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Twists of a General Class of L-Functions by Highly Ramified Characters

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Abstract. It is shown that given a local L-function defined by Langlands-Shahidi method, there exists a highly ramified character of the group which when is twisted with the original representation leads to a trivial L-function.

The purpose of this short note is to prove a general lemma on twists by highly ramified characters of all the L-functions which are obtained from the Langlands-Shahidi method [2], [4], [5], [6]. The lemma generalizes Proposition 5.1 of [1] to many other L-functions and seems to be useful in applications [3]. The idea of the proof is simple and relies on basic general facts in representation theory. Its bulk, if any, is due to its remarkable generality.

Let F be a p-adic local field of characteristic zero. Denote by O its ring of integers and let P be the unique maximal ideal of O. Let q be the cardinality of the field O/P and normalize an absolute value $|\ |$ on F such that the absolute value of a generator of P equals q^{-1}.

Let $\mathbf{G}$ be a quasisplit connected reductive algebraic group over F. Fix a Borel subgroup $\mathbf{B} = \mathbf{TU}$ of $\mathbf{G}$ with unipotent radical $\mathbf{U}$ and a maximal torus $\mathbf{T}$. Let $\mathbf{A}_0$ be the maximal split torus of $\mathbf{T}$. Denote by $W(\mathbf{A}_0)$ the Weyl group of $\mathbf{A}_0$. Let Δ be the set of simple roots of $\mathbf{A}_0$ in $\mathbf{U}$. For each subgroup $\mathbf{H}$ of $\mathbf{G}$, we use H to denote the group of F-points of $\mathbf{H}$.

Let $\mathbf{P}$ be a maximal parabolic subgroup of $\mathbf{G}$ such that $\mathbf{N} \subset \mathbf{U}$. Let χ be the generic character of U defined via a non-trivial character ψ_F of F (*cf.* [4]). Fix an irreducible admissible χ-generic representation (*cf.* [4]) π of $M = \mathbf{M}(F)$.

Next let r be the adjoint action of ${}^LM$, the L-group of $\mathbf{M}$, on the Lie algebra ${}^L\mathfrak{n}$ of the L-group ${}^LN$ of $\mathbf{N}$. Decompose $r = \bigoplus_{i=1}^m r_i$ according to the order of eigenvalues of ${}^LA$, the L-group of the split center $\mathbf{A}$ of $\mathbf{M}$, in ${}^L\mathfrak{n}$ as in [5]. Let $s \in \mathbb{C}$. For each i, $1 \le i \le m$, let $L(s, \pi, r_i)$ denote the L-function attached to π and r_i as in [5]. Finally, let $X^*(\mathbf{M})_F$ denote the subgroup of F-rational characters of $\mathbf{M}$. The purpose of this note is to prove the following useful lemma:

Main Lemma 1 *There exists a rational character $\xi \in X^*(\mathbf{M})_F$ of $\mathbf{M}$ such that: Given any irreducible admissible χ-generic representation π of $M = \mathbf{M}(F)$, there exists a character η of F^* so that $L\big(s, \pi \otimes (\eta \cdot \xi), r_i\big) \equiv 1$, $1 \le i \le m$. Moreover η can be replaced by any character of F^* whose conductor is larger than that of η.*

The lemma seems to have useful applications in the theory of automorphic forms (*e.g.* [3]) as soon as the general method developed in [2], [4], [5], [6] is used. In particular, in the special case of $\mathbf{G} = \mathrm{GL}_{n+m}$, $\mathbf{M} = \mathrm{GL}_n \times \mathrm{GL}_m$, the lemma is equivalent to the Proposition 5.1 of [1] which uses a completely different method.

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The idea of the proof of Lemma 1 is quite simple, although it becomes cumbersome due to its generality.

Let $\theta \subset \Delta$ be a subset of simple roots so that

$$\pi \hookrightarrow \operatorname{Ind}_{M_\theta N_\theta}^{M} \sigma \otimes \mathbf{1},$$

where $\mathbf{P}_\theta = \mathbf{M}_\theta \mathbf{N}_\theta$, $\mathbf{N}_\theta \subset \mathbf{U} \cap \mathbf{M}$, and σ is an irreducible supercuspidal χ-generic representation of M_θ.

