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# Conjugacy Classes and Binary Quadratic Forms for the Hecke Groups

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*Abstract.* In this paper we give a lower bound with respect to block length for the trace of non-elliptic conjugacy classes of the Hecke groups. One consequence of our bound is that there are finitely many conjugacy classes of a given trace in any Hecke group. We show that another consequence of our bound is that class numbers are finite for related hyperbolic $\mathbb{Z}[\lambda]$-binary quadratic forms. We give canonical class representatives and calculate class numbers for some classes of hyperbolic $\mathbb{Z}[\lambda]$-binary quadratic forms.

## 1 Introduction

In [3], Fine describes an algorithm that produces a representative of each conjugacy class of the modular group with trace less than or equal to a given bound. Schmidt and Sheingorn [13] observed that Fine's algorithm generalizes from the modular group to the Hecke groups. A key idea in each setting is to find a representative of every conjugacy class that can be written as a product of elements of a standard set of generating elements. In this paper we verify Schmidt and Sheingorn's observation and calculate a lower bound for the trace of a conjugacy class in terms of the length of the product of generators for the class representative (the block length). An immediate corollary of this result is that there are finitely many conjugacy classes of a given trace in any Hecke group.

In [9], the second author develops a theory of reduction of hyperbolic $\mathbb{Z}[\lambda]$-binary quadratic forms, where $\lambda$ is the minimal translation in the associated Hecke group. Equivalence classes of these binary quadratic forms correspond to conjugacy classes of associated Hecke groups, so a corollary of our main result gives a lower bound on the discriminant of hyperbolic $\mathbb{Z}[\lambda]$-binary quadratic forms. We use this to show that class numbers are finite for these forms, and we calculate some of those class numbers. We describe a procedure that determines a unique reduced $\mathbb{Z}[\lambda]$-binary quadratic form for every equivalence class with discriminant less than a given bound.

Our work with quadratic forms is motivated by the problem of characterizing rational period functions for automorphic integrals on the Hecke groups. The second author uses $\mathbb{Z}[\lambda]$-binary quadratic forms to give a partial solution to this problem in [2], and uses that solution to prove a Hecke correspondence theorem between related automorphic integrals and Dirichlet series in [10]. A full characterization of rational period functions for automorphic integrals on the Hecke groups would

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generalize the characterization of rational period functions for modular integrals on the modular group, which was completed by Choie and Zagier [1], and Parson [8]. The Choie–Zagier–Parson result uses properties of classical quadratic forms, which correspond to elements of the modular group as described in [14].

In Section 2 we give background definitions and facts about the Hecke groups and related continued fractions and binary quadratic forms. In Section 3 we present results about conjugacy classes in Hecke groups, including the lower bound that is our main result, Theorem 3.6. In Section 4 we apply our result to the problem of class numbers for hyperbolic $\mathbb{Z}[\lambda]$-binary quadratic forms.

# 2 Background

## 2.1 Hecke Groups

Let $\lambda$ be a positive real number, and put $S = S_\lambda = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The *Hecke groups* are the groups $G_p = G(\lambda_p) = \langle S_\lambda, T\rangle/\{\pm I\} \subseteq \mathrm{PSL}(2,\mathbb{R})$, where $\lambda = \lambda_p = 2\cos(\pi/p)$, for $p \geq 3$ is an integer. Erich Hecke showed that these values of $\lambda$ are the only ones between 0 and 2 for which $G(\lambda)$ is discrete [4]. (If $\lambda \geq 2$, then $G(\lambda)$ is discrete but has a simpler group structure.)

Throughout this paper we fix the integer $p \geq 3$ and the positive real number $\lambda = \lambda_p = 2\cos(\pi/p)$.

The Hecke groups are projective matrix groups, so they are isomorphic to groups of linear fractional transformations. We will use both ways of thinking about group elements. Elements of $G_p$ have entries in $\mathbb{Z}[\lambda_p]$, so $G_p$ is a subgroup of $\mathrm{PSL}(2,\mathbb{Z}[\lambda_p])$. One of the Hecke groups is the modular group $G_3 = \mathrm{PSL}(2,\mathbb{Z})$, however $G_p \subsetneq \mathrm{PSL}(2,\mathbb{Z}[\lambda_p])$ for $p > 3$.

Let $U = ST = \left(\begin{smallmatrix} \lambda & -1 \\ 1 & 0 \end{smallmatrix}\right)$. For each $p \geq 3$, $G_p$ has the group relations $T^2 = U^p = I$. Each Hecke group $G_p$ is the free product of the cyclic group of order $p$ generated by $U$, and the cyclic group of order 2 generated by $T$.

We let $\mathrm{Tr}(M)$ denote the trace of $M \in G_p$. An element $M \in G_p$ is *hyperbolic* if $|\mathrm{Tr}(M)| > 2$, *parabolic* if $|\mathrm{Tr}(M)| = 2$, and *elliptic* if $|\mathrm{Tr}(M)| < 2$.

Because $G_p$ is discrete, the *stabilizer* of any complex number $z$ in $G_p$, $\mathrm{stab}(z) = \{M \in G_p \mid Mz = z\}$ is a cyclic subgroup of $G_p$ [5]. Thus the fixed point sets of any two nontrivial elements of $G_p$ are identical or disjoint, and all nontrivial elements of a stabilizer have identical fixed points. We designate fixed points as *hyperbolic*, *parabolic*, or *elliptic* according to the elements fixing them. Hyperbolic elements have two real fixed points, one attracting and one repelling. Parabolic elements have a single real fixed point, and elliptic elements have two complex conjugate fixed points. We say that an element is *primitive* if it generates the stabilizer of each of its fixed points.

