. Let V_n be a $2n$-dimensional rational vector space with basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ and L_n its lattice of subspaces. Consider the following four subspaces of V_n given by sets of generators: $A_n = \langle f_1, \ldots, f_n \rangle$, $B_n = \langle e_1, \ldots, e_n \rangle$, $C_n = \langle (e_2 + f_1), \ldots, (e_n + f_n) \rangle$, $D_n = \langle (e_1 + f_1), \ldots, (e_n + f_n) \rangle$. In the terminology of Gelfand and Ponomorev [4] this is a quadruple $S_3(2n, -1)$ of defect -1 over the rationals. Inspection yields that there is an orthocomplementation on L_n with $A'_n = B_n$. For a nontrivial ultraproduct L with constants A, B, C, D of the ortholattices L_n with constants A_n, B_n, C_n, D_n let U be the subortholattice generated by A, C, D. As a lattice U is generated by $\mathscr{C} = \{A, B, C, D, C', D'\}$.

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Then, we embed L (as a lattice) in the lattice of subspaces of a suitable vector space in which there exist complementary subspaces E and F such that $X = X \cap E + X \cap F$ for all X in $\mathscr{E}$. By Lemma 1.2 in Poguntke [10] we obtain that $\alpha X = X \cap E$ defines a lattice homomorphism on U. Due to modularity $(\alpha X)^* = \alpha(X')$ is well defined and $M = \alpha(U)$ becomes with $1_M = E$, $0_M = 0$, and the operation * an ortholattice homomorphic image of U. In particular, it is generated by A, C, D.

From the L_n M inherits the following relations:

$$\alpha B = \alpha A^*, \qquad \alpha A + \alpha D = \alpha B + \alpha D = 1$$
$$\alpha A \alpha C = \alpha A \alpha D = \alpha B \alpha C = \alpha B \alpha D = \alpha C \alpha D = 0.$$

Moreover, we choose E such that in addition

$$\alpha A + \alpha C = \alpha D + \alpha C = 1 \quad \text{and} \quad \alpha B + \alpha C < 1.$$

For that, we have to find a vector space representation of L in which the elements of $\mathscr{E}$ can be described, effectively. This is achieved by an axiomatic correspondence in the sense of Mal'cev [9] between lattices with distinguished elements, vector spaces, and coordinate descriptions. The same idea has been applied in [5].

An axiomatic correspondence. Let I_n be the set of nonzero integers z with $-n \le z \le n$ equipped with the constants 1 and n, the relation $\le$, the operation $z \mapsto -z$, and the partial operation $z \mapsto z+1$ defined for $1 \le z < n$. Let $\mathbf{Q}$ denote the field of all rational numbers and V_n the $\mathbf{Q}$-vector space of all maps from (the set) I_n into $\mathbf{Q}$. Let $\kappa_n : V_n \times I_n \to \mathbf{Q}$ be the function which picks the coordinates: $\kappa_n(f, i) = f(i)$. Finally, let L_n be the lattice of all subspaces of V_n with the euclidean orthocomplementation $'$ and ϕ_n the relation describing subspaces: $f\phi_n U$ iff $f \in U$. Now, consider the subspaces of V_n given as follows:

$f\phi_n A_n$ iff $\kappa_n(f, i) = 0$ for $i \le -1$

$f\phi_n B_n$ iff $\kappa_n(f, i) = 0$ for $i \ge 1$

$f\phi_n D_n$ iff $\kappa_n(f, 1) = \kappa_n(f, -i)$ for all i

$f\phi_n C_n$ iff $\kappa_n(f, 1) = \kappa_n(f, -n) = 0$ and $\kappa_n(f, -i) = \kappa_n(f, i+1)$ for $1 \le i < n$.

Clearly, $A'_n = B_n$ and

$$f\phi_n D'_n \quad \text{iff} \quad \kappa_n(f, i) = -\kappa_n(f, -i) \quad \text{for all } i$$
$$f\phi_n C'_n \quad \text{iff} \quad \kappa_n(f, -i) = -\kappa_n(f, i+1) \quad \text{for} \quad 1 \le i < n.$$

Let $(V, L, A, B, C, D, \phi, I, K, \kappa)$ be a nontrivial ultraproduct of the multibase structures $(V_n, L_n, A_n, B_n, C_n, D_n, \phi_n, I_n, \mathbf{Q}, \kappa_n)$. Due to the Theorem of Łos a first order statement holds in the ultraproduct if it holds in all but finitely many factors. Therefore, we may consider V as a K-subspace of K^I with κ yielding the components and L as a sublattice of the subspace lattice of V with

$U=\{f \mid f\phi U\}$. Also, the X in $\mathscr{E}$ are described the same way the X_n are. Now, let J be the subset of I which is generated by 1 under the operators $i \mapsto -i$ and $i \mapsto i+1$ $(i \geq 1)$—this can be though of as the nonzero integers. Let E and F be the subspaces of K^I consisting of all maps which vanish outside J and inside J, respectively. Then $E \oplus F = K^I$ and $X = X \cap E + X \cap F$ for X in $\mathscr{E}$ are immediate and so is $A \cap E + C \cap E = E$. Also $E \neq B \cap E + C \cap E$ since $f(1)=0$ for all f herein. Finally, to show $E = C \cap E + D \cap E$ define for given f in E

$$d(1)=d(-1)=f(1), \qquad c(1)=0, \qquad c(-1)=f(-1)-f(1)$$

and, recursively,

$$c(i+1)=c(-i), \qquad d(i+1)=d(-i-1)=f(i+1)-c(i+1),$$
$$c(-i-1)=f(i+1)-d(i+1)$$

for i in J and $c(i)=d(i)=0$, else. Then $c \in C \cap E$, $d \in D \cap E$, and $f=c+d$.

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