There exists the natural restriction map from $X^*(\mathbf{M})_F$ into $X^*(\mathbf{M}_\theta)_F$, sending $\xi \mapsto \xi_\theta \in X^*(\mathbf{M}_\theta)_F$. Then

$$(\operatorname{Ind}_{M_\theta N_\theta}^{M} \sigma) \otimes (\eta \cdot \xi) \cong \operatorname{Ind}_{M_\theta N_\theta}^{M} \big(\sigma \otimes (\eta \cdot \xi_\theta)\big)$$

and therefore

$$\pi \otimes (\eta \cdot \xi) \hookrightarrow \operatorname{Ind}_{M_\theta N_\theta}^{M} \big(\sigma \otimes (\eta \cdot \xi_\theta)\big).$$

Let $\tilde{w}_0$ be the longest element in $W(\mathbf{A}_0)$ modulo that of the Weyl group of $\mathbf{A}_0$ in $\mathbf{M}$. Then $\tilde{w}_0$ sends the unique simple root in $\mathbf{N}$ to a negative root, while $\tilde{w}_0(\theta) \subset \Delta$.

Decompose $\tilde{w}_0$ as $\tilde{w}_0 = \tilde{w}_{n-1} \cdots \tilde{w}_1$ with respect to θ as in Lemma 2.1.1 of [6]. Then for each j there exists a unique root $\alpha_j \in \Delta$ such that $\tilde{w}_j(\alpha_j) < 0$. For each j, $2 \leq j \leq n-1$, let $\bar{w}_j = \tilde{w}_{j-1} \cdots \tilde{w}_1$ with $\bar{w}_1 = 1$. Let $\theta_{j+1} = \tilde{w}_j(\theta_j)$, $\theta_1 = \theta$ and for each j, let $\Omega_j = \theta_j \cup \{\alpha_j\}$. Then $\mathbf{M}_{\Omega_j}$ contains $\mathbf{M}_{\theta_j}$ as a maximal Levi subgroup. Moreover $\bar{w}_j \colon \mathbf{M}_\theta \cong \mathbf{M}_{\theta_j}$ and $\bar{w}_j \colon \mathbf{A}_\theta \cong \mathbf{A}_{\theta_j}$, their split components, and $\bar{w}_j(\sigma) = \sigma_j$ becomes an irreducible supercuspidal representation of M_{θ_j}. Finally, if

$$\gamma_i(s, \pi, \psi_F, \tilde{w}_0) = \varepsilon(s, \pi, r_i, \psi_F) L(1-s, \pi, \tilde{r}_i) / L(s, \pi, r_i),$$

then by Theorem 3.5, part 3, of [5],

$$\gamma_i(s, \pi, \psi_F, \tilde{w}_0) = \prod_{j \in S_i} \gamma_{i(j)}\big(s, \bar{w}_j(\sigma), \psi_F, \tilde{w}_j\big),$$

where $i(j)$ and S_i are as in [5] and the factors on the right are defined the same way for the triple $(M_{\Omega_j}, M_{\theta_j}, \sigma_j)$. Here $\varepsilon(s, \pi, r_i, \psi_F)$ is the corresponding root number defined in [5].

A similar identity holds if π and σ are replaced by $\pi \otimes (\eta \cdot \xi)$ and $\sigma \otimes (\eta \cdot \xi_\theta)$, respectively.

To prove the main lemma it would be enough to show that η and ξ can be chosen in such a way that each $\gamma_{i(j)}\Big(s, \bar{w}_j\big(\sigma \otimes (\eta \cdot \xi_\theta)\big), \psi_F, \tilde{w}_j\Big)$ becomes a monomial in q^{-s} for each j. For, then the L-functions $L\big(s, \pi \otimes (\eta \cdot \xi_\theta), r_i\big) \equiv 1$ for tempered π and consequently the same holds for arbitrary π by further induction and analytic continuation.