We will always take the trace to be positive, which we may do because the Hecke groups are projective. Trace is invariant under conjugation, so elements of a conjugacy class in $G_p$ all have the same trace and all have the same designation as hyperbolic, parabolic, or elliptic.

We say that complex numbers $z_1$ and $z_2$ are $G_p$-*equivalent* if there exists $M \in G_p$ such that $Mz_1 = z_2$. $G_p$-equivalence is an equivalence relation, so $G_p$ partitions

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complex numbers into equivalence classes. Two fixed points are $G_p$-equivalent if and only if the linear fractional transformations fixing them are conjugate to each other in $G_p$. Thus equivalence classes of numbers contain either all fixed points of the same kind, or no fixed points.

### 2.2 $\lambda$-continued Fractions

We will use a modification of Rosen's continued fractions [11], which are closely associated with the Hecke groups.

For real $\alpha$ we put $\alpha_0 = \alpha$ and define $r_j = [\frac{\alpha_j}{\lambda}] + 1$ and $\alpha_{j+1} = \frac{1}{r_j\lambda - \alpha_j}$ for $j \geq 0$. Then $\alpha_j = r_j\lambda - \frac{1}{\alpha_{j+1}}$ for $j \geq 0$ and

$$\alpha = r_0\lambda - \cfrac{1}{r_1\lambda - \ddots} = [r_0; r_1, \dots]$$

is the $\lambda_p$-*continued fraction* ($\lambda_p$-CF or $\lambda$-CF) for $\alpha$. An *admissible* $\lambda$-CF is one that arises from a finite real number by this algorithm.

An admissible $\lambda_p$-CF has at most $p-3$ consecutive ones in any position but the beginning, and has at most $p-2$ consecutive ones at the beginning [9]. Schmidt and Sheingorn [13] show that a real number is a fixed point of $G_p$ if and only if it has a periodic $\lambda$-CF; the number is parabolic if its $\lambda_p$-CF has period

$$\Lambda_3 = [\overline{2, \underbrace{1, \dots, 1}_{p-3}}],$$

and hyperbolic if its $\lambda_p$-CF has any other period.

### 2.3 $\mathbb{Z}[\lambda]$-binary Quadratic Forms

We consider indefinite binary quadratic forms ($\lambda_p$-BQFs or $\lambda$-BQFs) $Q(x, y) = Ax^2 + Bxy + Cy^2$ of discriminant $D = B^2 - 4AC > 0$, where $A, B, C \in \mathbb{Z}[\lambda]$. We also denote each $\lambda$-BQF $Q(x, y)$ by $Q = [A, B, C]$.

Elements of the corresponding Hecke group act on $\lambda$-BQFs by $(Q \circ M)(x, y) = Q(ax + by, cx + dy)$ for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_p$. This action preserves discriminants. We say that two $\lambda$-BQFs $Q$ and $Q'$ are $G_p$-*equivalent*, and write $Q \sim Q'$, if there exists a $V \in G_p$ such that $Q' = Q \circ V$. This is an equivalence relation, so $G_p$ partitions the $\lambda$-BQFs into equivalence classes of forms of the same discriminant.

### 2.4 Matrices, Forms, and Fixed Points

A key tool in [9] is an explicit isomorphism between primitive, hyperbolic elements of $G_p$, certain indefinite $\lambda$-BQFs, and hyperbolic fixed points.

Under that isomorphism a primitive hyperbolic element (which we take to have positive trace) $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_p$ corresponds to the indefinite form $\rho(M) = Q_M = [c, d - a, -b]$ with discriminant $D = (a + d)^2 - 4$. Images of hyperbolic elements of $G_p$ are indefinite forms. We say that the forms in the range of $\rho$ are *hyperbolic* $\lambda$-BQFs. Two hyperbolic $\lambda$-BQFs are $G_p$-equivalent if and only if the corresponding elements of $G_p$ are conjugate to each other. Thus $\lambda$-BQFs in the range of $\rho$ are in equivalence classes that correspond to conjugacy classes in $G_p$.

The second part of the isomorphism in [9] associates the hyperbolic form $Q = [A, B, C]$ with discriminant $D$ equal to the real number $\sigma(Q) = \alpha_q = \frac{-B+\sqrt{D}}{2A}$. The associated matrix $M = (\sigma \circ \rho)^{-1}(\alpha)$ is primitive, hyperbolic, and has $\alpha$ as an attracting fixed point [9, Lemma 4].

This isomorphism includes all hyperbolic fixed points. If $M \in G_p$ is hyperbolic and $\rho(M) = Q_M = [A, B, C]$ and $\sigma(Q_M) = \alpha = \frac{-B+\sqrt{D}}{2A}$, then $M^{-1}$ is also hyperbolic and $\rho(M^{-1}) = -Q_M = [-A, -B, -C]$ and $\sigma(-Q_M) = \alpha' = \frac{-B-\sqrt{D}}{2A}$. The fixed point $\alpha'$ is repelling for $M$ but attracting for $M^{-1}$. We call $\alpha'$ the *Hecke conjugate* of $\alpha$.

Equivalence classes of hyperbolic fixed points correspond to conjugacy classes of corresponding matrices, as well as to equivalence classes of hyperbolic $\lambda$-BQFs.

We can use $\lambda$-continued fractions to make explicit the map $\tau = (\sigma \circ \rho)^{-1}$ from hyperbolic fixed points to primitive hyperbolic elements of $G_p$. The following lemma is Lemma 6 in [9].

***Lemma 2.1*** *Fix $p \geq 3$ and $\lambda = \lambda_p = 2\cos(\pi/p)$. Suppose that*

$$\alpha = [r_0; r_1, \ldots, r_n, \overline{r_{n+1}, \ldots, r_{n+m}}]$$

*is a hyperbolic fixed point of $G_p$. Then the corresponding primitive linear fractional transformation is given by $\tau(\alpha) = RPR^{-1}$, where $R = S^{r_0}TS^{r_1}T\cdots S^{r_n}T$ and $P = S^{r_{n+1}}TS^{r_{n+2}}T\cdots S^{r_{n+m}}T$.*

We could extend this isomorphism to include primitive parabolic elements of $G_p$, certain semidefinite $\lambda$-BQFs, and parabolic fixed points, but we will not need that for our purposes. In this paper we will only consider $\lambda$-BQFs that are hyperbolic according to this isomorphism.

### 2.5 Reduction of $\lambda$-BQFs

The theory of reduction of hyperbolic $\mathbb{Z}[\lambda]$-binary quadratic forms in [9] uses $\lambda$-CFs and the isomorphism between primitive hyperbolic elements of $G_p$, hyperbolic indefinite $\lambda$-BQFs, and hyperbolic fixed points.

We say that a real number $\alpha$ is a $G_p$*-reduced* number if the $\lambda_p$-CF expansion of $\alpha$ is purely periodic with period other than $\Lambda_3$. A $G_p$-reduced number $\alpha$ is hyperbolic, and we say that the associated hyperbolic $\lambda$-BQF is $G_p$*-reduced*. Every hyperbolic equivalance class of $\lambda$-BQFs contains finitely many reduced forms that correspond to the cyclic permutations of the associated $\lambda$-CF period. Thus the reduced $\lambda$-BQFs in a hyperbolic equivalence class comprise a cycle.

## 3 Conjugacy Classes in Hecke Groups

Fine shows in [3] that the non-elliptic conjugacy classes in $\Gamma(1)$ each have a representative that is a cyclically reduced word in $S$ and $STS$. Fine uses this to give a lower bound for trace in terms of the "block length" of these products. Schmidt and Sheingorn [13] observe that Fine's algorithm generalizes to the Hecke group $G_p$, using the generators $V_j = U^{j-1}S$ for $1 \le j \le p-1$, where $U = ST = \left(\begin{smallmatrix} \lambda & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We first show that every conjugacy class in $G_p$ has a representative that is a product of these $V_j$.

***Lemma 3.1*** *Fix $p \ge 3$, put $\lambda = \lambda_p$, and let $V_j = U^{j-1}S$ for $1 \le j \le p-1$. Then every non-elliptic element $M \in G_p$ is conjugate in $G_p$ to a product of the $V_j$, that is, $M$ is conjugate to*

$$W = V_{j_1} V_{j_2} \cdots V_{j_t},$$

*where $1 \le j_k \le p-1$ for $1 \le k \le t$, and $t \in \mathbb{Z}^+$. This product is unique, except for cyclic permutations.*

**Proof** We first suppose that $M \in G_p$ is hyperbolic and let $\alpha$ be its attracting fixed point. Then $\alpha$ has a periodic $\lambda$-CF expansion of the form

$$\alpha = [r_0; r_1, \dots, r_n, \overline{r_{n+1}, \dots, r_{n+m}}],$$

where the period is not $\Lambda_3$. If $M$ is primitive we have by Lemma 2.1 that $M$ is conjugate to $P = S^{r_{n+1}} T S^{r_{n+2}} T \cdots S^{r_{n+m}} T$. We let $r_{a_1}, r_{a_2}, \dots, r_{a_q}$ be the entries in the period $[\overline{r_{n+1}, \dots, r_{n+m}}]$ that are greater than 1; we let $b_0$ be the number of leading ones and we let $b_k$ be the number of consecutive ones following $r_k$ for $1 \le k \le q$. Then

$$P = (ST)^{b_0} S^{r_{a_1}} T (ST)^{b_1} S^{r_{a_2}} T (ST)^{b_2} \cdots S^{r_{a_q}} T (ST)^{b_q}.$$

The restrictions on the number of consecutive ones in an admissible $\lambda$-CF (see [9, Lemma 3]) mean that $b_k \le p-3$ for $1 \le k \le q-1$ and $b_0 + b_q \le p-3$. We calculate that $P$ is conjugate to

$$\begin{aligned} W &= \big((ST)^{b_0} S\big)^{-1} P \big((ST)^{b_0} S\big) \\ &= S^{r_{a_1}-1} T (ST)^{b_1} S^{r_{a_2}} T (ST)^{b_2} \cdots S^{r_{a_q}} T (ST)^{b_0+b_q} S \\ &= S^{r_{a_1}-2} (ST)^{b_1+1} S \cdot S^{r_{a_2}-2} (ST)^{b_2+1} S \cdots S^{r_{a_q}-2} (ST)^{b_0+b_q+1} S \\ &= V_1^{r_{a_1}-2} V_{b_1+2} V_1^{r_{a_2}-2} V_{b_2+2} \cdots V_1^{r_{a_q}-2} V_{b_0+b_q+2}, \end{aligned} \tag{3.1}$$

so $M$ is conjugate to $W$. We note that $2 \le b_k + 2 \le p-1$ for $1 \le k \le q-1$ and $2 \le b_0 + b_q + 2 \le p-1$, so $W$ is a product of the $V_j$, $1 \le j \le p-1$.