But $\gamma_{i(j)}\Big(s, \bar{w}_j\big(\sigma \otimes (\eta \cdot \xi_\theta)\big), \psi_F, \tilde{w}_j\Big)$ becomes a monomial in q^{-s} as soon as the representation of M_{Ω_j} induced from any unramified twist of the representation $\bar{w}_j\big(\sigma \otimes (\eta \cdot \xi_\theta)\big)$ of M_{θ_j} is irreducible. More precisely, the irreducibility implies that the local coefficient $C_{\bar{\chi}}\Big(s\tilde{\alpha}_j, \bar{w}_j\big(\tilde{\sigma} \otimes (\eta^{-1} \cdot \xi_\theta)\big), \tilde{w}_j\Big)$ (*cf.* [5]) is a monomial in q^{-s} which then implies the same fact about $\gamma_{i(j)}\big(s, \bar{w}_j(\sigma \cdot \xi_\theta), \psi_F, \tilde{w}_j\big)$. Here one only needs to use Proposition 7.3 of [5] which implies that no cancellations take place among factors of γ's appearing in the local coefficient (Theorem 3.5 of [5], equation (3.11)).

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For representations of M_{Ω_j}, induced from unramified twists of $\bar{w}_j\big(\sigma \otimes (\eta \cdot \xi_\theta)\big) = \sigma_j \otimes (\eta \cdot \xi_{\theta_j})$, to become irreducible, it is enough to have

$$\tilde{w}_j\big(\sigma_j' \otimes (\eta \cdot \xi_{\theta_j})\big) \not\cong \sigma_j' \otimes (\eta \cdot \xi_{\theta_j}), \tag{1}$$

where σ_j' denotes an arbitrary unramified twist of $\sigma_j = \bar{w}_j(\sigma)$ and $\xi_{\theta_j} = \bar{w}_j(\xi_\theta)$. Observe that we may assume $\tilde{w}_j(\mathbf{A}_{\theta_j}) = \mathbf{A}_{\theta_j}$.

Now, by taking central characters in (1), it is enough to show that there exists a choice of ξ and η such that

$$\tilde{w}_j(\omega_j')(a)\eta\Big(\xi_{\theta_j}\big(\tilde{w}_j(a)\big)\Big) = \omega_j'(a)\eta\big(\xi_{\theta_j}(a)\big)$$

or

$$\eta \cdot \xi_{\theta_j}\big(\tilde{w}_j(a)a^{-1}\big) = \omega_j'\big(a\tilde{w}_j(a^{-1})\big) \tag{2}$$

does not hold for all $a \in A_{\theta_j}$, where ω_j' is the central character of σ_j'. Moreover the same is true if η is replaced with another character of higher conductor than η.

Consider the exact sequence

$$0 \to \mathbf{A}_{\theta_j}' \to \mathbf{A}_{\theta_j} \to \tilde{w}_j(\mathbf{A}_{\theta_j})\mathbf{A}_{\theta_j}^{-1} \to 0$$

in which the one before last arrow is defined by

$$a \mapsto \tilde{w}_j(a)a^{-1}.$$

Its kernel $\mathbf{A}_{\theta_j}'$ consists of all a with $\tilde{w}_j(a) = a$. It contains $\mathbf{A}_{\Omega_j}$. Set

$$\mathbf{A}_{\theta_j}^1 = \tilde{w}_j(\mathbf{A}_{\theta_j})\mathbf{A}_{\theta_j}^{-1} = \{a \in \mathbf{A}_{\theta_j} \mid \tilde{w}_j(a) = a^{-1}\},$$

where the last equality is easy to check. Observe that

$$\mathbf{A}_{\theta_j}^1 \cong \mathbf{A}_{\theta_j}/\mathbf{A}_{\theta_j}'$$

is connected.

We first specify ξ. Let $\mathfrak{n}$ be the Lie algebra of $\mathbf{N}$ and set

$$\xi(m) = \det\big(\mathrm{Ad}(m) \mid \mathfrak{n}\big),$$

$m \in \mathbf{M}$. Then $\xi \in X^*(\mathbf{M})$ and defines $\xi_\theta \in X^*(\mathbf{M}_\theta)$ by restriction. Define $\xi_j = \xi_{\theta_j} \in X^*(\mathbf{M}_{\theta_j})$ as before by

$$\xi_j\big(\bar{w}_j(m)\big) = \xi_\theta(m),$$

$m \in \mathbf{M}_\theta$. We need:

Lemma 2 $\xi_j \mid \mathbf{A}_{\theta_j}^1 \neq 1$.