Next we suppose that $M \in G_p$ is parabolic. If $M$ fixes $\infty$, and if $M$ is primitive, then $M = S$ or $M = S^{-1}$. But $S = V_1$ and $S^{-1}$ is conjugate to $V_{p-1} = TS^{-1}T$, so in either case $M$ is conjugate to one of the $V_j$.

If the fixed point of a parabolic $M$ is a finite number $\beta$, then $\beta$ has a periodic $\lambda$-CF expansion of the form [13, Lemma 3]

$$\beta = [r_0; r_1, \dots, r_n, \overline{2, \underbrace{1, \dots, 1}_{p-3}}].$$

If $M$ is primitive we have that $M$ is conjugate to $P = S^2T(ST)^{p-3}$, and we calculate that $P$ is conjugate to $W = S^{-1}PS = (ST)^{p-2}S = V_{p-1}$, so $M$ is conjugate to $W$.

If a non-elliptic element $M$ is not primitive, then $M = R^k$ where $k$ is a positive integer and $R \in G_p$ is primitive. Then $R$ is conjugate to a product of the $V_j$ as above, which implies that $M$ is also conjugate to a product of the $V_j$, which is $k$ times as long.

The uniqueness follows from the fact that $G_p$ is the free product of $\langle U\rangle$ and $\langle T\rangle$. This implies that the product is unique up to cyclic permutation. [6, Theorem 1.4] ■

We are ready to formally define the idea of "block length" in this context.

***Definition 3.2*** Fix $p \geq 3$ and put $\lambda = \lambda_p$. If $W = V_{j_1}V_{j_2}\cdots V_{j_t}$ where $V_j = U^{j-1}S$ for $1 \leq j \leq p-1$ we say that $W$ has *block length* $t$.

We need to know several things about the entries of the $V_j$ for $1 \leq j \leq p-1$. We will state an alternative version of Lemma 10 in [13], and give our own proof.

***Lemma 3.3*** *Fix $p \geq 3$ and put $\lambda = \lambda_p$. Let $V_j = U^{j-1}S$ and put $a_j = \frac{\sin(j\pi/p)}{\sin(\pi/p)}$ for $j \in \mathbb{Z}$. Then for $j \in \mathbb{Z}$ we have*

(i) $a_j = \lambda a_{j-1} - a_{j-2}$,
(ii) $V_j = \left(\begin{smallmatrix} a_j & a_{j+1} \\ a_{j-1} & a_j \end{smallmatrix}\right)$,
(iii) $a_1 = a_{p-1} = 1$, *and*
(iv) $a_j \geq \lambda$ *for* $2 \leq j \leq p-2$.

**Proof** We use the well-known fact [7, 12] that powers of $U$ satisfy

$$U^j = \begin{pmatrix} a_{j+1} & -a_j \\ a_j & -a_{j-1} \end{pmatrix}, \tag{3.2}$$

for $j \in \mathbb{Z}^+$. In fact, it is easy to show that (3.2) holds for all integers $j$. Then

$$\begin{aligned} U^j &= U^{j-1}U \\ &= \begin{pmatrix} a_j & -a_{j-1} \\ a_{j-1} & -a_{j-2} \end{pmatrix} \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a_j\lambda - a_{j-1} & -a_j \\ a_{j-1}\lambda - a_{j-2} & -a_{j-1} \end{pmatrix}, \end{aligned}$$

so $a_j = \lambda a_{j-1} - a_{j-2}$, which is (i).

For (ii) we calculate that

$$\begin{aligned} V_j &= U^{j-1}S \\ &= \begin{pmatrix} a_j & -a_{j-1} \\ a_{j-1} & -a_{j-2} \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \end{aligned}$$

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$$= \begin{pmatrix} a_j & a_j\lambda - a_{j-1} \\ a_{j-1} & a_{j-1}\lambda - a_{j-2} \end{pmatrix}$$
$$= \begin{pmatrix} a_j & a_{j+1} \\ a_{j-1} & a_j \end{pmatrix}.$$

We note that (iii) is immediate. For (iv) we first calculate that

$$a_2 = \frac{\sin(2\pi/p)}{\sin(\pi/p)} = 2\cos(\pi/p) = \lambda.$$

Then the restriction $2 \le j \le p-2$ implies that $2\pi/p \le j\pi/p \le (p-2)\pi/p$, so $\sin(2\pi/p) = \sin\big((p-2)\pi/p\big) \le \sin(j\pi/p)$, which gives us $\frac{\sin(2\pi/p)}{\sin(\pi/p)} \le \frac{\sin(j\pi/p)}{\sin(\pi/p)}$, or $\lambda \le a_j$. ∎

It is convenient to observe that for every $p \ge 3$ we have

$$V_1 = S = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix},$$
$$V_2 = US = \begin{pmatrix} \lambda & \lambda^2 - 1 \\ 1 & \lambda \end{pmatrix},$$
$$\vdots$$
$$V_{p-2} = U^{p-3}S = \begin{pmatrix} \lambda & 1 \\ \lambda^2 - 1 & \lambda \end{pmatrix},$$
$$V_{p-1} = U^{p-2}S = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}.$$

For $1 \le j \le p-1$ the entries of $V_j$ are non-negative, with 0 occurring only in the lower left entry of $V_1$ and in the upper right entry of $V_{p-1}$.

We also need several facts about the entries of products of the $V_j$ for $1 \le j \le p-1$.

***Lemma 3.4*** *Fix $p \ge 3$, put $\lambda = \lambda_p$, and let $V_j = U^{j-1}S$ for $1 \le j \le p-1$. Let $W$ be a product of the $V_j$. Then*

(i) *$W$ has non-negative entries, with 0 occurring only in the lower left entry if $W = V_1^n$ or in the upper right entry if $W = V_{p-1}^n$, and*

(ii) *every nonzero entry of $W$ is greater than or equal to 1.*

**Proof** Our proof is by induction on the block length of $W$. For the basis step, if $W$ has block length 1, then $W = V_j$ for $1 \le j \le p-1$ and (i) and (ii) are both true by Lemma 3.3.

For the induction step we suppose that (i) and (ii) are true for all products of the $V_j$ of block length $k$ for $k \in \mathbb{Z}^+$. Suppose that $W$ has block length $k+1$, so $W =$

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$W_k V_j$ where $W_k$ has block length $k$ and satisfies (i) and (ii). We write $W_k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and calculate that

$$\begin{aligned} W &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_j & a_{j+1} \\ a_{j-1} & a_j \end{pmatrix} \\ &= \begin{pmatrix} aa_j + ba_{j-1} & aa_{j+1} + ba_j \\ ca_j + da_{j-1} & ca_{j+1} + da_j \end{pmatrix}. \end{aligned}$$

If $W_k$ is not $V_1^k$ or $V_{p-1}^k$, then $a, b, c, d \geq 1$, and every entry of $W$ is also nonzero and greater than or equal to 1.

If $W_k = V_1^k = \left(\begin{smallmatrix} 1 & k\lambda \\ 0 & 1 \end{smallmatrix}\right)$, then

$$W = \begin{pmatrix} a_j + k\lambda a_{j-1} & a_{j+1} + k\lambda a_j \\ a_{j-1} & a_j \end{pmatrix}.$$

All of these entries are positive unless $j = 1$; in this case $a_{j-1} = a_0 = 0$ and $W = V_1^{k+1}$.

If $W_k = V_{p-1}^k = \left(\begin{smallmatrix} 1 & 0 \\ k\lambda & 1 \end{smallmatrix}\right)$, then

$$W = \begin{pmatrix} a_j & a_{j+1} \\ k\lambda a_j + a_{j-1} & k\lambda a_{j+1} + a_j \end{pmatrix}.$$

All of these entries are positive unless $j = p - 1$; in this case $a_{j+1} = a_p = 0$ and $W = V_{p-1}^{k+1}$. ■

We will also use Lemma 11 from [13], which we state as the following.

***Lemma 3.5*** (Schmidt & Sheingorn) *Fix $p > 3$, put $\lambda = \lambda_p$, and let $h = [p/2]$. Put $V_j = U^{j-1}S$ for $1 \leq j \leq p-1$. Then we have*

(i) $\mathrm{Tr}(V_j) > \mathrm{Tr}(V_{j-1})$ *for $2 \leq j \leq h$, and*
(ii) $\mathrm{Tr}(V_j) = \mathrm{Tr}(V_{p-j})$ *for $1 \leq j \leq h$.*

Lemmas 3.3 and 3.5 together mean that $V_1$ and $V_{p-1}$ are parabolic, while $V_j$ is hyperbolic for $1 < j < p - 1$.

The following theorem generalizes Fine's key lemma [3] to the context of Hecke groups.

***Theorem 3.6*** *Fix $p \geq 3$, put $\lambda = \lambda_p$, and let $V_j = U^{j-1}S$ for $1 \leq j \leq p-1$. Suppose that $W \in G(\lambda)$ is a product of the $V_j$ of block length $n$, and that $W$ is not $V_1^n$ or $V_{p-1}^n$. Then $\mathrm{Tr}(W) \geq n\lambda$.*

**Proof** Our proof is by induction on the block length of $W$. For the basis step, if $W$ has block length 1, then $W = V_j$ for $2 \leq j \leq p-2$, so

$$\mathrm{Tr}(W) = 2a_j \geq 2\lambda \geq \lambda,$$

by Lemma 3.3.

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For the induction step we suppose that the trace is at least $k\lambda$ for all products of the $V_j$ (other than $V_1^k$ or $V_{p-1}^k$) of block length $k$ for $k \in \mathbb{Z}^+$. Suppose that $W$ has block length $k+1$, so $W = W_k V_j$ where $W_k$ has block length $k$. We write $W_k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and calculate that

$$W = \begin{pmatrix} aa_j + ba_{j-1} & aa_{j+1} + ba_j \\ ca_j + da_{j-1} & ca_{j+1} + da_j \end{pmatrix},$$

as in the proof of Lemma 3.4. We consider several cases.

Case 1: Suppose that $W_k \neq V_1^k$ and $W_k \neq V_{p-1}^k$. We use the facts that $a_j \geq 1$ from Lemma 3.3 and $b, c \geq 1$ from Lemma 3.4 to calculate that

$$\begin{aligned} \operatorname{Tr}(W) &= aa_j + ba_{j-1} + ca_{j+1} + da_j \\ &\geq a + d + a_{j-1} + a_{j+1} \\ &\geq k\lambda + a_{j-1} + a_{j+1}, \end{aligned}$$

by the induction hypothesis. But $a_{j-1} + a_{j+1} \geq \lambda$ for $1 \leq j \leq p-1$ by part (iv) of Lemma 3.3, so $\operatorname{Tr}(W) \geq (k+1)\lambda$.