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Proof Suppose $\xi_j\big(\tilde{w}_j(a_j)\big) = \xi_j(a_j)$, $\forall a_j \in \mathbf{A}_{\theta_j}$ which makes sense since we have assumed $\tilde{w}_j(\mathbf{A}_{\theta_j}) = \mathbf{A}_{\theta_j}$. Then

$$\begin{aligned}\xi_j(m_j) &= \xi_j\big(\bar{w}_j(m)\big)\\ &= \det\big(\mathrm{Ad}(m) \mid \mathfrak{n}\big)\\ &= \det\big(\mathrm{Ad}(m_j) \mid \mathfrak{n}_j\big),\end{aligned}$$

where $\mathfrak{n}_j = \bar{w}_j(\mathfrak{n})$, $\forall m_j = \bar{w}_j(m) \in \mathbf{M}_{\theta_j}$. We therefore have

$$\Big\langle \sum_{X_\alpha \in \mathfrak{n}_j} \alpha, \tilde{w}_j(a_j)\Big\rangle = \Big\langle \sum_{X_\alpha \in \mathfrak{n}_j} \alpha, a_j\Big\rangle \quad (\forall a_j \in \mathbf{A}_{\theta_j})$$

or

$$\Big\langle \sum_{X_\alpha \in \mathfrak{n}_j} \tilde{w}_j(\alpha), a_j\Big\rangle = \Big\langle \sum_{X_\alpha \in \mathfrak{n}_j} \alpha, a_j\Big\rangle.$$

Thus

$$\sum_{X_\alpha \in \mathfrak{n}_j} \tilde{w}_j(\alpha) = \sum_{X_\alpha \in \mathfrak{n}_j} \alpha.$$

Write:

$$\alpha = \alpha_j + \sum_{\beta \in \theta_j} m_\beta^\alpha \beta,$$

with $m_\beta^\alpha \in \mathbb{Z}$. Then

$$\sum_{X_\alpha \in \mathfrak{n}_j} \alpha = n_j \alpha_j + \sum_\alpha \sum_{\beta \in \theta_j} m_\beta^\alpha \beta,$$

where n_j is a positive integer. On the other hand

$$\sum_{X_\alpha \in \mathfrak{n}_j} \tilde{w}_j(\alpha) = n_j \tilde{w}_j(\alpha_j) + \sum_\alpha \sum_{\beta \in \theta_j} m_\beta^\alpha \tilde{w}_j(\beta).$$

But $\tilde{w}_j(\alpha_j) = -\alpha_j$ and $\tilde{w}_j(\beta) \neq \pm\alpha_j$. Equality

$$n_j \alpha_j + \sum_\alpha \sum_\beta m_\beta^\alpha \beta = -n_j \alpha_j + \sum_\alpha \sum_\beta m_\beta^\alpha \tilde{w}_j(\beta)$$

is impossible since α_j and $\beta \in \Delta$ and $n_j > 0$. This proves the lemma.

Corollary *$\xi_j \colon \mathbf{A}_{\theta_j}^1 \to \mathbb{G}_m$ is onto.*

Proof By Lemma 2 this is a non-constant morphism of a connected variety (of dimension 1) into $\mathbb{G}_m$.

Proof of Main Lemma 1 By Corollary $\xi_j\big(\tilde{w}_j(A_{\theta_j})A_{\theta_j}^{-1}\big)$ is open in F^*. Choose $\ell_j \in \mathbb{Z}^+$ such that $1 + P^{\ell_j} \subset \xi_j\big(\tilde{w}_j(A_{\theta_j})A_{\theta_j}^{-1}\big)$. Take η with conductor larger than the larger of ℓ_j and

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the conductor of ω_j, the central character of σ_j, which is the same for all σ'_j, for all j. This completes the proof.

Corollary of Lemma 1 *There exist a positive integer N such that for every character η of F^* with conductor larger than N, the function*

$$\gamma(s, \pi_\eta, r_i, \psi_F) = \varepsilon(s, \pi_\eta, r_i, \psi_F) L(1 - s, \pi_\eta, \tilde{r}_i)/L(s, \pi_\eta, r_i),$$

$\pi_\eta = \pi \otimes (\eta \cdot \xi)$, *is a monomial in* q^{-s}.

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