Case 2: Suppose that $W_k = V_1^k$. Then $W_k = \left(\begin{smallmatrix} 1 & k\lambda \\ 0 & 1 \end{smallmatrix}\right)$ and $V_j \neq V_1$ (or we would have $W = V_1^{k+1}$). We calculate that $W = \left(\begin{smallmatrix} a_j + k\lambda a_{j-1} & * \\ * & a_j \end{smallmatrix}\right)$, so $\operatorname{Tr}(W) = 2a_j + k\lambda a_{j-1}$. Now $a_j \geq 1$ and $a_{j-1} \geq 1$ for $2 \leq j \leq p-1$, so $\operatorname{Tr}(W) \geq 2 + k\lambda \geq (k+1)\lambda$.

Case 3: Suppose that $W_k = V_{p-1}^k$. Then $W_k = \left(\begin{smallmatrix} 1 & 0 \\ k\lambda & 1 \end{smallmatrix}\right)$ and $V_j \neq V_{p-1}$ (or we would have $W = V_{p-1}^{k+1}$). We calculate that $W = \left(\begin{smallmatrix} a_j & * \\ * & k\lambda a_{j+1} + a_j \end{smallmatrix}\right)$, so $\operatorname{Tr}(W) = 2a_j + k\lambda a_{j+1}$. Now $a_j \geq 1$ and $a_{j+1} \geq 1$ for $1 \leq j \leq p-2$, so $\operatorname{Tr}(W) \geq 2 + k\lambda \geq (k+1)\lambda$.

In every case we have that $\operatorname{Tr}(W) \geq (k+1)\lambda$, so the result follows by induction. ∎

***Corollary 3.7*** *For each $p \geq 3$, $G_p$ has finitely many hyperbolic conjugacy classes of any given trace.*

**Proof** By Lemma 3.1 every conjugacy class of a given trace contains an element that is a product of the matrices $V_j = U^{j-1}S$, $1 \leq j \leq p-1$. Hyperbolic conjugacy classes cannot contain $V_1^n$ or $V_{p-1}^n$, which are parabolic, so by Theorem 3.6 the trace provides a bound on the block length of any product of the $V_j$ in classes of a given trace. There are finitely many such products, so there are a finite number of conjugacy classes of a given trace. ∎

## 4 Application to $\mathbb{Z}[\lambda]$-binary Quadratic Forms

We would like to define and study class numbers of $\mathbb{Z}[\lambda]$-binary quadratic forms. For $\lambda > 1$ the presence of nontrivial units in $\mathbb{Z}[\lambda]$ appears to make the general class number problem intractable. Nevertheless if we restrict our attention to hyperbolic $\lambda$-BQFs we have the following result.

***Corollary 4.1*** *Fix $p \geq 3$ and put $\lambda = \lambda_p$. There are finitely many distinct equivalence classes of hyperbolic $\mathbb{Z}[\lambda]$-binary quadratic forms of any given discriminant. Moreover, if a hyperbolic $\lambda$-BQF of discriminant $D$ corresponds to an element of $G_p$ of block length $n$, then $D \geq n^2\lambda^2 - 4$.*

**Proof** Every hyperbolic equivalence class of $\lambda$-BQFs of discriminant $D$ is associated with a hyperbolic conjugacy class of in $G_p$ of trace $T = \sqrt{D+4}$. By Corollary 3.7 there are finitely many such classes. The bound in Theorem 3.6 gives the bound on the discriminant. ∎

***Definition 4.2*** Fix $p \geq 3$ and put $\lambda = \lambda_p$. The number of distinct hyperbolic equivalence classes of $\mathbb{Z}[\lambda]$-binary quadratic forms with discriminant $D > 0$ is the *$G_p$-class number of $D$*, denoted $h_{p,D}$.

We now describe a procedure for calculating class numbers of hyperbolic $\lambda$-BQFs. Given a discriminant bound $D^*$, we use Theorem 3.6 and Corollary 4.1 to find a unique reduced $\lambda$-BQF for every hyperbolic equivalence class of discriminant $D \leq D^*$. We calculate class numbers by counting these equivalence class representatives.

Given $p$ and $D^*$, we first list all products of generators of block length $n \leq \sqrt{D^*+4}/\lambda$ that have the form (3.1). By the proof of Lemma 3.1, every hyperbolic conjugacy class in $G_p$ contains an element of this form. We discard cyclic permutations of previously listed products because they correspond to other reduced $\lambda$-BQFs in the same equivalence class. We also discard $V_1^m$ and $V_{p-1}^m$ because they are parabolic. For each remaining product we calculate that

$$\begin{aligned} W &= V_1^{m_1} V_{j_1} V_1^{m_2} V_{j_2} \cdots V_1^{m_\ell} V_{j_\ell} \\ &= S^{m_1} V_{j_1} S^{m_2} V_{j_2} \cdots S^{m_\ell} V_{j_\ell} \\ &= S^{m_1} U^{j_1-1} S^{m_2+1} U^{j_2-1} \cdots S^{m_\ell+1} U^{j_\ell-1} S \\ &= S^{m_1+1} T(ST)^{j_1-2} S^{m_2+2} T(ST)^{j_2-2} \cdots S^{m_\ell+2} T(ST)^{j_\ell-2} S. \end{aligned}$$

We recall that $m_k \geq 0$ and $2 \leq j_k \leq p-1$ for $1 \leq k \leq \ell$. Then $W$ is conjugate to

$$M = SWS^{-1} = S^{m_1+2} T(ST)^{j_1-2} S^{m_2+2} T(ST)^{j_2-2} \cdots S^{m_\ell+2} T(ST)^{j_\ell-2},$$

which corresponds to the attracting fixed point

$$\alpha = [\overline{m_1+2; \underbrace{1,1,\ldots,1}_{j_1-2}, m_2+2, \underbrace{1,1,\ldots,1}_{j_2-2}, \ldots, m_\ell+2, \underbrace{1,1,\ldots,1}_{j_\ell-2}}].$$

Now $m_j + 2 \geq 2$ for $1 \leq j \leq \ell$ and $0 \leq j_k - 2 \leq p-3$ for $1 \leq k \leq \ell$, so the $\lambda$-CF for $\alpha$ is admissible.

We use the map $\rho$ to find the corresponding reduced $\lambda$-BQFs; our list contains a single representative $\lambda$-BQF for every hyperbolic equivalence class of discriminant $D \leq D^*$. For each $D$ we count equivalence class representatives of discriminant $D$ to determine the class number $h_{p,D}$.

We include tables with some results for $p = 4, 5$ and $6$. In order to highlight patterns and to save space we have used exponents to indicate certain repetitions in a continued fraction period. For example, $[\overline{2; (2,1)^3}] = [\overline{2; 2, 1, 2, 1, 2, 1}]$.

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- Table 1 includes every $\lambda_4$-BQF that corresponds to a product of block length 3 or less, and every form with discriminant less than 100.
- Table 2 includes every $\lambda_5$-BQF that corresponds to a product of block length 2 or less, and every form with discriminant less than 100.
- Table 3 includes every $\lambda_6$-BQF that corresponds to a product of block length 1, and every form with discriminant less than 100.

| discriminant $D$ | class number $h_{4,D}$ | associated period $[\overline{r_i}]$ | reduced form $[A, B, C]$ | block length $n$ |
|---|---|---|---|---|
| 4 | 1 | $[\overline{2}]$ | $[1, -2\sqrt{2}, 1]$ | 1 |
| 12 | 1 | $[\overline{3, 1}]$ | $[\sqrt{2}, -6, 3\sqrt{2}]$ | 2 |
| 14 | 2 | $[\overline{3}]$ | $[1, -3\sqrt{2}, 1]$ | |
| | | $[\overline{2, 2, 1}]$ | $[3, -7\sqrt{2}, 7]$ | |
| 28 | 2 | $[\overline{4}]$ | $[1, -4\sqrt{2}, 1]$ | 3 |
| | | $[\overline{2, (2, 1)^2}]$ | $[5, -12\sqrt{2}, 13]$ | |
| 32 | 2 | $[\overline{4, 1}]$ | $[\sqrt{2}, -8, 4\sqrt{2}]$ | |
| | | $[\overline{3, 1, 2, 1}]$ | $[2\sqrt{2}, -12, 7\sqrt{2}]$ | |
| 46 | 2 | $[\overline{5}]$ | $[1, -5\sqrt{2}, 1]$ | 4 |
| | | $[\overline{2, (2, 1)^3}]$ | $[7, -17\sqrt{2}, 19]$ | |
| 60 | 2 | $[\overline{5, 1}]$ | $[\sqrt{2}, -10, 5\sqrt{2}]$ | |
| | | $[\overline{3, 1, (2, 1)^2}]$ | $[3\sqrt{2}, -18, 11\sqrt{2}]$ | |
| 68 | 4 | $[\overline{3, 2, 1}]$ | $[3, -10\sqrt{2}, 11]$ | 3 |
| | | $[\overline{3, 1, 2}]$ | $[3, -8\sqrt{2}, 5]$ | |
| | | $[\overline{6}]$ | $[1, -6\sqrt{2}, 1]$ | 5 |
| | | $[\overline{2, (2, 1)^4}]$ | $[9, -22\sqrt{2}, 25]$ | |
| 94 | 2 | $[\overline{7}]$ | $[1, -7\sqrt{2}, 1]$ | 6 |
| | | $[\overline{2, (2, 1)^5}]$ | $[11, -27\sqrt{2}, 31]$ | |
| 96 | 5 | $[\overline{3, 2}]$ | $[2\sqrt{2}, -12, 3\sqrt{2}]$ | 3 |
| | | $[\overline{2, 2, 2, 1}]$ | $[5\sqrt{2}, -24, 12\sqrt{2}]$ | |
| | | $[\overline{4, 1, 2, 1}]$ | $[2\sqrt{2}, -16, 10\sqrt{2}]$ | 4 |
| | | $[\overline{6, 1}]$ | $[\sqrt{2}, -12, 6\sqrt{2}]$ | 5 |
| | | $[\overline{3, 1, (2, 1)^3}]$ | $[4\sqrt{2}, -24, 15\sqrt{2}]$ | |
| $\vdots$ | | | | $\vdots$ |

*Table 1*: Discriminants and class numbers for $\lambda_p$-BQFs, $p = 4$, $\lambda = \sqrt{2}$.

| discriminant $D$ | class number $h_{5,D}$ | associated period $[\overline{r_i}]$ | reduced form $[A, B, C]$ | block length $n$ |
|---|---|---|---|---|
| $4\lambda \approx 6.47$ | 2 | $[\overline{2}]$ | $[1, -2\lambda, 1]$ | 1 |
| | | $[\overline{2, 1}]$ | $[\lambda, -2\lambda - 2, 2\lambda]$ | |
| $7\lambda + 6 \approx 17.33$ | 1 | $[\overline{3, 1, 1}]$ | $[\lambda, -3\lambda - 3, 3\lambda + 2]$ | 2 |
| $9\lambda + 5 \approx 19.56$ | 2 | $[\overline{3}]$ | $[1, -3\lambda, 1]$ | |
| | | $[\overline{2, 1, 2, 1, 1}]$ | $[2\lambda + 1, -7\lambda - 4, 6\lambda + 3]$ | |
| $15\lambda + 6 \approx 30.27$ | 2 | $[\overline{3, 1}]$ | $[\lambda, -3\lambda - 3, 3\lambda]$ | |
| | | $[\overline{2, 2, 1, 1}]$ | $[\lambda + 2, -7\lambda - 3, 5\lambda + 4]$ | |
| $16\lambda + 12 \approx 37.89$ | 2 | $[\overline{4}]$ | $[1, -4\lambda, 1]$ | 3 |
| | | $[\overline{2, 1, (2, 1, 1)^2}]$ | $[3\lambda + 2, -12\lambda - 6, 10\lambda + 6]$ | |
| $20\lambda + 16 \approx 48.36$ | 2 | $[\overline{4, 1, 1}]$ | $[\lambda, -4\lambda - 4, 4\lambda + 3]$ | |
| | | $[\overline{3, 1, 1, 2, 1, 1}]$ | $[2\lambda, -6\lambda - 6, 7\lambda + 4]$ | |
| $25\lambda + 21 \approx 61.45$ | 2 | $[\overline{5}]$ | $[1, -5\lambda, 1]$ | 4 |
| | | $[\overline{2, 1, (2, 1, 1)^3}]$ | $[4\lambda + 3, -17\lambda - 8, 14\lambda + 9]$ | |
| $32\lambda + 16 \approx 67.78$ | 2 | $[\overline{4, 1}]$ | $[\lambda, -4\lambda - 4, 4\lambda]$ | 3 |
| | | $[\overline{2, (2, 1, 1)^2}]$ | $[2\lambda + 3, -12\lambda - 6, 10\lambda + 7]$ | |
| $33\lambda + 21 \approx 74.40$ | 1 | $[\overline{2, 2, 1}]$ | $[2\lambda + 1, -7\lambda - 4, 4\lambda + 3]$ | 2 |
| $36\lambda + 32 \approx 90.25$ | 1 | $[\overline{6}]$ | $[1, -6\lambda, 1]$ | 5 |
| $39\lambda + 30 \approx 93.10$ | 2 | $[\overline{5, 1, 1}]$ | $[\lambda, -5\lambda - 5, 5\lambda + 4]$ | 4 |
| | | $[\overline{3, 1, 1, (2, 1, 1)^2}]$ | $[3\lambda, -9\lambda - 9, 11\lambda + 6]$ | |
| $\vdots$ | | | | $\vdots$ |

*Table 2*: Discriminants and class numbers for $\lambda_p$-BQFs, $p = 5$, $\lambda = \frac{1+\sqrt{5}}{2}$.

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| discriminant $D$ | class number $h_{6,D}$ | associated period $[\overline{r_i}]$ | reduced form $[A, B, C]$ | block length $n$ |
|---|---|---|---|---|
| 8 | 2 | $[\overline{2}]$ | $[1, -2\sqrt{3}, 1]$ | 1 |
| | | $[\overline{2, 1, 1}]$ | $[2, -4\sqrt{3}, 5]$ | |
| 12 | 1 | $[\overline{2, 1}]$ | $[\sqrt{3}, -6, 2\sqrt{3}]$ | |
| 21 | 1 | $[\overline{3, 1, 1, 1}]$ | $[\sqrt{3}, -9, 5\sqrt{3}]$ | 2 |
| 23 | 2 | $[\overline{3}]$ | $[1, -3\sqrt{3}, 1]$ | |
| | | $[\overline{2, 1, 1, 2, 1, 1, 1}]$ | $[5, -11\sqrt{3}, 17]$ | |
| 44 | 4 | $[\overline{3, 1, 1}]$ | $[2, -6\sqrt{3}, 8]$ | |
| | | $[\overline{2, 2, 1, 1, 1}]$ | $[4, -10\sqrt{3}, 16]$ | |
| | | $[\overline{4}]$ | $[1, -4\sqrt{3}, 1]$ | 3 |
| | | $[\overline{2, 1, 1, (2, 1, 1, 1)^2}]$ | $[8, -18\sqrt{3}, 29]$ | |
| 45 | 2 | $[\overline{3, 1}]$ | $[\sqrt{3}, -9, 3\sqrt{3}]$ | 2 |
| | | $[\overline{2, 1, 2, 1, 1, 1}]$ | $[3\sqrt{3}, -21, 11\sqrt{3}]$ | |
| 60 | 2 | $[\overline{4, 1, 1, 1}]$ | $[\sqrt{3}, -12, 7\sqrt{3}]$ | 3 |
| | | $[\overline{3, 1, 1, 1, 2, 1, 1, 1}]$ | $[2\sqrt{3}, -18, 11\sqrt{3}]$ | |
| 71 | 2 | $[\overline{5}]$ | $[1, -5\sqrt{3}, 1]$ | 4 |
| | | $[\overline{2, 1, 1, (2, 1, 1, 1)^3}]$ | $[11, -25\sqrt{3}, 41]$ | |
| 96 | 2 | $[\overline{4, 1}]$ | $[\sqrt{3}, -12, 4\sqrt{3}]$ | 3 |
| | | $[\overline{2, 1, (2, 1, 1, 1)^2}]$ | $[5\sqrt{3}, -36, 20\sqrt{3}]$ | |
| $\vdots$ | | | | $\vdots$ |

*Table 3*: Discriminants and class numbers for $\lambda_p$-BQFs, $p = 6$, $\lambda = \sqrt{3}$.

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